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Exploring the Foundations and Evolution of Mathematical Thought

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Truth shall prevail despite all attempts at suppression.
No subject loses more than mathematics
by any attempt to dissociate it from its history.
James Whitbread Lee Glaisher (1848–1928, England)
Dedicated to my teachers who taught me how to stand in the mathematical world:
Swami Dayal Nigam (1924–2009, India)
Vangipuram Lakshmikantham (1924–2012, India-USA)
Günther Hämmerlin (1928–1997, Germany)
Roberto Conti (1923–2006, Italy)

Foreword

This is an impressive book that provides a comprehensive treatment of a plethora of mathematical ideas and results from different civilizations and cultures before the time of Pythagoras to the present. Since numbers have held the attention of humans from the dawn of civilization, the history of mathematics is intertwined with the history of civilization. And among the numerous mathematical luminaries in history, Pythagoras occupies a lofty position as one of the most influential thinkers—both in terms of his mathematics and his philosophy. Thus Professor Agarwal has done a great service by writing this book focusing on the intellectual contributions of Pythagoras, and providing a global and historical perspective by discussing the mathematical developments before his time, and touching upon a variety of significant problems that have engaged, and continue to engage, many of the most gifted scientific minds of later generations up to the present.

Pythagoras was both a mathematician and a philosopher. He had a number of students and followers (the Pythagoreans), and his teachings influenced the development of mathematics and philosophy throughout the Mediterranean region for several centuries. Prof. Agarwal provides in Chap. 1, a detailed account of the fascinating life, work, and teachings of Pythagoras, describing also what the Greek philosopher-mathematician learned from other cultures during his travels.

The name of Pythagoras is most famously associated with *The Pythagoras Theorem* which states that for a right-angled triangle, the square on the hypotenuse is the sum of the squares on the other two sides. Pythagoras neither discovered nor proved this theorem, but he and his pupils were interested in right-angled triangles with positive integer sides. The Pythagoras theorem is central to all developments in geometry including the study of the lengths of curves, since distance between two points in two and higher dimensional spaces is determined to using this fact for right-angled triangles. In Chap. 5, Prof. Agarwal provides a detailed and thorough account of the history of the Pythagoras theorem, and gives five different proofs of it. And with the Pythagorean equation with integer solutions as a starting point, Prof.

Agarwal analyzes in Chap. 6 more general Diophantine equations including Fermat's assertion whose resolution after 300 years is one of the crowning achievements of twentieth-century mathematics.

The Pythagoreans also studied sequences of integers associated with geometrical figures, such as triangular numbers, squares, pentagonal numbers, and so on, and more generally, figurate numbers. Chapter 7 is a detailed treatment of such figurate numbers and certain number theoretic questions involving figurate numbers. The proof of the irrationality of $\sqrt{2}$ is attributed to Pythagoras in classic number theory textbooks such as that by Hardy and Wright. Regardless of who first proved the irrationality of $\sqrt{2}$, it was this realization that began the theory of irrational numbers, a subject that remains an active area of research to this day. In general, it is very difficult to confirm the irrationality of a given number. The irrationality and the transcendence of π were established only in the nineteenth century, and that finally settled in the negative one of the three problems of Greek antiquity, namely to construct using only a ruler and compass, a square equal in area to a given circle. The final Chap. 8 of the book is an account of some major developments in the study of irrational and transcendental numbers.

In summary, this book has a lot to offer—mathematically, historically, and even philosophically. It is written in a style that would appeal to lay persons, yet has a substantial amount on the history of various mathematical developments that will be useful even for researchers. We should be thankful that Prof. Agarwal, a very prolific and reputed researcher in the field of differential equations, has spent so much time in writing this book in the area of mathematical history, which experts and non-experts will definitely enjoy.

Professor Krishnaswami Alladi

Preface

Eric Temple Bell (1883–1960, USA) in his treatise [61] *The Magic of Numbers* remarked, “If one man more than another is to be credited with starting the mathematical and physical sciences on their course from antiquity to the present it is Pythagoras. And if western civilization means the technology and commerce of recurrent industrial revolutions detonated by the application of experiment and mathematics to the physical word, Pythagoras was its prime mover. All this is on the strictly scientific side. On the side of purely intellectual activity, the numerology (number mysticism) of Pythagoras and his Brotherhood is the source of essential germinal ideas in metaphysics of the sciences of Plato of Athens (around 427-347 BC, Greece).” According to Aristotle (around 384–322 BC, Greece) “The so-called Pythagoreans, who were the first to take up mathematics, not only advanced this subject, but saturated with it, they fancied that the principles of mathematics were the principles of all things.” Bertrand Arthur William Russell (1872–1970, England-USA) in [440] *A History of Western Philosophy* contends that the influence of Pythagoras on Plato and others was so great that he should be considered the most influential philosopher of all time. He concludes that “I do not know of any other man who has been as influential as he was in the school of thought.”

Besides philosophy, the following two attributes are due to Pythagoras: The explicit recognition that proof by deductive reasoning offers a foundation for the structures of number and form (in the sense we still know and follow it), and the daring conjecture that nature can be understood by human beings through mathematics, and that mathematics is the language most adequate for idealizing the complexity of nature into appropriable simplicity. Once deductive mathematics was accepted as real mathematics all the saints and sages (ancient philosophers, who by study, experiments, concentration of minds, and perhaps intuition [unreliable source of knowledge], arrived at the fixation of certain laws governing life) previous work was condemned or called trivial applied to practical problems such as land surveying, commerce, and counting.

To glorify Pythagoras achievements, and as a whole of Greeks, on all prior mathematical works several damaging remarks have been written, for example, according to Henry James Sumner Maine (1822–1888, Scotland-France), “Except the blind forces of nature, nothing moves in this world which is not Greek in its origin”; Walter William Rouse Ball (1850–1925, England) [53] wrote “Oriental mathematics may be an interesting curiosity, but Greek mathematics is the real thing”; “The Hindoos, like the Chinese, have pretended that they are the most ancient people on the face of the earth, and that to them all sciences owe their creation. But it is probably that these pretensions have no foundation; and in fact no science or useful art (except a rather fantastic architecture and sculpture) can be definitely traced back to the inhabitants of the Indian peninsula prior to the Aryan invasion”; Godfrey Harold Hardy (1877–1947, England) recorded “The Greeks are not clever schoolboys or scholarship candidates, but fellows at another college”; John Edensor Littlewood (1885–1977, England) noted “Compared with what the Greeks achieved, the mathematics of Egypt and Babylonia is the scrawling of children just learning to write, as opposed to great literature. These civilizations barely recognized mathematics as a distinct discipline, so that for over a period of 4000 years hardly any progress was made in the subject”; Anthropologist Ralph Linton (1893–1953, USA) stated hypothetically that “... if Albert Einstein (1879-1955, Germany-USA) had been born into a primitive tribe which was unable to count beyond three, life–long application to mathematics probably would not have carried him beyond the development of a decimal system based on fingers and toes”; in 1967, Árpád Szabó (1913–2001, Hungary) writes “Before the development of Greek culture the concept of deductive science was unknown to the Eastern people of antiquity. In the mathematical documents which have come down to us from these peoples, there are no theorems or demonstrations, and the fundamental concepts of deduction, definition, and axiom have not yet been formed. These fundamental concepts made their first appearance only with the Greek mathematics.” Most importantly, he proclaimed that if one means by a proof any explanatory note that serves to convince and to enlighten, then one finds an abundance of proofs in ancient mathematical texts other than those of the Greeks (it also means that mathematics was existing before

Greeks). Further, “We must not forget that what constitutes ‘proof’ varies from culture to culture, as well as from age to age.” In addition, Bell in [61] writes “A proof that convinces the greatest mathematicians of one generation may be glaringly fallacious or incomplete to a schoolboy of a later generation.”

We also note that the history of most mathematical subjects often traces a long way back in the timeline, where a process of slow evolution and introduction of new ideas led to some major discovery, which affects the foundations of modern mathematics. In fact, most of the times, ideas existed in the past, and were even applied in problem solving; however, it took a long time period to generalize the theory and systematically prove these ideas. Thus, mathematics has never been a single person process. For example, we will see that Isaac Newton (1642–1727, England) used simple interpolation in 1665 to generalize millennia-old binomial expansion, which was proved by Niels Henrik Abel (1802–1829, Norway) only in 1826; several results of Leonhard Euler (1707–1783, Switzerland) are based on simple (sometimes tedious) calculations, which were proved several years later; Srinivasa Ramanujan (1887–1920, India) compiled nearly 3900 results (mostly identities and equations) during his short lifetime, a small number of these results were actually false and some were already known, in recent years most of his claims have now been proven correct. George Gheverghese Joseph (born 1928, India) in his book [289] of 1991 focused mainly on the achievements of Kerala (India) in astronomy and mathematics and the transmission of mathematics from India to Europe. In his book “Beyond Numeracy” of 1992, John Allen Paulos (born 1945, USA) tells this story: “A German merchant of the fifteenth century asked an eminent professor where he should send his son for a good business education. The professor responded that German universities would be sufficient to teach the boy addition and subtraction, but he would have to go to Italy to learn multiplication and division. Before you smile indulgently, try multiplying or even just adding the Roman numerals CCLXIV, MDCCCIX, DCL, and MLXXXI without first translating them.” While Paulos provides no source for this story, there seems to be some truth as the whole of Europe was waking up from the dark ages between the fifth and fourteenth centuries.

In 2005, Vangipuram Lakshmikantham (1924–2012, India-USA), Srinivasa Leela (born 1936, India-USA), and Jonnalagadda Vasundhara Devi (born 1964, India) in [329] focused on the origin of the mathematics and corrected the chronology which was distorted by Western historians of mathematics. They have specially reported several pre-Pythagoras accomplishments of Indians for which historians have credited to Pythagoras and other Europeans. Their work was further extended and explained by Agarwal and Sen in 2014, see [14]. This book also lights on the very humanity of almost 400 mathematicians, their mode of thought, and struggle in their achievement. David Gray (USA) in his article *Indic Mathematics: India and the Scientific Revolution* of 2011 writes "The study of mathematics in the West has long been characterized by a certain ethnocentric bias, a bias which most often manifests not in explicit racism, but in a tendency toward undermining or eliding the real contributions made by non-Western civilizations. The debt owed by the West to other civilizations, and to India in particular, go back to the earliest epoch of the 'Western' scientific tradition, the age of the classical Greeks, and continued up until the dawn of the modern era, the renaissance, when Europe was awakening from its dark ages." He concludes by asserting that "the role played by India in the development (of the scientific revolution in Europe) is no mere footnote, easily and inconsequentially swept under the rug of Eurocentric bias. To do so is to distort history, and to deny India one of its greatest contributions to world civilization."

In Chap. 1, we present a comprehensive study of Pythagoras, Pythagoreanism, and the early Pythagoreans through an analysis of the many representations of the individual and his followers, allowing them to complement and critique each other. This includes major events and struggles in his life since birth till death, details of his philosophy (The Golden Verses and Symbols), and dramatic mathematical and astronomical achievements which made him immortal. We shall also report the origin of most of his accomplishments with supporting statements of distinguished scholars.

In mathematics one of the major contributions of Pythagoras is to give *divine significance* to most of the natural numbers, and an attempt to find mathematical explanations for everything in the Universe in terms of numbers (natural and rational) including in geometry. In Chap. 2, we begin with natural numbers whose very origin is a mystery; however, it is generally perceived that they have in some philosophical sense a natural/divine existence independent of man. This is followed by the number sense which is intuitive understanding of the natural numbers, their magnitude, their patterns and relationships, and how they are affected by the basic operations (addition, subtraction, multiplication, and division). We shall exhibit that number sense is not only the natural ability of primitive man and children, but also there are recorded incidences of birds, animals, insects, and aquatic creatures who show through their behavior a rudimentary number sense, namely, comparing/sorting. Next, we shall provide the origin of negative numbers, and Brahmagupta's (born 30 BC, India) treatment of positive and negative numbers in terms of "fortunes" (dhana) and "debts" (rina), also his rules for dealing with negative numbers (very similar to those we still use today). We shall convince the reader that only through continuous effort and struggle from the middle of the nineteenth century negative numbers received their relevance logically across the world.

This is followed by the origin of zero to whom the status of a number was given by Hindus. Its discovery took place within an environment that was at once mystical, philosophical, religious, cosmological, mythological, and metaphysical. Brahmagupta defined zero as the result of the subtraction of a number by itself, and laid down the basic rules; however, he struggled when it came to division by zero. In fact, it took several centuries to realize that mathematically 0/0 is neither meaningful nor meaningless, it is indeterminate, and it may have any value but only in the limiting sense. Most importantly the number zero led to the decimal system. From the thirteenth century, when calculations could be performed "in writing," slowly the importance of zero and the place-value system was recognized all over the world, and prominent mathematicians and philosophers started to understand their importance and making constructive comments.

In Sect. 2.7, we shall mainly preset several examples from physics, mathematics, games, and puzzles where large numbers appear in a very natural process. Large numbers will appear in later chapters routinely. One of the major struggles in mathematics has been to accept that infinity is a legitimate concept. In Sect. 2.8, we shall begin with Hindu mythology according to which zero is also a term Ananta, which means infinite (infinite void or void infinite), and in Hindu philosophy God is infinite and within us. The infinite remains the same, even though the infinite Universe which has no beginning or end has come out of it, for details see Lakshmikantham [330]. For general reading see the exceptional book [439] of Rudy Rucker (born 1946, USA). We shall discuss Jainas classification of numbers into three groups enumerable, innumerable, and infinite (nearly infinite, truly infinite, and infinitely infinite). We shall carefully define and illustrate potential and actual infinity. We shall show that from the beginning Greek philosophers and mathematicians refused to accept or confused with the concept of infinity and this continued till eighteenth century. In fact, during this period several prominent mathematicians perpetrated all sorts of blunders, made false proofs and drew incorrect conclusions. Finally, Georg Ferdinand Ludwig Philipp Cantor (1845–1918, Russia-Germany) during 1871–84 systematically mathematized the concept of infinity. His classification of countable and uncountable sets became a turning point in whole of mathematics. In this section we shall also introduce infinitely small numbers or infinitesimals, which eventually led to the discovery of calculus.

Section 2.9 deals with number mysticism, which is based on the idealistic belief that numbers are not only symbols of reality, but the final substance of real things, and possess spiritual and magical powers. While the origin of number mysticism is unknown, but it is believed that it started along with the birth of natural numbers. For Pythagoras only first ten numbers were of spiritual significance (some claim first 50) and some human attribute. We shall discuss a special geometric arrangement of the numbers ten, which Pythagoreans called Tetraktys and recognized it as fate, the Universe, the heaven, and even God, and honored it by never gathering in groups larger than ten. We shall also discuss about numerology, which is an offshoot of number mysticism and to this day persists in otherwise unaccountable omens and superstitions in most of the religions. In Sect. 2.10, we have collected several numbers which have some special properties. This includes palindromic numbers, and magic squares which have been considered strong talismans against evil, and possession of a magic square was thought to insure health and wealth. Finally, in Sect. 2.11, we have introduced complex numbers. This includes their origin, basic rules, representations, Euler's most curious formula, and roots of unity.

To make this book accessible to wider audiences, in Chap. 3, some basic questions which are vaguely discussed in existing books have been clearly explained and embellished through interesting examples from several diverse fields. These questions will also pave the way to appreciate the later chapters. To summarize, we shall show that despite of numerous attempts from primordial to modern leading philosophers and mathematicians, the word mathematics is too subtle to define exactly; however, a mathematics teacher and a mathematician can be differentiated and defined assuredly. We shall reveal that history of mathematics deepens our respect for human cultures and collaboration across time regardless of their location, and presents us with role models. We shall also exhibit the human nature of mathematicians who are very often believed to be bizarre individuals. We shall detail basic prerequisites for the deductive mathematics such as a mathematical statement and a mathematical definition. We shall rigorously define axioms and list them for geometry, natural numbers, fields, and sets. We shall establish that occasionally eliminating or changing an axiom from the earlier assumed axioms has led to altogether new mathematics, which is as consistent as earlier, and often more useful.

Then we shall define only that segment of logic that is necessary in mathematics. This prepares us to define the terms theorem/result/proposition, lemma, and corollary, which are the heart of whole mathematics. Even an obvious proposition in mathematics without its proof is meaningless so we shall carefully study the term mathematical proof. Then we shall discuss several widely used methods to prove theorems and illustrate each with elementary, but of paramount interest, examples. In mathematics there are many innocent looking problems for which classical mathematical proofs are not within the reach of humans. For one of such problems, namely, four color theorem, a major breakthrough came in 1976 with the assistance of electronic computer. Since then such proofs have been added in the vocabulary of mathematics as computer-based proofs, and have been successfully applied to several unsolved problems. This has meticulously filled the gap between mathematicians and computer scientists. However, among mathematicians there is a disagreement whether to accept computer-based proofs 100%. Certainly, such proofs provide guidance in understanding the problem better, but loses the flavor of classical mathematics.

An example that disproves a mathematical statement (shows that it is false) is called a counterexample. It is beyond doubt that often the construction of a counterexample is challenging. We shall provide a few simple examples to clear up this important concept in mathematics. Next we shall take up one of the most demanding questions in mathematics "can proofs be exact." We shall conclude that today's proof of a theorem is never permanent, within a few years (sometimes several years) it is modified/simplified/generalized, and later (often) you as well as your proof is being criticized. Contemplating this in mind, we shall mention several proofs that are excessively long for which mathematicians are searching for shorter proofs. A mathematical statement that has not yet been rigorously proved is called a conjecture. We shall cite and explain several conjectures, some of which are challenging from the last several years. A statement for which different valid logical arguments lead to different conclusions (namely true and false) is called a paradox. We shall discuss several paradoxes, some of which are entertaining. We shall also discuss in detail four paradoxes of Zeno of Elea (around 495–435 BC, Greece) which require the acceptance of infinity. While deciding of bad, good, and beautiful mathematics is individualistic, several mathematicians/philosophers have tried to response conclusively. We have tried to recognize the difference between bad, good, and beautiful mathematics through simple examples. In the last Sect. 3.20, we shall take up mainly three classical problems of antiquity. We shall show that Euclidean tools are not enough to solve these problems. The most important aspect of these problems is that the failure of solving these problems has led to substantial amount of new and deeper mathematics.

In Chap. 4, we shall study subsets of natural numbers. We shall begin with the sets of prime and composite numbers whose union is the set of natural numbers. In Sect. 4.2, we shall discuss Eratosthenes of Rhodes' (around 276–194 BC, Greece) method known as Sieve of Eratosthenes which is apparently the first methodical attempt to separate the primes from the set of natural numbers; Ramanujan highly composite numbers; Square spirals of Stanislaw Marcin Ulam (1909–1984, Poland-USA) and his co-workers; Two jewels in number theory proved by Euclid of Alexandria (around 325–265 BC, Egypt-Greece), namely, Fundamental Theorem of Arithmetic which ensures every integer $n \geq 2$ is either prime or can be expressed as a product of primes (thus prime numbers are the "atoms" of the natural numbers), and Infinity of Prime Numbers (which makes their study fascinating); Theorem of Peter Gustav Lejeune Dirichlet (1805–1859, France) which ensures every arithmetic sequence $a + nd,\ n = 1, 2, \cdots$ in which *a* and *d* are relatively prime (no common factors other than 1) contains an infinitude of primes; Present status of Joseph Louis François Bertrand's (1822–1900, France) assertion that between any number and its double there exists at least one prime; and palindromic primes.

In Sect. 4.3, we shall provide easily verifiable divisible tests by certain integers, especially for all primes up to 50, which help in confirming for a given number of reasonable size to be composite. In Sect. 4.4, we shall examine Pére Marin Mersenne (1588–1648, France) numbers and primes denoted as $M_n = 2^n - 1,\ n \geq 1$. We shall affirm that $M_{82589933}$ is the largest known prime. It is not known whether there exist infinitely many Mersenne primes, if every Mersenne number is square free, and if there are infinitely many composite Mersenne numbers. An integer $n \geq 2$ is said to be perfect (the nomenclature is due to Pythagoras) if it is equal to the sum of its proper divisors (excluding itself and including 1). In Sect. 4.5, we shall prove Euclid's result which provides the construction of all even perfect numbers, and its stronger version due to Euler. The largest known even perfect number is $2^{82,589,932}(2^{82,589,933} - 1)$.

In 1640, the father of modern number theory, Pierre de Fermat (1601–1665, France), also known as *the prince of amateurs and mischievous genius* (see Michael Sean Mahoney, 1939–2008, USA [356,357]), conjectured that Fermat numbers $F_n = 2^{2^n} + 1,\ n \geq 0$ without exception are prime. In Sect. 4.6, we shall follow Euler to show that $F_5 = 641 \times 6700417$ and hence composite. In fact, no other Fermat primes F_n with $n > 4$ have been found. In Sect. 4.7, we shall provide the proof of Fermat's Little Theorem: If *p* is prime and *a* any positive integer, then *p* divides $a^p - a$. We shall also show that the converse of this result does not hold. This innocent looking result turned out to be fundamental for the progress of number theory. A desire of every number theorist is to find a function $f(n)$ that yields only prime numbers, and the sequence of primes so obtained is infinite. Some known attempts which are only of theoretical importance have been discussed in Sect. 4.8. John Wilson's (1741–1793, England) Theorem states: If *n* is a prime, then the quantity $((n-1)!+1)/n$ is a whole number. Joseph Louis Lagrange (1736–1813, Italy-France) not only completed John Wilson's result: *n* is prime iff (both necessary and sufficient) *n* divides $(n-1)!+1,$ but also proved it; however, his proof uses complicated arguments. In Sect. 4.9, we shall give an elementary proof of the complete result, and because of occurrence of $n!$ in the result we conclude that this result is also only of theoretical interest.

In number theory Christian Goldbach's (1690–1764, Prussia-Russia) Conjecture: Every even $n>2$ is the sum of two, not necessarily distinct, primes, and is widely known for its simplicity in stating and complexities in proving. In Sect. 4.10, we shall summarize the efforts made in settling Goldbach's conjecture. Primes of the form *p* and $p+2$ are called twin primes. For these primes the famous conjecture is: There are infinitely many twin primes. In Sect. 4.11, we shall provide the present status of this conjecture. In Sect. 4.12, we shall consider one of the most important function in number theory, namely, $\pi(x)$, which represents the number of primes less than or equal to a given number $x.$ Karl Friedrich Gauss (1777–1855, Germany) conjectured that $\pi(x)$ is asymptotically equal to the ratio $x/\ln x.$ His conjecture now known as the Prime Number Theorem was independently proved by Jacques Salomon Hadamard (1865–1963, France) and Charles de la Vallée Poussin (1866–1962, Belgium). Since then, several proofs of prime number theorem have been offered, some of these we shall summarize. A pair of integers in which each is the sum of the divisors of the other is called an Amicable Pair, or the Friendly Pair. In Sect. 4.13, we shall discuss Thabit ibn Qurra's (826–901, Turkey-Iraq) general formula which leads to certain types of amicable pairs, and its generalization due to Euler. Unfortunately, their results require the primality of three numbers in advance. Although more than $1,227,319,870$ amicable pairs are known, theoretically it is not known if the number of amicable pairs is finite or infinite.

In Sects. 4.14 and 4.15, we shall respectively discuss Fibonacci (Leonardo of Pisa, around 1170–1250, Italy) and François Édouard Anatole Lucas (1842–1891, France) numbers. For these numbers we shall provide recurrence relations, explicit solutions, identities, and generating functions. We shall notice that Fibonacci numbers occur in nature in many surprising ways. It has been conjectured that there are infinitely many Fibonacci as well as Lucas primes. In Sect. 4.16, we shall provide the construction of Golden Section/Ratio (also known as Divine Proportion) $\varphi = (1+\sqrt{5})/2,$ and show its connection with Fibonacci and Lucas numbers. The number φ is found in nature, art, architecture, poetry, music, and of course mathematics. Psychologists have shown that the golden ratio subconsciously affects many of our choices, such as where to sit as we enter a large auditorium, where to stand on a stage when we address an audience, and so on.

The main aim of Sect. 4.17 is to discuss Gauss Law of Quadratic Reciprocity, which he called the gem of arithmetic, and remained fascinated by it throughout his life. In fact, out of 246 known proofs of this law 8 belongs to Gauss. In Sect. 4.18, we shall prove that there are infinite number of primes of the form $4n-1$ and $4n+1;$ any number of the form $4n+3$ cannot be expressed as a sum a^2+b^2 of two perfect squares; and Fermat's Two Square Theorem: if *n* is a prime number, then it can be expressed as a unique (except the order) sum of two squares iff either $n=2$ or $n=4k+1.$ Fermat's this result is cited in any discussion of mathematical beauty. In Sect. 4.19, we shall sate and partially prove Adrien-Marie Legendre's (1752–1833, France) Three-Square Theorem: An integer *n* can be represented as the sum of three squares of integers, i.e., $n=a^2+b^2+c^2$ iff *n* is not of the form $n=4^h(8k+7)$ for nonnegative integers *h* and $k.$ In this result the representation is not necessarily unique. In Sect. 4.20, we shall prove Lagrange's Four-Square Theorem: Every positive integer can be written as the sum of four integer squares. In this result, the representation is also not necessarily unique.

Keeping in mind that the converse of Fermat's Little Theorem does not hold, a composite number *n* is called Carmichael Number (after

Robert Daniel Carmichael, 1879–1967, USA) provided *n* divides $b^n - b$ for all integers $b.$ In Sect. 4.21, we shall provide a characterization of Carmichael numbers. In Sect. 4.22, we shall discuss the importance of the numbers 714 and 715, and the new mathematics that has emerged from these numbers. In Sect. 4.23, we shall discuss Bell Primes, Marie-Sophie Germain (1776–1831, France) Primes, Balanced Primes, Ferdinand Gotthold Max Eisenstein (1823–1852, Germany) Real Primes, Primorial Primes, Fortunate Numbers, Good Primes, Denis Arthur Higgs (1932–2011, England), and Ramanujan Primes, which are special subsets of prime numbers that have been studied with great interest. In Sect. 4.24, we shall conclude this chapter by answering the necessity to find next larger prime number. It is interesting to note that a few prime numbers were known almost 22,000 years back; Hindus had adequate knowledge of prime, perfect, and amicable numbers, much before the days of Pythagoreans; and Fibonacci numbers were known to Hindus by the name matrameru during 500 BC.

An ever fresh result in geometry is Pythagoras (or Pythagorean) Theorem: If *a* and *b* are the lengths of the two legs of a right triangle, and *c* is the length of the hypotenuse, then the sum of the areas of the two squares on the legs equals the area of the square on the hypotenuse, i.e., $a^2 + b^2 = c^2.$ This equation has been ranked very high among all mathematical equations, and appreciated throughout the history for its simplicity and variety of applications. In Chap. 5, we shall provide its origin which is at least 5200 years old. For Pythagorean theorem almost 500 different proofs are known; out of these we shall provide five which are elementary and have historical importance. Among these we include a proof owing to President James Abram Garfield (1831–1881). We shall also furnish the converse of Pythagorean theorem. Then we shall detail five important generalizations of Pythagorean theorem which were contributed by Hippocrates of Chios (around 470 BC, Greece), Alexandrian Claudius Ptolemaeus (Ptolemy, around 90–168, Egypt-Greece), Pappus of Alexandria (around 290–350, Egypt, was either Greek or a Hellenized Egyptian), ibn Qurra, and the Law of Cosines which first appeared in Euclid’s Book II (Propositions 12 and 13) and explicitly stated by Jemshid al-Kashi (around 1380–1429, Persia). Next, we shall generalize Pythagorean theorem in vector spaces, and show how it encompasses for rectangular solids. We shall also prove three abstract results which are due to Jean Paul de Gua de Malves (1713–1785, France), D.R. Conant (USA) and W.A. Beyer (USA), and Eisso Atzema (USA). Finally, we shall discuss Pythagorean theorem in non-Euclidean geometry. Specifically, we shall present spherical law of cosine which was recorded in the first book on Astronomy *Surya Siddhanta*, hyperbolic law of cosine which was first known to Franz Adolph Taurinus (1794–1874, Germany), Pythagorean theorem in Riemannian geometry which was first given by George Friedrich Bernhard Riemann (1826–1866, Germany) in his doctoral address in 1854, and give reason why Pythagorean theorem fails in Elliptic geometry. We shall conclude this chapter with 11 historical problems and an example that requires Pythagorean theorem.

A set of three positive integers $a, b,$ and *c* which satisfies Pythagorean relation $a^2 + b^2 = c^2$ is called *Pythagorean triple* and written as an ordered triple $(a, b, c).$ A triangle whose sides form a Pythagorean triple is called a Pythagorean triangle, which is clearly a right triangle. A Pythagorean triple (a, b, c) is said to be *primitive* if a, b, c have no common divisor other than $1.$ In Chap. 6, we shall make a systematic investigation of primitive Pythagorean triples. In Sect. 6.2, we shall show that Hindus, Babylonians, Egyptians, and Chinese were having ample knowledge of Pythagorean triples several centuries before Pythagoras. In Sect. 6.3, we shall provide Euclid's proposition which gives the characterization of all primitive Pythagorean triples. This proposition was later proved by several mathematicians; we shall break the proof in six parts and give complete details. In this section we shall also furnish a table of primitive Pythagorean triples with $c \leq 1000.$ In Sect. 6.4, for the primitive Pythagorean triples we shall provide 36 elementary results which can be considered as the modern beginning of the number theory. For example, we shall show that in a primitive Pythagorean triple (a, b, c) either *a* or *b* is divisible by $3,$ either *a* or *b* is divisible by $4,$ and either $a, b,$ or *c* is divisible by $5,$ and hence the product *ab* is divisible by $12,$ and the product *abc* is divisible by $60.$ As an another example, we shall show that perimeter of a primitive Pythagorean triangle and its area are the same only for the Pythagorean triple $(5, 12, 13).$

In Sect. 6.5, we shall provide triples ensuring the construction of right-angled triangles whose sides are rational numbers. For this, we shall assume that a rational side or rational hypotenuse is given in advance. A Heronian triangle (a, b, c) has integer sides whose area is also an integer. Clearly, every Pythagorean triple is a Heronian triple, and hence there are infinitely many primitive Heronian triples; however, the converse is not true. In Sect. 6.6, for a given Heronian triangle we shall provide Brahmagupta's proportional condition which the triple (a, b, c) must satisfy, and for a given triple (a, b, c) sufficient conditions so that it is a Heronian triangle. A congruent number is a positive integer that is equal to the area of a rational right triangle. In Sect. 6.7, we shall list first ten congruent numbers and provide the simplest rational right triangle for the congruent number $157.$ So far, to decide if a given positive integer is congruent remains an open number-theoretic problem.

Fermat's claim of 1637 that the equation $a^n + b^n = c^n$ has no positive integer solutions for $a, b,$ and *c* if $n > 2$ is known as Fermat's Last Theorem. In Sect. 6.8, we shall record the continuous struggle of several outstanding mathematicians for 350 years to prove this result, until Andrew John Wiles (born 1953, England) resolved it in 1994. For this, he employed known theories from many branches of mathematics; his original 200-page-long proof (it would be 1000 pages if all details are provided) was published in 1995 after condensing it to 129 pages. Apparently only very few people understand Andrew Wiles's proof, and the world is waiting for a simpler proof. A tuple of four integers a, b, c and *d* such that $a^2 + b^2 + c^2 = d^2$ is called Pythagorean quadruple, and (a, b, c, d) is called primitive if the greatest common divisor of its numbers is $1.$ In Sect. 6.9, we shall provide a few characterizations for the construction of Pythagorean quadruple. In Sect. 6.10, we shall report several identities which not only generalize Pythagorean quadruple but also parameterizes the sum of three cubes into a cube, i.e., of the form $x^3 + y^3 + z^3 = c^3.$ In an effort to generalize Fermat's Last Theorem, in 1769, Euler conjectured that $x_1^k + x_2^k + \cdots + x_n^k = c^k$ implies $n \geq k.$ From Sect. 6.10 it follows that Euler's conjecture holds for $k = 3.$ In Sect. 6.11, we shall provide counterexamples to show that his conjecture is not true for $k = 4$ and $k = 5.$ For $k \geq 6$ the validity of the conjecture is unknown. We shall also provide several examples for $4 \leq k \leq 8$ which support Euler's conjecture. Finally, in Sect. 6.12, we shall discuss Eugéne Charles Catalan (1814–1894, Belgium-France) and Subbayya Sivasankaranarayana Pillai (1901–1950, India) conjectures. Catalan conjecture confirms that the only solution in natural numbers of the equation $x^a - y^b = 1$ for $a, b > 1,\ x, y > 0$ is $x = 3, a = 2,\ y = 2, b = 3.$ Pillai's conjecture (which is a generalization of Catalan's conjecture) says for fixed positive integers A, B, C the equation $Ax^n - By^m = C$ has only finitely many solutions (x, y, m, n) with $(m, n) \neq (2, 2).$ So far for the Pillai's conjecture the number of solutions has been calculated only for some particular cases.

Figurative numbers are numbers that can be represented in a geometric pattern, usually by dots/pebbles arranged in various regular and discrete patterns. It has been accepted that Pythagoreans were the first to study triangular and square figurative numbers. Nicomachus of Gerasa (around 60–120, Syria-Greece) in his book *Introduction to Arithmetic* (see [394]) of around 100 AD collected earlier works of Pythagoreans on natural numbers, and presented cubic figurative numbers (solid numbers). Since then, the study of figurative numbers continues to be a source of interest and motivation to both amateur and professional mathematicians. In Chap. 7, we shall study 34 different types of figurative numbers, starting with triangular numbers. For each type of figurative number, we shall provide: recurrence relation (which leads to an infinite sequence), the general term, various equalities, numerous properties, explicit relation with other numbers, necessary condition for a given number to be a figurative number, generating function, sum of first *n* and inverse of all terms of the sequence, and some possible applications. We shall also provide sums of first *n* positive integers with positive integer exponents, and some bounds when the exponents are positive fractions. Fermat in 1638 claimed that every positive integer is expressible as at most $k,$ *k*-gonal numbers (Fermat's Polygonal Number Theorem). His theorem was fully resolved in 1813 by Augustin-Louis Cauchy (1789–1857, France). A difficult triangular case (every positive integer is the sum of three or fewer triangular numbers) was disposed of by Gauss in 1796. In the literature, Gauss result is known as EGPHKA theorem, and he wrote it as EGPHKA! num $= \triangle + \triangle + \triangle$.

One of the greatest discoveries in the whole of mathematics is the invention of irrational numbers, and then their understanding. In Chap. 8, we shall demonstrate that Vedic Ascetics more than 5000 years back were unsuccessful in finding exact values of the numbers $\sqrt{2}$ and π. The ancient records (supported by great philosophers, mathematicians, and historians) stipulate that Vedic Ascetics were also definite that these numbers are incommensurable/irrational. We shall exhibit that the claim of the historians of mathematics that Pythagoras proved the irrationality of $\sqrt{2}$ is only conjectural. In fact, the first geometric proof of the irrationality of $\sqrt{2}$ appeared only in Meno (Socratic dialogue by Plato) almost two hundred years after Pythagoras. Since then several different proofs of the irrationality of $\sqrt{2}$ and in general for $\sqrt{N}$ for any natural number *N* which is not a perfect square have been given. We have provided some of these important proofs. The next major understanding of irrational numbers came from the scholars of the Islamic Middle East toward the end of the first millennium CE. They started treating irrational numbers as algebraic objects, and most importantly provided a geometric interpretation of rational numbers on a horizontal straight line. Since then research continues for the known as well as unknown/expected irrational numbers, their subset of transcendental numbers, and their computation to trillions of decimal places, we have detailed some of these advancements. We have also discussed Dedekind-Cantor axiom of the nineteenth century which provides geometric interpretation of all real numbers, and thus completes the Islamic work. Particularly, for the number π we have arranged individual's contributions chronologically to show that each continent of the world has contributed in this fascinating field of mathematics. We have also provided very simple proofs of the irrationality of *e* and e^2, and transcendence of *e* and π.

We conclude this book with the note that mathematically interesting sequences of numbers are those that continue without end. If the primes were finite, they would be of considerably less interest; and if it is established ultimately that the perfect numbers are finite, their interest will become merely historical. Odd and even numbers, the primes and composite numbers, the squares, the cubes, the curious pentagonal numbers, algebraic numbers, irrational numbers, transcendental numbers, all are infinite. These infinite sequences of numbers among the infinite sequences of the natural numbers first suggested the revolutionary idea which is cornerstone of the modern theory of the infinite.

We hope in future readers of this book will justify (at least remember) the statement of Archimedes of Syracuse (287–212 BC, Greece) “the man who first states a *theorem* (poses a problem) deserves as much credit as the man who first proves it.” The present mathematical knowledge has only reached its present high level through the labors of numerous centuries for which one cannot underestimate the influence of every culture, personality, philosophy, region, religion, society, and social status. Of course, the focus of mathematical scholarship has shifted from place to place throughout history.

The main purpose of this book is to create interest among students and teachers at all levels, and hopefully its content should be accessible even to non-mathematicians. In the book we have combined history, philosophy, religion, mathematics, and elementary computation. Only at few places we have used sophisticated mathematical terms, which readers can easily skip without any lack of consistency. We have completely avoided tedious proofs, but illustrated the importance of the results with simple examples. To make this collection stimulating, we have included amusing anecdotes, puzzles, and historical problems. Our book requires a certain degree of intellectual maturity and a willingness to do some thinking on one’s own.

A book of this nature cannot be written without deriving many valuable ideas from several sources. We express our indebtedness to all authors, too numerous to acknowledge individually, from whose specialized knowledge we have been benefitted. We have also immensely benefitted from several websites, especially *en.wikipedia.org* and *www-history.mcs.st-andrews.ac.uk*. Our sincere thanks to Number Theorists Heng Huat Chan (born 1967, Singapore), Carl Bernard Pomerance (born 1944, USA), and Stephen George Simpson (born 1946, USA) for clarifying doubts during the process of writing this book over the period of more than three years.

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1. Life and Teaching of Pythagoras

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1.1 Introduction

Pythagoras is one of the most unexplained personalities in history. He is among those individuals given the status of becoming a myth/omnipotent in his own lifetime. Since he followed typical oriental tradition (the knowledge was passed from one generation to the next mainly by word to mouth), whatever little we know about him is from the imaginations and a great many humorous and anecdotal fables thrown in by the historians who wrote (frequently contradict one another) and painted his picture hundreds of years after him, which continues even today. He has been called as mystic philosopher, master among masters, blend of genius and madness, mysterious, semidivine sage, divinity, God-like figure, whereas some have shown doubt that such a person ever existed. He is considered to be a remarkably significant figure in the advancement of mathematics, science, pre-Socratic philosophy (the study of the fundamental nature of knowledge, love of wisdom, reality, and existence, the word *philosophia* is due to him, and by this he meant the one who is attempting to find out), metaphysics (the term was named by a first century CE editor who assembled various small selections of works of Aristotle, it deals with the first principles of things, including abstract concepts such as being, knowing, substance, cause, identity, time, and space), metempsychosis (the transmigration/reincarnation at death of the Soul/Atma/Self of a human being or animal into a new body of the same or a different species), musica universalis (a perfectly harmonious music produced by the movement of celestial bodies but to be inaudible on the Earth), ethics, and politics, even though we know comparatively little about his own achievements (according to custom his followers referred all discoveries back in the name of their revered master).

Although a lot of these, as has been discovered in recent years, was already known several centuries before him, the Pythagorean legacy lasted well over more than two and a half millenniums and continues to provoke thoughts among scientists including mathematicians even today, starting from high school students. His philosophy appeared suddenly and unexpectedly in Einstein's formulation of the general theory of relativity. Today, Pythagoras is revered as a prophet by the Ahl-al-Tawhid or Druze faith along with philosopher Plato. Plato (meaning broad) is a nickname, his real name was Aristocles, and he died at a wedding feast. Alfred North Whitehead (1861–1947, England) argued that "In a sense, Plato and Pythagoras stand nearer to modern physical science than does Aristotle. The two formers were mathematicians, whereas Aristotle was the son of a doctor (student of Plato and teacher of Alexander the Great, 356–323, BC)." In this chapter we shall discuss various stages of Pythagoras's life starting from his birth to death, and those topics his name historians have associated willfully. Philosophy: Golden Verses (seventy-one lines), which are Human and Divine Virtues whose aim is to make a Good Men who becomes one of the gods; thirty-nine symbols that set high divinity, moral, and discipline; also, reincarnation and transmigration of the soul. Mathematics: numbers rule the universe; even and odd numbers; arithmetic (Foundation of all mathematics, pure or applied. It is the most useful of all sciences, and there is, probably, no other branch of human knowledge that is more widely spread among the masses. In Book VII of his master work *The Republic*, Plato advises: "We must endeavor to persuade those who are to be the principal men of the state to go and learn arithmetic."), geometry, and harmonic means; figurative numbers; consonant intervals; relations between mathematics and music; Pythagorean theorem; Pythagorean triples; incommensurability; regular polygon; and regular polyhedra (Platonic solid). Astronomy: Kosmos; twelve concentric spheres; Philia.

1.2 Life of Pythagoras

Pythagoras was born in around 582 BC (there is no certainty regarding the exact year) on the island of Samos (it was a thriving cultural hub known for its feats of advanced architectural engineering, including the building of the Tunnel of Eupalinos, and for its riotous festival culture; it was also a major center of trade in the Aegean Sea, south of Chios, north of Patmos and the Dodecanese, and off the coast of Asia Minor, where traders brought goods from the Near East), Samos is also the birthplace of the astronomer and mathematician Aristarchus (around 310–230 BC, Greece), who said that the Sun, and not the Earth, was the center of our Universe. Aristarchus is also known for the trigonometric inequality that states that if α and β are acute angles (i.e., between 0 and a right angle) and $\beta < \alpha$, then

$$\frac{\sin\alpha}{\sin\beta} < \frac{\alpha}{\beta} < \frac{\tan\alpha}{\tan\beta}.$$

In 1955, the town of Tigani on the island of Samos is renamed as Pythagoria. A statue of Pythagoras is built at the town's harbor. Pythagoras busts are in the Capitoline Museums, Rome, and in the Vatican Museums, Vatican City, showing him as a "tired-looking older man." Pythagoras was born to Pythais (his mother, a native of Samos), and Mnesarchus or Mnesarch (his father, an opulent merchant who came from the Phoenician city of Tyre; Diogenes Laërtius (around 180–240, Greece) in his book [326] published in 2013 mentions that he was a gem cutter or engraver). There is a story that Mnesarchus brought corn to Samos at a time of famine, and he was granted citizenship of Samos as a mark of gratitude. Iamblichus of Chalcis (around 245–325, Syria) for whom Emperor Julian (331—363, Constantinople) wrote "that he was posterior indeed in time, but not in genius, to Plato," tells the story that the Pythia (name of one of the highest priestesses of the Temple of Apollo at Delphi who also served as its oracle, also known as the Oracle of Delphi) prophesied that his pregnant mother would give birth to a male child supremely beautiful, wise, and of benefit to humankind. We are also told that he was born of the virgin Parthenis and fathered by the God Apollo, the Hellenic God of music, prophecy, and learning (blessed with a superhuman ancestry) for spreading Apollonian wisdom to the people.

When Pythagoras was born, people, in general, were superstitious and had strong beliefs in spirits and supernatural forces/occurrences. Religious cults (specific system of worship) were popular in that era. Only little is known of Pythagoras childhood, and in fact, all descriptions of his physical appearance such as he was six feet tall, his body was as perfectly formed as that of Apollo, in his presence everyone felt humble and afraid, are likely to be made up except that he had a striking birthmark on his thigh (some say he had golden thigh, a proof of his divinity). It is anticipated that he had two brothers, although some sources say that he had three. As a child, Pythagoras spent his early years in Samos but travelled widely with his father. There are accounts of Mnesarchus returning to Tyre with Pythagoras and that he was taught there by the Chaldeans (Semitic people of Chaldea who seized Babylon from the Assyrians in the seventh century BC, giving rise to the Neo-Babylonian or Chaldean dynasty 625–539 BC) and the learned men of Syria. It seems that he also visited Italy with his father.

Pythagoras parents wanted that their son got the best possible education. They arranged for the child's instruction in Lyre (kithara) playing, gymnastics, and painting. During his childhood as well as boyhood days, his mother and father both had immense influence on his highly receptive mind. They provided the most vital building block of his multifaceted character. Pythagoras learnt a lot in science and philosophy from Pherecydes of Syros (around 600–520 BC, Greece), his first teacher. Pherecydes authored a book on cosmogony (the branch of science that deals with the origin of the Universe, especially the solar system), derived from three divine principles, Zas, Cthonie and Chronos, known as the "Pentemychos." It formed a bridge between the mythological thought of Hesiod (around 750–650 BC, Greek) and pre-Socratic philosophy. Pythagoras remained in touch with him until Pherecydes death. At the age of 18, shortly after the death of his father, Pythagoras went to Lesbos, a Greek island located in the northeastern Aegean Sea.

In Lesbos, Pythagoras worked and learnt from (Sophos) Thales of Miletus (around 625–545 BC, Greece). According to Proclus Diadochus (410–485, Greece), the last major Greek philosopher, Thales was the first to go to Egypt in search of wisdom and knowledge. Thales is considered Greek's earliest naturalist, first who used demonstrative and deductive reasoning of Greek geometry, astronomer, philosopher, engineer, and one of the "seven wise men of antiquity" (others are: Pittacus of Mytilene (around 640–568 BC), Bias of Priene (fl. in sixth century BC), Solon of Athens (around 638–558 BC), Cleobulus of Lindos (fl. in 600 BC), Periander of Corinth (around 634–585 BC), and Chilon of Sparta (fl. around 555 BC). In mathematics his special case of the inscribed angle theorem is well known. He also stated that for any two equiangular triangles, the ratio of any two corresponding sides is always the same. Aristotle, known as the Father of Western Philosophy, records Thales teaching (perceived as a valuable gift and not as a hard duty) that, "all things are full of gods," and another ancient source attributes to him the statement, "the mind of the world is God and the whole is imbued with soul and full of sprits."

Thales is remembered by several apposite but probably apocryphal anecdotes. He astounded the Egyptians by calculating the height of a pyramid using proportionate right-angled triangles; diverted the river Halys, the frontier between Lydia and Persia, to enable a Lydian army under Croesus to cross; taught that a year contained about 365 days (Sun to complete its orbit about the Earth), and not twelve months of thirty days; believed that the Earth is a disc-like body floating on water; and correctly forecasted a plentiful olive crop one year after several bad crops, bought all olive presses around Miletus, and made a huge profit by renting them. He is believed to have predicted that a total solar eclipse would occur during May of 585 BC; however, it was not until long after the sixth century that Chaldean astronomers could give reasonably accurate predictions of eclipses of the Moon, though they were never able to predict that an eclipse of the Sun would be seen in a particular region. Thales was not the kindest of people. It is said that once when he was transporting some salt, which was loaded on mules, one of the animals slipped in a stream, dampened its load, and so caused some of the salt to dissolve. Finding its burden thus lightened it rolled over at the next ford to which it came. To break it of this trick, Thales loaded it with rags and sponges that absorbed the water, made the load heavier, and soon effectually cured it of its troublesome habit. Thales never married. When Solon asked why, Thales arranged a cruel ruse whereby a messenger brought Solon news of his son's death. According to Plutarch of Chaeronea (around 46–120, Greece), Solon then "…began to beat his head and to do say all that is usual with men in transports of grief." But Thales took his hand, and with a smile said, "These things, Solon, keep me from marriage and rearing children, which are too great for even your constancy to support, however, be not concerned at the report, for it is a fiction." Another favorite tale Plato told was that one night when Thales, while walking and stargazing, fell into a ditch, whereupon a pretty Thracian girl mocked him for trying to learn about the heavens, while he could not see what was lying at his feet. The following theory of Thales' is silly: He believed that this globe of lands is sustained by water and is carried along like a boat, and on the occasions when the Earth is said to quake, it is fluctuating because of the movement of the water. It is no wonder, therefore, that there is abundant water for making the rivers flow since the entire world is floating.

Thales introduced skepticism and criticism to Greek philosophy, which separates the Greek thinkers from those of earlier civilizations. His philosophy is called monism—the belief that everything is one. When Thales was asked what is most difficult, he said, "To know thyself." Asked what is most easy, he replied, "To give advice," and when asked what was the strangest thing he had ever seen, he answered "An aged tyrant." Thales founded the Ionian School, which continued to flourish until about 400 BC. Anaximander of Miletus (around 610–547 BC, Greece) and Anaximenes of Miletus (around 585–525 BC, Greece) were his meritorious students. The importance of the Ionian School for philosophy and the philosophy of science is without dispute. The most influential lessen Pythagoras learned from Thales was the value of time in the pursuit of knowledge. Iamblichus informs us that Pythagoras refrained from wine and meat for the rest of his life. He used to eat only moderate portions of simple foods, his periods of sleep were short, his mind sharp, his spirit pure, and the health of his body excellent. Next, Pythagoras studied with Anaximander, who was the first of the Greeks to explain the origins of man, and one of the earliest thinkers of exact sciences, and interested in geometry and cosmology (the science of the origin and development of the Universe). Anaximander believed that from fish evolved man, and gods were born, but the time is long between their birth and their death. Anaximander produced the first map of the known, inhabited world, and taught that heavenly bodies travel along concentric celestial spheres. Both the personalities had profound impact on the thinking process about the visible and invisible worlds of young Pythagoras.

According to Iamblichus, on the recommendation of Thales, who himself visited Egypt to learn mathematics, Pythagoras came to acquire knowledge mainly with the priests at Memphis who were renowned for their wisdom, at the age of 23, and stayed in Egypt until he was 44. There is some evidence that Pythagoras went to Egypt with a letter of introduction written by the tyrant Polycrates who was then controlling the city of Samos. Polycrates had an alliance with Egypt, and there were therefore strong links between Samos and Egypt at that time. During these 21 years, he visited many temples and took part in many discussions with the priests (according to Porphyry of Tyre, 233–305, Lebanon, Pythagoras was refused admission to all the temples except the one at Diospolis (i.e., Thebes) where he was accepted into the priesthood after completing the rites necessary for admission, the first foreigner ever to do so). The writer Antiphon of Rhamnos (around 480–411 BC, Greece) claimed in his lost work *On Men of Outstanding Merit* that Pythagoras learned to speak Egyptian from the Pharaoh Amasis II himself. In Egypt he accumulated and absorbed mathematics, medicine, herbalism (practice of the medicinal and therapeutic use of plants) and was instructed in the stages of the evolution of the soul. He was introduced to the Egyptian sciences of architecture (mainly system of measuring) and music and admitted into the most secret mystery rituals. For his rest of the life he also adopted the secrecy of the Egyptian priests, refusal to eat beans (Aristotle relates that abstention from beans is advised either because they resemble privy parts, or because they are like the gates of Hades, or because they are destructive, or because they are like the nature of the Universe, or, finally, because they are oligarchical, being used in the choice of rulers by lot; whereas others have added beans generate belching, indigestion, and bad dreams, or because a bean has the shape of a human head, or dead friend's souls were inside the beans) and animal flesh, refusal to wear cloths made from animal skins, and striving for purity. From Egyptian priests, he also advanced his knowledge of geometry that he was already acquainted with the teachings of Thales and Anaximander (geometry was systematically studied by Egyptian priests because the periodical inundations of the Nile River obliterated property lines). According to a Christian theologian Clement of Alexandria (150–215, Greece), Pythagoras was a disciple of Soches, an Egyptian arch prophet.

When Pythagoras was 44 years old, Cambyses II, the king of Persia, invaded Egypt. Polycrates abandoned his alliance with Egypt and sent 40 ships to join the Persian fleet against the Egyptians. After Cambyses had won the Battle of Pelusium in the Nile Delta and had captured Heliopolis and Memphis, Egyptian resistance collapsed. The historian Herodotus of Halicarnassus (around 484–425 BC, Greece) portrays Cambyses as a mad king who committed many acts of sacrilege during his stay in Egypt, including the slaying of the sacred Apis calf. Pythagoras was taken prisoner and taken to Babylon (the largest and most sophisticated city in that part of the world in those days). Soon after, Polycrates was killed and Cambyses died either by committing suicide or as the result of an accident. Iamblichus writes that Pythagoras "…was transported by the followers of Cambyses as a prisoner of war. Whilst he was there he gladly associated with the Magoi (priests in Zoroastrianism and the earlier religions of the western Iranians) …and was instructed in their sacred rites and learnt about a very mystical worship of the gods. He also reached the acme of perfection in arithmetic and music and the other mathematical sciences (cluster of intertwined subjects) taught by the Babylonians…." From magi he also mastered most advanced astronomy (in astronomy we observe, record our observations, reduce them whenever possible to numerical statements, and frame hypotheses to correlate what we observe). In Babylonia, he learned "the motion of God is circular, God's body is like light, and his soul is like truth" (however, according to Zhmud [548] stories of his travels to Egypt and other lands are probably spurious).

Pythagoras returned to Samos when he was about 55. There he confided to a friend: "I am a different man, I am reborn. Through this purification, my center of being has changed. Now truth is not a concept to me, but a life." In Samos the people welcomed him warmly and possessed by his nobility and passion. They were very curious to see him after hearing marvelous deeds (mesmerism) that he had subdued a razing huge bear by simply "the power of his voice" and "influence of touch" at the beast and commanding it in the name of Apollo to abstain, and converse with rivers (once the spirit of a river arose a voice, “Pythagoras, I greet thee”). In Samos Pythagoras started an organization which he called the semicircle. Iamblichus records that “…he formed a school in the city [of Samos], the ‘semicircle’ of Pythagoras, which is known by that name even today, in which the Samians hold political meetings. They do this because they think one should discuss questions about goodness, justice and expediency in this place which was founded by the man who made all these subjects his business. Outside the city he made a cave the private site of his own philosophical teaching, spending most of the night and daytime there and doing research into the uses of mathematics….” Supposedly, the school became so renowned that the brightest minds in all of Greece came to Samos to hear Pythagoras teach. During that period, Darius of Persia (around 550–486 BC) was in control of Samos, but this conflicts with the writings of Porphyry and Laërtius who claimed that Polycrates was still in control of Samos when Pythagoras returned there.

The political affairs of Pythagoras native city were not conducive to philosophy, so in less than two years he moved to Croton (now Crotone, on the heal of southern Italy, then the wealthiest and most learned city of Magna Graecia inhabited largely by Greeks). Iamblichus gives two reasons for him leaving. First Pythagoras tried to use his symbolic method of teaching which was similar in all respects to the lessons he had learnt in Egypt. The Samians were not very keen on this method and treated him in a rude and improper manner. Second, he was dragged into all sorts of diplomatic missions by his fellow citizens and forced to participate in public affairs in which he was least interested. Iamblichus further narrates a magical feat, within a few days after Pythagoras arrived at Croton, he saw fishermen pulling in a large haul of fish. Pythagoras told them how many fish they had. When they tallied their catch, they were stunned to learn that he was correct. When Pythagoras ordered the men to return the fish to the sea, they thought it best to comply. As if by a further miracle, not one fish died. He was welcomed eagerly in Croton and served as an advisor to elites and gave them frequent advice. There he started a semi-religious school that was mainly involved in teaching and learning of mathematics (more a humanity than a science), music, philosophy, astronomy, and their relationship with religion. Several commentators have affirmed that Pythagorean School was in the house of Milo of Croton (sixth century

BC, Italy) who was a loyal Pythagorean, and by profession army commander and wrestler who secured many victories in the most important athletic festivals of ancient Greece. Aristotle called Milo "an excessive eater" as he used to eat nine kilograms of meat and the same amount of bread every day, and drank ten liters of wine.

Initially, Pythagoras had a group of as many as 600 talented students of all ranks, male and female (who broke a law which forbade their going to public meetings). Each member of this group had already established personal controls of self-discipline. The Pythagorean Order (also known as putative Pythagoreanism movement) was largely a mystical organization (like the earlier Orphic cult which proposed new doctrine by which to live, preaching temperance, communal living and domination, emphasized transmigration of souls after death into new bodies, and most importantly government by only very few enlightened), in some ways a monastery, rather than a school as we know today. He divided those who attended his school into two main sects akousmatikoi (from akousma, "esoteric teachings") and mathematikoi (from mathematikos, "scientific"), whom we prefer to term as probationers and Pythagoreans (Pythagoras followers, the term coined by Herodotus) [however, Zhmud [548] writes the distinction between mathematikoi and akousmatikoi was a much later fabrication]. Some historians have claimed that Pythagorean School consisted of two groups of disciples: the insiders ("Esoteric Pythagoreans") and the outsiders (the Exoteric, also called Pythagoristae). Dicaearchus of Messana (around 350–285 BC, Greece) one of Aristotle's students wrote that the majority were probationers who had to maintain silence (called echemythia) for five years. Porphyry stated in his *Vita Pythagorae*: "What Pythagoras said to his associates there is no one who can tell for certain, since they observed a quite unusual silence."

Pythagoras's teaching methods were followed by the Hugh of Saint Victor (1096–1141, Duchy of Saxony-France), a leading theologian and writer on mystical theology (the study of the nature of God and religious belief), who in his *The Didascalion* (a comprehensive early encyclopedia, as well as commentaries on the Scriptures and on the Celestial Hierarchy of Pseudo-Dionysius I) writes "for seven years, according to the number of the seven liberal arts (grammar, rhetoric, and logic [the trivium] and geometry, arithmetic, music, and astronomy [the quadrivium, which was long considered to constitute a necessary and sufficient course of study for a liberal education]), no one of his pupils dared ask the reason behind statements made by him; instead, he was to give credence to the words of the master until he had heard him out, and then, having done this, he would be able to come at the reason of those things himself." At no time they were allowed (akousmata: a list of rules laid down) to wear garments made of wool, touch a white rooster, and eat meats (modern scholars doubt that he ever advocated for complete vegetarianism, this term was coined in the West by 1847) or beans. There is a extensively repeated story that a group of Pythagoreans was murdered when they chose to die at the hands of their enemies rather than escape capture by hiding in a field of beans. Another story is that in the town of Tarentum he observed an ox in a pasture feeding on green beans. He advised the herdsman to tell his ox that it would be better if he ate other kinds of food. The herdsman laughed, remarking he did not know the language of oxen, but if Pythagoras did, he was welcome to tell him so himself. Pythagoras approached the ox and whispered into the ear for a long time. The ox never again ate beans and lived to a very old age near the temple of Hera in Tarentum, where he was treated as sacred. If any of his pupil failed to survive their low-protein diet, he could take solace in the belief that he would be reborn again, perhaps in another form.

Pythagoreans had all things in common, holding the same philosophical and political beliefs, engaged in the same pursuits, their food and dress were simple, their discipline severe, their mode of life arranged to encourage self-command, temperance, purity, and obedience, were taught by Pythagoras himself. The beliefs that Pythagoras held were: (1) at its deepest level, reality is mathematical in nature, (2) philosophy can be used for spiritual purification, (3) the soul can rise to union with the divine, (4) both the Universe and man, the macrocosm and microcosm, are constructed on the same harmonic proportions (also known as The Golden Ratio, see Sect. 4.16), Leonardo da Vinci's (1452–1519, Italy) famous drawing of the Vitruvian man of 1490 can be called Pythagorean in its examination of harmony between microcosm and macrocosm), (5) all existing objects were fundamentally composed of form and not of material substance, (6) certain symbols have a mystical significance, in particular, ten-pointed *tetraktys* or tetrad (a triangular figure consisting of ten points arranged in four rows: one, two, three, and four points in each row, which is the geometrical representation of the fourth triangular number, see Sects. 2.9 and 7.2) that symbolizes all the visible and invisible dimensions of creation, (7) all brothers of the order should observe mutual regards, devotion to each other, strict loyalty, and secrecy (not to reveal sensitive information that could be misinterpreted without proper training and might compromise the society), (8) excess brings lust, intoxication, and uncontrolled emotions, which drive men and women into abyss, (9) greed brings envy, theft, and exploitation, (10) all disease is caused by indigestion, (11) remove your shoes before worship, and (12) hard physical work is a slow poison destructive of creative thinking. He also advised that it is best to make love to women in winter, but not in the summer, and only occasionally in autumn and spring.

Pythagoras warned his disciples that when they prayed, they should not pray for themselves; that when they asked things of the gods they should not ask things for themselves, because no man knows what is good for him and it is for this reason undesirable to ask for things which, if obtained, would only prove to be injurious. Pythagoras advocated to them following several advantages of knowledge over other earthly possessions (see Seven Great Bible Verses About Earthly Possessions) such as wealth, fame, beauty, power, and strength: Knowledge benefits not only the individual, but also society; without knowledge, we are unable to enjoy the benefits of other goods; knowledge is not diminished when used or given away; most men and women, by birth or nature, lack the means to advance in wealth and power, but all have the ability to advance in knowledge; unlike the body, which in spite of all our care will decay and die, knowledge lasts throughout life, and for some, brings immortal fame; knowledge always leads men and women to serve others. He adds “the most cleansing of all purifications from the taints of many lives is the pursuit of knowledge for its own sake.”

Relating complete mankind to a gathering at the Olympic Games, Pythagoras proclaimed “Men are of three kinds; the lowest come to the Games to buy and sell; the next higher to compete; the highest come to look on, so it is with life.” For Pythagoras, the dyad (something that consists of two elements or parts) was the source of opposites that brings harmony on the Earth. The following ten dyads of Pythagoras have been preserved by Aristotle in his Metaphysics: Limited and Unlimited, as ultimate principles, or truths; numerical oddness and evenness are equated with Limited and Unlimited; one and plurality (many); right and left; male and female; motionlessness and movement; straight and crooked; light and darkness; good and evil; and square and oblong. It is ambiguous whether an ultimate One, or Monad (a term meaning unit used by Pythagoras to signify a variety of entities from a genus to God) was presented as the cause of the two categories. In ancient Chinese philosophy (before 600 BC) dyads was known as Yin and Yang, which represents the belief that the world arises out of the interplay of opposite forces. Within the human body, traditional Chinese physicians believe, Yin and Yang forces must be balanced. Acupuncture, the insertion of needles into specific spots on the body to relieve pain or disease, a practice developed many centuries ago in China is believed to "negate," or unblock, the pathways in the body in order to restore balance and health.

According to Pythagoras east was more important than the west, the morning than the evening, the beginning than the end, city planners than city builders, and gods were more worthy of honor than demigods, heroes, and men. He advised that one should be mindful of the gods at all times, giving them praise, and granting them credit for what appear to be our own accomplishments. Pythagoras taught that nothing happens by chance or fortune, but everything takes place according to a divine plan. This is depicted by an episode about the Pythagorean, Thymaridas of Tarentium (400–350 BC, Greece) who was sailing from his country. His companions were there for the purpose of bidding him farewell. He was about to embark when one of his companions said to him, "Thymaridas, I pray that the gods grant you all you desire." He responded, "Pray for better things, my friend. Pray that what happens to me may conform to the desires of the gods." Thymaridas knew that whatever the outcome of his journey, it would be wrong to question the wisdom of divine providence. Pythagoras also placed particular emphasis on the importance of physical exercise (he used to stand on one leg), therapeutic dancing, daily morning walks along scenic routes, and athletics. He has been quoted as saying, "No man is free who cannot command himself."

In Croton amongst his most attentive auditors was Theano (around 546 BC, Greece), the young and beautiful daughter of Pythonax of Crete (an Orphic philosopher and physician) who was a great supporter of Pythagoras. According to Porphyry, in spite of the disparity of their ages, Pythagoras married her when he was 60. The couple had three children (some say seven): two sons, Mnesarchus (named after his grandfather) and Telauges (who apparently died while still a young man), and a daughter, Damo (according to Iamblichus, she married to Meno the Crotonian). Pythagoras regarded the bearing and raising of children as sacred responsibilities. He maintained that this was a duty of men and women to the gods, so that their worship might continue into future generations. There is a strong conviction that Theano wrote several texts including a treatise on the principle of the golden mean (she told Hippodamus of Thutrium 498–408 BC, Greece, that her treatise *On Virtue* contained the doctrine of the golden mean) the felicitous middle between the extremes of excess and deficiency, but all are lost. She is also credited for not only inspiring Pythagoras during the years of his life but after his death continued to promulgate his doctrines. She also wrote a biography of her husband, but unfortunately it is also lost.

Pythagoras cult known as the *secret brotherhood* reached its peak in Croton and spread in many southern Italian cities including some parts of the Middle East. The members of brotherhood exclusively and systematically applied deductive reasoning in solving mathematical problems. A distinctive badge of this brotherhood, by which they could recognize each other, was the beautiful star pentagram/Pentalpha (like a star in the flag, and the headquarters building of defence, of the United States)—a fit symbol of the mathematics (see Fig. 1.1). It was also Pythagorean emblem of health. In several cultures pentagon is still used as a symbol of faith. Further, over the period of 2500 years scientists have discovered its numerous mathematical properties. Édouard Schuré (1841–1929, France) in [455], Bell, and others have related the following incident to illustrate the bond of fellowship between Pythagoreans: “One of them who had fallen upon sickness and poverty was kindly taken in by an innkeeper. Before dying he traced a few mysterious signs (the pentagram, no doubt) on the door of the inn and said to the host, ‘Do not be uneasy, one of my brothers will pay my debts.’ A year afterwards, as a stranger was passing by this inn he saw the signs and said to the host, 'I am a Pythagorean; one of my brothers died here; tell me what I owe you on his account."'

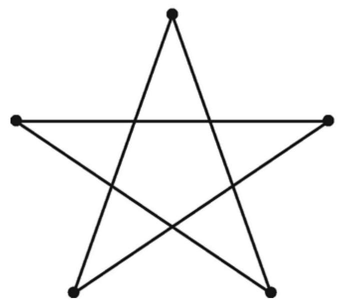

Fig. 1.1 Pentagram

Around 510 BC Croton attacked and defeated several neighboring colonies in southern Italy, including Caulonia, Terina, and Sybaris, and there are some propositions that Pythagoras was involved in these combats. Then, around 509 BC, Cylon a noble from Croton led a revolt against the Pythagoreans. Iamblichus quotes (perhaps due to Porphyry): "Cylon, a Crotoniate and leading citizen by birth, fame and riches, but otherwise a difficult, violent, disturbing and tyrannically disposed man, eagerly desired to participate in the Pythagorean way of life. He approached Pythagoras (around 509 BC), then an old man, but was rejected because of the character defects just described. When this happened Cylon and his friends vowed to make a strong attack on Pythagoras and his followers. They started with false propaganda which disgruntled the minds of the common people against the Pythagoreans. Thus a powerfully aggressive zeal activated Cylon and his followers to persecute the Pythagoreans to the very last man." While Iamblichus himself does not accept this version and argues that the attack by Cylon was a minor affair, it is believed that in the decisive moment of the revolt, a meeting house was set on fire as 40 Pythagoreans were debating inside. Some accounts state that Pythagoras also died in the fire; others have it that he died of grief, sorrowing over how difficult a task it was to elevate humanity. However, the general agreement is that Pythagoras with some of disciples escaped to Tarentum and reestablished themselves at once as a philosophical and mathematical society, and they continued to flourish for another several years, and then they had to move again, this time to Metapontus where he lived four years before he died at the age of 99 in 481 BC.

According to tradition, Pythagoras was buried in Metapontus. After the death of Pythagoras Crotonians repented of losing the great philosopher and master. It was apparently Aristaeus of Croton (6th-5th BC, Greece), a Pythagorean, who according to Iamblichus married his widow Theano, helped in reestablishing the Pythagorean community in Croton, and became its head. The successors to Aristaeus were Mnesarchus (the son of Pythagoras), Boulagoras, Gartydas of Croton, Aresas of Lucania, Diodorus of Aspendus, Philolaus of Croton (around 480–405 BC), and Archytas of Tarentum (around 428–347 BC) who saved Plato when his life was in peril at the court of Dionysius I or Dionysius the Elder (around 432–367 BC, Greece), the tyrant of Syracuse. Philolaus is said to have published a detailed summary of Pythagorean philosophy and science (see Burkert [109] and Huffman [269]), whereas Archytas besides being a mathematician also known for his enlightened attitude in his treatment of slaves and in the education of children (see Huffman [269,270]). Iamblichus provides a list of 235 famous Pythagoreans, seventeen of whom are women. In this list some of the distinguished Pythagoreans who pursued mathematics are Hippasus of Metapontum (around 530–450 BC, Greece) made his discovery of the existence of irrational numbers (cannot be written as the fraction of two integers); Philolaus systematized Pythagoras's number theory; Theodorus of Cyrene (around 431 BC, Libya-Greece) was the mathematical teacher of Plato, he showed how to construct a line segment of length $\sqrt{n}$ (the sign $\sqrt{\ }$ was introduced in 1525 by Christoff Rudolff, 1499–1545, Germany-Austria) for any positive integer n; and Archytas solved the problem of doubling the cube with a geometric construction.

It is strange historians have not mentioned even a single Pythagorean who followed footsteps of Pythagoras in practicing supernaturalism, mysticism, or superstitiousness. The new era of Pythagoreanism initiated by Plato's disciples to claim that the separation of layers of reality and positing a pair of abstract ultimate principles are not the innovations of Plato but belong to the very core of Pythagoreanism. The brotherhood virtually disappeared in the late third century BC. However, his teachings underwent a major revival in the first century BC among Middle Platonists, coinciding with the rise of Neopythagoreanism. In 1917, an underground basilica, known as *Porta*

Maggiore Basilica, was discovered in Rome. It is dated to the first century BC and believed to be the only historical meeting place for the neo-Pythagoreans. Pythagorean teachings as portrayed in Ovid's Metamorphoses (Bk XV:60–142) influenced the modern vegetarian movement. To keep Pythagoras immortal, bibliographers have repeatedly added the following spurious stories:

During a single day, Pythagoras was present in Metapontum in Italy, and in Tauromenium in Sicily, instructing disciples in both places, although these cities are separated by land and sea, by some 200 miles. Some say he was able to travel this distance by means of a golden dart given to him by Abaris the Hyperborean (a legendary sage, healer, and priest of Apollo).

One year, when Pythagoras traveled to Olympia for the athletic games, he met with a group of friends and fell into a discussion of prophecies, omens, and divine signs. He took the position that men of piety continually receive messages from the gods if they but attune themselves to their calling. Flying over his head at that moment was an eagle, who, at his signal, turned, descended, and perched on Pythagoras's arm. After stroking her awhile and continuing his conversation, he released her.

A shepherd reported that he had heard the sound of chanting coming from the tomb of a Pythagorean teacher. A student of the deceased, without hesitation, asked what the song was, not questioning at all the possibility that the dead might sing.

Pythagoras is believed to have had a remarkable wheel by means of which he could predict future events, and to have learned hydromancy from the Egyptians. He believed that brass had oracular powers, because even when everything was perfectly still there was always a rumbling sound in brass bowls.

The historians have misled the world by reporting that Pythagoras was a contemporary of Lord Gautama Buddha "The Enlighten One" (the historical founder of Buddhism), who actually lived during (1887–1807 BC). In fact, Pythagoras lived during the time of Chinese philosopher and writer Lao Tse (Born 601 BC and departed to the West in 531 BC). Several renowned authors and philosophers have strongly suggested that toward the end of Egypt's visit, priests recommended Pythagoras to go to India (Greater Bharat) for mastering mathematics, astronomy, music, mysticism, Yoga, various religious doctrines, and philosophy. He made his way through Persia to India, where he continued his education under the Ajivikas, Jaina, and Brahmin priests (learned men, teachers, interpreters of Hindu life, temple priests, astrologers, gurus, pandits specializing in sacred law and Vedic exegesis, indispensable at every wedding and funeral, and occupying the most exalted niche in the Indian caste system) of the university of ancient Taxila (founded before tenth century BC) which was located in the city of Taxila (modern day Pakistan), on the eastern bank of the Indus river. Sophist Lucius Flavius Philostratus (around 170–250, Greece) claims that Pythagoras also studied under Hindu sages in India. In the book *India in Greece* published in 1852, in England, by the Greek historian Edward Pococke (1604–1691) reports that Pythagoras, who taught Buddhist philosophy, was a great missionary. His name indicates his office and position; Pythagoras in English is equivalent to putha-gorus in Greek and Budha-guru (or Prithviguru) in Sanskrit (one of the oldest languages of the world which was literary instrument of mathematicians, astronomers, and all Hindu scholars), which implies that he was a Buddhist spiritual leader (this explains the close parallels between Indian and Pythagorean teaching, living, philosophy, and religion). However, in spite of huge population of Buddhist in China (around 245 million), the part of the title of the book of Swetz and Kao *Was Pythagoras Chinese? …* [502] cannot be justified.

In [132], Henry Thomas Colebrooke (1765–1837, England) has shown that the doctrines of Pythagoras were rooted in India. He says: "Adverting to what has come to us of the history of Pythagoras, I shall not hesitate to acknowledge an inclination to consider the Grecian to have been indebted to Indian instructors." In [452], Leopold von Schröder (1851–1920, Estonia-Austria) says Pythagoras was "the bringer of Indian traditions to Greece." In [283], James Jeans (1877–1946, England) was the first to point out the striking similarities between the theories of Pythagoras and those of Sankhya system (one of the six $\bar{a}$ stika schools of Hindu philosophy which was composed before 1200 BC). Pythagoras's emphasis on number, i.e., Sankhya, James says, indicates his Indian inspiration. Frank Higgins in his book [262] maintains that Pythagoras remained in India for several years and learned from the Brahmins of Elephanta and Ellora. He further adds that the name of Pythagoras is still preserved in the records of the Brahmins as Yavancharya, the Ionian Teacher. Burkert [109] and Kahn [291] in their books have stated that the Greek and the Indians "had to meet regularly at the New year festival at Persepolis." Bernabé and Mendoza [68] in their work mention about a curious instance of a meeting of Socrates (around 470–399 BC, Greece, he was sentenced to death by the drinking of a mixture containing poison hemlock, because he was found guilty of corrupting the minds of the youth of Athens and of impiety "not believing in the gods of the state") with an Indian, transmitted or invented by Aristoxenus of Tarentum (around 360–300 BC, Greece) [fragment 53]. An analysis of this meeting has been documented by Lacrosse [323]. The following quotations are also well known:

Francois Marie Arouet Voltaire (1694–1778), one of the greatest French writers and philosophers: "I am convinced that everything has come down to us from the banks of Ganga—Astronomy, Astrology, and Spiritualism. Pythagoras went from Samos to Ganga 2600 years ago to learn Geometry. He would not have undertaken this journey had the reputation of the Indian science had not been established before."

Thomas Stearns Eliot (1888–1965), American-British poet, Nobel Laureate (1948): "I am convinced that everything has come down to us from the banks of the Ganga—Astronomy, Astrology, Spiritualism, etc. It is very important to note that some 2,500 years ago at the least Pythagoras went from Samos to the Ganga to learn Geometry but he would certainly not have undertaken such a strange journey had the reputation of the Brahmins' science not been long established in Europe."

If we do not agree with the fact that Pythagoras was in India for several years, at least we can certainly say that he was influenced by ideas from India, transmitted via Persia and Egypt.

1.3 Philosophy of Pythagoras

In view of Heraclitus of Ephesus (around 535–475 BC, Greece), nicknamed as Weeping Philosopher, Pythagoras pursued research and inquiry more assiduously than any other man. He compounded his wisdom from polymathy and bad arts. Pythagoras had a profound influence in the writings of Plato (it is very likely a copy of Philolaus' work came into the possession of Plato), for example, *Timacus* and *Phaedo* exhibit Pythagorean teachings. Mainly Plato advocated that true knowledge could be acquired only through philosophical contemplation of abstract ideas and not through observation of the accidental and imperfect things in the real world. Athenian rhetorician Isocrates (436–338 BC, Greece) states that, above all else, Pythagoras was known as the eponymous founder of a new way of life, while Iamblichus, in particular, presents the “Pythagorean Way of Life” in [277,278] as a pagan alternative to the Christian monastic communities of his own time. Godfrey Higgins (1772–1833, England) in [263] finds several striking circumstances in which the history of Pythagoras agrees with the history of Jesus. Johannes Reuchlin (1455–1522, Germany) synthesized Pythagoreanism with Christian theology and Jewish Kabbalah, arguing that Kabbalah and Pythagoreanism were both inspired by Mosaic tradition and that Pythagoras was therefore a kabbalist. Aristotle and his disciples Dicaearchus, Aristoxenus and Heraclides Ponticus (around 387–312 BC, Turkey-Greece) wrote treatises on Pythagoras, which are no longer extant.

According to Hermippus of Smyrna (third century BC, Greece) Pythagoras was familiar with and an admirer of Jewish customs and wisdom, and he introduced many points of Jewish law into his philosophy. Ion of Chios (around 490–420 BC, Greece) and Empedocles of Acragas (around 495–435 BC, Greece) expressed appreciation for Pythagoras in their poems. Herodotus described Pythagoras as “not the most insignificant” (important sophist) of Greek sages and states that Pythagoras taught his followers how to attain immortality. Pythagorean ideas on mathematical perfection also impacted ancient Greek art. Publius Nigidus Figulus (around 98–45 BC, Greece) founded the neopythagoreanist School of Philosophy. Pythagoras appears as a character in the last book of Ovid’s Metamorphoses (first published in eighth century), where Ovid has him expound upon his philosophical viewpoints. Pythagoras continued to be regarded as a great philosopher throughout the Middle Ages and his philosophy had a major impact on scientists such as Nicolaus Copernicus (1473–1543, Poland), Galileo Galilei (1564–1642, Italy), Johannes Kepler (1571–1630, Germany-Czechoslovakia), and Newton. The Pythagoras crater located near the northwestern limb of the Moon was named in his honor. The Pythagoras

Awards, established in 2009, are given annually to Bulgarian nationals by the Ministry of Science and Education of Bulgaria in recognition for outstanding scientific achievements. These awards are considered as the Bulgarian Nobel Prizes.

The Golden Verses, written in dactylic hexameter (seventy-one lines in the form of rhythmic poetry in Greek), are believed to have been known in the third century BC, and some claim to have been composed by Empedocles. These fragments are traditionally attributed to Pythagoras and constitute an important source of Pythagorean doctrines and philosophy and are assumed to be read each morning and evening by disciples. These verses have enjoyed great popularity and widely distributed also quoted throughout since then, especially after their translations first in modern Greek: Chrysea Epē, then in Latin: Aurea carmina, next in French: Vers d'or in 1706 by André Dacier (1651–1722, France) and in 1813 by Antoine Fabre d'Olivet (1768–1825, France), and finally in English by several authors, e.g., in 1707 by Nicholas Rowe (1674–1718, England) and in 1995 by Johan Carl Thom (born 1954, South Africa). The following comprehensive list of self-explanatory Golden Verses (with minor changes) has been given at several places, especially in the book of Firth [190]:

1. First worship the immortal gods, as they are established and ordained by the Law.

2. Reverence the Oath, and next the Heroes, full of goodness and light.

3. Honor likewise the Terrestrial Daemons by rendering them the worship lawfully due to them.

4. Honor likewise your parents, and those most nearly related to you.

5. Of all the rest of mankind, make him your friend who distinguishes himself by his virtue.

6. Always give ear to his mild exhortations, and take example from his virtuous and useful actions.

7. Avoid as much as possible hating your friend for a slight fault.

8. [And understand that] Power is a near neighbor to necessity.

9. Know that all these things are as I have told you; and accustom yourself to overcome and vanquish these passions.

10. First gluttony, then sloth, sensuality, and anger.

11. Do nothing evil, neither in the presence of others, nor privately.

12. But above all things respect yourself.

13. In the next place, observe justice in your actions and in your words.

14. And do not accustom yourself to behave yourself in anything without rule, and without reason.

15. But always make this reflection, that it is ordained by destiny that all men shall die.

16. And that the goods of fortune are uncertain; and that just as they may be acquired, they may likewise be lost.

17. Concerning all the calamities that men suffer by divine fortune.

18. If you suffer, suffer in patience, and resent them not.

19. But do your best to remedy them, and bear in Mind.

20. And consider that fate does not send the greatest portion of these misfortunes to good men.

21. There are many sorts of reasonings among men, good and bad.

22. But do not be disturbed by them, not allow them to harass you.

23. But if falsehoods be advanced, hear them with mildness, and arm yourself with patience.

24. Observe well, on every occasion, what I am going to tell you.

25. Do not let any man either by his words, or by his deeds, ever seduce you.

26. Nor entice you to say or to do what is not profitable for yourself.

27. Consult and deliberate before you act, that you may not commit foolish actions.

28. For it is the part of a miserable man to speak and to act without reflection.

29. But do the thing which will not afflict you afterward, nor oblige you to repentance.

30. Never do anything which you do not understand.

31. But learn all you ought to know, and by that means you will lead a very pleasant life.

32. In no way neglect the health of your body.

33. But give it drink and food in due measure, and also the exercise of which it needs.

34. Now by measure, I mean what will not discomfort you.

35. Accustom yourself to a style of living which is neat and decent, but not luxurious.

36. Avoid anything which might give rise to envy.

37. And do not be prodigal out of season, like someone who does not know what is decent and honorable.

38. Neither be covetous nor niggardly; a due measure is excellent in these things.

39. Do only the things that cannot hurt you, and deliberate before you do them.

40. Never allow sleep to close your eyelids, after you went to bed.

41. Never sleep before going over the acts of the day in the mind.

42. In what have I done wrong? What have I done? What have I omitted that I ought to have done?

43. If in this examination you find that you have done wrong, reprove yourself severely for it.

44. And if you have done any good, rejoice.

45. Practice thoroughly all these things; meditate on them well; you ought to love them with all your heart.

46. This will put you in the way of divine virtue.

47. I swear it by he who has transmitted into our souls the sacred quaternion, the source of nature, whose cause is eternal.

48. But never begin to set your hand to any work, until you have first prayed the gods to accomplish what you are going to begin.

49. When you have made this habit familiar to you.

50. You will know the constitution of the Immortal gods and of men.

51. Even how far the different beings extend, and what contains and binds them together.

52. You shall likewise know that according to Law, the nature of this Universe is in all things alike.

53. So that you shall not hope what you ought not to hope; and nothing in this world shall be hidden from you.

54. You will likewise know that men draw upon themselves their own misfortunes voluntarily, and of their own free choice.

55. Unhappy they are! They neither see nor understand that their good is near them.

56. Few know how to free themselves from their own misfortune.

57. Such is the fate that blinds humankind, and takes away his senses.

58. Some, like wheels, are carried in one direction, some in another, pressed down by ills innumerable.

59. For fatal strife, natural, pursues them everywhere, tossing them up and down; nor do they perceive it.

60. Instead of provoking and stirring it up, they ought to avoid it by yielding.

61. Oh! Jupiter, our Father! If you would deliver men from all the evils that oppress them.

62. Show them what fate is about to overtake them.

63. But be of good heart, the race of man is divine.

64. Sacred nature reveals to them the most hidden mysteries.

65. If she imparts to you her secrets, you will easily perform all the things which I have ordained you.

66. And by the healing of your soul, you will deliver it from all evils, from all afflictions.

67. But you should abstain from the food, which we have forbidden, which in the purifications and in the deliverance of the soul.

68. And in the deliverance of thy soul, decide between the courses open to you, and thoroughly examine all things.

69. Leave yourself always to be guided and directed by the understanding that comes from above, and that ought to hold the reins.

70. And when, after having deprived yourself of your mortal body, you arrived at the most pure Aither (the primordial God of light and the bright, blue ether of the heavens).

71. You shall be a God, immortal, incorruptible, and Death shall have no more dominion over you.

Primary emphasis of these verses is on Human Virtues, whose aim is the making of Good Men; and Divine Virtues, so that a Good Men becomes one of the gods. In the introduction of Firth's book, Annie Besant (1847–1933, England) has clearly stated that "Pythagoras brought from India the wisdom of the Buddha, and translated it into Greek thought, adding to its austere grandeur the beauty characteristic of Greece, as Grecian art made tenderer the stern outlines of Indian sculpture. Those whose thought runs on Greek lines will here find the oldest wisdom garbed in Grecian grace, retaining the beauty of simplicity, and adding the fairness of form." Firth's book also contains notes on The Golden Verses from the commentaries of Hierocles of Alexandria (active around 430, Greece), and the related work of Democrates (Greece), Demophilus (Greece), Joannes Stobaeus (fl. in fifth century, Greece), Sextus Empiricus (around 160–210, Greece), and Iamblichus (see [277]). Firth's book also gives details of thirty-nine *symbols/symbola/aphorisms of Pythagoras* that were first gathered and interpreted in detail by Iamblichus in his book [276] and translated from Greek into English by Thomas Taylor (1758–1835, England) in 1818. These well-documented symbols that we denote in Roman numbers (still used today, mainly for decorative purposes: on clock faces, for chapter numbers in books, and so on) set high divinity, moral, philosophy, and discipline, and according to commentators of Pythagoras are considered favorite method of instruction used in the

Pythagorean School in Croton. Wolfgang Amadeus Mozart (1756–1791, Austria) incorporated Masonic and Pythagorean symbolism into his opera The Magic Flute.

Symbol I. When going to the temple to adore Divinity neither say nor do any thing in the interim pertaining to the common affairs of life: This Symbol preserves a divine nature such as it is in itself pure and undefiled; for the pure it won't to be conjoined to the pure. It also causes us to introduce nothing from human affairs into the worship of Divinity: for all such things are foreign from, and contrary to, religious worship. This Symbol also greatly contributes to science; for in divine science, it is necessary to introduce nothing of this kind, such as human conceptions, or those pertaining to the concerns of life. We are exhorted to nothing else, therefore, by these words than this: that we should not mingle sacred discourses and divine actions with the instability of human manners. (One should have exclusive and unswerving devotion: undivided adherence when the whole mind, heart, and soul are given to God.)

Symbol II. Neither enter into a temple negligently, nor in short adore carelessly, not even though you should stand at the very doors themselves: With the preceding this symbol also accords. For if the similar is friendly and allied to the similar, it is evident that since the gods have a most principal essence among wholes, we ought to make the worship of them a principal object. But he who does this for the sake of anything else gives a secondary rank to that which takes the precedence of all things, and subverts the whole order of religious worship and knowledge. Besides, it is not proper to rank illustrious goods in the subordinate condition of human utility, nor to place our condition in the order of an end, but things more excellent, whether they be works or conceptions, in the condition of an appendage.

Symbol III. Sacrifice and adore unshod: An exhortation to the same thing may also be obtained from this Symbol. For it signifies that we ought to worship the gods and acquire a knowledge of them orderly and modestly, and in a manner not surpassing our condition on the Earth. It also signifies that in worshipping them, and acquiring this knowledge, we should be free from bonds, and properly liberated. But the Symbol exhorts that sacrifice and adoration should be performed not only in the body, but also in the energies of the soul; so that these energies may neither be detained by passions, nor by the imbecilities of the body, nor by generation, with which we are externally surrounded. But everything pertaining to us should be properly liberated and prepared for the participation of the gods.

Symbol IV. Disbelieve nothing wonderful concerning the gods, nor concerning divine dogmas: This Symbol in like manner exhorts to the same virtue. For this, dogma sufficiently venerates and unfolds the transcendency of the gods. Affording us a viaticum and recalling to our memory that we ought not to estimate divine power from our judgment. But it is likely that some things should appear difficult and impossible to us, in consequence of our corporeal subsistence, and from our being conversant with generation and corruption; from our having a momentary existence; from being subject to a variety of diseases; from the smallness of our habitation; from our gravitating tendency to the middle; from our somnolence, indigence and repletion; from our want of counsel and our imbecility; from the impediments of our soul, and a variety of other circumstances, although our nature possesses many illustrious prerogatives. At the same time, however, we perfectly fall short of the gods, and neither possess the same power with them, nor equal virtue. This Symbol, therefore, in a particular manner introduces the knowledge of the gods, as beings who are able to affect all things. On this account it exhorts us to disbelieve nothing concerning the gods. It also adds, nor about divine dogmas, that is to say, these belonging to the Pythagoric philosophy. For these being secured by discipline and scientific theory are alone true and free from falsehood, being corroborated by all various demonstration accompanied with necessity. The same Symbol is also capable of exhorting us to the science concerning the gods; for it urges us to acquire a science of that kind through which we shall be in no respect deficient in things asserted about the gods. It is also able to exhort the same things concerning divine dogmas and a disciplinative progression. For disciplines alone give eyes to and produce light about all things in him who intends to consider and survey them. For from the participation of disciplines, one thing before all others is effected, that is to say, a belief in the nature, essence, and power of the gods, and also in those Pythagoric dogmas that appear to be prodigious to such as have not been introduced to, and are uninitiated in, disciplines. So that the precept disbelieve not is equivalent to participate, and acquire, those things through which you will not disbelieve; that is to say, acquire disciplines and scientific demonstrations.

Symbol V. Declining from the public ways, walk in unfrequented paths: We believe that this Symbol also contributes to the same thing as the preceding. For this exhorts us to abandon a popular and merely human life; but thinks fit that we should pursue a separate and divine life. It also signifies that it is necessary to look above common opinions; but very much to esteem such as are private and arcane; and that we should despise merely human delight; but ardently pursue that felicitous mode of conduct which adheres to the divine will. It likewise exhorts us to dismiss human manners as popular, and to exchange for these the religious cultivation of the gods, transcending a popular life. (Those who desire wisdom must seek it in solitude.)

Symbol VI. Abstain from Melanurus (black termination) for it belongs to the terrestrial gods: This Symbol also is allied to the preceding. Other particulars therefore pertaining to it we shall speak of in our discourse about the Symbols. So far then as it pertains to exhortation it admonishes us to embrace the celestial journey, to conjoin ourselves to the intellectual gods, to become separated from a material nature, and to be led, as it were in a circular progression to an immaterial and pure life. It further exhorts us to adopt the most excellent worship of the gods, and especially that which pertains to the primary gods. Such, therefore, are the exhortations to the knowledge and worship of Divinity.

Symbol VII. Govern your tongue before all other things, following the gods: For it is the first work of wisdom to convert reason to itself and to accustom it not to proceed externally, but to be perfected in itself and in a conversion to itself. But the second work consists in following the gods. For nothing so perfects the intellect as, when being converted into itself, it at the same time follows Divinity. (This Symbol warns man that his words, instead of representing him, misrepresent him, and that when in doubt as to what he should say, he should always be silent.)

Symbol VIII. The wind is blowing, adore the wind: This Symbol is also a token of divine wisdom. For it obscurely signifies that we ought to love the similitude of the divine essences and powers, and when their words accord with their energies, to honor and reverence them with the greatest earnestness. (The fiat of God is heard in the voice of the elements, and that all things in Nature manifest through harmony, rhythm, order, or procedure the attributes of the Deity.)

Symbol IX. Cut not fire with a sword: This Symbol exhorts to prudence. For it excites in us an appropriate conception with the respect to the propriety of not opposing sharp words to a man full of fire and wrath, nor contending with him. For frequently by words, you will agitate and disturb an ignorant man and will yourself suffer things dreadful and unpleasant. Heraclitus also testifies to the truth of this Symbol, for he says, "It is difficult to fight with anger; for whatever is necessary to be done, benefits the soul." For many by gratifying anger have changed the condition of the soul, and have made death preferable to life. But by governing the tongue and being quiet, friendship is produced from strife, the fire of anger being extinguished, and you yourself will not appear to be destitute of intellect. (Do not vex with sharp words a man swollen with anger.)

Symbol X. Remove yourself from every vinegar bottle: The truth of the preceding is testified by the present Symbol. For it exhorts to prudence and not to anger; since that which is sharp in the soul and which we call anger is deprived of reasoning and prudence. For anger boils like a kettle heated by the fire, being attentive to nothing but its own emotions, and dividing the judgment into minute parts. It is proper therefore that the soul being established in quiet should turn from anger, which frequently attacks itself as if it touched sounding brass. Hence it is requisite to suppress this passion by the reasoning power.

Symbol XI. Assist a man in raising a burden; but do not assist him in laying it down: This Symbol exhorts to fortitude, for whoever takes up a burden signifies an action of labor and energy; but he who lays one down, of rest and remission. So that the Symbol has the following meaning. Do not become either to yourself or another the cause of an indolent and effeminate mode of conduct; for every useful thing is acquired by labor. But the Pythagoreans celebrate this Symbol as Herculean, thus denominating it from the labors of Hercules. For, during his association with men, he frequently returned from fire and everything dreadful, indignantly rejecting indolence. For rectitude of conduct is produced from acting and operating, but not from sluggishness. (Aid the diligent but never to assist those who seek to evade their responsibilities, for it is a great sin to encourage indolence.)

Symbol XII. When stretching forth your feet to have your sandals put on, first extend your right foot; but when about to use a foot bath, first extend your left foot: This Symbol exhorts to practical prudence, admonishing us to place worthy actions about us as right-handed; but entirely to lay aside and throw away such as are base, as being left-handed.

Symbol XIII. Speak not about Pythagoric concerns without light: This Symbol exhorts to the possession of intellect according to prudence. For this is similar to the light of the soul, to which being indefinite it gives bound, and leads it, as it were, from darkness into light. It is proper, therefore, to place intellect as the leader of everything beautiful in life, but especially in Pythagorean dogmas; for these cannot be known without light. (Do not attempt to interpret the mysteries of God and the secrets of the sciences without spiritual and intellectual illumination.)

Symbol XIV. Step not beyond the beam of the balance: This Symbol exhorts us to the exercise of justice, to the honoring equality and moderation in an admirable degree, and to the knowledge of justice as the most perfect virtue, to which the other virtues give completion, and without which none of the rest are of any advantage. It also admonishes us that it is proper to know this virtue not in a careless manner, but through theorems and scientific demonstrations. But this knowledge is the business of no other art and science than the Pythagorean philosophy alone, which in a transcendent degree honors disciplines before everything else.

Symbol XV. Having departed from your house, turn not back; for the furies will be your attendants: This Symbol also exhorts to philosophy and self-operating energy according to intellect. It clearly manifests too and predicts, that having applied yourself to philosophy, you should separate yourself from everything corporeal and sensible, and truly meditate upon death, proceeding, without turning back, to things intelligible and which always subsist according to the same and after a similar manner, through appropriate disciplines; for journeying is a change of place; and death is the separation of the soul from the body.

But we should philosophize truly and without sensible and corporeal energies, employing a pure intellect in the apprehension of the truth of things, which knowledge, when acquired, is wisdom. But having applied yourself to philosophy, turn not back nor suffer yourself to be drawn to former objects and to corporeal natures together with which you were nourished. For by so doing you will be attended by abundant repentance, in consequence of being impeded in sane apprehensions by the darkness in which corporeal natures are involved. But the Symbol denominates repentance, the furies. (It is better to know nothing about Divinity than to learn a little and then stop without learning all.)

Symbol XVI. Being turned to the Sun, make not water: The exhortation of this Symbol is as follows: Attempt to do nothing which is merely of an animal nature; but philosophize, looking to the heavens and the Sun. Let the light of truth also be your leader, and remember that no abject conceptions must be admitted in philosophy; but ascend to the gods and wisdom through the survey of the celestial orbs. Having likewise applied yourself to philosophy and purified yourself by the light of truth which is in it; being also, converted to a pursuit of this kind, to theology, to physiology, and so astronomy, and to the knowledge of that cause which is above all these; no longer do anything of a merely brutal nature.

Symbol XVII. Wipe not a seat with a torch: This Symbol also exhorts the same thing. For since a torch is of a purifying nature in consequence of its rapid and abundant participation of fire, in the same manner as what is called Sulfur, the Symbol not only exhorts not to defile it, since it is itself abstergent of defilements, nor to oppose its natural aptitude by defiling that which is an impediment to defilement; but rather that we should not mingle the peculiarities of wisdom with those of the merely animal nature. For a torch through the bright light it emits is compared to philosophy; but a seat through its lowly condition to the merely animal nature.

Symbol XVIII. Nourish a cock; but sacrifice it not; for it is sacred to the Sun and the Moon: This Symbol advises us to nourish and strengthen the body and not neglect it, dissolving and destroying the mighty tokens of union, connection, sympathy, and consent of the world. So that it exhorts us to engage in the contemplation and philosophy of the Universe. For though the truth concerning the Universe is naturally occult, and sufficiently difficult of investigation, it must, however, at the same time, be enquired into and investigated by man, and especially through philosophy. For it is truly impossible to be discovered through any other pursuit. But philosophy, receiving certain sparks, and as it were viatica, from nature, excites and expands them into magnitude, rendering them more conspicuous through the disciplines which it possesses. Hence, therefore, we should philosophize. (A warning against the sacrifice of living things to the gods is given, because life is sacred and man should not destroy it even as an offering to the Deity. Further, it is advised to man that the human body should be nurtured and protected.)

Symbol XIX. Sit not upon a bushel: The Symbol may be considered more Pythagorically, beginning from the same principles with those above. For since nutriment is to be measured by the corporeal and animal nature, and not by a bushel, do not pass your life in indolence nor without being initiated into philosophy; but dedicating yourself to this, rather provide for that part of you which is more divine, which is soul, and much more for the intellect which soul contains; the nutriment of which is measured, not by a bushel, but by contemplation and discipline.

Symbol XX. Nourish not that which has crooked nails: This Symbol also in a more Pythagorean manner advises us to communicate and impart and prepare others to do so, accustoming them to give and receive without depravity and abundantly; not indeed receiving everything insatiably and giving nothing. For the physical organization of animals with crooked nails is adapted to receive rapidly and with facility, but by no means to relinquish what they hold, or impart to others, through the opposition of the nails in consequence of their being crooked; just as the fish called crang, are naturally adapted to draw anything to themselves with celerity, but relinquish it with difficulty, unless by turning from, we avoid them. But hands were indeed suspended from us by nature, that through them we might both give and receive, and the fingers, also, are naturally attached to the hands, straight and not crooked. In things of this kind, therefore, we must not imitate animals with crooked nails, since we are fashioned by our maker in a different way, but should rather be communicative and impart to each other, being exhorted to a thing of this kind by the fabricators of names themselves, who denominated the right hand more honorable than the left, not only from receiving but from being capable of imparting. We must act justly therefore, and through this philosophize. For justice is a certain retribution and remuneration equalizing the abounding and deficient by reciprocal gifts.

Symbol XXI. Cut not in the way: This Symbol manifests that truth is one, but falsehood multifarious. But this is evident from hence, that what any particular thing is can be predicted only in one way, if it be properly predicted; but what it is not, may be predicted in infinite ways. Philosophy, too, appears to be a path or way. The Symbol therefore says, choose that philosophy, and that path to philosophy in which there is no division, and in which you will not dogmatize things contradictory to each other, but such as are stable and the same with themselves, being established by scientific demonstration through disciplines and contemplation; which is the same thing as if it said, philosophize Pythagorically. And this is indeed possible. But the philosophy which proceeds through things corporeal and sensible, and which is employed by the moderns even to satiety, which likewise considers Divinity, qualities, the soul, the virtues, and in short, all the most principal causes of things to be body—this philosophy easily eludes the grasp and is easily subverted. And this is evident from the various arguments of its advocates. On the other hand, the philosophy that proceeds through things incorporeal, intelligible, immaterial, and perpetual, and which always subsist according to the same, and in a similar manner, and never, as far as possible to them, admit either corruption or mutation, being similar to their subjects—this philosophy is the artificer of firm, stable, and undeviating demonstrations. The precept, therefore, admonishes us when we philosophize, and proceed in the way pointed out, to fly from the snares of, and avoid all connection with, things corporeal and multifarious, but to become familiar with the essence of the incorporeal natures, which at all times are similar to themselves, through the truth and stability which they naturally contain.

Symbol XXII. Receive not a swallow into your house: This Symbol admonishes as follows: Do not admit to your dogmas a man who is indolent, who does not labor incessantly, and who is not a firm adherent to the Pythagorean sect, and endued with intelligence; for these dogmas require continued and most strenuous attention, and an endurance of labor through the mutation and circumvolution of the various disciplines which they contain. But it uses the swallow as an image of indolence and an interruption of time, because this bird visits us for a certain part of the year, and for a short time becomes as it were our guest; but leaves us for the greater part of the year and is not seen by us. (This warns the seeker after truth not to allow drifting thoughts to come into his mind nor shiftless persons to enter into his life. He must ever surround himself with rationally inspired thinkers and with conscientious workers.)

Symbol XXIII. Wear not a ring: This Symbol is an exhortation to the Pythagorean doctrine: A ring embraces those that wear it after the manner of a bond; and a peculiarity of it is neither to pinch nor pain the wearer, but in a certain respect to be accommodated and adapted to him. But the body is a bond of this kind to the soul. The precept, therefore, wears not a ring, is equivalent to, Philosophize truly, and separates your soul from its surrounding bond. For philosophy is the meditation of death and the separation of the soul from the body. Betake yourself, therefore, with great earnestness to the Pythagorean philosophy, which through intellect separates itself from all corporeal natures, and is conversant through speculative disciplines with things intelligible and immaterial. Liberate yourself also from sin and from those occupations of the flesh which draw you aside from, and impede the philosophic energy; likewise, from superabundant nourishment and unseasonable repletion, which confine the soul like a bond and incessantly introduce a crowd of diseases, and interruptions of leisure.

Symbol XXIV. Inscribe not the image of God in a ring: This Symbol, conformably to the foregoing conception, employs the following exhortation: Philosophize, and before everything consider the gods as having an incorporeal subsistence. For this is the most principal root of the Pythagorean dogmas, from which nearly all of them are suspended, and by which they are strengthened even to the end. Do not, therefore, think that the gods use such forms as are corporeal, or that they are received by a material subject and by body as a material bond, like other animals. But the engravings in rings exhibit the bond which subsists through the ring, its corporeal nature, and sensible form, and the view, as it were, of some partial animal which becomes apparent through the engraving; from which especially we should separate the genus of the gods as being eternal and intelligible, and always subsisting according to the same and in a similar manner, as we have particularly, most fully, and scientifically shown in our discourse concerning the gods.

Symbol XXV. Behold not yourself in a mirror by the light of a lamp: This Symbol advises us in a more Pythagorean manner to philosophize, not be taking ourselves to the imaginations belonging to the senses, which produce indeed a certain light about our apprehensions of things; but this light resembles that of a lamp, and is neither natural nor true. It admonishes us, therefore, rather to betake ourselves to scientific conceptions about intellectual objects, from which a most splendid and stable purity is produced about the eye of the soul, resulting from all intellectual conceptions and intelligibles, and the contemplation about these, and not from corporeal and sensible natures. For we have frequently shown that these are in a continual flux and mutation and do not in any manner subsist stably and similar to themselves, so as to sustain a firm and scientific apprehension and knowledge in the same manner as the objects of Intellectual vision.

Symbol XXVI. Be not addicted to immoderate laughter: This Symbol shows that the passions are to be subdued, Recall, therefore, into your memory right reason, and be not inflated with prosperity nor abject in calamity; being persuaded that no worthy attention takes place in either of these. But this Symbol mentions laughter above all the passions, because this alone is most conspicuous, being, as it were, a certain efflorescence and inflammation of the disposition proceeding as far as to the face. Perhaps, too, it admonishes us to abstain from immoderate laughter, because laughter is the peculiarity of man with respect to other animals; and hence he is defined to be a risible animal. It is shown, therefore, by this precept that we should not firmly adhere to the human nature, but acquire by philosophizing an imitation of divinity to the utmost of our power; and withdrawing ourselves from this peculiarity of man, prefer the rational to the risible in the distinction and difference which we make of him with respect to other animals.

Symbol XXVII. Cut not your nails at a sacrifice: The exhortation of this Symbol pertains to friendship. For of our relations and those allied to us by blood, the nearest of kin are brothers, children, and parents, who resemble those parts of our body which when taken away produce pain and mutilation by no means trifling, such as fingers, hands, ears, nostrils, and the like. But others who are distantly related to us, such as the daughters of cousins, or the sons-in-law of uncles, or others of this kind, resemble those parts of our body from the cutting off of which no pain is produced, such as hair, nails, and the like. The Symbol, therefore, wishing to indicate those relations who have been for a time neglected by us through the distance of their alliance, employs the word nails, and says: Do not entirely cast off these; but if at sacrifices, or any other time, you have neglected them, draw them to you, and renew your familiarity with them.

Symbol XXVIII. Offer not your right hand easily to everyone: The meaning of this Symbol is, Do not draw up, nor endeavor to raise, by extending your right hand, the unadopted and uninitiated. It also signifies that the right hand is not to be given easily even to those who have for a long time proved themselves worthy of it through disciplines and doctrines, and the participation of continence, the quinquennial silence, and other probationary trials. (This warns the disciple to keep his own counsel and not offer wisdom and knowledge to such as are incapable of appreciating them.)

Symbol XXIX. When rising from the bed clothes, roll them together and obliterate the impression of the body: This Symbol exhorts that, having applied yourself to philosophy, in the next place you should familiarize yourself with intelligible and incorporeal natures. Rising therefore from the sleep and nocturnal darkness of ignorance, draw off with you nothing corporeal to the daylight of philosophy, but purify and obliterate from your memory all the vestiges of that sleep of ignorance. (Those who had awakened from the sleep of ignorance into the waking state of intelligence must eliminate from their recollection all memory of their former spiritual darkness.)

Symbol XXX. Eat not the heart: This Symbol signifies that it is not proper to divulse the union and consent of the Universe. And still further it signifies this, be not envious, but philanthropic and communicative; and from this it exhorts us to philosophize. For philosophy alone among the sciences and arts is neither pained with the goods of others, nor rejoices in evils of neighbors, these being allied and familiar by nature, subject to the like passions, and exposed to one common fortune; and evinces that all men are equally incapable of foreseeing future events. Hence it exhorts us to sympathy and mutual love, and to be truly communicative, as it becomes rational animals. (You should not vex yourself with grief.)

Symbol XXXI. Eat not the brain: This Symbol also resembles the former; for the brain is the ruling instrument of intellectual prudence. The Symbol, therefore, obscurely signifies that we ought not to dilacerate nor mangle things and dogmas which have been the objects of judicious deliberation. But these will be such as have been the subject of intellectual consideration, becoming thus equal to objects of a scientific nature. For things of this kind are to be surveyed, not through the instruments of the irrational form of the soul, such as the heart and the liver; but through the pure rational nature. Hence to dilacerate these by opposition, is inconsiderate folly; but the Symbol rather exhorts us to venerate the fountain of intelligence and the most proximate organ of intellectual perception, through which we shall possess contemplation, science, and wisdom; and by which we shall truly philosophize, and neither confound nor obscure the vestiges which philosophy produces.

Symbol XXXII. Indignantly turn from your excrements and the parings of your nails: The meaning of this Symbol is as follows: Despise things which are connascent with you, and which in a certain respect are more destitute of soul, since things which are more animated are more honorable. Thus, also when you apply yourself to philosophy, honor the things which are demonstrated through soul and intellect without sensible instruments, and through contemplative science, but despise and reject things which are opined merely through the connascent instruments of sense without intellectual light, and which are by no means able to acquire the perpetuity of intellect.

Symbol XXXIII. Receive not Erythinus (a fish of a red color): This Symbol seems to be merely referred to the etymology of the name. Receive not an unblushing and impudent man, nor on the contrary one stupidly astonished, and who in everything blushes and is humble in the extreme through the imbecility of his intellect and dianötic power (power of the soul which reasons scientifically). Hence this also is understood: Be not yourself such a one.

Symbol XXXIV. Obliterate the mark of the pot from the ashes: This Symbol signifies that he who applies himself to philosophy should consign to oblivion the confusion and grossness which subsists in corporeal and sensible demonstrations and that he should rather use such as are conversant with intelligible objects. But ashes are here assumed instead of the dust in the tables, in which the Pythagoreans completed their demonstrations.

Symbol XXXV. Draw not near to that which has gold, in order to produce children: The Symbol does not here speak of a woman, but of that sect and philosophy which has much of the corporeal in it, and a gravitating tendency downward. For gold is the heaviest of all things in the Earth, and pursues a tendency to the middle, which is the peculiarity of corporeal weight; but the term to draw near not only signifies to be connected with, but always to approach toward, and be seated near, another.

Symbol XXXVI. Honor a figure and a step before a figure and a tribolus: The exhortation of this Symbol is as follows: Philosophize and diligently betake yourself to disciplines, and through these, as through steps, proceed to the thing proposed; but reject the progression through those things that are honored and venerated by the many. Prefer also the Italic philosophy, which contemplates things essentially incorporeal, to the Ionic, which makes bodies the principal object of consideration.

Symbol XXXVII. Abstain from beans: This Symbol admonishes us to beware of everything which is corruptive of our converse with the gods and divine prophecy. (See above.)

Symbol XXXVIII. Transplant mallows indeed in your garden; but eat them not: This Symbol obscurely signifies that plants of this kind turn with the Sun, and it thinks fit that this should be noticed by us. It also adds transplant, that is to say, observe its nature, its tendency toward, and sympathy with, the Sun; but rest not satisfied, nor dwell upon this, but transfer, and as it were transplanting your conception to kindred plants and pot herbs, and also to animals that are not kindred, to stones, and rivers, and in short to natures of every kind. For you will find them to be prolific and multiform, and admirably abundant; and this to one who begins from the mallows, as from a root and principle, is significant of the union and consent of the world. Not only, therefore, do not destroy or obliterate observations of this kind, but increase and multiply them as if they were transplanted.

Symbol XXXIX.

Abstain from animals: This Symbol exhorts to justice, to all the honor of kindred, to the reception of similar life, and to many other things of a like kind. From all this, therefore, the exhortatory type through symbols becomes apparent, which contains much in it of the ancient and Pythagorean mode of writing.

Pherecydes and so his student Pythagoras, and later Alcmaeon of Croton (fl. fifth century BC, Greece) and Epictetus (around 50–135 Greece) were devout believer in the doctrines of metempsychosis and said the soul is invisible, immortal, and a string-like self-moving number, especially its reincarnation after death into a new body (the end of life is death and the end of death is life) of the same or a different species until it is liberated (unite with the Divine, Moksha, or Nirvana) by means of spiritual purification. However, the notion of metempsychosis was coined thousands of years before in India: Sage Veda Vyasa, born 3374 BC, India, along with his disciples Jaimani, Paila, Sumanthu, and Vaisampayana codified the four Vedas [Veda means wisdom, knowledge, or vision]; *Rigveda* contains hymns and prayers to be recited during the performance of rituals and sacrifices, and it mentions that in the earliest age of the gods, existence was born from nonexistence; *Samaveda* contains melodies to be sung on suitable occasions; *Yajurveda* contains sacrificial formulas for ceremonial occasions, and the concept of metempsychosis with details similar to Greeks; and *Atharvaveda* is a collection of magical formulas and spells. According to Plato, the souls of cowards are reincarnated in the bodies of women; the souls of the stupid animate four-footed beasts and birds; and an utterly worthless soul, being unworthy to breathe pure air, must content itself with the body of a fish. For Pherecydes and Pythagoras the number of souls in the Universe is fixed, and the present life is a punishment for the sins of some previous existence. Pherecydes propounded his teaching on the soul in terms of a pentemychos (five-nooks, or five hidden cavities), the most likely origin of the Pythagorean use of the pentagram. This teaching is referenced by Xenophanes of Colophon (around 560–478 BC, Greece), the Greek poets Ion of Chios, and Herodotus. Pythagoras believed that the soul of man is divided into three parts intelligence (nous), reason (phren), and passion (thumos). Female Pythagorean Aesara of Lucania (fourth or third century BC, Greece) is also known for her theory of the tripartite soul, which she believed consisted of mind, spiritedness, and desire. Further, according to Ebenezer Cobham Brewer (1810–1897, England), three parts of the soul are: the ethereal, luminous, and celestial, the soul in a state of bliss in the stars; the Luminous, the soul that suffers punishment of sin after death; The terrestrial, what vessel the soul occupies on the Earth. Pythagoras also believed that soul encloses the heart and the brain; intelligence and reason are found in the brain, and passion is based in the heart. He also advised that to be strong of soul rather than strong of body.

Philostratus and Laërtius claimed that Pythagoras knew not only who he was himself, but also who he had been. In fact, he remembered previous four lives in detail: First he had been Euphorus (a young hero who died in the Trojan ranks after having wounded Patroclus), secondly Aethalides (son of Hermes, the Greek God of the Word, of thought and magic, the swift-moving messenger of the Divine and guardian of souls in the Afterlife, who gave blessings to Aethalides that he can recollect everything that happened in this life, as well as the life beyond the grave and on into new incarnations), thirdly philosopher Hermotimus (who recognized and demonstrated the shield of Euphorus in the temple of Apollo), fourth Pyrrhus (a fisherman from Delos), and now he was Pythagoras. He was also able to recognize the souls of others. Once, upon seeing a dog being severely beaten, he rushed over and restrained the dog's owner, saying, "You must stop this. I know from the sound of his cries that within this animal is the soul of my late friend Abides who died in Memphis twelve years ago." Another time, he identified the soul of the legendary Phrygian King Midas inhabiting the body of Myllias, a citizen of Croton. At Pythagoras's urging, Myllias traveled to Asia to perform expiatory rites at Midas' tomb. Leonardo da Vinci wrote down in one of his notebooks an anecdote about Pythagoras's belief in the transmigration of the soul. An imaginative conjecture is that Pythagoras returned in the fifth century as Merlin, then as the English philosopher Francis Bacon (1561–1626, England). After an incarnation as the Count Saint-Germain (1691 or 1712–1784, died in Germany), his next life was supposedly Hermann Gring in the twentieth century.

We shall not go any further into the philosophical theories of the Pythagoreans but shall devote mainly with their mathematical accomplishments as has been documented by the historians of mathematics and show that most of it was already known several centuries earlier, and we shall also present in simplest possible terms current extensions of their investigations.

1.4 Mathematics of Pythagoras

Pythagoras instead of worshiping the dead as he did with his Egyptian teachers gave "divine significance" to most numbers and attempted to find mathematical explanations for everything in the Universe in terms of numbers including in geometry "possibly the most mischievous misreading of nature in the history of human error" (see Bell [61]). He paid homage to every numerical relationship such as equation and inequality (arithmetic then). He believed that maturity began at age 14, marriage occurred in the 21st year, and the normal life span ended at 70 years. Pythagoras always advised *Reason Answer All* (reason is immortal, all else is mortal), and his motto was *All is Number*, "Numbers Rules the Universe," "Number is the Ruler of Forms and Ideas and the Cause of gods and Demons," "Number is the Wisest." It has been claimed that from the writings of Orpheus (legendary musician, poet, and prophet in Greek mythology), Pythagoras learned that the eternal essence of number is the source of immortality, and from this he reasoned that the fundamental nature of the gods is numerical. Pythagoras perception was supported by several later writers: Philolaus in his work added "All things which can be known have number; for it is not possible that without number anything can be either conceived or known." For Plato, numbers were "the highest degree of knowledge" and constituted the essence of outer and inner harmony. Nicomachus "All things that have been arranged by nature according to a workmanlike plan appear, both individually and as a whole, as singled out and set in order by Foreknowledge and Reason, which created all according to Number, conceivable to mind only and therefore wholly immaterial; yet real; indeed, the really real, the eternal." Plotinus (204/5–270, Egypt-Italy) proclaimed "Number exists before objects which are described by number. The variety of sense objects merely recalls to the soul the notion of number." Saint Augustine (354–430, Algeria) had written in his *City of God* that "we must not despise the science of numbers, which, in many passages of Holy Scripture, is found to be of eminent service to the careful interpreter." Neither has it been without reason numbered among God’s praises: “Thou hast ordered all things in number, and measure, and weight.” In the fifth century, Anicius Manlius Severinus Boethius (around 475–526, Italy) declared “All things, do appear to be formed of numbers.”

Nicholas of Cusa (1401–1464, Germany-Italy) advanced a step beyond Pythagoras: The numbers were not the things their generation from the Monad symbolized, but merely an image, comprehensive to a finite mind, of the reality known only to the deity. John Dee (1527–1608, England) said “All things (which from the very first original being of things, have been framed and made) do appear to be formed by reason of numbers. For this was the principle example or pattern in the mind of Creator.” Bishop George Berkeley (1685–1753, Ireland) accorded number the status of reality. For Gauss “the prince of mathematicians,” “Number alone of all mathematical concepts was a necessity of rational thought, if not actually a creation of the mind.” In 1801, he asserted that number is purely a psychic reality, a free creation of mind. Further, James Clerk Maxwell (1831–1879, Scotland) aforementioned “Thus number may be said to rule the whole world of quantity, and the four rules of arithmetic may be regarded as the complete equipment of the mathematician.” Julius Wilhelm Richard Dedekind (1831–1916, Germany) said: “The numbers are a free creation of human mind.” According to Oswald Arnold Gottfried Spengler (1880–1936, Germany): “Number is the symbol of causal necessity. Like the conception of God, it contains the ultimate meaning of the world–as–nature. The existence of numbers may therefore be called a mystery, and the religious thought of every Culture has felt their impress.” Hermann Minkowski (1864–1909, Russian-Germany) felt “Integers are the fountainhead of all mathematics.” Paul Erdös (1913–1996, Hungary) once said, “I know numbers are beautiful. If they aren’t beautiful, nothing is.”

Pythagoras identified some human attribute to most numbers (see Sect. 2.9). The multiplication table that we know of today is sometimes attributed to Pythagoras; however, the oldest known multiplication tables were used by the Babylonians more than 4000 years back. Pythagoreans used deductive methods to prove ten basic results about even and odd numbers (originally this classification is due to Jainas), mainly they employed definitions of unit and even and odd numbers, and geometry. These results can be listed as follows: If as many even numbers as we please be added together, the whole is even; if as many odd numbers as we please be added together, and their multitude be even, the whole will be even; if as many odd numbers as we please be added together, and their multitude be odd, the whole will also be odd; if from an even number an even number be subtracted, the remainder will be even; if from an even number an odd number be subtracted, the remainder will be odd; if from an odd number an odd number be subtracted, the remainder will be even; if from an odd number an even number be subtracted, the remainder will be odd; if an odd number by multiplying an even number makes some number, the product will be even; if an odd number by multiplying an odd number makes some number, the product will be odd; if an odd number measures an even number, it will also measure the half of it. Everything in the world we live in involves evenness and oddness. Evenness can be defined as a term that means a balance between equal measurements, fair quantities, and neutral stability. On the other hand, oddness can be described in terms as being an uneven element, irregular varieties, and an additional essence. Pythagoreans also worked with Prime, Perfect, and Amicable Numbers (see Sect. 4.13). Thus they planted the seeds that would grow into modern number theory (earlier higher arithmetic). Some authors with caution have also credited him for the indirect method also. Pythagoras divided the mathematical subjects into four disciplines: numbers absolute (arithmetic), numbers applied (music), magnitudes at rest (geometry), and magnitudes in motion (astronomy).

For a given pair of two positive numbers a and $c, \ a < c$ any number b between a and c in some sense is called a *mean* (or average) of a and $c.$ Pythagoreans considered following ten means b of a and $c:$

1. $(b-a)/(c-b) = a/a \Longrightarrow b=(a+c)/2,$ known as Arithmetic Mean (AM)

2. $(b-a)/(c-b) = a/b \Longrightarrow b = \sqrt{ac},$ known as Geometric Mean (GM)

3. $(b-a)/(c-b) = a/c \Longrightarrow b=2ac/(a+c),$ known as Harmonic Mean (HM)

4. $(b-a)/(c-b) = c/a \Longrightarrow b = (a^2+c^2)(a+c),$

5. $(b-a)/(c-b) = b/a \Longrightarrow b = [(c-a) + \sqrt{(c-a)^2 + 4a^2}]/2,$

6. $(b-a)/(c-b) = c/b \Longrightarrow b = [-(c-a) + \sqrt{(c-a)^2 + 4c^2}]/2,$

7. $(c-a)/(b-a) = c/a \Longrightarrow b = a(2 - a/c),$

8. $(c-a)/(c-b) = c/a \Longrightarrow b = (c-a) + a^2/c,$

9. $(c-a)/(b-a) = b/a,\ a < b \Longrightarrow b = [a + \sqrt{(a^2 + 4a(c-a)}]/2,$

10. $(c-a)/(c-b) = b/a,\ a < b \Longrightarrow b = (c-a).$

It is rather easy to show that in all above ten means $0 < a < b < c.$ In mathematics, AM, GM, and HM (first it was called the subcontrary mean) are classical and appropriately used for situations when the average of rates is desired. These three means were defined by Pythagoras (Archytas reports these means in his work by name) by ratios/proportions because of their importance in geometry and music. There is a general consensus that Pythagoras brought the statement known as *perfect or musical proportion* $a/AM = HM/c$ or $AM/GM = GM/HM,$ or $AM : GM :: GM : HM,$ from Babylon to Croton. This relation immediately gives an alliance between AM, GM, and HM, namely, $GM = \sqrt{AM \times HM},$ i.e., GM is the geometric mean of AM and HM. This relation delighted and mystified the ancient Greeks including Plato. Further, these means satisfy the inequalities $AM \geq GM \geq HM,$ where equality holds if and only if $a = c.$ It also follows that HM can be expressed as the reciprocal of the AM of the reciprocals of the given numbers. A simple calculation also shows that $1/[H-a] + 1/[H-b] = 1/a + 1/b.$ It is also easy to show that there exists a number *k* such that $a = HM + a/k$ and $HM = c + c/k.$ Since 8 is the HM of 12 and $6,$ Philolaus called the cube a "geometrical harmony." Pappus in his *Synagoge* (*Mathematical Collection*) provided a construction for the harmonic mean of the segment *OA* and *OB* as follows. On the perpendicular to *OB* at *B* lay off $BC = BD,$ and let the perpendicular to *OB* at *A* meet *OC* at the point $E.$ Join $ED,$ and let *H* be the point at which *ED* cuts $OB.$ Then, $h = OH$ is the desired harmonic mean between $a = OA$ and $c = OB,$ see Fig. 1.2. In fact, from the similarity of triangles *OAE* and $OBC,$ as well as of triangles *HAE* and $HBD,$ we infer that

$$\frac{a}{c} = \frac{AH}{HB}, \quad \frac{a}{c} = \frac{AE}{BC} = \frac{AE}{BD} = \frac{AH}{HB} = \frac{OH - a}{c - OH}.$$

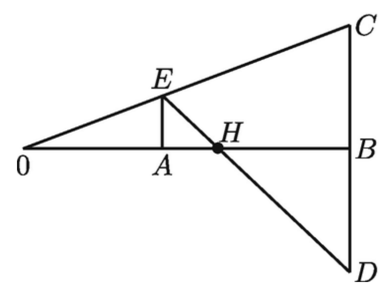

Fig. 1.2 Construction of harmonic mean

The study of the Pythagorean means is closely related to the study of majorization and Schur convex (Issai Schur, 1875–1941, Russia) functions. The harmonic and geometric means are concave symmetric functions of their arguments, and hence Schur concave, while the arithmetic mean is a linear function of its arguments, so both concave and convex. For a set of *n* positive real numbers $a_1, a_2, \cdots, a_n,$ AM, GM, and HM are respectively defined as $\frac{a_1+a_2+\ldots+a_n}{n}$, $\sqrt[n]{a_1 \cdot a_2 \cdots a_n},$ and $\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}}.$ Again it follows that $AM \geq GM \geq HM.$ Similar to AM, GM, and HM, there are Arithmetic Progression (AP), Geometric Progression (GP), and Harmonic Progression (HP), which are defined as follows:

A sequence of the form $a, a+d, a+2d, a+3d, \cdots, a+(n-1)d, a+nd, a+(n+1)d, \cdots$ is called AP. The numbers are said to form an AP with first term *a* and the common difference $d.$ It is clear that the step from each number to the next is the same number $d.$ It is to be noted that $a+nd=(a+(n-1)d+a+(n+1)d)/2,$ i.e., each number after the first is half of its left and right neighbors. For this reason, each number after the first is called the AM of its immediate predecessor and immediate successor. *Sulbasutras* (around 3200 BC) are considered to be appendices to the Vedas. Only seven Sulbasutras are extant, named for the sages who wrote them: Apastamba, Baudhayana (born 3200 BC), Katyayana, Manava, Maitrayana, Varaha, and Vidhula. The four major Sulbasutras, which are mathematically the most significant, are those composed by Baudhayana, Manava, Apastamba, and Katyayana. In India, Vedic mathematics was traditionally taught through aphorisms or *Sutras*. A Sutra is a thread of knowledge, a theorem, a ground norm, a repository of proof. Sulbasutras (see Agarwal and Sen [14]) contain several examples of AP. In Rhind Papyruses (around 1850 and 1650 BC) out of 87 mathematical problems two problems deal with APs and seem to indicate that Egyptian scriber Ahmes (around 1680–1620 BC) knew how to sum such series. For example, Problem 40 concerns an AP of five terms. It states: divide 100 loaves among 5 men so that the sum of the three largest shares is 7 times the sum of the two smallest ($x+(x+d)+(x+2d)+(x+3d)+(x+4d)=100,\ 7[x+(x+d)]=(x+2d)+(x+3d)+(x+4d),\ x=10/6,\ d=55/6$). There is a discussion of AP in the works of Archimedes (for most of his works, see Thomas Little Heath (1861–1940, England, [256]), Hypsicles of Alexandria (fl. around 200 BC, Greece), Brahmagupta, Diophantus of Alexandria (around 250, was either a Greek, a Hellenized Babylonian, an Egyptian, a Jew, or a Chaldean), Zhang Qiujian (around 430–490, China), Bhaskara II or Bhaskaracharya (working 486, India), Alcuin of York (around 735–804, England-France), Dicuilus (fl. 825, Ireland), Fibonacci (Leonardo of Pisa, around 1170–1250, Italy), Johannes de Sacrobosco (around 1195–1256, England), Levi ben Gershon (1288–1344, France). Abraham De Moivre (1667–1754. England) predicted the day of his own death. He found that he slept 15 minutes longer each night, and summing the arithmetic progression, calculated that he would die on November 27, 1754, the day that he would sleep all 24 hours.

A sequence of the form $a, ar, ar^2, ar^3, \cdots, ar^{r-1}, ar^n, ar^{n+1}, \cdots$ is called GP. Here, $a\neq 0$ is called the scale factor and $r\neq 0$ the common ratio. If we denote the general term as $a_n=ar^{n-1}$, then GP satisfies the difference equation (recurrence relation) $a_{n+1}=aa_n,\ n\geq 1.$ It is also interesting to see that $a_{n+1}^2 = a_n a_{n+2}$. It seems first appearance of GP is in Sulbasutras. GPs are also found in Babylonian tablets dating back to 2100 BC. GPs were repeatedly used by Plato.

A sequence of the form

$1/a, 1/(a+d), 1/(a+2d), 1/(a+3d), \cdots, 1/(a+(n-1)d), 1/(a+nd), 1/(a+(n+1)d), \cdots$

is called HP, i.e., a sequence obtained by taking the reciprocals of an arithmetic progression. Here, $a \neq 0$ and $-a/d$ is not a natural number. If we denote the general term as $a_n = 1/(a+(n-1)d)$, then it follows that $2a_n a_{n+2}/(a_n + a_{n+2}) = a_{n+1}$, i.e., each number after the first is the harmonic mean of its immediate neighbors. Pythagoras himself is credited to use first HPs in his music theory.

Eurytus (around 400 BC, Greece) was a disciple of Philolaus and Laërtius. He persuaded further Philolaus' teaching to discover connections between objects and numbers by employing extremely simple method. Archytas confirms that Eurytus announced a certain number of pebbles are the same as in a group of people, and in a herd of horses. Aristotle extended Eurytus' observation to geometrical figures by a fixed number of pebbles/points/dots arranged in a specific way. For example, three points arranged in a certain way are necessary and sufficient to define a triangle, four to define a quadrangle, and so forth. In the literature, natural numbers that are portrayed in orderly geometrical configuration of points and have been labeled as *Pythagoreans Figurative Numbers*. We shall discuss such numbers in detail in Chap. 7.

First major power of numbers that Pythagoras accomplishments was in acoustics. According to a well-recorded fable, one day while walking down a street he heard melodious sound coming from a blacksmith's shop. He went into the shop and anxiously heard the sound of hammers striking a piece of iron on an anvil (a heavy iron block with a flat top, concave sides, and typically a pointed end, on which metal can be hammered and shaped). His first perception was that the difference in tone might be due to the strength of the workers, so he requested them to exchange hammers. He noticed that the disparity in resonance was not due to the men, not from the force of the stroke, not the shape of the hammer, not the changes in the beaten iron, but rather from the vibrations of metal sheets followed by the weights of the hammers which were six, eight, nine, and twelve pounds. He found that the hammers whose weights were in a ratio of $1:2$, i.e., $1/2$ (the six and twelve pounds) produced the interval of an octave, those in a ratio of $3:4$ (the nine and twelve pounds) gave the sound of a fourth, and those in a ratio of $2:3$ (the eight and twelve pounds) produced the musical interval known as a fifth. (The names octave, fourth, and fifth come from the position of these intervals in the musical scale, called consonant intervals, Pythagoras called them "perfect intervals." There is a claim that the 45,000-year-old Divje Babe Flute used a diatonic scale; however, there is no proof or consensus of it even being a musical instrument. There is evidence that the Sumerians and Babylonians used a version of the diatonic scale. Almost 9,000-year-old flutes found in Jiahu, China, indicate the evolution, over a period of 1,200 years, of flutes having 4, 5, and 6 holes to having 7 and 8 holes, the latter exhibiting striking similarity to diatonic hole spacings and sounds.) Pythagoras further realized that the ratios of these three pleasing intervals were all derived from the numbers one, two, three, and four. Pythagoras then experimented with bells; water-filled glasses; pipes cutting them to precisely varying lengths; stretched strings in musical instruments, specially Lyre and Monochord; triangles; and pans of varying sizes. In all cases he found the same general relationship: the more massive an object, the lower the pitch of its sound. With this discovery, Pythagoras advocated that simple ratios of whole numbers rule the laws of musical harmony—and generally, the entire universal phenomena. Perhaps this was the first time a natural phenomenon was described in terms of a precise quantitative expression (the first recoded facts of mathematical physics), and it was an understandable extrapolation that "Numbers Rules the Universe" and that the "essence" of all things is number.

Pythagoras applied his achievement to tuning the stringed instruments so that they would consistently produce musically consonant intervals. In musical tuning, the *Pythagorean comma* is the small interval (or comma) between two enharmonically equivalent notes such as C and B. Archytas gave the numerical ratios for the intervals of the tetrachord on three scales, the enharmonic, the chromatic, and the diatonic. He held that sound was due to impact and that higher tones correspond to quicker, and lower tones to slower, motion communicated to the air. However, Aristoxenus, like his friend and teacher Xenophilus of Chalcidike in Thrace (fourth century BC), who died in Athens at the age of 105, maintained that the true method of determining intervals was by the ear, not by numeral ratio, and the dominant notes of the Universe are proportion, order, and harmony. Pythagoras was an excellent musician, use to play Lyre, sing songs, and often recite from the Odyssey and the Iliad of Homer (fl. ninth or eighth century BC). He used music as a means to control such passions as sadness, anger, lust, despondency, envy, and pride. He was able to pacify both animals and people; in fact, he can be regarded as the founder of music therapy.

One anecdote of Pythagoras reports that when he encountered some drunken youths trying to break into the home of a virtuous woman, he sang a solemn tune with long spondees and the boys' "raging willfulness" was quelled. Another anecdote we are told is: One night, while walking through Croton and observing the skies, he encountered a young man from Tauromenia. Although they were strangers, the man accosted Pythagoras in the street as he passed by and refused to stop when asked to do so. Later that night, further inflamed by drinking and music, the Tauromenian set out to burn down the house of another man. When Pythagoras encountered the man a second time, he ordered a flute player who stood by the change his Phrygian song into a spondaic rhythm. The young man's rage was immediately repressed and he was persuaded to return home peacefully. According to Iamblichus, music featured as an essential organizing factor of his life: The disciples would sing hymns to God Apollo together regularly, they used the Lyre to cure illness of the soul or body, and poetry recitations occurred before and after sleep to aid the memory. The relations between mathematics and music, first discovered by Pythagoras, had a great impact in Renaissance (fourteenth century to seventeenth century) Europe, where cathedrals were designed according to the musical proportions $2:1,\ 3:2,$ and $4:3.$ In 1618, Robert Fludd (1574–1637, England) sketched *mundane Monochord* (also called celestial or divine monochord), which shows God's hand tuning a giant monochord, a string stretched along a sounding board on which the planetary orbits are superimposed over the intervals of the musical scale, see the book [361] of Eli Maor (born 1937, Israel-USA).

It was in Egypt, Babylonia, or India where he probably learnt that the square on the hypotenuse (the longest side of a right triangle) of a right-angled triangle equals the sum of the squares on the other two sides. Here the square on the hypotenuse means the geometrical square constructed on the side, and the sum of two squares is equal to a third square meant that the two squares could be cut up and reassembled to form a square identical to the third square. This theorem is known to the world by his name, i.e., the Pythagoras (or Pythagorean) Theorem, although it was neither discovered nor proved by Pythagoras (see Chap. 5). The converse of this theorem that led to positive integers that satisfy this relation are known as Pythagorean Triples; however, these triples were known and used long before Pythagoras (see Chap. 6). A most obvious right-angled triangle of all-with one legs leads to the *incommensurable* or *irrational* (not logical or reasonable) number $\sqrt{2},$ whose finding is also attributed to the Pythagoreans. In Chap. 8, we will see that the irrationality of $\sqrt{2}$ was discussed several centuries before him. Pythagorean theorem is often cited as the beginning of mathematics in western world. Pythagoreans also knew that the sum of the angles of a triangle is equal to two right angles and that a regular polygon (figure whose sides and angles are all equal) with *n* sides has sum of interior angles $2n-4$ right angles and sum of exterior angles is equal to four right angles. According to Proclus for geometry, Pythagoreans had a conventional phrase "a figure and a platform, not a figure and a sixpence," by which they implied that the geometry that is deserving of study is that which, at each new theorem, sets up a platform to ascend by and lifts the soul on high instead of allowing it to go down among the sensible objects and so become subservient to the common needs of this mortal life. Proclus, quoting from Eudemus of Rhodes (around 350–290 BC, Greece), Pythagoras "changed the study of geometry into the form of a liberal education, for he examined its principles to the bottom and investigated its theorems in an intellectual manner." Thus, his naming geometry as historia, or inquiry, is convincing.

For Pythagoreans geometry consisted of studying the various forms —squares, triangles, circles, etc., and the relationships between them and their parts. They related numbers/algebra to geometry; for example, to multiply $2\times 2,$ they constructed a square with each side equal to 2 units. The area of this square, $4,$ is equal to the product of its sides, i.e., they constructed figures of a given area. This construction they used to solve equations such as $a(a-x)=x^2$ by geometrical means. The Pythagoreans noted that when we attempt to tile a floor with square tiles, we succeed because the meeting point of four right-angled corners leaves no space, that is, four right angles add up to $360°$. Their next observation was that six equilateral triangles (see Fig. 1.3) meeting at a point also leave no space.

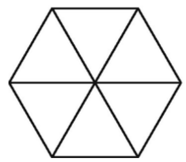

Fig. 1.3 Six equilateral triangles meeting at a point

A geometric solid whose faces are all identical, i.e., regular polygons, meeting at the same three-dimensional angles, is called a solid regular polygon or regular polyhedra. There are only five possible regular polyhedra (see Fig. 1.4):

	Name	Number of Vertices = V	Number of Edges = E	Number of Faces = F
1	Tetrahedron (or pyramid)	4	6	4
2	Hexahedron (or cube)	8	12	6
3	Octahedron	6	12	8
4	Icosahedron	12	30	20
5	Dodecahedron	20	30	12

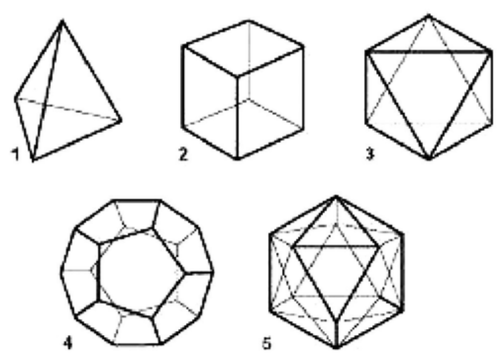

Fig. 1.4 Regular polyhedra (platonic solids)

Historians of mathematics have speculated different priorities about the invention of regular polyhedra, e.g., the first three were known to the

Egyptians, whereas Pythagoras discovered the remaining two; Pythagoras may have been familiar or invented first, second, and fifth, whereas the discovery of the third and fourth belongs to Theaetetus (around 417–369 BC, Greece); Proclus credits Pythagoras with the discovery of all five. Let us believe that Pythagoras knew first four regular polyhedra (it is also believed that among these regular polyhedra Pythagoras asserted that of all solids the sphere is the one perfect, and thus the most beautiful figure of all), and Theaetetus not only discovered the fifth, he also gave a mathematical description of all five and may have been responsible for the first known proof that no other regular polyhedra exist. In the literature the entire group of five regular polyhedra is better known as *Platonic solid* because Plato in his dialogue *Timaeus* described a familiar construction of these solids from the appropriate regular polygons, and assigned the first four identical with the four primal elements (fire, Earth, air, water) of all material bodies: tetrahedron, with its sharp points and edges, to the element fire; hexahedron, with its four-square regularity, to the element Earth; octahedron, its minuscule components are so smooth that one can barely feel it, to the element air; icosahedron, flows out of one's hand when picked up, as if it is made of tiny little balls, to the element water.

The faces of all but the hexahedron are triangles. This blemish was easily obliterated by splitting each face of the hexahedron into two triangles by a diagonal of the square. To earlier Pythagoreans dodecahedron was the most mysterious of the solids; however, Plato also assigned to dodecahedron, with 12 pentagonal faces (also referred to as twelve primordial gods in Greek mythology, and twelve convolutions of the human brain), to the heavens with its 12 constellations. Thus, his systematic development of a theory of the Universe is based on these five regular polyhedra. Aristotle added a fifth element, *ether* (which could be neither seen, tasted, smelled, weighted, not touched) the interpenetrative substance permeating all of the other elements and acting as a common solvent and common denominator of them (Hindus called it *Akasha*) and proposed that the heavens were made of this element, but he had no interest in matching it with Plato's fifth solid. Aristaeus the Elder (around 370–300 BC, Greece) wrote on the five regular solids and on conic sections. Euclid in his *Elements* (A systematic and logical compilation of the works based on his experience and achievements of his predecessors in the three centuries just past, consisting 13 books [chapters or parts] with 465 propositions [in mathematics a proposition is a statement of interest which we prove] on plane and solid geometry, and number theory. This work set the trends how mathematics is written and studied even today. Since 1482, *Elements* has appeared in more editions than any work other than Bible, and it has been translated into countless languages. However, in some of the propositions there are certain gaps as he tacitly used some unstated assumptions.) last Book XIII has completely mathematically described the Platonic solids and their properties. This work has been considered as the crown of the entire work. Of all Platonic solids the tetrahedron encloses the smallest volume for its surface, while the icosahedron encloses the largest.

The symmetry, structural integrity, and beauty of Platonic solids have inspired architects, artists, and artisans from ancient Egypt to the present. Kepler in his 1596 publication *Mysterium Cosmographicum* offered a model of the Solar System in which Platonic solids were installed one another and separated by a series of inscribed and circumscribed spheres. According to him the distance relationships between the six planets (Mercury, Venus, Earth, Mars, Jupiter, and Saturn) known at that time could be perceived in terms of the Platonic solids enclosed within a sphere that represented the orbit of Saturn. The solids were arranged with the innermost being the octahedron, followed by the icosahedron, dodecahedron, tetrahedron, and finally the hexahedron, thereby dictating the structure of the solar system and the distance relationships between the planets by the Platonic solids. Thus, Kepler resuscitated the idea of using the Platonic solids to explain the geometry of the Universe in his first model of the cosmos. Kepler was very proud of his model and said he valued it more than the Electorate of Saxony. He decorated his diagram of the tetrahedron with the drawing of a bonfire; hexahedron with a carrot, a tree, and miscellaneous gardening implements; octahedron with clouds and birds; icosahedron with a lobster and some fish; and dodecahedron with the Sun, Moon, and star. However, later he had to discard his idea, but it led to his three laws of orbital dynamics, the first of which was that the orbits of planets are ellipses rather than circles, which changed completely the course of physics and astronomy.

1.5 Astronomy of Pythagoras

Pythagoras invented the word *kosmos* or *cosmos* (not Carl Sagan, 1934–1996, as reconfirmed by American physicist and Nobel laureate Leon Lederman, 1922–2018) to refer to everything in our Universe, from human being to the Earth to the whirling stars overhead. Kosmos is an untranslatable Greek word that imposes upon the observed heavens exact laws of harmony and adornment. According to Pythagoras, the secrets of the kosmos, many of which are not perceptible to common human senses, are disclosed by pure thought and reasoning through a process that can be expressed in terms of numbers and relationships involving numbers. According to Pythagoras, kosmos is formed of twelve concentric spheres. The farthest sphere is the fixed stars embedded like tiny jewels, where God and deities live (in some communities this thought process continues even today). Next, at equal distance from each other in descending order, there are seven planets (fate-deciding gods in astrology) Saturn, Jupiter, Mars, Venus (highly revered planet because it was the only planet bright enough to cast a shadow, it is called "the morning star" as it is visible before sunrise, and as "the evening star" it shines forth immediately after sunset), Mercury, the Sun, and the Moon (Pythagoras referred to the Sun and the Moon as gods, as these heavenly bodies are the origin of the principle that is the cause of living things). Below the sphere of the Moon are the spheres of fire, air, and water. These planets and essential spheres rotate around the Earth in a circular (the most perfect of all shapes) motion, and the Earth revolves in a stationary position at the center of the Universe. In this ordering, the farthest sphere was perfect in all aspects, and then each was down to the sphere of the Moon. Below the Moon there was increasing disorder, and the Earth was regarded as the least perfect of all spheres.

Pythagoras established that day and night were a result of the Earth's revolution, and the change in seasons was due to the tilt of the Earth's axis relative to the Sun. He taught that the Moon shines by the reflected light of the Sun and thus implied the nature of solar and lunar eclipses. From Pythagoras originated the doctrine of the "harmony of the spheres," a theory according to which the heavenly bodies emit constant tones that correspond to their distances from the Earth. Pythagoras reputedly claimed to hear the "music of the spheres" and thus produce an inaudible symphony. According to Porphyry, Pythagoras taught that out of nine, seven Muses [the word museum derives from the Muses] (daughters of Zeus and Mnemosyne): Calliope (epic poetry), Clio (history), Euterpe (flutes and music), Thalia (comedy and pastoral poetry), Melpomene (tragedy), Terpsichore (dance), Erato (love poetry and lyric poetry), Polyhymnia (sacred poetry), Urania (astronomy) were actually the seven planets singing together. The notion music of the spheres became an inspiration for many Renaissance scientists including Kepler who spent/wasted almost thirty years of his life seeking to discover the laws of planetary motion in musical harmony, before formulating his three laws about the motion of the planets. However, nothing is known about Pythagoras's teachings regarding the influence of planetary movement on human behavior (astrology).

When Pythagoras was asked why humans exist, he said, "to observe the heavens," and he used to claim that he himself was an observer of nature, and it was for the sake of this that he had passed over into life. When the same question was asked to Anaxagoras (around 500–428 BC, Greece) he quoted the same words of Pythagoras and added "To observe the heavens and the stars in it, as well as the Moon and the stars', since everything else at any rate is worth nothing." However, Burkert condemns all earlier commentators of Pythagoras. He writes (see Burkert [109]) "Pythagoras was a charismatic political and religious teacher, the number philosophy attributed to him was really an innovation by Philolaus. Pythagoras never dealt with numbers at all, let alone made any noteworthy contribution to mathematics. The only mathematics the Pythagoreans ever actually engaged in was simple, proof less arithmetic, but that these arithmetic discoveries did contribute significantly to the beginnings of mathematics." Bernabé and Mendoza [68] after their instructive comparison of the Pythagorean and Vedic cosmogony, with little hesitation, affirm that some of these ideas "arrived to the Greeks through direct contact between wise men and priests; but it is also probable that some of them were transmitted through secondary channels, for example by way of traders, soldiers and slaves, in the same way that other fables and folk-tales travel from one culture to another." See also the concluding remarks in Afonasin and Afonasina [8] and the book of Dahlquist [145].

Philia is an ancient Greek word that is usually translated as friendship or affection, and its opposite is phobia. Aristotle gives following examples of philia: young lovers, lifelong friends, cities with one another, political or business contacts, parents and children, fellow voyagers and fellow soldiers, members of the same religious society, or of the same tribe, and a cobbler and the person who buys from him. But, to Pythagoras, philia is a cosmic force that attracts all the elements of nature into harmonious relationships. It helps to preserve the order of planets as they move across the sky, and encourage men and women, once their souls have been purified, to help one another. He taught that each person has a responsibility to observe the law of philia in every aspect of life, in particular, cultivates friendship between: gods and men, the body and the three parts of the soul, citizens and states, and husband, wife, children, and neighbors.

Pythagoreans honored friendship so highly is exhibited by the following anecdote told by Aristoxenus, and after him Marcus Tullius Cicero (106–43 BC, Italy), Diodorus Siculus (90–30 BC, Greece), and several others: Pythagoreans Pythias and Damon traveled to Syracuse during the reign of the tyrannical Dionysius I. Pythias was accused of conspiring to assassinate the tyrant and sentenced to death. Pythias requested if his execution is unavoidable, he may be allowed to return home one last time to settle his affairs and bid his family farewell. Dionysius I refused his request, convinced that, once released, Pythias would flee and never return. But then Damon offered himself as a hostage in Pythias' absence, and when the tyrant insisted that, should Pythias not return by an appointed day and time, Damon would be executed in his stead, Damon agreed and Pythias was released. Pythias promised to return on the appointed day and time came and went, Dionysius I called for Damon's execution—but just as the executioner was about to kill Damon, Pythias returned. Confessing to his friend for the delay, Pythias clarified that on the passage back to Syracuse pirates had captured his ship and thrown him overboard, but that he swam to shore and made his way back to Syracuse as quickly as possible, arriving just in time to save his friend. Dionysius I was so astonished by and pleased with their unquestioning loyalty that he pardoned Pythias. The tyrant then sought to become their third friend, but it was denied. Another version (which makes more sense) says that it was planned by

Dionysius I and his courtiers to test Pythagoreans live up to their reputation.

1.6 Cup of Pythagoras

A Pythagorean cup (also known as the greedy cup) looks like a normal drinking cup, except it was designed to hold an optimal amount of wine, forcing people to consume only in moderation—a virtue of great regard among ancient Greeks. The cup has a central column in it, which is positioned directly over the stem of the cup and over a hole at the bottom of the stem. A small open pipe runs from this hole almost to the top of the central column, where there is an open chamber. The chamber is connected by a second pipe to the bottom of the central column, where a hole in the column exposes the pipe to (the contents of) the bowl of the cup. When the cup is filled, wine rises through the second pipe up to the chamber at the top of the central column. As long as the level of the wine does not rise beyond the level of the chamber, the cup functions as normal. If the level rises further, however, the entire wine spills through the chamber into the first pipe and out of the bottom. Most modern toilets operate on the same mechanism: When the water level in the bowl rises high enough, a siphon is created, which empties the bowl.

1.7 Doctrine of Pythagoras

Frank Higgins [262] provides an excellent summary of the Pythagorean doctrine as follows: "Pythagoras's teachings are of the most transcendental importance to Masons, inasmuch as they are the necessary fruit of his contact with the leading philosophers of the whole civilized world of his own day, and must represent that in which all were agreed, shorn of all weeds of controversy. Thus, the determined stand made by Pythagoras, in defense of pure monotheism, is sufficient evidence that the tradition to the effect that the unity of God was the supreme secret of all the ancient initiations is substantially correct. The philosophical school of Pythagoras was, in a measure, also a series of initiations, for he caused his pupils to pass through a series of degrees and never permitted them personal contact with himself until they had reached the higher grades. According to his biographers, his degrees were three in number. The first, that of *Mathematicus*, assuring his pupils proficiency in mathematics and geometry, which was then, as it would be now if Masonry were properly inculcated, the basis upon which all other knowledge was erected. Secondly, the degree of *Theoreticus*, which dealt with superficial applications of the exact sciences, and, lastly, the degree of *Electus*, which entitled the candidate to pass forward into the light of the fullest illumination which he was capable of absorbing":

- Eli Maor, in his book [361], mentions that during his recent visit to Pythagoria, he noticed that in Samos, Pythagoras is a household name, the main square is named after him, as is a street, a high school, and at least one Hotel. Martin Bohner (born 1966, Germany-USA) visits Samos every few years, he also hikes up to the cave where Pythagoras used to hide, seen the tunnel, his statue, and purchased a few Pythagorean cups.
- For further speculative details of Pythagoras's life, philosophy, mathematics, and astronomy, see Bamford [54], Day [157], De Vogel [161], Dreyer [170], Friedricks [199], Von Fritz [201], Gorman [217], Leslie [343], McClain [364], O'Meara [400], Philip [409], Riedweg [433], Strohmeier and Westbrook [499], Taylor [507], and Thesleff [508].

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2. Numbers and Number Mysticism

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2.1 Introduction

Consciously/unconsciously/subconsciously in every aspect of our life we use numbers. In fact, number is the key to all arts, sciences, technologies, and all philosophies. According to Plato "Arithmetic has a very great and elevating effect, compelling the soul to reason about abstract number," Aristotle "The elements of numbers are the elements of things, and that the whole heaven is a harmony and a number. He further says, Pythagoreans considered number as the substance of all things," John Locke (1632–1704, England) "The simple modes of number are of all other the most distinct; even the least variation, which is a unit, making each combination as clearly different from that which approacheth nearest to it, as the most remote, two being as distinct from one, as two hundred, and the idea of two as distinct from the idea of three, as the magnitude of the whole Earth from that of a mite," and Auguste Comte (1798–1857, France) "There is no inquiry which is not finally reducible to a question of numbers; for there is none which may not be conceived of as consisting in the determination of quantities by each other, according to certain relations."

In the domain of the history of numbers, at least, human intelligence is universal and that the progress has been achieved in the mental, cultural and collective endowment of the whole humankind. Owing to the genius of the Indian arithmeticians that three significant ideas were combined: nine numerals which gave no visual clue as to the numbers they represented and which constituted the prefiguration of our modern numerals, the discovery of the place-value system, which was applied to these nine numerals, making them dynamic numerical signs, and the invention of zero and infinity and their enormous operational potential, see Dutta [175]. In India numbers were in search of enlightenment. As far as we could go in the past, the concepts of natural numbers, rational and irrational numbers, zero, infinity, and the place-value system among others referred in mathematical portion of *Vedic Samhitas* (a collection of all four Vedas). A dice made of fired clay dating from 3000 BC has been found in northern Iraq; however, the ancient players did not have symbols for numbers, that is why they marked the sides of the dice with dots. In this chapter mainly we shall study natural, negative, and rational numbers, number zero, and legitimacy of infinity. We shall also discuss properties and mystic aspect of natural numbers that is specially the union with the divine or sacred. The last section will be devoted to complex numbers. The irrational numbers we shall study in detail in Chap. 8.

2.2 Natural Numbers

From early times numbers were words, which refer to collections of objects. Around 4000 BC, special symbols for numbers appeared in the river valleys of the Indus (which was home to more than five million people), the Nile, the Tigris and Euphrates, and the Yangtze. As these symbols evolved, their utility slowly improved, and they began to assist man in assimilating, combining, distinguishing, remembering, and expressing big numbers. The next step forward was the creation of names for the numbers and (due to applications to practical problems) the basic operations. The signs for these operations $+, -, \times,$ and $\div,$ respectively, first appeared in the works of Nicole Oresme (1320–1382, France) in 1360, Johann Widmann (1462–1500, Czechoslovakia) in 1489, John Napier (1550–1617, Scotland) in 1618, and Johann Heinrich Rahn (1622–1676, Switzerland) in 1659. In antiquity, counting was considered a talent as mystical and arcane as casting spells and calling the gods by name. In the Egyptian *Book of the Dead*, when a dead soul is challenged by Aqen, the ferryman who conveys departed spirits across a River Styx in the netherworld, Aqen refuses to allow anyone aboard “who does not know the number of his fingers.” The soul must then recite a counting rhyme to tally his fingers, satisfying the ferryman.

Almost from the last 2500 years philosophers have been unsuccessful in providing satisfactory answer to the questions like “what is an integer” and “in what sense do integers exist?” However, at least we can say numbers are abstract objects of the mind that allow us to understand our world and our lives with greater clarity. The numbers $1, 2, 3, 4, \cdots$, in general represented as $n,$ have been called as *natural numbers* or *positive integers* because it is generally perceived that they have in some philosophical sense a natural/divine existence independent of man. We will never know if there existed a genius who, when, where, and how invented or introduced these natural numbers (the very origin of numbers is a mystery), but it is generally accepted that these numbers came down to us, ready-made, from an antiquity most of whose aspects are preserved in folklore rather than in historical documents. In India, the earliest written first nine integers in Sanskrit appeared in the Vedic Samhita and the Brahmana (commentaries on the Vedas) as follows: eka (1), dvi (2), tri (3), catur (4), panca (5), sat (6), sapta (7), asta (8), and nava (9). Here we also find the derived decuple terms for the first nine multiples of ten: dasa (10), vimsati (20), trimsat (30), catvarimsat (40), pancasat (50), sasti (60), saptati (70), asiti (80), and navati (90); also terms for powers of ten up to Koti (ten million).

While natural numbers are primarily used for counting finite collections of objects, there is hardly any aspect of our life in which natural numbers do not play a significant-though generally hidden part, e.g., time you woke up today, your birthday, your phone number. In fact, natural numbers are building blocks of all sciences and technologies. *Number Theory*, which mainly deals with properties, relationships, speculations/conjectures, and answers/formulas to amusing problems of natural numbers, without the need for a real life application—in fact, mathematics done purely for the sake of doing mathematics—has been classified as *Pure Mathematics*. According to Hardy “The theory of numbers, more than any other branch of mathematics, began by being an experimental science. Its most famous theorems have all been conjectured, sometimes a hundred years or more before they were proved; and they have been suggested by the evidence of a mass of computations.” Isaac Albert Barnett (1894–1974, England-USA) said: To discover mathematical talent, there is no better course in elementary mathematics than number theory. Since antiquity, number theory has captivated the best minds of every era. An important feature of number theory is that challenging problems can be formulated in very simple terms; however, hidden within their simplicity is complexity. Some of these problems have been instrumental in the development of large parts of mathematics. Amateurs and professionals are on an almost equal footing in this field. The set (Bernhard Placidus Johann Nepomuk Bolzano, 1781–1848, Czech Republic, introduced the notion of set [in German: Menge]) of all natural numbers is denoted as $\mathcal{N}$.

2.3 Number Sense

This faculty can be defined as an intuitive understanding of the integers, their magnitude, their patterns and relationships, and how they are affected by the basic operations. A child between twelve and eighteen months is able to recognize if an object is removed from or added to a small collection. This simple number sense, the ability to distinguish "plenty" and "few" without counting, is a useful tool for a conscious being and a fundamental ability of humans. For primitive man and children, mathematics is simply a comparison of small collections. So far as nonhuman living beings are concerned, there are recorded incidences of birds, animals, insects, and aquatic creatures who show through their behavior a rudimentary number sense, namely, comparing/sorting. Birds have shown that they can be trained to determine the number of seeds in different piles of seeds. Birds navigate accurately over hundreds of miles of open sea, and, according to one theory, can use the stars for guiding them at night. It was recently reported that five penguins were taken by airplane from their home at Wilkes Station to Mcmurdo Sound, in Antarctica. When they were released, three of them waddled more than 2000 miles across the bleak and monotonous ice to return to their native rookery. Mothers of domestic animals and other animals have shown that they can definitely determine and perceive when one of their young is missing from the group. Animals such as dogs, horses, and elephants are displayed to have the abilities to add, subtract, and count. *Clever Hans* a horse owned by Wilhelm von Osten (Germany) became famous around 1904 for performing arithmetical and other intellectual tasks. Then a wealthy German businessman named Karl Krall (1863–1929) announced he had trained three other horses which were smarter than Hans. These horses could not only solve complex mathematical calculations and recognize people, he said, they could also transmit the correct answer to questions via the newly invented telephone. Nobel Laureate, Maurice Polydore Marie Bernard Maeterlinck (1862–1949, Belgium) went to inspect these educated German horses and concluded, "You rub your eyes and ask yourself in the presence of what new creature you stand. You look for some trace, obvious or subtle, of the mystery. You feel yourself attacked in your innermost citadel where you held yourself impregnable." The behavior of insects, such as the solitary wasp, indicates some basic mathematical understanding. The mother wasp lays her eggs in individual cells and provides each egg with a number of live caterpillars, on which the young will feed on when they hatch. Honey bees can keep track of directions by using the Sun, and they use the state of polarization of the light to tell them where the Sun is when it is cloudy. Marine mammals could be far more skilled at math than was ever thought possible before, for example, Dolphins may use complex nonlinear mathematics when hunting. Salmon find their way back to the streams where they were born, very possibly identifying their native location because it smells to them like home. Mice learn from experience what turns to make in order to "run" a maze and to end up at the place where food will reward them. It is observed that a cat makes no objection when she was relieved of two of her six kittens, but was plainly distressed when she was deprived of three.

2.4 Rational Numbers

A positive *rational number* is defined as the exact ratio/fraction/quotient of two positive integers $p/q,$ where $q \neq 0.$ Here *p* is called the numerator (the numberer), whereas *q* the denominator (namer). The ancient Chinese called the numerator "the son" and the denominator "the mother." It is very likely that the notion of rational numbers also dates to prehistoric times, but owing to the lack of a good notation in the beginning, they were not treated as numbers. Around 4000 BC, rational numbers were used to measure various quantities, such as length, weights, and time in the Indus river valley. Thus, then rational numbers were sufficient for all practical measuring purposes. The Babylonians used elementary arithmetic operations for rational numbers as early as 2000 BC. We also find ancient Egyptians texts describing how to convert general fractions into their special notation. Ancient Greek and Indian mathematicians made studies of the theory of rational numbers, as part of the general study of number theory, see *Elements* and *Sthananga Sutra* (around third century). Note that rational numbers are dense, i.e., between two rational numbers (no matter how close they are) there is a rational number. The set of all positive rational numbers is denoted as $\mathcal{Q}^+$.

2.5 Negative Numbers

Throughout the ancient history negative solutions of linear and quadratic equations have been called as absurd/ugly/unpleasant solutions. First systematic use of negative numbers in mathematics for finding the solutions of determinate and indeterminate systems of linear equations of higher order with both positive and negative numbers appeared in the commentary by Liu Hui (around 220–280, China) on the *Jiuzhang Suanshu* (Nine Chapters on the Mathematical Art), see Shen et al. [475], which came into being in the Eastern Han Dynasty (202 BC–220 AD) and is believed to have been originally written around 1000 BC, much before Han Dynasty; however, multiplication of negative numbers never used. In appreciation, the historian Jean-Claude Martzloff (1943–2018, France) theorized that the importance of duality in Chinese natural philosophy made it easier for the Chinese to accept the idea of negative numbers (see [362]). Brahmagupta in his treatise *Brahma-sphuta-siddhanta* (Brahma's correct system) treated positive numbers in the sense of "fortunes" (dhana) and negative numbers as "debts" (rina), which we still use in taxes, income, loans, and other things. Clearly, fortunes and debts are equalled to the inverse of the other. These terms continued until the twelfth century. Brahmagupta also set rules for dealing with negative numbers (very similar to the methods we still use today). He also found negative solutions of quadratic equations.

Bhaskara II gave $x = 50$ and $x = -5$ as the roots of the equation $x^2 - 45x = 250,$ but cautioned, "The second value in this case is not to be taken, for it is inadequate; people do not approve of negative roots." He also expressed: The square of a positive number, as also that of a negative number, is positive; and the square root of a positive number is two-fold, positive, and negative; there is no square root of a negative number, for a negative number is not a square. Ibn Yahya al-Maghribi Al-Samawal (around 1130–1180, Iraq) stepped away from fortunes and debts and produced an algebra with positive, negative numbers, and an "empty power." Michael Stifel (1486–1567, Germany) in his famous work *Arithmetica Integra* (1554) dealt with negative numbers where he called them numeri absurdi (absurd numbers). Ch'in Chiu-Shao (around 1202–1261, China) also known as Qin Jiushao, on the counting board used black and red colors for negative and positive numbers. Fra Luca Bartolomeo de Pacioli (1447–1517, Italy) is credited with a knowledge of the rule of signs on such evidence as $(7-4)(4-2)=3\times 2=6.$ In 1545, Girolamo Cardano (1501–1576, Italy) in his book *Ars Magna* (crown of all algebra up to his time) recognized negative numbers and stated their rules, specially "minus times minus gives plus." Despite this, he was still reluctant to use negative numbers in practical calculations. In fact, he called negative solutions "fictitious," whereas positive solutions were deemed "true."

In 1572, Rafael Bombelli (1526–1573, Italy) claimed that he understood the rules of addition in such instances as $m-n,$ where m, n are positive integers. François Viéte (1540–1603, France) rejected negative roots. Thomas Harriot (1560–1621, England) in his book *Artis analyticae praxis* (Practice of the Analytic Art), which appeared posthumously in 1631, allowed negative number in an equation, but refused to admit negative roots. Albert Girard (1595–1632, France) was among the first to note the geometric meaning of a negative solution to an equation: "The minus solution is explicated in geometry by retrograding; the minus goes backward where the plus advances." He even gave an example of a geometric problem whose algebraic translation has two positive and two negative solutions and noted on the relevant diagram that the negative solutions were to be interpreted as being laid off in the direction opposite that of the positive ones. René Descartes (1596–1650, France) avoided using negative quantities, which he called "false," on the grounds that nothing should be less than "nothing" though he did not forbid their use. John Wallis (1616–1703, England) rejected the idea that a negative number is less than nothing but accepted the view that it is something greater than infinity. He argued that since the ratio $a/0,$ when *a* is positive, is infinite, then, when the denominator is changed to a negative number, as in a/b with *b* negative, the ratio must be greater than infinity. Wallis is credited for giving some geometric meaning to negative numbers by inventing the number line. In 1659, Johann van Waveren Hudde (1629–1704, Netherland) used a letter to denote a positive or a negative number reluctantly. Newton assigned affirmative motion as progression and negative motion as regression. This paved the way for him to progress on his path of physics. The formal infinite binomial expansion, i.e.,

$$(a+b)^r=\sum_{k=0}^{\infty}\binom{r}{k}a^{r-k}b^k=a^r+ra^{r-1}b+\frac{r(r-1)}{2!}a^{r-2}b^2+\frac{r(r-1)(r-2)}{3!}a^{r-3}b^3+\cdots; \tag{2.1}$$

here,

$$\binom{r}{k} = \frac{r(r-1)\cdots(r-k+1)}{k!}, \tag{2.2}$$

applied to $(1-2)^{-1}$ gives $-1 = 1+2+4+8+16+\cdots$, a meaningless result which did not astonish Euler [during his seventy-six years there was scarcely an aspect of mathematics which he did not leave more systematized, then he had found it]. Further, dividing $1-x$ into *x* and $x-1$ into $x,$ and then adding the results obtained the ridiculous result found by Euler

$$\begin{aligned}\frac{x}{1-x}+\frac{x}{x-1} &= x(1-x)^{-1}+(1-1/x)^{-1}\\ &= \cdots+\frac{1}{x^2}+\frac{1}{x}+1+x+x^2+\cdots = 0\end{aligned}$$

for all *x* different from 0 and $1.$ Clearly, for $r=n,$ a nonnegative integer, (2.2) gives the *binomial coefficients*

$$\binom{n}{k} = \begin{cases} \dfrac{n!}{k!\,(n-k)!} = \dfrac{n(n-1)(n-2)\cdots(n-k+1)}{k!}, & k\le n\\ 0, & k>n \end{cases} \tag{2.3}$$

and the infinite expansion (2.1) reduces to the finite expansion known as *binomial theorem* (first explicitly appeared in the writings of Brahmagupta, see Biggs [71], and then in Halayudha's *meruprastaara*, around 975)

$$\begin{aligned}(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k &= a^n + na^{n-1}b+\frac{n(n-1)}{2!}a^{n-2}b^2\\ &+\frac{n(n-1)(n-2)}{3!}a^{n-3}b^3+\cdots+b^n.\end{aligned} \tag{2.4}$$

The ancient Greeks did not have to deal with negative numbers because their math was founded on geometric ideas. In fact, Diophantus in one of the problems of his collection had a negative four as the formal solution which he considered "absurd" (he is also credited to be the first Greek mathematician who frankly recognized fractions as numbers). Unfortunately, in Britain pessimistic attitude toward negative numbers continued till the eighteenth century, in fact, William Frend (1757–1841, England) took the view that negative numbers did not exist, whereas his contemporary Francis Maseres (1731–1824, England) in 1759 wrote that negative numbers "darken the very whole doctrines of the equations and make dark of the things which are in their nature excessively obvious and simple." He came to the conclusion that negative numbers were nonsensical. Even as late as in the nineteenth century, the English mathematician Augustine De Morgan (1806–1871) said, "We have showed the symbol $\sqrt{-a}$ to be void of meaning, or rather self contradictory and absurd. Nevertheless, by means of such symbols, a part of algebra is established which is of great utility." However, from the middle of the nineteenth century, negative numbers received their relevance logically across the world. Today, negative numbers are used in everyday life without even realizing, e.g., temperature, purchases, electrical charges, and withdrawals.

2.6 Zero as a Number

In Sanskrit Sunya is a word with meaning "zero," "nothing," "empty," or "void." George Joseph [289] suggests that its origin can be traced in Rigveda. Sunya for several centuries constituted the central element of a mystical and religious philosophy which had become a way of thinking. Sunya is derived from the root, svi, meaning "hollow." It is also the root word for sunyata, which means "emptiness" or "nothingness." Sunya has many synonyms: Abhra (atmosphere), Akasha, Ambara (space), Ananta (immensity of space), Antariksha (space), Gagana (canopy of heaven is represented by human beings either by a semicircle or by a complete circle; graphically the little circle for zero might have originated from this perception), Jaladharapatha (voyage on water), Kha (empty space), Nabha (sky, atmosphere), Nabhas (sky, atmosphere), Purna (fullness, wholeness, integrity, completeness), Randhra (hole), Shunya (void), Vindu/Bindu (point or dot, represents the Universe in its non-manifest form, before its transformation into the world of appearances), Vishnupada (foot of Lord Vishnu), Vyant (sky), Vyoman (sky, space), and hence represents the Universe.

The idea of "nothingness" is fundamental in Buddhist philosophy also, and it is called as *Madhyamaka* (middle way) which teaches that every thing in the world is empty, impermanent, impersonal, painful, and without original nature. According to Buddha in Sunya there is no form, no sensation, no idea, no volition, no consciousness, no eyes, no ears, no nose, no tongue, no body, no mind, no color, no noise, no smell, no taste, no contact, no elements, no ignorance, no knowledge, and no aging or death. Thus, the Indian concept of zero far surpassed the heterogeneous notions of vacuity, nihilism, nothingness, insignificant, absence, and nonbeing of Greek and Latin philosophies. For ancient Hindus Sunya was not the number zero, rather it was a mechanical device to indicate an empty space; however, with Sunya the symbol zero, $\odot$ (a dot inside a circle) had been invented; but the number zero was yet to be discovered. Unparalleled "Archimedes rejected zero, which is the bridge between the realms of the finite and the infinite, a bridge that is absolutely necessary for calculus and higher mathematics." The mathematical and astronomical tablets from the Seleucid era (312–63, BC) employ a numerical system which includes a separation mark for zero. The zero occurs either at the beginning of a number or within it, but never at the end.

Bakhshali Manuscript (around 200 BC) was found in 1881 in the village Bakhshali in Gandhara, near Peshawar, North-West India (present-day Pakistan). The very word manuscript comes from the Latin words meaning "written by hand." It has been held at the Bodleian Libraries (England) since 1902. It is written in an old form of Sanskrit on birch barks of which only about 70 (mutilated) are available. This manuscript gives various algorithms and techniques for a variety of problems, such as computing square roots, dealing with negative numbers (using $+$ as a negative sign), and finding solutions of quadratic equations. The text contains negative numbers and zero as well as several numerical entries expressed in numerical symbols. Ptolemy used a circular symbol for zero at the end of a number. A black dot on a third-century manuscript has been identified by Oxford University as the mathematical symbol for zero. A dot was also used for zero in the Sharada system of Kashmir (eight to twelve centuries) and in the vernacular notations of Southeast Asia. The point is the most insignificant geometrical figure, constituting as it does the circle reduced to its simplest expression, its center.

The Muslims translated the Hindu's Sunya as sifr (khala, faragh, [it led to the word cipher which means secret code]), and in east (Asia, Baghdad) they also represented it by dot "$\cdot$," whereas in west (North Africa, Spain) by circle "$\circ$." Zero found its way to Europe through the Moorish conquest of Spain, but then the sifr was assimilated to a near-homophone in Latin, zephyrus, meaning "the west wind" and, by rather convenient extension, a mere breath of wind, a light breeze, or—almost—nothing. Fibonacci in his treatise *Liber Abaci* (The Book of Calculation) of 1202 wrote of the symbol as zephirum, the term remained in use in Italy until the fifteenth century, except in Venetian dialect it was called zefiro, and its contraction finally became zero. Jordanus Nemorarius (1225–1260, Germany) changed it to Spanish word cifra. In various spelling, the term sifra, cyfra, cyphra, zyphra, tzphra, etc., continued to be used to mean zero by some mathematicians for many centuries. In the translation of Maximus Planudes' (1260–1305, Bithynia) work *Psephophoria kata Indos* (Methods of Reckoning of the Indians) in 1340, the symbol was called as Tzipha, and this form was still used as late as the sixteenth century. In Italian it was called Zenero, Cenero, and Zephiro. Since the fourteenth century, zero has been used as shown in the records of 1491 by Calnadri and of 1494 by Pacioli. The Latin word nulla appears in Italian translations of Muslim writings of the twelfth century, and also in French by Nicolas Chuquet (around 1445–1488, France) in his work *Triparty en la science des nombres* (he also used negative numbers as exponents but referred to them as absurd numbers), and German of the fifteenth century. The English word cipher was used for zero by Adrian Metiers (France) in 1611, Pierre Hérigone (1580–1643, France) in 1634, Bonaventura Francesco Cavalieri (1598–1647, Italy) in 1643, and Euler in 1783, even though the more modern German word Ziffer had been introduced. For zero several other names are in common use such as nil, aught, nada, naught, nix, nought, zilch, zip, zot, and the list continues. In modern uses zero is denoted as 0, which is an interesting return to the Greek name omicron, although negative numbers and zero were impossible in Greek geometry, for a line cannot be zero or less in length. Aczel [2] in his book advocates that zero first appeared in Cambodia (then a part of Greater Bharat), which cannot be believed. Jean Chevalier (1906–1993, France) and Alain Gheerbrant (1920–2013, France) in their collection of 1982 *Dictionary of Symbols* write "In drawings and pictograms, the canopy of heaven is universally represented either by a semicircle or by a circular diagram or by a whole circle. The circle has always been regarded as the representation of the sky and of the Milky Way as it symbolizes both activity and cyclic movements. Thus the little circle, through a simple transposition and association of ideas, came to symbolise the concept of zero for the Indians."

To recognize the unique understanding of zero among the positive and negative numbers, we arrange them in order. The positive numbers to the right and negative numbers to the left, with every consecutive distance 1. But then the distance between -1 and 1 is twice the distance between any other pair, and hence they are not consecutive. Thus, to preserve the regularity between -1 and $1,$ the Hindu scholars filled the gap by giving the status of a number to zero 0 (logically at home with the natural numbers)

$$\cdots -5, -4, -3, -2, -1, 0, +1, +2, +3, +4, +5, \cdots .$$

Among the integers, zero is unique, being neither negative nor positive, provides mirror images of natural numbers $-n = 0 - n,$ only number which has an infinite number of divisors, and lies between two odd numbers -1 and 1 and hence even, also above all the most basic concept in the progress of all modern sciences, astronomy, industries, abstract mathematics, specially, modern algebra (cornerstone of modern mathematics). The set of integers including positive, negative, and zero is denoted as $\mathcal{Z},$ and the set of all rational numbers is represented by $\mathcal{Q}.$ It is clear that $\mathcal{Z} \subset \mathcal{Q}.$

The problem of setting the rules to consider negative and zero as numbers with regard to the basic operations Brahmagupta (he defined zero as the result of the subtraction of a number by itself, and for zero used the symbol a dot underneath numbers) laid down the following rules: A debt plus or minus zero is a debt, a fortune plus or minus zero is a fortune, zero plus or minus zero is a zero, a debt subtracted from zero is a fortune, a fortune subtracted from zero is a debt, from a fortune or debt if you subtract itself you obtain zero, the product of zero multiplied by a debt or fortune is zero, the product of zero multiplied by zero is zero, the product or quotient of two fortunes is one fortune, the product or quotient of two debts is one fortune, the product or quotient of a debt and a fortune is a debt, and the product or quotient of a fortune and a debt is a debt. Brahmagupta, however, struggles when it comes to division by zero: Fortune or debt when divided by zero is a fraction the zero as denominator. Zero divided by fortune or debt is either zero or is expressed as a fraction with zero as numerator and the finite quantity as denominator, and zero divided by zero is zero. Clearly, he is wrong in claiming that zero divided by zero is zero.

Bhaskara II called zero as *Ganita Chakra Chudamani* (the gem in the circle of mathematicians), further in his treatise *Bijaganita* (Basic Arithmetic) writes: A quantity divided by zero becomes a fraction the denominator of which is zero. This fraction termed an infinite quantity (Euler viewed that $2/0$ is twice as big as $1/0$). In this quantity consisting of that which has zero for its divisor, there is no alteration, though many may be inserted or extracted; as no change takes place in the infinite and immutable God when worlds are created or destroyed, though numerous orders of beings are absorbed or put forth. Bhaskara II also could not divide zero by zero, in fact, he could set the problem. “What number is it, which multiplied by zero, and added to half itself, and multiplied by three, and divided by zero, amounts to the given number 63”? He was simply thinking of the equation

$$\frac{3(0x + \frac{1}{2}0x)}{0} \ = \ 63,$$

which by factoring out the zeros in the numerator and “canceling” becomes $3x + \frac{3}{2}x = 63,$ an equation whose solution is 14. However, he did correctly state other properties of zero such as $0^2 = 0$ and $\sqrt{0} = 0.$ Mahavira (817–875, India) who said “Whatever object exists in this moving and non-moving world, cannot be understood without the base of Ganit, i.e. mathematics” in his text *Ganita-Sara-Samgraha*.

In appreciation of *Bijaganita* the mathematician and historian of mathematics David Eugene Smith (1860–1944, USA) commented “$\cdots$ oriental mathematics possesses a richness of imagination, an interest in problem solving and poetry, all of which are lacking in the treatises of the West, although abounding in the works of China and Japan.” The Arabs were particularly attracted by the poetical, the rhetorical, and the picturesque in the Hindu treatment of the subject, more than by the abstract approach. Mahavira brings himself to the point of admitting that one could not divide by zero writes: A number remains unchanged when divided by zero. Thus, Hindu mathematicians and astronomers were unsuccessful in denying Aristotle’s command “we must not divide by zero,” which he realized in connection with speed through a vacuum. Fallacies resulting from division by 0 are rarely presented in simple form that they may be detected at a glance. For this, we assume $A + B = C,$ and $A = 5$ and $B = 3.$ Multiplying both sides of the equation $A + B = C$ by $(A + B),$ we obtain $A^2 + 2AB + B^2 = C(A + B).$ Rearranging the terms, we have $A^2 + AB - AC = -AB - B^2 + BC.$ Factoring out $(A + B - C),$ we get $A(A + B - C) = -B(A + B - C).$ Now dividing both sides by $(A + B - C),$ i.e., dividing by zero, we get $A = -B,$ or $A + B = 0,$ which is evidently absurd. In fact, even today mathematically,

$0/0$ is neither meaningful nor meaningless, it is *indeterminate*, and it can have any numerical value. (There are seven indeterminate forms which are typically considered in the literature: $0/0, \infty/\infty, 0 \times \infty, \infty - \infty, 0^\circ, 1^\infty$, and ∞°.) In conclusion, zero is the only number which can be divided by every other number, and the only number which can divide no other number. In fact, if any mathematical model represents the physical problem exactly, then division by exact zero must not be occurring in the model. Recently, Saitoh [443] in his book has tried to show division by zero is possible in a natural sense, but it is too early for the mathematical community to appreciate its repercussions.

Sage Veda Vyasa is also the narrator of the epic *Mahabharata* of which *The Bhagavad Gita* (meaning "The Song of the Supreme") is a part. In this epic there is a phrase which says a combination of nine (digits) always (suffices) for any count (or calculation). Aryabhata (born 2765 BC, India) whose legacy continues to baffle mathematicians and astronomers, for details of his astonishing contributions, see Agarwal and Sen [14] and Keller [304], invented a unique method of recording numbers which required perfect understanding of zero and the place-value system. A decimal system was already in place in India during the Harappan period, which emerged before 2600 BC along the Indus River valley, as indicated by an analysis of Harappan weights and measures. In fact, weights corresponding to ratios of $0.05,\ 0.1,\ 0.2,\ 0.5,\ 1,\ 2,\ 5,\ 10,\ 20,\ 50,\ 100,\ 200,$ and 500 have been identified. A bronze rod marked in units of 0.367 inches points to the degree of precision demanded in those times. *Puranas* (meaning ancient, is a vast genre of Indian literature, dated before 2000 BC, and texts were composed primarily in Sanskrit during 350–750) are named after Hindu deities such as Brahma (the creator), Vishnu (the preserver), Shiva (the destroyer), and Shakti (Shiva's power), and legends, heroes, astronomy, and philosophy that contain religious aspects. In the Puranas, nine ordinary names of numbers and the term shunya play a great importance in the place-value system. In *Agnipurana*, the eighth text, it is written that "after the place of the units, the value of each place is ten times that of the preceding place," i.e., from left to right in units, tens, hundreds, $\cdots$ positions (if zero is added to the right of the representation of a given number, the value of the number is multiplied by ten). Similarly, in the *Shivapurana*, it is explained that usually "there are eighteen positions for calculation," the text also points out that "the Sages say that in this way, the number of places can also be equal to hundreds." Thus any number *r* can be written as in *decimal expansion*

$$r \; = \; \sum_{p=0}^{m} a_{m-p} \cdot 10^{m-p} + \sum_{q=1}^{n} b_q \cdot 10^{-q}, \tag{2.5}$$

where a_p, b_q are integers 0 to $9,$ and m, n are positive integers. Here the first summation is referred to as the integer part, whereas the second summation as the decimal part.

According to Kaplan [296] Sumerian system (around 2000 BC), which was found by the Babylonians, had place-value system (using 60 symbols) including placeholder zero as a pair of angled wedges to represent an empty number column, but they never developed the idea of zero as a number. However, Seife [458] believes that the wedges represented only an empty space. Daivajna Varahamihira (working 123 BC) in his *Pancha-Siddhanta* mentioned the use of zero in mathematical operations. Mayans (1500 BC-900 AD), who lived in Central America, occupying the area, which today is southern Mexico, Guatemala, and northern Belize, developed zero as a placeholder around 350 AD and used it to denote a placeholder in a vertical place-value system (using 20 symbols), which is considered one of their cultures greatest achievements. The Mayans started numbering days with the number zero. The place-value system and zero were discovered in the middle of the reign of the Gupta Dynasty, whose empire stretched the whole length of the Ganges Valley and its tributaries from 240 to approximately 535, known as the classical period. Sarvanandi's (around 458, India) *Lokavibhaga* (Parts of the Universe), the Jaina cosmological text, contains names of the first nine numbers, uses the place-value system (with the term Sthanakramad) and zero sometimes expressed in word-symbols.

Bhaskara II mentions a tradition, according to which zero and the place-value system were invented by the God Brahma. In other words, these notions were so well established in Indian thought and tradition that at this time they were considered to have always been used by humans, and thus to have constituted a "revelation" of the divinities. In the fifth century, the first nine Indian numerals taken from the Brahmi notation (initiated during the reign of the Buddhist King Asoka 1472–1436 BC, and in the Buddhist inscriptions of Nana Ghat and Nasik) began to be used with the place-value system and the decimal system and were completed by a sign in the form of a little circle or dot which constituted zero. Thus, Indian place-value system was born out of a simplification of the *Sanskrit place-value system* as a consequence of the suppression of the word-symbols for the various powers of ten. Xiahou Yang (around 400–470, China) noted that to multiply a number by $10, 100, 1000,$ or 10000 all that needs to be done is that the rods on the counting board are moved forward by $1, 2, 3,$ or 4 decimal places. Similarly to divide by $10, 100, 1000,$ or 10000 the rods are moved backward by $1, 2, 3,$ or 4 decimal places. From the sixth century onward, the use of the place-value system and zero began to appear frequently in documents from India and Southeast Asia. Jinabhadra Gani (529–589, India), a Svetambara Jain monk, in his notable work *Brihatkshetrasamasa* gives an expression for the number 3200400000000 in the simplified Sanskrit system using the place-value system, which proves that he was well acquainted with zero and the place-value system.

In 594, the date of the donation charter engraved on copper of Dadda III of Sankheda, in Gujarat (India), uses nine numerals according to the place-value system. In 598, a Shaka inscription date in Cambodia is written in Sanskrit word-symbols according to the place-value system. Severus Sebokht (575–667, Syria) in 662 wrote "I shall not now speak of the knowledge of the Hindus, …of their subtle discoveries in the science of astronomy—discoveries even more ingenious than those of the Greeks and Babylonians—of their rational system of mathematics, or of their methods of calculation which no words can praise strongly enough—I mean the system using nine symbols. In 683, Shaka vernacular inscriptions of Kedukan Bukit in Malaysia and of Trapeang Prei in Cambodia are written using the place-value system including zero. In 687, a Shaka inscription date in Champa (Vietnam, Indianized civilization of Southeast Asia) is expressed using Sanskrit word-symbols according to the place-value system. The poet Subandhu (seventh century, India) made direct references to the Indian zero (in the form of a dot) as a mathematical processing device. In the seventh century, Hindu scholars introduced to Islamic and Arabic mathematicians further west the ideas of zero and place-value system. A Sanskrit inscription of 732 from Java (Indonesia, [then a part of Greater Bharat] contains the Shaka date which is expressed using the place-value system and word-symbols of the Indian astronomers.

Gautama Siddhanta (different from Gautama Siddhartha, the Buddha), Chinese Buddhist astronomer of Indian origin, author of a work on astronomy and astrology entitled *Kaiyuan Zhanjing* (718–729), where he describes zero, the place-value system, and Indian methods of calculation, replacing the counting rods (used by mathematicians for calculation in ancient East Asia, specially in China). Lalla (around 720–790, India) in his two volumes work *Shishyadhividdhidatantra* (On the Computation of the Positions of the Planets) and (On the sphere) used abundant usage of the place-value system by means of Sanskrit numerical symbols. Abu Jafar Mohammed Ibn Musa al-Khwarizmi (780–850, Khwarazm-Iraq) around 820 wrote a treatise in Arabic which is entitled as *Hindu-Arabic Numerals* (the word Hindu-Arabic came from the Catholic authorities to the counting systems borrowed from the Islamic world), it describes the Indian place-value system based on the numerals $1, 2, 3, 4, 5, 6, 7, 8, 9,$ and 0 (these ten figures in Arabic were called *zephirum*), but his original work has been lost; however, its Latin translation *Algoritmi de numero Indorum* (al-Khwarizmi on the Hindu Art of Reckoning), a name given to the work by Baldassarre Boncompagni (1821–1894) in 1857, has survived. Al-Khwarizmi also developed quick methods for multiplying and dividing numbers (it freed the use of abacus/swan-pan), which are known as a corruption of his name (while between the tenth and fifteenth centuries *algorithm* was synonymous with positional numeration, today it applies to any mathematical procedure consisting of finite or an indefinite number of steps, each step applying to the result of the one preceding it; in particular, the role of the algorithm is fundamental to the use of mathematical machines). This Latin translation after several centuries became crucial in the introduction of Hindu-Arabic numerals to medieval Europe.

In 830, Govindaswami (around 800–860, India), in his famous treatise which is a commentary on the *Mahabhaskariya* of Bhaskara I, illustrates many examples of using a place-value system. Mahavira in 850, Haridatta (fl. around 683, India) in his *Grahacaranibandha*, and Shankaranarayana (around 840–900, India) in his notably text *Laghubhaskariyavivarana* of 869 frequently used the place-value system. The inscriptions of Gwalior temple (875–876, India) use zero (in the form of a little circle) and the nine numerals (in Nagari) according to the place-value system. Toward the end of the ninth century the philosopher Shankaracharya (700–750) made a direct reference to the Indian place-value system in his work *Sharirakamimamsabhashya* (great commentary on the Vedantasutra). Abu Arrayhan Muhammad ibn Ahmad al-Biruni (973–1048, Uzbekistan–Afganistan, travelled to India and testified that the Indian attainments in mathematics were unrivalled and unsurpassed) in his works used Indian place-value system and methods of calculations. A manuscript *Codex*

Vigilanus of 976 (now kept in a museum of Madrid) by Vigila, a monk in a Spanish monastery records: The Indians have an extremely subtle intelligence, and when it comes to arithmetic, geometry, and other such advanced disciplines, other ideas must make way for theirs. The best proof of this is the nine figures with which they represent each number no matter how large. Shripati (fl. in 1045, India) works include *Siddhantashekhara* in which the place-value system of the Sanskrit numerical system is used frequently.

Rabbi Abraham ben Meir Ibn Ezra (around 1089–1167, Spain) wrote three treatises on numbers which helped to bring the Indian symbols, place-value system, and ideas of decimal fractions to the attention of some of the learned people in Europe. In his work ibn Ezra called zero as galgal (meaning wheel or circle). Despite ibn Ezra's books, these ideas were not accepted among European mathematics for several more centuries. Fibonacci in his *Liber Abaci* (the first Western books to describe the Hindu-Arabic numeral system) states that "The nine Indian numerals are …with these nine and with the sign $0,$ which in Arabic is sifr, any desired number can be written." This is considered as an important link between the Hindu-Arabic number system and the European mathematics. In 1247, Chiu-Shao wrote *Shushu Jiuzhang* (Mathematical treatise in nine sections) which uses the symbol 0 for zero. A little later, in 1303, Zhu Shijie (1249–1314, China) wrote *Siyuan Yujian* (Jade mirror of the four elements), which again uses the symbol 0 for zero. After the Mongol period (1206–1368), the circle-zero Chinese used as an ordinary figure.

By the fifteenth century, the numerals were showing up on coins and gravestones, and zero appeared in the middle of every Renaissance painting. In 1585, Simon Stevin (1548–1620, Belgium–Netherland) published a 36-page booklet, *La Thiende* (The Tenth) in Leiden, which is an account of decimal fractions and their daily use. By the end of 1600s, this booklet became a turning point, as Europeans realized the simplicity of Hindu-Arabic numerals over the cumbersome Roman numerals which were in common usage in Europe (Napier was the first to use comma or point to separate decimals from the integers). In particular, zero began to come into widespread use as it played a fundamental role in Descartes Cartesian coordinate system, and in calculus, developed independently by Newton and Gottfried Wilhelm von Leibniz (1646–1716, Germany). Calculus paved the way for physics, engineering, computers, and much of financial and economic theory. Zero/vanishing point in paintings forms part of a linear perspective scheme. It is the point in fictive space which is supposed to appear the furthest from the viewer—the position at which all receding parallel lines meet.

The discovery of zero took place within an environment which was at once mystical, philosophical, religious, cosmological, mythological, and metaphysical (for a detailed history of zero, its role in life, and mathematics (see Sen and Agarwal [466]). In India, the use of zero and the place-value system has been a part of the way of thinking for so long that people have gone as far as to use their principal characteristics in a subtle and very poetic form in a variety of Sanskrit verses. While the importance of zero and the place-value system was well recognized and used in Asia, it took several centuries for Europeans to fully understand their simplicity and practicability. In fact, the local merchants of Pisa (the home city of Fibonacci), the trading class, neglected Fibonacci's *Liber Abaci*. They were more interested in prosperity and did not want to be bothered with giving up Roman numerals and adopting a zero. However, Fibonacci's mathematician friends realized immense superiority of the new number system and slowly over time gave up the Roman numerals. After Fibonacci's work, some regions of Europe believed that the number zero was evil, and "not of God." Therefore, Europeans banned the number zero from being used, but people would still use "sifr" in secret code cipher.

Recently, Annewies van der Hoek (born 1943, USA) said "Medieval religious leaders in Europe did not support the use of zero, they saw it as satanic. God was in everything that was. Everything that was not was of the devil." In popular language, words derived from sifr soon came to be associated not with figures in general but with "nothing" in particular: In thirteenth-century Paris, a "worthless fellow," was called *cyfre d'angorisme* or cifre en algorisme, i.e., "an arithmetical nothing." Similarly, a recent example appearing in the newspapers is "a lousy nothing divided by nothing." In appreciation to place-value system, Severus Sebokht wrote "subtle discoveries" of Indian astronomers as being "more ingenious than those of the Greeks and the Babylonians" and "their valuable methods of computation which surpass description" and then goes on to mention the use of nine numerals. From the thirteenth century, when calculations could be performed "in writing," slowly the importance of zero and the place-value system was recognized all over the world, and prominent mathematicians and philosophers started making constructive comments:

Simon Jacob (1510–1564, Germany) says: "It is true that it (the abacus) is useful for everyday calculations, where we often need to calculate sums, subtract or add, but in technical work, which is rather more complicated, an abacus is often an encumbrance. I am not claiming that such calculations cannot be carried out on the lines [of the abacus] but, just as a man without baggage has the advantage over one who is heavily loaded, so calculating with figures has the advantage over calculating on the lines."

Simon Stevin in his *"l" Arithmétique* of 1558 writes that Arithmetic is the science of numbers and number is that which explains the quantity of each thing. He makes a point that number represents quantity, any type of quantity at all. Number is no longer to be only a collection of units, as defined by Euclid.

Pierre Simon de Laplace (1749–1827, France): "It is India, that gave us the ingenious method of expressing all numbers by means of ten symbols, each receiving a value of position as well as an absolute value, a profound and important idea which appears so simple to us now that we ignore its true merit. But its very simplicity, the great ease which it has lent to all computations, puts our arithmetic in the first rank of useful inventions; and we appreciate the grandeur of this achievement the more when we remember that it escaped the genius of Archimedes and Apollonius of Perga (around 262–200 BC, Greece), two of the greatest men produced by antiquity."

George Bruce Halsted (1853–1922, USA): "The importance of the creation of the zero mark can never be exaggerated. This giving to airy nothing, not merely a local habitation and a name, a picture, a symbol, but helpful power, is the characteristic of the Hindu race whence it sprang. It is like coining the Nirvana (the supreme state of nonexistence, reincarnation and absorption of the being in the Brahman) into dynamos. No single mathematical creation has been more potent for the general on-go of intelligence and power."

Whitehead: "The point about zero is that we do not need to use it in the operations of daily life. No one goes to buy zero fish. It is in a way the most civilized of all the cardinals, and its use is only forced on us by the needs of cultivated modes of thought."

Swami Vivekananda (1863–1902, born Narendranath Datta, India): "… the ten numerals, the very cornerstone of all present civilization, were discovered in India, and are, in reality, Sanskrit words."

Einstein: "The horizon of many people is a circle with a radius of zero. They call this their point of view."

Tobias Dantzig (1884–1956, USA): "In the history of culture the discovery of zero will always stand out as one of the greatest single achievements of the human race."

Bibhuti Bhusan Datta (1888–1958, India): “The Hindus adopted the decimal scale vary early. The numerical language of no other nation is so scientific and has attained as high a state of perfection as that of the ancient Hindus. In symbolism they succeeded with ten signs to express any number most elegantly and simple. It is this beauty of the Hindu numerical notation which attracted the attention of all the civilised people of the world and charmed them to adopt it.” He also writes “From Arabia, the numerals slowly marched towards the West through Egypt and Northern Arabia; and they finally entered Europe in the eleventh century. The Europeans called them the Arabic notations, because they received them from the Arabs. By the Arabs themselves, the Eastern as well as the Western has unanimously called them the Hindu figures (Al-Arqan-Al-Hindu).”

Jekuthial Ginsburg (1889–1957, USA): “The Hindu notation was carried to Arabia about 770 AD by a Hindu Scholar named Kanaka who was invited from Ujjain to the famous Court of Baghdad by the Abbaside Khalif Al-Mansur, Kanka taught Hindu astronomy and mathematics to the Arabian scholars; and, with the help, they translated into Arabic the *Brahma-Sphuta-Siddhanta* of Brahmagupta. The recent discovery by the French savant M.F. Nau proves that the Hindu numerals were well-known and much appreciated in Syria about the middle of the seventh century AD.”

Arthur Llewellyn Basham (1914–1986, England) observed: “Most of the great discoveries and inventions of which Europe is so proud would have been impossible without a developed system of mathematics, and this in turn would have been impossible if Europe had been shackled by the unwieldy system of Roman numerals. The unknown man who devised the new system was from the world’s point of view, after the Buddha, the most important son of India. His achievement, though easily taken for granted, was the work of an analytic mind of the first order, and he deserves much more honor than he has so far received.”

Dalai Lama (born 1935, China): “A zero itself is nothing, but without a zero you cannot count anything; therefore, a zero is something, yet zero.”

John Horton Conway (1937–2020, England) and Richard Kenneth Guy (1916–2020, England–Canada): Waclaw Sierpinski (1882–1969), the great Polish mathematician… was worried that he had lost one piece of his luggage. “No, dear!” said his wife. “All six pieces are here.” “That can’t be true,” said Sierpinski, “I’ve counted them several times, zero, one, two, three, four, five.”

Stephen William Hawking (1942–2018, England): “The concept of emptiness is now central to modern physics: the entire known Universe is seen as zero sum game.”

Agarwal and Sen [14] “In the world, although there are about four thousand languages of which several hundred are widespread, there are only several dozen alphabets and writing systems that represent them; however, we can safely say that there is but one single place-value system that uses zero and nine numerals, symbolically $0, 1, 2, 3, 4, 5, 6, 7, 8, 9,$ by which any number can be expressed and understood rather easily. This gift of India has truly united the Universe in the language of numbers.”

Sen and Agarwal [466] “As with the invention of the wheel, modern science and technology would not be possible without the humble zero. Mathematics would be no more than a bookkeeper’s art if zero were to be banished to the forbidden zone of the intellect.”

Sen and Agarwal [466] “In the real world (of physics and chemistry), the phantom of zero haunts and taunts all those who seek the truth behind the visible and tangible. There is a limit to which you can lower the temperature of anything, and that limit is called “Absolute Zero.” Then there are other types of “zero” in nature: the “quantum zero” and the “relativistic zero.” Both of these zeros are bizarre and counter-intuitive. The first tells that a vacuum (empty space) is not only non-empty but is also “full of force.” The other says that the Universe as we understand it could be populated by many so-called “black holes” (in mathematical lingo “singularities”) within which the known laws of physics appear to break down. Like dividing by zero, attempts to unveil the mysteries of these physical zeros often bring about what appear to be paradoxes (see Sect. 3.18) to the finite mind. And then there is the “most tantalizing” and “most unfathomable” zero of all in the Universe: the birth of our own Universe at the “zeroth” hour of the “Big Bang.”

Peter Gobets, secretary of the ZerOrigIndia Foundation (The Zero Project), said “The Indian (or numerical) zero, widely seen as one of the greatest innovations in human history, is the cornerstone of modern mathematics and physics, plus the spin-off technology.” He also remarked “The numeral and concept of zero, imported from India, has manifested in various ways. So commonplace has zero become that few, if any, realize it astounding role in the lives of every single person in the world.”

Van der Hoek: “From this philosophy, we think that a numeral to use in mathematical equations developed, we are looking for the bridge between Indian philosophy and mathematics.”

Librarian Richard Ovenden (born 1964, England) said “the discovery was of vital importance to the history of mathematics and the study of early South Asian culture. These surprising research results testify to the subcontinents rich and longstanding scientific tradition.”

Marcus Peter Francis du Sautoy (born 1965, England) said, “Today we take it for granted that the concept of zero is used across the globe and is a key building block of the digital world. But the creation of zero as a number in its own right, which evolved from the placeholder dot symbol found in the Bakhshali manuscript, was one of the greatest breakthroughs in the history of mathematics”:

- For the question “What does Prob(A)$= 0$ mean?” The most probable answer is “A in an impossible event.” But that is not true. Consider a circular disk (see Fig. 2.1) which is calibrated between 0 and $100,\ 100$ coincides with zero. At the center of the disc, there is an elastic pointer. If we push the pointer, it rotates for a sufficiently long time and finally stops.

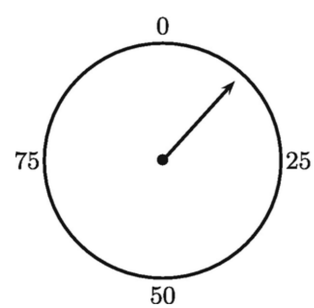


Fig. 2.1 Prob(A) $= 0$

Prob [the pointer will stop between 10 and 20]$= 10/100$.
Prob [the pointer will stop between 10 and 15]$= 5/100$.
Prob [the pointer will stop between 10 and 11]$= 1/100$.
Prob [the pointer will stop between 10 and 10.1]$= 1/1000$.
and finally
Prob [the pointer will stop at 10]$= 0$.

Does this mean that the pointer will never stop at 10? No. The same is true for the pointer stopping at any particular point. This shows that Prob(A)$= 0$ indicates that A is very very rare or an “almost impossible” event, as it is called, i.e., [P(A)$= 0$]$\Rightarrow A$ is an impossible event.

2.7 Large and Small Numbers

Positive numbers which are significantly larger than those typically used/required in everyday life appear frequently in fields such as mathematics, cosmology, cryptography, and statistical mechanics. In particular, large numbers have immense applicability in science and technology, especially with the advent of silicon technology-based high-speed digital computers. Modern names of large numbers are Million 10^6, Billion 10^9, Trillion 10^{12}, Quadrillion 10^{15}, Quintillion 10^{18}, Sextillion 10^{21}, Septillion 10^{24}, Octillion 10^{27}, Nonillion 10^{30}, Decillion 10^{33}, Undecillion 10^{36}, Duodecillion 10^{39}, Tredecillion 10^{42}, Quattuordecillion 10^{45}, Quindecillion 10^{48}, Sexdecillion 10^{51}, Septendecillion 10^{54}, Octodecillion 10^{57}, Novemdecillion 10^{60}, Vigintillion 10^{63} (so far all powers of the form $3(n+1),\ 1 \leq n \leq 20$), Googol 10^{100}, Centillion 10^{303}, and Googolplex $10^{10^{100}}$ (which if written will easily fill space out to the most distant visible star). Some of these names were introduced by Jehan Adam (fl. in fifteenth century, France) and Chuquet, whereas googol and googolplex were invented by a nine-year-old child Milton Sirotta in 1938.

In the Bhagavad Gita, Lord Krishna manifested himself as the entire Cosmos—of 2 trillion Galaxies and Multiverses as his corporeality—this is known as Visvarupa Darsana—all appearing in a singularity of concentrated light. In the epic *Mahabharata* the combined number of warriors and soldiers in both armies was approximately 3.94 million; however, some authors claim that one billion 660 million and $20{,}000$ men fall in this battle and $240{,}165$ escaped (which is obviously an exaggeration). Almost 3500 BC the Egyptians had far outgrown the primitive inability to think boldly in terms of large numbers. A royal mace of about that time records the capture of 120,000 human prisoners, 400,000 oxen, and 1,422,000 goats. The large number $12{,}960{,}000 = 60 \times 60 \times 60 \times 60$ attracted Babylonians in numerology, which in the literature known as Plato's *Nuptial Number* by which perhaps he advocated eugenics, astrology, and a strange number mysticism. For Aryabhata, the fundamental period was the *Mahayuga* of 4,320,000 years, the last quarter of which, the *Kaliyuga*, began in 3102 BC For Brahmagupta, the fundamental period was the *Kalpa* of 1000 *Mahayugas*. In Hinduism one of the main religious epic Ramayana (written in Sanskrit around 300 BC by the sage Valmiki) contains $24{,}000$ couplets divided into seven books. It uses a scale of hundred thousand (10^5) to mention terms up to 10^{55} (mahaugha).

Buddhist work *Lalitavistara Sutra* also composed around 300 BC lists numbers in multiples of 100 from koti (10^7) (in Sanskrit another meaning of koti is “supreme”) up to tallakshanaam (10^{53}). Hindus also reveled in large numbers, especially in their pantheon ($330{,}000{,}000$ deities) and their mythical chronology. However, the correct meaning of this is that there are only 33 supreme divinities: Eight Vasus (deities of material elements)—Dyaus (sky), Prithvi (Earth), Vayu (wind), Agni (fire), Nakshatra (stars), Varuna (water), Surya (Sun), Chandra (Moon); Twelve Adityas (personified deities)—Indra (Shakra), Aryaman, Tvashtr, Varuna, Bhaga, Savitr, Vivasvat, Amsha, Mitra, Pushan, Daksha, Vishnu (this list sometimes varies in particulars); Eleven Rudras, consisting of Aja, Ekapada, Ahirbudhanya, Tvasta, Rudra, Hara, Sambhu, Trayambaka, Aparajita, Ishana, and Tribhuvana; Two Ashvins (or Nasatyas), twin solar deities. The Jainas had an interest in the enumeration of very large numbers, which was intimately tied up with their philosophy of time and space. They could conceive of such huge units of time as $756 \times 1011 \times 840{,}000{,}028$ days, which was termed as Sirsaprahelika. Jains texts also use auxiliary terms like sahassa (thousand) and koti and use them to write large numbers. An anonymous Jaina work *Amalasiddhi* has terms for all powers of 10 up to 10^{96} (da sa-ananta). In the Jaina text Anuyogadvara-sutra (around 100 BC) the total number of human beings is described variously as “the product of 2^{64} and 2^{32}” (i.e., 2^{96}), as a number which “can be divided by two 96 times,” and it is also said that this number, “when expressed in terms of denominations like koti-koti (10^{14}),” “occupies 29 places (sthana).” Indeed, the number $2^{96}(= 79{,}228{,}162{,}514{,}264{,}337{,}593{,}543{,}950{,}336)$ requires precisely 29 digits in the decimal notation.

The ancient Chinese used the words for large numbers in a poetic manner to create new words. Xu Yue, Wade-Giles Hsü Yüeh (around 185–227, China), in one of his books provides three methods of assigning the powers of 10 up to $10^{4{,}096}$ to traditionally established terms for “large numbers” and allusively mentions a method of indefinite generation of even larger numbers. The world population was estimated 7.79 billion for the year 2020, it was projected as 7.87 billion as of July 1, 2021, and it was expected to be 8 billion by 2023, around 140 million babies are born every year in the world, and about 60 million people die every year. The cosmologists believed that the Universe came into being as a highly dynamic event between 10,000 and 20,000 million years ago, Big-Bang theory suggests Universe was born about 13.7 billion years ago, whereas in recent years the present age of the Universe is assessed as 155.52×10^{12}. The age of the Earth has been estimated as $4{,}600{,}000{,}000$ years (Erdös once gave a lecture called "My First Two-and-a-Half Billion Years in Mathematics." His justification? "When I was a child the Earth was said to be two billion years old. Now scientists say it is four and a half billion. So that makes me two and a half billion). An educated guess is that there are of the order of 130 million important books in the world.

In real world applications we have: There are more than 10^{19} molecules in a cubic centimeter of gas under normal conditions; the total number of atoms in the Universe is 3×10^{74}; there are 86 billion neurons and 85 billion nonneuronal cells in a normal human brain, and in the whole body about 37.2 trillion cells (every second a single cell divides to make two new cells, which is a unique ability for living organisms, these cells join forces to form organs); scientists at CERN's Large Hadron Collider may have created the world's hottest man-made temperature, forming a quark–gluon plasma that could have reached temperatures of 5.5 trillion degrees Celsius or 9.9 trillion Fahrenheit, which is already about 250,000 times hotter than the center of the Sun; at standard room temperature and pressure, we take approximately 25 sextillion molecules every time in a breath; in our body each red blood cell carries about 270 million molecules of hemoglobin; Avogadro's number (the number of atoms, molecules, etc., in a gram mole (gram molecule) of every chemical substance) $N_k = 6.022 \times 10^{23}$; Loschmidt's constant (the number of molecules in one cubic meter of a gaseous substance under ordinary conditions of temperature and pressure) $n_0 = 2.6867775 \times 10^{25}$; the speed of light in vacuum, denoted as c, is $299{,}792{,}458$ meters (186282 mi) per second; distance from the Earth to Moon is $238{,}900$ mi; distance from the Earth to Mars is about 140 million mi; distance from the Earth to Sun is 93.417 million mi; distance from the Earth to nearest star Proxima Centauri is about 4.24 light years away (one light year is approximately 6 trillion miles); some astronomers estimate that there are over eleven billion planets in the Milky Way (its radius is 52,850 light years); James Webb Space Telescope, which was launched on December 25, 2021, shows a portion of the center of our galaxy (a huge collection of gas, dust, and billions of stars and their solar systems) about 25,000 light years.

Before the discovery in physics of a whole array of "new" elementary particles, Arthur Stanley Eddington (1882–1944, England) in 1931 estimated the number of charged particles in the universe as 10^{42} and in 1938 gave a strange and powerful argument that, among other things, led to a value for the total number of protons and the same number of electrons in the observable Universe. This number (known as Eddington's number) was
$15,747,724,136,275,002,577,605,653,961,181,555,468,044,717,914,527,$
$116,709,366,231,425,076,185,631,031,296 \simeq 136 \times 2^{256} \simeq 1.57 \times 10^{79}$.
Several other large known numbers are: Hubble Constant (after Edwin Powell Hubble, 1889–1953, USA) 70(km/s)/Mpc, Mpc stands for megaparsec (3.09×10^{19} km); Dirac large numbers hypothesis (after Paul Adrien Maurice Dirac, 1902–1984, England) leads to 10^{42} [hypothesis is a supposition or proposed explanation made on the basis of limited evidence as a starting point for further investigation]; Chandrasekhar limit (after Subrahmanyan Chandrasekhar (1910–1995, India–USA) 2.765×10^{30} kg; and von Weizsäcker ratio (after Carl Friedrich Freiherr von Weizsäcker, 1912–2007, Germany), 10^{120}.

At freezing temperature, planetoids circle the Sun at speeds of 3 miles per minute, thus taking $30,000,000$ years to orbit the Sun. The number of words printed since the Guttenberg Bible in 1456 until the 1940's is about 10^{16}. It is estimated that about 117 billion members of our species have ever been born on the Earth. Probiotics up to 100 billion CFU (Colony-Forming Unit) supplement are available in the market. In 2021, the USA government spent 6.82 trillion dollars. Based on retail sales generated in the financial year 2020, Walmart Inc. was by far the world's leading retailer with retail revenues reaching over 559 billion dollars. On September 19, 2019 and again on November 14, 2022, Amazon founder Jeffrey Preston Bezos (born 1964, USA) pledged to donate majority of his net worth, currently $124 billion, to fight climate change and unify humanity. In 2022, Americans' holdings of corporate equities and mutual fund shares fell to $33 trillion at the end of the second quarter, down from $42 trillion at the start of the year. The list of losers includes Elon Reeve Musk (born 1971, South Africa, USA) about $115 billion, Bezos $85 billion, and Mark Elliot Zuckerberg (born 1984, USA) $78 billion. In February 2023, Gautam Shantilal Adani (born 1960, India) shed more than $100 billion in days.

Statista (German company) data suppliers estimate that the USA has 280.54 million smartphone users in 2020. The term *perihelion* refers to the point in the orbit of a planet or other astronomical body, at which it is closest to the Sun: For the planets in the solar system Mercury, Venus, Earth, Moon (satellite planet), Mars, Jupiter, Saturn, Uranus, Neptune, and Pluto, it is, respectively, $46.0,\ 107.5,\ 147.1,\ 0.363,\ 206.7,\ 740.6,\ 1357.6,\ 2732.7,\ 4471.1,\ 4436.8$ million kilometers. Currently, FBI's (Federal Bureau of Investigation) master criminal fingerprint file contains the records of approximately 47 million individuals, while civil file represents approximately 30.7 million individuals. Magic Kingdom at Disney World in Florida was the world's most-visited theme park in 2018, with a whopping 20.8 million visitors. New York city has two million rats, thriving on the streets, in sewers, in both abandoned and un-abandoned buildings, in the parks, in the subways, in shoe stores, and in restaurants:

- In his treatise *The Psammites*, or *Sand Reckoner*, Archimedes says: "There are some, King Gelon, who think that the number of sand grains is infinite in multitude; and I mean by sand not only that which exists about Syracuse and the rest of Sicily, but all the grains of sand which may be found in all the regions of the Earth, whether inhabited or uninhabited. Again there are some who, without regarding the number as infinite, yet think that no number can be named which is great enough to exceed that which would designate the number of the Earttis grains of sand. And it is clear that those who hold this view, if they imagined a mass made tip of sand in other respects as large as the mass of the Earth, including in it all the seas and all the hollows of the Earth filled up to the height of the highest mountains, would be still more certain that no number could be expressed which would be larger than that needed to represent the grains of sand thus accumulated. But I will try to show that of the numbers named by me some exceed not only the number of grains of sand which would make a mass equal in size to the Earth filled up in the way described, but even equal to a mass the size of the Universe." Archimedes concluded that the diameter of the Universe was no more than 10^{14} stadia (about 2 light years), and that it would require no more than 10^{63} grains of sand to fill it.
- Archimedes's Cattle Problem, also known as bovinum problema, or Archimedes' reverse was found in old Greek and Latin manuscripts in 1773 by Gotthold Ephraim Lessing (1729–1781, Germany). It is originally a Greek poem of forty-four lines which has been translated in

English as follows: "The Sun God had a herd of cattle consisting of bulls and cows, one part of which was white, a second black, a third spotted, and a fourth brown. Among the bulls, the number of white ones was one half plus one third the number of the black greater than the brown; the number of the black, one quarter plus one fifth the number of the spotted greater than the brown; the number of the spotted, one sixth and one seventh the number of the white greater than the brown. Among the cows, the number of white ones was one third plus one quarter of the total black cattle; the number of the black, one quarter plus one fifth the total of the spotted cattle; the number of spotted, one fifth plus one sixth the total of the brown cattle; the number of the brown, one sixth plus one seventh the total of the white cattle. What was the composition of the herd?" If we let in integers W, X, Y, Z (the number of white, black, spotted, and brown bulls) and w, x, y, z (the number of white, black, spotted, and brown cows), then mathematically the problem leads to solving a system of seven linear equations with eight unknowns (Diophantine equations)

$$W = \frac{5}{6}X + Z, \quad X = \frac{9}{20}Y + Z, \quad Y = \frac{13}{42}W + Z,$$

$$w = \frac{7}{12}(X + x), \quad x = \frac{9}{20}(Y + y), \quad y = \frac{11}{30}(Z + z), \quad z = \frac{13}{42}(W + w).$$

This system of equations is indeterminate and has infinitely many solutions. The smallest solution in integers is
$W = 10366482, X = 7460514, Y = 7358060, Z = 4149387, w = 7206360, x = 4893246, y = 3515820, z = 5439213,$
which gives a total of $50{,}389{,}082$ cattle. The other solutions are integral multiples of these. In 1880, Carl Ernst August Amthor (1845–1916, Germany) tackled a more intricate cattle problem which requires that $W + X$ is a square number and $Y + Z$ a triangular number (see Sect. 7.2). He used logarithmic tables to find the smallest solution as 7.76×10^{206544} cattle, far more than could fit in the observable Universe. In 1965, Williams et al. [535] used a combination of the IBM 7040 and IBM 1620 computers to show that the solutions to this problem are numbers with 206544 or 206545 digits.

- Chess is a brain-storming game and distinctly different from any game of chance. According to legend, chess/shaturanja was invented (around 1500 years back) by Grand Vizier Sissa Ben Dahir, and given as a gift to King Shirham of India. It has also been claimed that the origin of chess dates beginning of the seventh century written in Pahlavi (Persian) and Harshacharita (Sanskrit). Chess then found its way to China and to Europe through the Arab countries. Historians believe that chess, nicknamed the royal game, was employed in India to teach the military strategy to Indian princes. Only two players play chess on a chessboard which is a 8×8 checkered board with a black square in each player's lower left corner. Each player controls their own army of 16 chessmen. The player who controls white moves first. The goal is to capture the opponent's king. The capture is known as checkmate. The checkmate occurs once the king is under attack, and can neither move nor be helped by its own army of chessmen. Chess is a zero sum game, i.e., one person's gain is equivalent to another's loss, so the net change in wealth or benefit is zero. It has not been proved feasible to estimate the total number of games of chess; but it has been shown that the number is less than $10^{10^{7.5}}$. Further, the computational complexity of the deterministic chess algorithm is exponential, needing the evaluation of 10^{120} possible moves starting from the first (initial) move.

The King Shirham was so delighted with his Grand Vizier that he offered him any reward he requested, provided that it sounded reasonable. The Grand Vizier requested the following: "Just one grain of wheat on the first square of a chessboard. Then put two on the second square, four on the next, then eight, and continue, doubling the number of grains on each successive square, until every square on the chessboard is reached." Ostensibly, King Shirham underestimated the number of grains and laughed at Sissa because he had asked such a small gift. When he had someone to calculate the total number of grains, it took more than a week before he came back with the solution. King Shirham undoubtedly became very pale when he got the answer: The aggregated number of grains on all squares of a chessboard would be

$$1 + 2 + 2^2 + \cdots + 2^{63} \;=\; 2^{64} - 1 = 18{,}446{,}744{,}073{,}709{,}551{,}615 \;\simeq\; 10^{19.3}.$$

This is the harvest of all the wheat of the world, of several decades. Obviously, King Shirham could not fulfill Grand Vizier's desire. The moral of this tale is even intelligent people in haste overconfidence can be easily deluded. The above tale was recorded in 1256 by Iibn Khallik$\overline{\text{a}}$n (1211–1282, Iran), and it has been retold in several languages.

Another version of this fable is in the fifteenth century, Ambalappuzha Sree Krishna Swamy Temple (Kerala/India) dedicated to Lord Krishna was built by the local ruler Chembakasserry Pooradam Thirunal-Devanarayanan Thampuran who was chess enthusiast. One day Lord Krishna appeared in the form of a sage in the court of the king and challenged him for a game of chess. The king gladly accepted his invitation, and to make the game more interesting, and confident in his playing abilities, the king asked the sage to choose a prize if he won. Sage reluctantly said being a man of a few material needs, all he needed was a few grains of rice. The number of rice itself shall be determined on the chessboard in the following manner. One grain of rice shall be placed in the first square, two grains in the second square, four in the third square, and so on with each successive square doubling the amount of its predecessor. The king considered the requested reward insignificant given the vast riches in his empire. The king obviously lost the game, and the sage appealed for the agreed-upon prize. As he started putting grains of rice to the chessboard, the king quickly realized the true nature of the sage's demand. The royal stockpiles soon ran out of grains of rice and the king determined he would never be able to repay the debt. Upon seeing the dilemma, the sage appeared to the king in his true-form and told the king that he did not have to pay the debt immediately but could pay him over time. The king would serve paal payasam (pudding made of rice and milk) in the temple freely to the pilgrims every day until the debt was paid off. According to Thomas Henry Huxley (1825–1895, England), the chessboard is the world; the pieces are the phenomena of the universe; the rules of the game are what we call the laws of nature.

Shen Kuo (1031–1095, Japan) besides grain problem also considered the problem to calculate the number of positive configurations of a *weiqi* board (that is, of a *go-board*) of size $19 \times 19 = 361$ points of intersection in which each point may be unoccupied, occupied by a white pawn, or occupied by a black pawn. His solution is 3^{361}, which is approximately the same as 1.74×10^{172} :

- In a lore of India there is a tale about a stone, a cubic mile in size, a million times harder than a diamond. Every million years a holy man visits the stone to give it the lightest possible touch. How long does it take to wear the stone away? On the basis of reasonable estimates of the wear from each touch, this works out to be the order of 10^{35} years. If you calculate the number of "atoms" of carbon in cubic mile of density 10^6 times that of the ordinary diamond, you get a number of the order of 10^{45}. We have put the word atom in quotes, because one is really dealing with an interlocked crystal structure: But "the lightest possible touch" would seem to imply removing one atom at each touch. Removing one every 10^6 years then indicates that 10^{51} years would perhaps be required.

- Another story in which a large number plays the chief role also comes from India and pertains to the problem of the "End of the World" (Rouse Ball [53]), the historian of mathematical fancy tells the story in the following words: It is interesting to compare this purely legendary prophecy of the duration of the Universe with the prediction of modern science. According to the present theory concerning the evolution of the Universe, the stars, the Sun, and the planets, including our Earth, were formed about $3{,}000{,}000{,}000$ years ago from shapeless masses. We also know that the "atomic fuel" that energizes the stars, and in particular our Sun, can last for another $10{,}000{,}000{,}000$ or $15{,}000{,}000{,}000$ years. Thus the total life period of our Universe is definitely shorter than $20{,}000{,}000{,}000$ years, rather than as long as the $58{,}000$ billion years estimated by Indian legend! But, after all, it is only a legend!
- The strength of the traditional Japanese samurai sword is legendry. The master sword maker prepares the blade by heating a bar of iron until it is white hot, then folding it over, and pounding it smooth. He does this 15 times. Each time the metal is folded, the layers of steel are doubled (a geometric sequence). For a sword of 15 folds, the blade contains 2^{15}, or $32{,}768$ layers of steel.
- The Tower of Brahma is a romantic legend constructed by Lucas in 1883, as an enhancement to the popular game he invented, The Tower of Hanoi (amusingly, the original version of this problem, dating back to ancient Tibet). According to the legend of the Tower of Brahma, in the Indian city of Benares/Varanasi, beneath a dome that marked the center of the world, is to be found a brass plate in which are set three diamond needles, "each a cubit high and as thick as the body of a bee." Brahma placed 64 disks of pure gold on one (first) of these needles at the time of creation. Each disk is a different size, and each is placed so that it rests on top of another disk of greater size, with the largest resting on the brass plate at the bottom and the smallest at the top. Within the temple are priests whose job is to transfer all the gold disks from their original needle to one of the others (third), without ever moving more than one disk at a time. No priest can ever place any disk on top of a smaller one, or anywhere else except on one of the needles. When the task is done,

and all 64 disks have been successfully transferred to another needle, “tower, temple, and Brahmins alike will crumble into dust, and with a thunder-clap the world will vanish.” If m_k denotes the minimum number of moves required to move *k* disks from the first needle to the third needle, then The Tower of Hanoi Problem mathematically leads to the difference equation

$$m_{k+1} \;=\; m_k + 1 + m_k \;=\; 2m_k + 1, \quad m_1 = 1, \quad k \geq 1,$$

whose solution is $m_k = 2^k - 1.$ Thus, the number of steps required to transfer all the disks is again the same number $2^{64} - 1,$ so if a priest makes one move a second, night and day, the solution would require slightly more than fifty-eight thousand billion years.

- In 1626, Peter Minuit (1580–1638, Germany) purchased Manhattan Island (the nucleus of New York City) from the Indians for trade goods worth a mere 60 guilders = \$24. If he could have invested this money with an annual rate of 5% compounded quarterly in 2021 ($2021 - 1626 = 395,$ i.e., for 1580 quarters), its worth could have been enormous. In fact, if we denote P_n the investment after *n*th quarter, then it follows that

$$P_{n+1} \;=\; P_n + 0.0125P_n \;=\; (1.0125)P_n, \quad n = 0, 1, 2, \cdots, \quad P_0 = 24,$$

and its solution is $P_n = 24(1.0125)^n$, which for $n = 1580$ gives approximately \$8023458723, whereas if the rate was 8%, then approximately \$929999308299777.

- Second-order indeterminate equations, of the form $Nx^2 + 1 = y^2$ where *N* is an integer, were first discussed by Brahmagupta. For their solution, he employed his “Bhavana” method and showed that they have infinitely many solutions. Unfortunately, it has been recorded that Fermat was the first to assert that such equations have infinitely many solutions. In the literature these equations mistakenly known as Pell’s equations. In fact, the English mathematician John Pell (1611–1685) has nothing to do with these equations. Euler mistakenly attributed to Pell a solution method that had in fact been found by another English mathematician, William Viscount Brouncker (1620–1684, Ireland), in response to a challenge by Fermat. Bhaskaracharya used his method *cakravala* or “cyclic process” for finding integer solutions of an equation like $61x^2 + 1 = y^2;$ the smallest pair of integers x, y satisfying this equation turns out to be $x = 226{,}153{,}980, \; y = 1{,}766{,}319{,}049.$ This problem was again solved by Fermat in 1657. The equation $1141x^2 + 1 = y^2$ was also solved by Fermat, and its smallest solution is $x = 30{,}693{,}385{,}322{,}765{,}657{,}197{,}397{,}208,\ y = 1{,}036{,}782{,}394{,}157{,}223{,}963{,}237{,}125{,}215.$

- Ramanujan, the genius who was one of the greatest mathematicians of our time and the mystic for whom "a mathematical equation had a meaning because it expressed a thought of God." Once, Hardy went to see Ramanujan when he was in a nursing home and remarked that he had traveled in a taxi with a rather dull number, 1729, at which Ramanujan exclaimed, "No, Hardy, 1729 is a very interesting number. It is the *smallest number* that can be expressed as the sum of two cubes in two different ways ($1729 = 1^3 + 12^3 = 9^3 + 10^3$), and the next such number is very large." We are told Ramanujan was endowed with an astounding memory and remembered the idiosyncrasies of the first 10000 integers to such an extent that each number became like a personal friend to him. The significance of the number 1729 was first noticed in 1657 by Bernard Frénicle de Bessy (1604–1674, France). In mathematics, the smallest integer that can be expressed as a sum of two *positive* integer cubes in *n* distinct ways is called the *n*th taxicab number (*n*th Hardy-Ramanujan number) and denoted as $\mathrm{Ta}(n).$ From this definition, in spite of $1{,}009{,}736 = 96^3 + 50^3 = 93^3 + 59^3$, it follows that $1{,}009{,}736$ is not a taxicab number. In 1938, Hardy and Edward Maitland Wright (1906–2005, England) established that such numbers exist for all positive integers $n,$ and their proof was easily programmed to generate such numbers. But, unfortunately their proof, and hence the computation, makes no claim of these numbers to be the smallest possible. Until now only 6 taxicab numbers with the help of computer are known:

$$
\begin{aligned}
\mathrm{Ta}(1) &= 2 = 1^3 + 1^3 \\
\mathrm{Ta}(2) &= 1729 = 1^3 + 12^3 = 9^3 + 10^3 \\
\mathrm{Ta}(3) &= 87{,}539{,}319 = 167^3 + 436^3 = 228^3 + 423^3 = 255^3 + 414^3 \\
\mathrm{Ta}(4) &= 6{,}963{,}472{,}309{,}248 = 2421^3 + 19{,}083^3 = 5436^3 + 18{,}948^3 \\
&= 10{,}200^3 + 18, = 072^3 = 13{,}322^3 + 16{,}630^3 \\
\mathrm{Ta}(5) &= 48{,}988{,}659{,}276{,}962{,}496 = 38{,}787^3 + 365{,}757^3 \\
&= 107{,}839^3 + 362{,}753^3 = 205{,}292^3 + 342{,}952^3 = 221{,}424^3 + 336{,}588^3 \\
&= 231{,}518^3 + 331{,}954^3 \\
\mathrm{Ta}(6) &= 24{,}153{,}319{,}581{,}254{,}312{,}065{,}344 = 582{,}162^3 + 28{,}906{,}206^3 \\
&= 3{,}064{,}173^3 + 28{,}894{,}803^3 = 8{,}519{,}281^3 + 28{,}657{,}487^3 \\
&= 16{,}218{,}068^3 + 27{,}093{,}208^3 = 17{,}492{,}496^3 + 26{,}590{,}452^3 \\
&= 18{,}289{,}922^3 + 26{,}224{,}366^3
\end{aligned}
$$

- The problem of finding integer solutions of the Diophantine equation $x^3 + y^3 + z^3 = k$, where *k* equals any integer from 1 to 100 attracted interest from the works of Mordell [377], and Miller and Woollett [372]. Clearly, if x, y, z all are positive, then none of these can be greater than 4. For example, $1^3 + 1^3 + 1^3 = 3$, $1^3 + 2^3 + 4^3 = 73$, and $2^3 + 3^3 + 4^3 = 99$. However, only positive integers do not give solutions for all values of $1 \le k \le 100$, and so the search began for x, y, z in $\mathcal{Z}$. But, even then the stubborn cases $k = 33$ and 42 remain unsolved and had to wait more than 65 years. In fact, recently Booker [79] and Booker and Sutherland [80] provided solutions for several values of *k* (beyond 100 also) in terms of large numbers:

$$569,936,821,221,962,380,720^3 + (-569,936,821,113,563,493,509)^3 + (-472,715,493,453,327,032)^3 = 3$$

$$8,866,128,975,287,528^3 + (-8,778,405,442,862,239)^3 + (-2,736,111,468,807,040)^3 = 33$$

$$(-80,538,738,812,075,974)^3 + 80,435,758,145,817,515^3 + 12,602,123,297,335,631^3 = 42$$

$$(-385,495,523,231,271,884)^3 + 383,344,975,542,639,445^3 + 98,422,560,467,622,814^3 = 165$$

$$143,075,750,505,019,222,645^3 + (-143,070,303,858,622,169,975)^3 + (-6,941,531,883,806,363,291)^3 = 579$$

$$(-14,219,049,725,358,227)^3 + 14,197,965,759,741,571^3 + 2,337,348,783,323,923^3 = 795$$

$$(-74,924,259,395,610,397)^3 + 72,054,089,679,353,378^3 + 35,961,979,615,356,503^3 = 906.$$

- The 15 Puzzle (Gem Puzzle, Boss Puzzle, Game of Fifteen, Mystic Square, etc.) consists of a square shallow box of wood or metal which holds 15 little square blocks numbered from 1 to 15. There is actually room for 16 blocks in the box so that the 15 blocks can be moved about and their places interchanged. The number of conceivable positions is $16! = 20{,}922{,}789{,}888{,}000$. A problem consists of bringing about a specific arrangement of the blocks from a given initial position, which is usually the normal position given in Fig. 2.2.

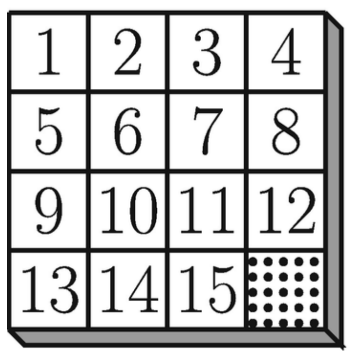


Fig. 2.2 The 15 Puzzle

This puzzle was invented as early as 1874 by a postman Noyes Palmer Chapman (1820–1892, USA); however, Samuel Loyd (1841–1911, USA) claimed from 1891 until his death that he invented the 15 puzzle. This puzzle became a craze specially in USA, Germany, and France (as such whole of Europe); it was played in the streets, in factories, and in the royal places (just like smart phone today). To garner attention to the 15 puzzle tournaments were staged and huge prizes were offered for the solution(s), but apparently no one ever won any of these prizes. In 1879, William Woolsey Johnson (1841–1927, USA) and William Edward Story (1850–1930, USA) proved that from any given initial order only half of all the conceivable positions can actually be obtained. In conclusion, there are about ten trillion starting positions which give success and about ten trillion lead to failure. This problem has been generalized in several different directions:

- The *n*-Queens Puzzle is an arrangement of *n* queens on an $n \times n$ chessboard so that no queen can attack another queen. This means that no two queens can be placed in the same row, in the same column, or on the same diagonal. In 1848, Max Friedrich William Bezzel (1824–1871, Germany) proposed the eight queens puzzle and was first fully solved by Franz Christian Nauck (1815–1874, Germany) in 1850. He showed that there are 92 configurations that kept the eight queens from each other's throats, with all but 12 of the solutions being fundamental solutions (differ only by the symmetry operations of rotation and reflection of the board are counted as one). A fundamental solution is presented in Fig. 2.3.

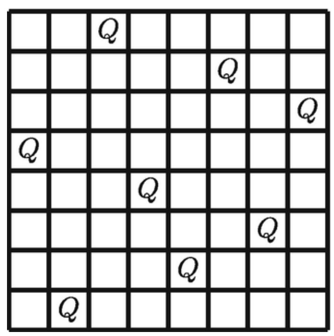


Fig. 2.3 8-Queens Puzzle

From trial it immediately follows that for $n = 2$ and $n = 3$ the puzzle has no solutions. For all natural numbers $n \geq 4$ the existence of solutions was established in 1874 by E. Pauls (Germany). However, the exact number of solutions is only known for $n \leq 27.$ The asymptotic growth rate of the number of solutions is $(0.143n)^n.$ For $n = 27$ all solutions are $234{,}907{,}967{,}154{,}122{,}528$, whereas the fundamental solutions are $29{,}363{,}495{,}934{,}315{,}694.$ This puzzle has attracted numerous number of mathematicians including Gauss, Adam Wilhelm Siegmund Günther (1848–1923, Germany), James Glaisher, and Edsger Wybe Dijkstra (1930–2002, The Netherlands):

- In 1974, Ernö Rubik (born 1944, Hungary) combined mathematics, art, and science to invent Rubik's Cube (originally called the Magic Cube), which is a three dimensions combination puzzle. Rubik's Cube is the world's best-known addictive puzzle and has fascinated fans since its launch in 1980. This cube has

$43{,}252{,}003{,}274{,}489{,}856{,}000 = 2^{27} \times 3^{14} \times 5^3 \times 7^2 \times 11$ combinations, but only one solution. In 2010, computer has shown that the Rubik's Cube can be solved in 20 moves.

- Bubonic plague, Cholera, HIV/AIDS, Influenza, and Smallpox are some of the most brutal killers in human history. The outbreaks of these diseases across international borders are defined as pandemic. The well-known pandemics are: "Plague of Justinian" during 542–750 from which half of Europe's population—almost 100 million died; "Black Death/Pestilence" during 1346–1353 causing the deaths of 75–200 million people, peaking in Europe from 1347 to 1351 and killing at least 25 million people, 30 to 60 percent of the European population; and "Third Plague Pandemic" during 1855–1960 taking life of almost 15 million people mainly in China and India. Donald Ainslie Henderson (1928–2016, USA) in his work "The eradication of smallpox – An overview of the past, present, and future" reports that during the twentieth century alone an estimated 300 million people died of the disease. "Spanish Flu Pandemic" during 1918–1919 killed 17 to 100 million people. "HIV/AIDS Pandemic" during 1981-present has approximately taken life of 36 million humans. "Hong Kong Flu" during 1968–1970 killed at least 1 million people. Beginning in December 2019, in the region of Wuhan, China, a new "novel" coronavirus began appearing in human beings. It has been named Covid-19. This new virus spreads incredibly quickly between people, due to its newness—no one on the Earth has an immunity to Covid-19. In March 2020 it was declared a pandemic, and by the end of that month, the world saw more than a half-million people infected and nearly 30,000 deaths. As of June 7, 2023, with Covid-19, a total of 767,750,853 people have infected and among these 6,941,095 have died, and also a total of 13,396,086,098 vaccine doses have been administered.

Like large numbers, positive numbers that are significantly smaller than those typically used in everyday life also appear frequently in real world applications. Modern names of small numbers are: One-Millionth 10^{-6}, One-Billionth 10^{-9}, One-Trillionth 10^{-12}, One-Quadrillionth 10^{-15}, One-Quintillionth 10^{-18}, One-Sextillionth 10^{-21}, One-Septillionth 10^{-24}, One-Octillionth 10^{-27}, One-Nonillionth 10^{-30}, and so on. Paramanu (Supreme Atom) is the smallest indivisible material particle and has a taste, odor, and color. This is different to our notion of the atom and is more like what we call a molecule, the smallest particle that constitutes a part of a compound body. The paramanu and the paramanu raja (or grain of dust of the first atoms) have long been the smallest units of length and weight in India. These are found in the *Lalitavistara*, where the paramanu corresponds to 0.000000287 mm and the paramanu raja 0.000000614 g. In view of quantum mechanics, the usual conception of continuous time does not extend to intervals shorter than 5×10^{-44} second. The Ludwig Boltzmann (1844–1906, Austria–Italy) constant (the number that relates the average energy of a molecule to its absolute temperature) $k = 1.3806503 \times 10^{-23}$ joules/kelvin; the Max Planck (1858–1947, Germany) constant *E* (energy of a vibrating molecule was quantized) is proportional to the frequency ν of vibration, i.e., $E = h\nu,$) here $h = 6.626176 \times 10^{-34}$ joule seconds. Newton's universal gravitational constant ($F = GMm/r^2,$ where *M* and *m* are the masses of two bodies separated by a distance *r*) $G = 6.672 \times 10^{-8}$ $cm^3 g^{-1} s^{-2}$. The distance between the nucleus and electron of a hydrogen atom is 0.0000000000005291772 meters. The weight of one atomic mass unit (a.m.u.) is 1.66×10^{-27} :

- From a well-shuffled deck of 52 cards, we deal off 13 cards, one after another, and then find that we have thus obtained a "specified hand," i.e., we have obtained (the order of obtaining them not being important) exactly the group of 13 cards we specified before dealing? The probability of this event will be

$$\frac{13}{52} \cdot \frac{12}{51} \cdot \frac{11}{50} \cdot \frac{10}{49} \cdot \frac{9}{48} \cdot \frac{8}{47} \cdot \frac{7}{46} \cdot \frac{6}{45} \cdot \frac{5}{44} \cdot \frac{4}{43} \cdot \frac{3}{42} \cdot \frac{2}{41} \cdot \frac{1}{40}$$
$$= \frac{1}{635,013,559,600} \simeq 0.000000000001574769.$$

  Hence, billions of hands are equally likely to occur, and the probability is very small.

- For one of the most popular lotteries in the United States, Mega Millions, your odds of winning are about 1 in 176 million. If you are playing a single-state lottery, like the California Super Lotto, your odds increase—to 1 in 42 million.
- In Hamlet there are about $27,000$ letters and spaces, and there being 101 keys on a standard computer keyboards (inputting character sets including alphabets, numbers, symbols, or functions) so that the chance that each time a monkey hitting the right key is $1/101$), the overall

probability of producing Hamlet by chance is unity divided by 101 raised to the power, $27{,}000$ i.e., $(101)^{27000}$, which is almost zero.

- In 1928 the New York Times carried a cabled story from Paris, dated February 21. It reported the fact that six persons had been found guilty of the “accidental” death of a M. Desnoyelles. He had been in a sanatorium, and a medicine had been prescribed for him. But the chief physician gave M. Desnoyelles a prescription really intended for another patient named Desmalles. Second, the chief physician failed to check that the prescription, dictated to a clerk, was written as he had ordered. Third, the intern who filled the prescription confused two drugs and introduced one of a poisonous nature. Fourth, the order for the prescription was mistakenly written on a slip used for medicines for internal use, rather than externally as the doctor intended. Fifth, the head pharmacist, who was supposed to check all prescriptions, was busy. He left the matter to his assistant, and she neglected to check. Sixth, an assistant “corrected” the error in name and wrote on the medicine that it was intended for M. Desnoyelles. Seventh, the intern who administered the medicine disregarded the indicated dose, handed the bottle to the patient, and instructed him to “take a good big drink.” The probability of this strange event is very rare.
- Life Magazine (Life, March 27, 1950, p. 19), also see Weaver [529], reported that all fifteen members of the choir of a church in Beatrice, Nebraska, due at choir at $7:20$ p.m., were late the evening of March 1, 1950. The minister and his wife and daughter had one reason (his wife delated to iron the daughter’s dress); one girl waited to finish a geometry problem; one could not start her car; another could not start her car; two lingered to hear the end of an especially exciting radio program; one mother and daughter were late because the mother had to call the daughter twice to wake her from a nap, and so on. The reasons seemed rather ordinary, but there were ten separate and quite unconnected reasons for the lateness of the fifteen persons. It was rather fortunate that none of the fifteen arrived on time at $7:20$ p.m., for at $7:25$ p.m. the church building was destroyed in an explosion. The members of the choir, *Life* reported, wondered if their delay was “an act of God.” The probability of this curious episode is very small.
- While many people fear the risks of air travel, your odds against dying in an airplane crash are relatively low, approximately 800,000 to 1. Your odds against dying in a car accident, approximately 5000 to 1, are much greater than in an airplane.

- It takes an electric impulse one-billionth of a second to travel 8 inches. One-billionth of a second has come to be called a *nanosecond*. Light travels one foot in one nanosecond. Computers today are built to perform millions of operations per second.

2.8 Infinity is a Legitimate Concept

Infinity (without end) is just a phantom of the mind (outside our range of detection) of something that has no end, it is not a real number, and it has tantalized and often troubled the mankind. Before 1874, infinity was not even considered a legitimate mathematical concept/necessity. However, now the concept of infinity, though not imposed upon us either by logic or by experience, is one of the most important concepts in mathematics, physics, statistics, and metaphysics. In Hindu mythologies and cosmologies, as we have noted, surprising thing is that zero is also a term *Ananta* that means infinite (infinite void or void infinite), limitlessness, endless, without an end, boundless; however, it becomes intelligible from the fact that Ananta refers to a huge celestial serpent *Adishesha or Anantashesha* representing eternity and the immensity of space all at once. It is shown resting on the primordial waters of original chaos. Lord Vishnu is lying on the serpent, between two creations of the world, floating on the "ocean of unconsciousness." The serpent is always represented as coiled up, in a sort of figure eight on its side (like the symbol ∞ [lemniscate], which was adopted in mathematics by Wallis for infinity in 1657, Romans used it to denote the number 1000, Voltaire described the ∞ as a "love knot," and he was skeptical about the sign making the idea of infinity any clearer) and theoretically has a thousand heads. It is considered to be the great king (Nagas) and lord of hell (Patala). Each time the serpent opens its mouth it produces an earthquake because there is a belief that the serpent also supported the world on its back. It is the serpent that at the end of each Kalpa, spits the destructive fire over the whole of creation. According to Hindu philosophy, God is infinite and within us, but seems to become limited by our body-mind complex giving a feeling of finitude, which is the cause of all limitations.

According to Vedas, the infinite remains the same, even though the infinite universe that has no beginning or end has come out of it, see Lakshmikantham [330]. From abstract zero to infinity was a single step which Hindu scholars took early and nimbly. In fact, vanishing point links zero and infinity. The "AUM" symbol (or OM) symbolizes the universe and the ultimate reality, and hence infinity. It is the most important Hindu symbols. At the dawn of creation, from emptiness first emerged a syllable consisting of three letters - A-U-M, where A stands for Brahma, U for Vishnu, and M for Shiva. Deities and qualities associated with Ananta also appear in Buddhism and Jainism, see Surya Prajnapti (400 BC). For where all causes concur by the blending and altering of atoms or elements in the physical universe, there their effects must also appear." Jainas classified numbers into three groups enumerable (lowest, intermediate, and highest), innumerable (nearly innumerable, truly innumerable, and innumerably innumerable), and infinite (nearly infinite, truly infinite, and infinitely infinite). The first group, the enumerable numbers, consisted of all the numbers from 2 (1 was ignored) to the highest. An idea of the "highest" number is given by the following extract from the Anuyoga Dwara Sutra: Consider a trough whose diameter is that of the Earth (100,000 yojana) and whose circumference is $316,227$ yojana. Fill it up with white mustard seeds counting one after another. Similarly, fill up with mustard seeds other troughs of the sizes of the various lands and seas. Still the highest enumerable number has not been attained. But once this number, call it $N,$ is attained, infinity is reached via the following sequence of operations

$$N+1, N+2, \ldots, (N+1)^2 - 1xxximg2stmathopenparexxxN+1)^2, (N+2)^2, \ldots, (N+1)^4 - 1xxximg2stmathopenparexxxN+1)^4, (N+2)^4, \ldots, (N+1)^8 - 1$$

and so on.

Jainas recognized five different kinds of infinity: infinite in one direction, infinite in two directions, infinite in area, infinite everywhere in space, and infinite everywhere in space and at all times. Thus, they were the first to discard the idea that all infinities were the same or equal, and to conceive two basic types of transfinite numbers, a concept, which was brought to Europe by Cantor in the late nineteenth century. Like Hindus all theologians and metaphysicians from Plotinus have supposed the God to be infinite (God's infinity is called transcend mundane infinity). In Chinese mythology "Turtles all the way down" is an expression of the problem of infinite regress. The saying alludes to the mythological idea of a World Turtle that supports the Earth on its back. It suggests that this turtle rests on the back of an even larger turtle, which itself is part of a column of increasingly large turtles that continues indefinitely.

Babylonians and Egyptians dealt at least subconsciously with the concept of the mathematical infinite. Ancient Greeks coined the terms *apeiron* (unbounded, indefinite, undefined, and formless) and *peras* (limit or bound), which are now labeled as *potentially infinite* and *actually infinite* (completed, definite, extended, or existential and consists of infinitely many elements), respectively. (We remark that unbounded is not necessarily infinite, e.g., by the theory of relativity the universe is unbounded and finite.) For Anaximander apeiron was the principle or main element composing all things, some sort of basic permanent substance, whereas for Plato apeiron was more abstract, having to do with indefinite variability. In fact, Plato had two infinities, the Great and the Small. The potential infinite is a group of numbers or group of "things" that continues without terminating. For example, when we list natural numbers as $1, 2, 3, 4,$ there always exists another number to proceed the one before; similarly a geometric line with a starting point could extend on without end. Also, an infinite sequence (reflecting the infiniteness of the material world in space and time) of divisions might start $1, 1/2, 1/4, 1/8, 1/16,$ but the process of division cannot be exhausted or completed (if it has any real meaning). From the fact that the process of adding or dividing never comes to an end ensures that these activities exist potentially, but not that the infinite exists separately. In view of infinitely divisibility each object would in principle contain a potentially infinite collection of particles; however, quantum mechanics rules out (often formulated in terms of an infinite-dimensional Hilbert space [after David Hilbert, 1862–1943, Germany], but these dimensions are more useful fictions than solid realities) this notion. Potential infinity can be observed in reality also, e.g., the Earth turning on its axis, planets revolving around the Sun, the cycle of generations, the accumulation of knowledge, and the list continues.

The actual infinity involves never-ending sets or "things" within a space that has a beginning and end; it is a sequence/series that is technically "completed" but consists of an infinite number of members. Zeno is remembered for his eponymous paradoxes (see Sect. 3.18) of motion that are rooted in deep questions about the nature of time and space and in some misconceptions about infinity. In fact, according to Aristotle, "Zeno's argument makes a false assumption in asserting that it is impossible for a thing to pass over or severally to meet with infinite[ly many] things in finite time." Some say that Zeno directed his paradoxes against Pythagoreans thought of space as the sum of points. Anaxagoras statement "There is no smallest among the small and no largest among the large; but always something still smaller and something still larger" stipulates zero and infinity. Ponticus proposed that cosmos is infinite. Aristotle handled the topic of infinity in physics and in metaphysics. According to him, actual infinities cannot exist in the real world because they are paradoxical (in an infinite collection, since it would have a proper part that was bounded by it and smaller than it, and yet infinite too, which is absurd). It is impossible to say that you can always "take another step" or "add another member" in a completed set with a beginning and end, unlike a potential infinite. According to him "Infinity turns out to be the opposite of what people say it is. It is not "that which has nothing beyond itself" that is infinite, but "that which always has something beyond itself'." He further argued that actual infinity is not applicable to geometry, so not relevant to mathematics, and hence only potential infinity is important.

Aristotle's rejection of the actual infinite refuted Zeno's paradoxes. Further, his influence continued for more than a millennium, which deferred a lot of major concepts in mathematics. According to Metrodoros of Chios (fl. fourth century BC, Greece), "To consider the Earth as the only inhabited world in the infinite universe is as absurd as to assert that in an entire field sown with millet, only one grain will grow. That the universe is infinite with an infinite number of worlds follows from the infinite number of causalities that govern it. If the universe were finite and the causes that caused it infinite, then the universe would be comprised of an infinite number of worlds. Eudoxus of Cnidus (around 400–347 BC, Greece) and Archimedes used potential infinity to develop a technique (infinite process), later known as the method of exhaustion, whereby area of curved figures was calculated by halving the measuring unit at successive stages until the remaining area was below some fixed value (the remaining region having been "exhausted"). For example, Archimedes found the area of a parabolic segment from the infinite series

$$1+\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\cdots\ ,$$

which has sum $4/3.$ He was able to find this sum by considering only the potential infinity of terms $1, 1+1/4, 1+1/4+1/4^2, 1+1/4+1/4^2+1/4^3, \cdots$ by showing that each of these terms is less than $4/3,$ and each number less than $4/3$ is exceeded by some term in the sequence. By avoiding actual infinity, both of them foreshadowed the concept of the limit (a decisive step in whole of mathematics, particularly in calculus), which had to wait more than two millennia.

Brahmagupta talked of infinity, defining it as the opposite/inverse of zero, a concept which reached Europe after more than a millennia (zero and infinity are simply opposite poles on the Riemann sphere). Seife [458] endorses that zero is powerful because it is infinity's twin. They are equal and opposite, yin and yang. They are equally paradoxical and troubling. In the Middle Ages an enormous amount of scholastic logic was expended on the theory of the infinite as it applies to Christian theology. In fact, Roger Bacon (1219/20–1292, England), Nicholas of Cusa (man is finite and can never attain the infinite, he may apprehend his existence through mathematics, the only truth of science), Thomas Digges (1546–1595, England), and Giordano Bruno (1548–1600, Italy) all believed that universe is infinite whose center is everywhere and circumference nowhere. Medieval thinkers were not limited to Aristotle's potential infinity. In particular, Gregory of Rimini (1300–1358, Italy) maintained, against Aristotle, that God or the Absolute could create an actually infinite stone. Gregory explained that God could do this by creating equal-sized bits of the stone at each of the times $t = 0, 1/2, 3/4, 7/8, \cdots$. John Baconthorpe (around 1290–1347, England) had argued that actual infinity exists in number, time, and quantity; Galileo said the continuum actually consists of infinitely many indivisibles; Blaise Pascal (1623–1662, France) claimed that like the infinity of numbers, humans are not able to conceive the infinity of God; and Leibniz was in favor of actual infinity. Seventeenth and eighteenth centuries mathematicians had little understanding of infinite series. They often applied, to such series, operations that hold for finite series but apply to infinite series only under certain restrictions. Not being aware of the restrictions, the laughable results were obtained. For example, Luigi Guido Grandi (1671–1742, Italy) in his book *Quadratura Circuli et Hyperbolae* of 1703 discussed the result of adding 1 and -1 alternately taken infinitely many times

$$1 + (-1) + 1 + (-1) + \cdots .$$

If we apply the rules of arithmetic (see Sect. 3.9), while calculating the sum, then we arrive at very strange results. Let us denote the sum as S. Then,

$$S \ = \ 1 + (-1) + 1 + (-1) + \cdots .$$

Associating 1 and -1 in pairs and using the property of 0, we easily get

$$S \ = \ (1 + (-1)) + (1 + (-1)) + \cdots \ = \ 0 + 0 + \cdots \ = \ 0.$$

Thus it seems reasonable to assume that the sum is 0. However if we associate the terms a little differently, we have

$$S \;=\; 1 + ((-1) + 1) + ((-1) + 1) + \cdots \;=\; 1 + 0 + 0 \cdots \;=\; 1.$$

If we use the following combination of associativity, commutativity, and 0, we have

$$\begin{aligned} S \;&=\; 1 + \{[(-1) + 1] + [(-1) + 1] + \cdots\} \\ &=\; 1 + \{[1 + (-1)] + [1 + (-1)] + \cdots\} \\ &=\; 1 + \{1 + [(-1) + 1] + [(-1) + 1] + \cdots\} \\ &=\; 1 + \{1 + 0 + 0 + \cdots\} \;=\; 2. \end{aligned}$$

Using similar arguments repeatedly, we can prove *S* to be equal to any natural number. Let us now write

$$\begin{aligned} S \;&=\; (1 + (-1)) + (1 + (-1)) + \cdots \\ &=\; ((-1) + 1) + ((-1) + 1) + \cdots \\ &=\; (-1) + \{(1 + (-1)) + (1 + (-1)) + \cdots\} \\ &=\; -1 + \{0 + 0 + \cdots\} \;=\; -1. \end{aligned}$$

Thus using different combinations of the rules applied above, we can show that *S* is equal to any integer—positive, negative, or zero. Now applying the binomial expansion (2.1), we have $1/(1+x) = 1 - x + x^2 - x^3 + \cdots$. Putting $x = 1$, we get $1/2 = 1 - 1 + 1 - 1 + \cdots$. Therefore, $S = 1/2$. If we multiply both sides by -1, we get

$$\begin{aligned} -\frac{1}{2} &= -1+1-1+1-\cdots \\ &= ((-1)+1)+((-1)+1)+\cdots \\ &= (1+(-1))+(1+(-1))+\cdots \\ &= 1+(-1)+1+(-1)+\cdots \;=\; S. \end{aligned}$$

Therefore, Grandi felt that any rational number could be derived from S. The "original" and "obvious" value of *S* is 0. Therefore, Grandi claims that the world is created out of nothing. This claim has been supported by Leibniz. In fact, he further added that Grandi's conclusion may sound metaphysical, but there is more metaphysical truth in mathematics than is generally believed. As an another example, let *S* denote the sum of the convergent series

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots .$$

Then

$$\begin{aligned} S &= \left(\frac{1}{1}-\frac{2}{3}\right)+\left(\frac{2}{3}-\frac{3}{5}\right)+\left(\frac{3}{5}-\frac{4}{7}\right)+\cdots \\ &= 1-\frac{2}{3}+\frac{2}{3}-\frac{3}{5}+\frac{3}{5}-\frac{4}{7}+\cdots \;=\; 1 \end{aligned}$$

since all terms after the first cancel out. Again

$$\begin{aligned} S &= \frac{1}{2}\left(\frac{1}{1}-\frac{1}{3}\right)+\frac{1}{2}\left(\frac{1}{3}-\frac{1}{5}\right)+\frac{1}{2}\left(\frac{1}{5}-\frac{1}{7}\right)+\cdots \\ &= \frac{1}{2}-\frac{1}{6}+\frac{1}{6}-\frac{1}{10}+\frac{1}{10}-\frac{1}{14}+\cdots \;=\; \frac{1}{2} \end{aligned}$$

since all terms after the first term cancel out. It follows that $1 = 1/2$. Similarly, consider the series

$$\ln 2 \;=\; 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots \;=\; 0.6931471\cdots .$$

If we rearrange these terms as we would be prompted to do in finite arithmetic, we obtain

$$\ln 2 \;=\; \left(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots\right)-\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots\right).$$

Thus,

$$\ln 2 \;=\; \left\{\left(1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots\right)+\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots\right)\right\}-2\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots\right)$$

$$=\;\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots\right)-\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots\right)\;=\;0.$$

Therefore, $\ln 2 = 0$.

According to Morris Kline (1908–1992, USA), "the ignorance about the nature of infinity was so universal, that, in the words of mathematicians, "Newton, Liebniz, Johann Bernoulli (1667–1748, Switzerland), Euler, Jean le Rond d'Alembert (1717–1783, France), Lagrange, and several eighteenth-century men struggled with infinite series. They perpetrated all sorts of blunders, made false proofs, and drew incorrect conclusions; they even gave arguments that, now with hindsight, we are obliged to call ludicrous." The majority of pre-modern thinkers agreed with the well-known quote of Gauss: "I protest against the use of infinite magnitude as something completed, which is never permissible in mathematics. Infinity is merely a way of speaking, the true meaning being a limit which certain ratios approach indefinitely close, while others are permitted to increase without restriction."

In physics infinities are believed/imagined and are limited to space (it spreads out infinitely in all directions, and galaxies fill all of the space throughout the entire infinite Universe; Bruno strongly advocated that the Universe is one, infinite, immobile…. It is not capable of comprehension and therefore is endless and limitless, and to that extent infinite and indeterminable, and consequently immobile), time (it was before us; provided we do not believe in Big Bang theory which physicists adhered took place about 15 billion years ago, it is with us, and will be after us; for which physicists are mostly silent), energy (it cannot be created or destroyed; so it will continue forever), divisibility (matter is infinitely divisible). However, many parts of mathematics demand that we accept some form of infinity in a definite sense. In fact, infinity is the lifeblood of mathematics, a substantial part of the abstract mathematics deals with infinite dimensions and infinite extensions, but we cannot conceptualize it.

Infinity occurs, for instance, as the number of mathematical points even on a finite continuous line/space (see Chap. 8) or as the size of the endless sequence of counting numbers $1, 2, 3, \cdots$. When we expand numbers in decimals, we get three different types of expansions: Nonterminating Decimal Numbers:
$0.3333\cdots$, $4.43333\cdots$, $5.34672310\cdots$; Recurring Decimals: $0.111\cdots$, $4.444444\cdots$, $5.232323\cdots$, $21.123123\cdots$; and Nonrecurring Decimals: $0.1223589\cdots$, $4.4782451\ \cdots$, $5.67245\cdots$. Thus, in the representation (2.5), *m* or/and *n* can be infinite. It is to be noted that any number *r* is rational if and only if its decimal part terminates or recurring, e.g., $13/7 = 857142, 857142, 857142, \cdots$. Hence, in the philosophy of mathematics, the existence of actual infinity is generally accepted.

The actual infinity is contrasted with potential infinity, in which a nonterminating process produces a sequence with no last element, and where each individual result is finite and is achieved in a finite number of steps. As a result, potential infinity is often formalized using the concept of limit. In 1350, Albert of Saxony (around 1320–1390, Germany) showed that one can take a proper subset of an infinite set and rearrange its elements so that it shows itself to be just as big and unbounded as the infinite set of which it is a proper part. He noted that if one has an infinitely long beam of wood, with equal width and depth, one can saw it up into equal-sized cubic blocks with which one can fill the whole of Euclidean three-dimensional space. (Surrounded the first block with $3^3 - 1$ more blocks, making a cube of side 3, then surrounded that cube with $5^3 - 3^3$ more blocks, making a cube of side 5; and so on.) In modern terminology, what Albert observed is that there is a one-to-one correspondence/mapping/function (it needs no counting, the method employed in antiquity, but formalized by Jains) between the set of triples $(n, 1, 1)$, with *n* a positive integer and the set of triple (a, b, c), with a, b, and *c* any integers. William Shakespeare (1564–1616, England) commented “A could be bounded in a nutshell, and count myself a king of infinite space.”

In the early 1600s Galileo proposed that “infinity should obey a different arithmetic than finite numbers.” According to him “part” means some but not all. In a finite collection there are always more things in the whole collection than in any one of its parts. Galileo in his *Dialogues*

Concerning the Two New Sciences, First Day observed that there is a one-to-one correspondence between the set $\mathcal{N}$ of positive integers and the subset *S* of $\mathcal{N}$ consisting of the squares of positive integers.

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & \cdots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 1^2 & 2^2 & 3^2 & 4^2 & 5^2 & \cdots \end{array}$$

This impelled Galileo to observe that even though there are many positive integers that are not squares, there are as many squares as there are positive integers. This led Galileo to be faced with a property of an infinite set that he found bothersome. There can be a one-to-one correspondence between a set and a proper subset of the set. While Galileo concluded correctly that the number of squares of positive integers is not less than the number of positive integers, he could not bring himself to say that these sets have the same number of elements (in the literature his observation is famous as Galileo's paradox). He concluded that the Attributes of Equality, Majority, and Minority have no place in Infinities, but only in terminate quantities. Similarly, we have $n \leftrightarrow 2n$ (all even integers), $n \leftrightarrow (2n-1)$ (all odd integers), and $n \leftrightarrow (1+(-1)^n(2n-1))/4$ (the set of all integers $\mathcal{Z}$)

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & \cdots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 0 & 1 & -1 & 2 & -2 & \cdots \end{array}$$

Newton in the seventeenth century boldly used the mathematical infinity. His conception of the deity is of some mathematical interest for its insistence on the infinite as a characteristic attribute of the supreme being. The deity, according to Newton,"is supreme, or most perfect. He is eternal and infinite, omnipotent and omniscient; that is, his duration reaches from eternity to eternity; his presence from infinity to infinity.... He is not eternity and infinity, but eternal and infinite; he is not duration or space, but he endures and is present. He endures forever and is everywhere present; and, by existence always and everywhere, he constitutes duration and space.... He is utterly void of all body and bodily figure..." William Blake (1757–1827, England) remarked "To see the world in a grain of sand. And a heaven in a wildflower: Hold infinity in the palm of your hand, And eternity in an hour."

The drastic change was initialized by Bolzano; he recognized that one-to-one correspondence between an infinite set and a proper subset of itself is common and was comfortable with this fact, contrary to Galileo's feelings. According to him a multitude which is larger than any finite multitude, i.e., a multitude with the property that every finite set (of members of the kind in question) is only a part of it, I will call an infinite multitude. What had baffled Galileo and excited Bolzano led to Dedekind a proper definition of an infinite set during the last part of the nineteenth century. According to him, a set *S* is infinite if it contains a proper subset that can be put in one-to-one correspondence with $S.$ This definition was the first hint that it may be possible to talk consistent mathematics about "the infinite," and without it, Cantor's theory of sets of points, fundamental in modern analysis, would not exist. In 1887, Dedekind published a proof that the mindscape (thought world) is infinite.

Finally, Cantor during 1871–84 systematically mathematized the works of Jainas of 200–875 AD on set theory (Jainas introduced several different types of sets, such as cosmological, philosophical, finite, infinite, transfinite, and variable sets. They called the largest set an omniscient set, and the conceptual set containing no elements was known as the null set. They also defined the concept of a union of sets and used the method of one-to-one correspondence for the comparison of transfinite sets. In order to determine the order of comparability of all sets, they considered fourteen types of monotone sequences.) to put infinity on a firm logical foundation and described a way to do arithmetic with infinite quantities useful to modern mathematics. According to Cantor, "A set is a Many that allows itself to be thought of as a One." He firmly said in mathematics actual infinities exist. Thus, the infinite sets $\{1, 2, 3, \cdots\}$ and $\{1, 1+1/4, 1+1/4+1/4^2, 1+1/4+1/4^2+1/4^3, \cdots\}$ are examples of actual infinity. The series $1+1/2+1/4+1/8+\cdots$ and $1+1/2+1/4+1/8+\cdots=2,$ respectively, are examples of potential and actual infinities. The series $1+1/2+1/3+1/4+\cdots$ known as *Harmonic Series* (the nomenclature comes from the overtones or harmonics in music) is an example of potential infinity. Oresme around 1350 showed that

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\cdots$$

$$> \ 1+\left(\frac{1}{2}\right)+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\left(\frac{1}{16}+\cdots+\frac{1}{16}\right)+\cdots$$

$$> \ 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots \quad \longrightarrow \quad \infty.$$

Thus, the series $1+1/2+1/3+1/4+\cdots=\infty$ is an example of actual infinity on the extended real number line $[-\infty,\infty].$ Cantor also defined the intellectual pursuit of the Absolute infinity as a form of soul's quest for God. Besides one-to-one correspondence between arbitrary infinite sets, he called two sets are *equivalent* or have the same *cardinal number* (cardinality, answers the question "How many"?), which in short written as $Card,$ provided there is a one-to-one correspondence between their respective members. Clearly, between two finite sets a one-to-one correspondence can be set if and only if they have the same number of elements. The cardinality of empty set $\emptyset$ is always taken as $0,$ whereas for an arbitrary set with *n* elements, e.g., $\{1,2,\cdots,n\}$ it is $n.$ For two finite sets *A* and $B,$ it is clear that
$Card(A\cup B)=Card(A)+Card(B)-Card(A\cap B)$ and
$Card(A\cap B)=Card(A)+Card(B)-Card(A\cup B).$ If a cardinal number is not finite, it is called *transfinite cardinal*. Cantor called a set *countable* (*denumerable*) if it is equivalent to the set $\mathcal{N}.$ The cardinal number of a denumerable set is denoted by $\aleph_0$ (the Hebrew first letter, called aleph not/zero), and it is not a positive integer at all. Countable infinite sets are considered of smallest size. Thus, from the above, it follows that the set of all even integers, the set of all odd integers, the set of all integers, and the set of all square numbers have the same cardinality $\aleph_0.$ Each member of a denumerable set can be assigned a fixed place (such as first, second, third,…), i.e., members can be written as sequences and therefore also called *ordinal numbers*. The following properties of countable sets are fundamental:

(R1). Every subset of a countable set is countable.

(R2). The union of a countable number of countable sets is also a countable set.

(R3). The Cartesian product of two countable sets *A* and *B* is countable.

(R4). The set of intervals with rational numbers as endpoints is countable.

For illustration we shall show (R3). We arrange the elements of the sets *A* and *B* as $A = \{a_1, a_2, \cdots\}$ and $B = \{b_1, b_2, \cdots\}$ so that $A \times B$ can be written as

$$\begin{array}{cccc} (a_1, b_1) & (a_1, b_2) & (a_1, b_3) & \cdots \\ (a_2, b_1) & (a_2, b_2) & (a_2, b_3) & \cdots \\ (a_3, b_1) & (a_3, b_2) & (a_3, b_3) & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{array}$$

Now we define a one-to-one mapping from $A \times B$ to $\mathcal{N}$ as follows:

$f(a_1, b_1) = 1,\ f(a_2, b_1) = 2,\ f(a_1, b_2) = 3,\ f(a_1, b_3) = 4,\ f(a_2, b_2) = 5,\ f(a_3, b_1) = 6,\ f(a_4, b_1) = 7,\ f(a_3, b_2) = 8,\ f(a_2, b_3) = 9,\ f(a_1, b_4) = 10, \cdots$

(draw the diagonal zigzag path).

Now we shall show that the set of all rational numbers $\mathcal{Q}$, which appears to be more numerous than the set of natural numbers, is countable. For this, we note that $\mathcal{Q} = \cup_{n=1}^{\infty} A_n$, where A_n is the set of rational numbers with denominator n, i.e., $A_n = \{0/n, -1/n, 1/n, -2/n, 2/n, \cdots\}$. Now each A_n is equivalent to $\mathcal{Z}$ and is thus countable. Hence, the set $\mathcal{Q}$ is the countable union of countable sets and hence from (R2) must be countable.

From this and (R1) it follows that the set of all rational numbers contained in any given interval is countable. However, these rational numbers cannot be arranged as an increasing sequence. This follows from the fact that there is no smallest rational number among the numbers exceeding a given rational number:

- In 1885, Carl Gustav Axel Harnack (1851–1888, Germany) considered $a_1, a_2, a_3, \cdots$ any list of real numbers, that is, the points on the real line $(-\infty, \infty)$ denoted as $\mathcal{R}$. He observed that these numbers can be covered by line segments of total length as small as we please, say ϵ. Just take a line segment of length ϵ, break it in half, and use a segment of length $\epsilon/2$ to cover the number a_1. Then break the remaining segment of length $\epsilon/2$ in half and use a segment of length $\epsilon/4$ to cover $a_2,$ and so on. Thus we cover a_n by $\epsilon/2^n.$ Hence all numbers $a_1, a_2, a_3, \cdots$ are covered by $\sum_{n=1}^{\infty} \epsilon/2^n = \epsilon.$ It follows that the numbers $a_1, a_2, a_3, \cdots$ do not include all real numbers. In fact, far from filling the whole line, the set of numbers $a_1, a_2, a_3, \cdots$ has the total length zero! In particular, the infinity of rationales have the total length zero.

- Now we shall demonstrate that if the elements of a set *A* specified by a finite number of parameters each of which can independently take on any value belonging to a countable set. Then, the set *A* is countable. For this, we write the elements of the set *A* as $a(\lambda_1, \lambda_2, \cdots, \lambda_n)$ where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are parameters. We assume, without loss of generality, that these parameters are natural numbers. For each $a(\lambda_1, \lambda_2, \cdots, \lambda_n) \in A$ we set $N(a) = \lambda_1 + \lambda_2 + \cdots + \lambda_n.$ It is clear that $N(a)$ is a natural number and $N(a) \geq n.$ Now given any $m \geq n$, let A_m denote the set of all elements of *A* for which $N(a) = m.$ It is clear that every set A_m is finite and $A = \cup_{m=n}^{\infty} A_m.$ Therefore, from (R2) the set *A* is countable.

  From the above result it immediately follows that the set $\mathcal{P}_n$ of all algebraic polynomials of a fixed degree *n* with rational coefficients

$$P_n(x) \;=\; a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n \quad a_0 \neq 0, \quad a_i \in \mathcal{Q}, \;\; 0 \leq i \leq n$$

  is countable. Thus, from (R2) we can conclude that the set of all algebraic polynomials $\mathcal{P} = \cup_{n=0}^{\infty} \mathcal{P}_n$ with rational coefficients is countable.

A real number is called *algebraic* if it is a zero of an algebraic polynomial with integer coefficients. For example, the number $\sqrt{7}$ is algebraic because it is a zero of the polynomial $x^2 - 7.$ Thus whole numbers, fractions, and their square roots, cube roots, and so on are algebraic. The imaginary number *i* is an algebraic number because it is a solution to the polynomial equation $x^2 + 1 = 0.$ Since each algebraic polynomial can have only a finite number of distinct real zeros, the countability of the set $\mathcal{P}$ immediately implies that the set of all algebraic numbers is countable. Real numbers which are not algebraic numbers are called *transcendental numbers*, e.g., the numbers $e, \; \pi$ (known as universal constants) are transcendental, see Chap. 8:

- In 1947 [206], George Gamow (1904–1968, Ukraine–USA) elucidated (R2) by considering Hilbert’s Grand Hotel/Paradox. There are rooms for a countable infinity of countable infinities of guests. Suppose, say, that the guests arrive on infinite buses numbered $1, 2, 3, \cdots$ and that each bus has guests numbered $1, 2, 3, \cdots$. The guests in bus 1 can be accommodated as follows: Put guest 1 in room 1, and then skip 1 room; put guest 2 in room 3, and then skip 2 rooms; put guest 3 in room 6, and then skip 3 room; $\cdots$ (the guests from the first bus are taking rooms numbers in the sequence of triangular numbers (see Sect. 7.2). Thus the first bus fills the rooms. After the first bus has been unloaded, the unoccupied rooms are in blocks of $1, 2, 3, \cdots$ rooms, so we can unload the second bus by putting its guests in the leftmost room of each block. After that, the unlocked rooms are again in blocks of $1, 2, 3, \cdots$ rooms, so we can repeat the process with the third bus, and so on. This also shows that $\aleph_0 + \aleph_0 = \aleph_0$ and $\aleph_0 \times \aleph_0 = \aleph_0$ (instead of respectively implying $\aleph_0 = 0$ and $\aleph_0 = 1$), in fact, for any integer $n \geq 1,\ \aleph_0^n = \aleph_0$. Thus, the rules of addition, multiplication, and exponential for denumerable sets are different from the cardinality of finite sets. In 1897, Friedrich Wilhelm Karl Ernst Schröder (1841–1902, Germany) and Felix Bernstein (1878–1956, Switzerland) proved that if *A* and *B* are sets such that $Card(A) \leq Card(B)$ and $Card(B) \leq Crad(A)$, then $Card(A) = Crad(B)$. To illustrate this result, we note that $Card(\mathcal{Q}) \leq Card(\mathcal{N})$ and $Card(\mathcal{N}) \leq Card(\mathcal{Q})$, and it follows that $Card(\mathcal{N}) = Card(\mathcal{Q}) = \aleph_0$.

Cantor further used the so-called diagonal argument (see Chap. 8) to show that for every transfinite cardinal there exists a larger transfinite cardinal, which he denoted by $\aleph_1;$ for every transfinite cardinal there exists a next larger transfinite cardinal, which he denoted by $\aleph_2;$ and so on. He proved that these alephs include all possible transfinite cardinals. Then, in 1874 he showed that the set of all real numbers $\mathcal{R}$ is uncountable (the quanta of light are unidentifiable and uncountable; electrons are unidentifiable but countable) and is of higher power than the set $\mathcal{N}$. Cantor denoted the number of the continuum (the size of real numbers) by the German character $\mathfrak{c}$. He could not establish where the number of the continuum stood among the alephs (known as continuum problem); however, he conjectured that $\mathfrak{c} = \aleph_1$, which is known as Cantor's continuum hypothesis (CH), and it is equivalent to $\aleph_1 = 2^{\aleph_0}$ (a finite set *S* with *n* elements contains 2^n all possible subsets, so that the cardinality of the set *S* is *n* and its so-called power set $P(S)$ is 2^n). In 1940, Kurt

Friedrich Gödel (1906–1978, Czech Republic–USA) proved that the continuum hypothesis could not be proven to be false, whereas in 1963, Paul Joseph Cohen (1934–2007, USA) proved that continuum hypothesis could not be proven to be true. Thus (CH) is undecidable, and hence it is possible to adopt this statement, or its negation. The undecidability of (CH) was already suspected in 1922 by Albert Thoralf Skolem (1887–1963). Sets that have a greater power than $\mathfrak{c}$ are known. We shall discuss details about these concepts in Chap. 8. Here we note that Medieval thinkers were aware of the paradoxical (here synonymous with contradiction) fact that line segments of varying lengths seemed to have the same number of points. This fact can be seen easily geometrically: For this, it suffices to show that there are as many points on the short line as there are on the long line. In Fig. 2.4, we take the line *AB* and the longer line $CD,$ place them parallel to each other, and join the ends *AC* and $BD.$ We extend *CA* and *DB* until they intersect at $O.$ It is then easy to see that any line drawn from *O* through the two lines *AB* and *CD* will intersect them at points *P* and $Q,$ respectively. For every point *Q* on the longer line, there will be a point *P* on the shorter which can be paired in one-to-one correspondence with it.

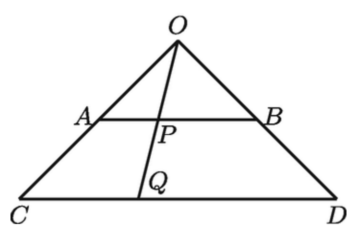


Fig. 2.4 One-to-one correspondence

The topic of infinitely small numbers or infinitesimals (means something distinguishable from zero, yet which is exceedingly small-so minute that no finite multiplication of it can be made of an observable size, except multiplication of infinite numbers with infinitesimals can yield real, detectable numbers) led to the discovery of calculus in the late 1600s by Newton and Leibniz (almost two hundred years before Cantor). Earlier to them rudimentary form of calculus was known in India and Islam (see Katz [300]). Archimedes application of the method of exhaustion possessed all the elements essential to an infinitesimal analysis. Newton introduced his theory of infinitesimals, to justify the calculation of derivatives (fluxions), or slopes of functions. In fact, to find the slope, i.e., the change in *y* over the change in *x* for a line touching a curve $y = f(x)$ at a given point (x, y), he found it useful to look at the ratio between *dy* and dx, where *dy* is an infinitesimal change in *y* produced by moving an infinitesimal amount *dx* from x. However, since infinitesimals were something new, Newton's contemporaries heavily criticized them. For example, the Italian Jesuit Girolamo Saccheri (1667–1733) rejected an improper use of the infinitesimals. Specially, on fluxions, Bishop Berkeley in his tract *The Analyst: or a discourse addressed to an infidel mathematician* of 1734 wrote a knowledgeable and extremely witty attack: "And what are these fluxions? The velocities of evanescent increments. And what are these same evanescent increments? They are neither finite quantities, nor quantities infinitely small, nor yet nothing. May we not call them ghosts of departed quantities?" His criticisms were well founded and important, as they focused the attention of mathematicians on a logical clarification of the calculus.

According to Bishop Berkeley "our only knowledge of this world is what comes to us through our senses, and he went so far as to say that physical objects only exist relative to the mind." Bishop Berkeley's objections are justified because a belief in the infinitesimal does not triumph easily. Later Euler added "If a nonnegative quantity was so small that is smaller than any given one, then it certainly could not be anything but zero. To those who ask what the infinity small quantity in mathematics is, we answer that it is actually zero. Hence there are not so many mysteries hidden in this concept as they are usually believed to be. These supposed mysteries have rendered the calculus of the infinitely small quite suspect to many people". Yet when one thinks boldly and freely, the initial distrust will soon mellow into a pleasant certainty. A majority of educated people will admit an infinite in space and time, and not just an "unboundedly large." But they will only with difficulty believe in the infinitesimal, despite the fact that the infinitesimal has the same right to existence as the infinitely large.

Simon Stevin was one of the first to open a new and productive period in the application of infinitesimals to mathematical problems. In 1748, Euler [180] published a seminal text in Latin which laid the foundations of mathematical analysis (calculus). In 1877, Paul David Gustav du Bois-Reymond (1831–1889, Germany) claimed that infinitesimal is a mathematical quantity and has all its properties in common with the finite. However, Cantor vehemently opposed to infinitesimals. Finally, the use of infinitesimal numbers gained a firm footing with the development of nonstandard analysis by the mathematician Abraham Robinson (1918–1974, Germany–USA) in the 1960s. His method is based on enlarging the real numbers to the set of *hyperreal numbers*. He completely denied the existence of any type of infinity. It is to be noted that Newton's calculus is about functions, whereas Leibniz's calculus is about relations defined by constraints. In Newton's calculus the concept of limit is built into every operation, whereas in Leibniz's calculus the limit is a separate operation. It was to be over 100 years, however, before calculus was to be made rigorous by Cauchy and Karl Theodor Wilhelm Weierstrass (1815–1897, Germany), in the sense that the derivative and the integral were formed in terms of limits, instead of terrifying infinitesimal analysis.

In the beginning Cantor's work on infinite sets provoked heated controversy. His claim that the infinite sets are unbounded offended the religious view of the time that God had created a complete universe, which could not be wholly comprehended by man. Cantor's revolutionary ideas were opposed with emotion, much of it blind and bitter. His former mentor, Leopold Kronecker (1823–1891, Germany), ridiculed Cantor's theory as a "scientific charlatan," a "renegade" and a "corrupter of youth." He prevented him from gaining a position at the University of Berlin. However, Cantor stood firm: "I was logically forced, almost against my will, because in opposition to traditions which had become valued by me, in the course of scientific researches, extending over many years, to the thought of considering the infinitely great, not merely in the form of the unlimitedly increasing… but also to fix it mathematically by numbers in the definite form of a "completed infinite." I do not believe, then, that any reasons can be urged against it which I am unable to combat." In connection with the mathematical infinite Luitzen Egbertus Jan Brouwer (1881–1966, Netherland) followed Kronecker, he also refused to admit that a proposition is either true or false unless some means for deciding which prescribed. In 1906, Jules Henri Poincaré (1854–1912, France) wrote that there was no actual infinity; he saw the Cantorians as being trapped by contradictions. He referred to Cantor's proofs and theories as mathematics' "grave disease." Ludwig Josef Johann Wittgenstein (1889–1951, Austrian–British) argued that Cantor's proofs of infinity were "ridden through and through with the pernicious idioms …" dismissing his work as "utter nonsense" that is both "laughable" and unquestionably "wrong." According to Abraham Robinson, infinite totalities do not exist in any sense of the word (i.e., either really or ideally). More precisely, any mention, or purported mention, of infinite totalities is, literally, meaningless. However, Weierstrass and Hilbert were impressed with Cantor's work and defended Cantor to his detractors. Bertrand Russell declared "the solution of the difficulties which formerly surrounded the mathematically infinite is probably the greatest achievement of which our age has to boast." Abraham Fraenkel (1891–1965, Germany–Israel) said "Thus the conquest of actual infinity may be considered an expansion of our scientific horizon no less revolutionary than the Copernican system or than the theory of relativity, or even of quantum and nuclear physics":

- Hilbert's remarks on infinity and Cantor's work are of exceptional importance: From time immemorial, the infinite has stirred men's emotions more than any other question. No other question has ever moved so profoundly the spirit of man; no other idea has so fruitfully stimulated his intellect; yet no other concept stands in greater need of clarification than that of the infinite. Indeed, shedding light on the infinite has been a daunting task for mathematicians and philosophers down through the ages. The infinite is nowhere realized. Neither is it present in nature nor is it admissible as a foundation of our rational thinking—a remarkable harmony between being and thinking. In a certain sense, mathematical analysis is a symphony of the infinite. Owing to the gigantic simultaneous efforts of Friedrich Ludwig Gottlob Frege (1848–1925, Germany), Dedekind and Cantor, the infinite was set on a throne and revelled in its total triumph. In its daring flight the infinite reached dizzying heights of success. From the paradise created for us by Cantor, no one will drive us out (the paradise of which Hilbert was speaking was, of course, a massively non-finitistic paradise, chock full of actual infinities). In 1904, the Royal Society awarded Cantor its Sylvester Medal (after James Joseph Sylvester, 1814–1897, England), the highest honor it can confer for work in mathematics.
- The simple differential equation $y' = ay$ has an infinite number of solutions $y = ce^{ax}$, where *c* is an arbitrary constant. The simple continuous function $y = x^2,\ x \in [-1, 1]$ can be made discontinuous at every point of the interval $[-1, 1]$. Mathematically interesting sequences of numbers are those that continue without end. The set of all convergent/divergent sequences/series is infinite.
- Infinity Void is a hyperrealistic metaverse (a virtual-reality space in which users can interact with a computer-generated environment and other users) with an aim to provide immersive experience to the users backed by powerful graphics.

The following citations about infinity are interesting:

- There are among the marvels that surpass the bounds of our imagination and that must warn us how gravely one errs in trying to reason about infinities by using the same attributes that we apply to finities. (Galileo)
- Mathematical motion is just an infinite succession of states of rest, i.e., mathematics reduces dynamics to a branch of statics. (d'Alembert)
- The known is finite, the unknown infinite; intellectually, we stand on an island in the midst of an illimitable ocean of inexplicability. Our business in every generation is to reclaim a little more land. (Huxley)
- Hilbert's famous address in memory of Weierstrass (use for infinity): "The infinite! No other question has ever moved so profoundly the spirit of man; no other idea has so fruitfully stimulated his intellect; yet no other concept stands in greater need of clarification than that of the infinite$\cdots$." "Our first naive impression of nature and matter is that of continuity. Be it a piece of metal or a volume of liquid, we invariably conceive it as divisible into infinity, and ever so small a part of it appears to us to possess the same properties as the whole."
- I saw … a quantity passing through infinity and changing its sign from plus to minus. I saw exactly how this happened … but it was after dinner and I let it go. (Winston Leonard Spencer Churchill, 1874–1965, England)
- There are at least two different ways of looking at the numbers: As a completed infinity and as an incomplete infinity… regarding the numbers as an incomplete infinity offers a viable and interesting alternative to regarding the numbers as a completed infinity, one that leads to great simplifications in some areas of mathematics and that has strong connections with problems of computational complexity. (Edward Nelson, 1932–2014, USA)
- Mathematics as we know it and as it has come to shape modern science could never have come into being without some disregard for the dangers of the infinite. (David Marius Bressoud, born in 1950, USA)

2.9 Number Mysticism of Pythagoras

Just as the Babylonians associated with each of their gods a number up to 60, the number indicating the rank of the God in the heavenly hierarchy (60 was the number of Anu, the god of heaven, and 30 was the number of

Sin, the lunar god), so did the Pythagorean theory was dominated by the idealistic belief that numbers were not only symbols of reality, but the final substance of real things, and possess spiritual and magical powers, which was called as "number mysticism" (to appreciate or dislike this it is necessary to enter into that condition of mind/thoughts which takes any analogy to represent a real bond). According to Iamblichus, Pythagoras once said that "number is the ruler of forms and ideas and the cause of gods and demons." According to Poincaré "Religious values vary with longitude and latitude. While $3, 7, 10, 13, 40$, and 60 were especially favored, we find practically every other number invested with occult significance in different places and at different times."

Pythagoreans regarded even numbers (a set of pebbles which could be arranged into two equal rows), represented as $2n,$ as unlimited, soluble, therefore ephemeral, feminine, weak (when divided they are empty in the center), and pertaining to the earthly nature (evil), and odd numbers including 1 (a set of pebbles that could not be arranged into two equal rows because there would always be a single pebble left over), represented as $2n+1,$ as limited, finite, indissoluble, masculine, partaking of celestial nature (good). There is a striking correspondence in Chinese mythology. They considered the odd numbers symbolized white, day, heat, Sun, fire; the even numbers, on the other hand, black, night, cold, matter, water, Earth. The numbers were arranged in a holy board, the Lo-Chou, which had magic properties when properly used. Everything in the world we live involves evenness and oddness. On the contrary to Pythagoreans, evenness can be defined as a term that means a balance between equal measurements, fair quantities, and neutral stability, whereas oddness can be described as an uneven element, irregular varieties, and an additional essence. In Hinduism, an amount of money is always gifted as multiple of 10 plus $1.$ For Pythagoras only first ten numbers were of spiritual significance (some claim first fifty) and some human attribute. However, since the numbers greater than 10 are the prolongation of first ten numbers, and the order and the finite takes antecedence over the unlimited and the infinite, it becomes clear that the properties of first ten numbers disclose not only the nature of all integers, but also the pattern of the universe as it exists in the mind of god. Ward Rutherford (born 1927, Channel Islands) writes in his account of the Pythagoreans' linkage of numbers with even the most abstract notions. Porta Maggiore Basilica as well as Hadrian's Pantheon in Rome was built based on Pythagorean number mysticism. Pythagoras devoid practical applications. In what follows, besides first ten numbers, we will also add interesting attributes of some selected numbers greater than ten.

Number 1, was called Monad/Ousia, it was acknowledged as all-good, all-wise, all-knowing, true happiness, harmony, order and friendship, eternal, reason, unchangeable, undifferentiated soul of the universe. Parmenides (around 480 BC, Greece), Xenophanes, Plato, Aristotle, and Plotinus followed Pythagoras to consider Monad as a term for God (of reason). Gnostics used the term Monad to refer to the most primal aspect of God. Monad is both male and female, odd and even, itself not a number, but the source and progenitor of numbers, $2 = 1 + 1,\ 3 = 2 + 1 = 1 + 1 + 1,$ and so on (creator itself cannot be creation, in fact, Aristotle reasonably observed, the measure is not measures but *the measure*), and hence origin of all things, the beginning and ending of all things, yet itself not knowing a beginning or ending. According to Bruno, "Monad was his pagan substitute for the deity sanctioned by the Aristotelian theologians." As late as 1547, one of the questions Ludovico Ferrari (1522–1565, Italy) sent to Niccoló Fontana (nickname Tartaglia, 1500–1557, Italy) as part of the challenge competition (see Agarwal and Sen [14]) was whether unity was a number. Tartaglia complained that the question did not have to do with mathematics but with metaphysics. He then hedged his answer by asserting that unity was a number "in potential" but not one "in actuality." However, Simon Stevin writes in *'l' Arithmétique* in Capitals THAT UNITY IS A NUMBER. His argument was "If from a number is subtracted no number, the number remains; but if from 3 we take, $1,\ 3$ does not remain; hence, 1 is not no number." According to the Vedic tradition there is only one manifesting sound indicative of the Supreme Being (Para-Brahman), and that is called "Om," as far as the human ears could capture it. Leibniz observed that 1 and 0 are the only digits in the binary scale of notation (Pingala, about 500 BC, India, a Sanskrit grammarian, presented the first known description of a binary numeral system, see Agarwal and Sen [14], and Nooten [397]. Leibniz tried to reunite the Protestant and Catholic churches. According to Laplace, Leibniz in binary arithmetic saw the image of Creation. He imagined that Unity represented God, and zero the void; that the Supreme Being drew all beings from the void, just as unity and zero express all numbers in the binary system of numeration. The story is that Leibniz communicated his idea to the Jesuit Grimaldi who was the president of the Chinese tribunal for mathematics in the hopes that it would help convert into Christianity the Emperor of China, who was said to be very fond of the Sciences.)

In geometry, Monad precedes every point, and because a point is the beginning of a line, a line of a flat surface, a surface of a three-dimensional solid, and so on. Number 1 has only one divisor namely one itself, all other numbers have at least two divisors namely themselves and one. Any number when multiplied by one yields a product that is itself and when divided, a quotient that is itself. There is one: Moon, Sun, Venus, $\cdots$. In a normal human body there is one: brain, heart, mouth, nose,$\cdots$. In science, hydrogen is the atomic number $1.$ The number 1 symbolizes the leadership and indicates the priority in a major accomplishment. George Washington (1732–1799) served as the first president of USA during 1789–1797. Alexey Arkhipovich Leonov (1934–2019, Soviet Union–Russia) was the first human to conduct a spacewalk on March 18, 1965. Neil Alden Armstrong (1930–2012, USA) was the first human to walk on the Moon on July 21, 1969. Valentina Tereshkova (born 1937) was the first Soviet Union woman in space in 1963. Sally Ride (1951–2012) was the first American woman in space in 1983. Queen Victoria (1819–1901, UK) became the first monarch to take up residence at Buckingham Palace. Theodore Roosevelt (1858–1919) was the first American to win the Nobel Peace Prize in 1906. Winston Churchill was the first Prime Minister of Great Britain to receive the Nobel Prize in Literature in 1953. Strengths, at nine letters long, is the longest word in the English language with only one vowel. According to Hindus, the first person to enter the home after house-warming ceremony has to be a person of good intentions and superior will so that the house gets all blessings from gods and everything goes favorably to the owner. In 1950s French mathematicians who worked under the collective pseudonym of “Monsieur Nicholas Bourbaki” embarked on the development of an encyclopedic description of all mathematics. They devoted 200 pages simply to introduce the innocent looking concept, the number 1.

Number 2, was called Dyad, the mark of matter, the senses, opinion, chaos. It signifies polarization, opposition, divergence, inequality, divisibility and mutability (for example, knower and known), the principle of exiting at one time in one way, and another time in another way. Ten foremost dyads of Pythagoras we have listed in Sect. 1.2. Dyad breaks away from the perfection and unity of Monad. The Monad is limited and expresses moderation, the Dyad is unlimited and expresses excess, defect and the capacity for infinity and indeterminacy. A Dyad is also considered the most basic and fundamental social group. The twin deities of the Hindu pantheon are Saranyu and Vivashvant (also called Dasra and Nasatya). They symbolize the nervous and vital forces and are supposed to respectively represent the morning star and the evening star. They are the offspring of horses, hence their name. These divinities are considered as the "Primordial couple" who appeared in the sky before dawn in a horse drawn chariot. Dyad was also associated with the head of the Egyptian Horus, and in Greek mythology Rhea "the mother of the gods" and therefore associated with Gaia and Cybele. In the Garden of Eden, there were two trees, namely, the tree of life and that of knowledge of good and evil that Adam and Eve were commanded not to eat from. In a normal human body, there are two: arms, ears, eyes, kidneys, legs, nostrils, $\cdots$. Two is the atomic number of Helium. Ancient Chinese proceeded a step further to Greeks, they also refused to consider two as a number, as it is the creator of all even numbers, see Solomon [489]. In the year 1905, only two automobiles were registered in the entire state of Missouri, and only two head-on automobiles collisions were registered.

Number 3, was called Triad, the first true number—a principle of everything that is whole and perfect, and according to Aristotle permits all things with a beginning, middle, and an end. From the duality a third element is implied, the act of knowing, the flow of consciousness. As Triad implies past, present, and future, it embodies wisdom and foresight; people act correctly when they consider all three parts of time. All knowledge falls under the Triad, and it was considered to hold powers of prophecy and providence. Consequently, God Apollo delivered his oracles from a tripod, and three libations are always offered to the gods. Three is the first and commonest of all innumerable trinities that have dominated religions since the dawn of history, for example, trinity of Hindu deities are Brahma, Vishnu, and Shiva, whereas their respective tridevi's are Saraswati (education, creativity, and music), Lakshmi (wealth, fortune, power, beauty and prosperity, and associated with Maya), and Parvati (power, nourishment, harmony, devotion, and motherhood), also trinities are Dharma–Artha–Kama and Satva–Raja–Tama; in Sumerian theology the oldest triad is Anu (gods of heaven), Enlil (Earth), and Enki (water); in Egyptian mythology the divine family is Osiris, Isis, and Horus; further there were three aspects to the Egyptian Sun God Khepri (rising), Re (midday), and Atum (setting); holy trinity in Christianity (God the Father, God the Son [Jesus Christ], and God the Holy Spirit); and there are three days entombment of Christ. In the Babylonian flood legend, Utnapishtim (the Babylonian Noah) simultaneously releases a dove, a raven, and a swallow. There are three gifts of the magi (Gold, Frankincense, and Myrrh).

According to ancient Greek mythology, the modern world began when the three brothers Zeus, Poseidon, and Hades rolled dice for the universe. On that occasion, Zeus won the first prize, the heavens, Poseidon took second prize, the seas, and Hades had to settle for hell. In Mohammedan religion, the divorce (used to) takes place on the utterance of the word "Talakh" three times. Triad infuses the world of matter with its three dimensions, and the world of the three-part soul. Triad in the human cycle is birth, life, and death. There are three aspects of the manifestation (creation, preservation, destruction), and three states of consciousness (awake, asleep, and dreaming). From the earliest times, sacrifices and vows have been repeated three times. There is a common superstition that events take place in threes and that the third time will be lucky. Primary colors are only three red, blue, and yellow (black and white are not considered colors). A Triad consists of three people and is considered more stable than a dyad because the third group member can act as the mediator during conflict. There are three receptacle worlds (receptacle of principles, receptacle of intelligences, and receptacle of quantities). Three is the atomic number of Lithium. Plato saw three as being symbolic of the triangle, the simplest spatial shape, and considered the world to have been built from triangles.

Number 4, was called Tetrad, depicting everything in the Universe, both natural and numerical. Tetrad represents justice (male virtue), because it was the first perfect square (ignoring the trivial 0^2 and 1^2), the product of equals (four dots make a square and even today we speak of "a square deal"). In Hinduism Lord Brahma has four faces (Chaturmukha) forming itself within and from the infinite circle, he also has four arms and he is often depicted holding one of the four Vedas in each hand. There are four: aims of human life (Dharma, Artha, Kama, and Moksha), stages to a man's life (brahmacharya, grihastha, banaprastha, and sannyasa), primary castes (Brahmana, Kshatriya, Vaishya, and Shudra), yugas (cycles of 4,320,000 years) [Satya, Dvapara, Treta, and Kali], and great kings (Vaishravana, Virupaksha, Virudhaka, and Dhritarashtra). The swastika symbol (see Freed and Freed [195]) is traditionally used in Hindu religions as a sign of good luck and signifies good from all four directions. In Hinduism, the sacred mountain Kailash has four sides from which four rivers flow to the four quarters of the world (the Ganges, Indus, Oxus, and Sita) dividing the world into four quadrants, Mount Meru buttressed by four terrestrial mountain ranges that extend in four directions, and between them lie four sacred lakes, through which the celestial river divides into four earthly rivers, which flow to the four corners and irrigate the four quadrants of the Earth. According to Egyptian cosmogony, the heavenly roof as supported by four pillars, mountains, or women at the cardinal points.

In Buddhism there are four noble truths (the truth of suffering, the truth of the cause of suffering, the truth of the end of suffering, and the truth of the path that leads to the end of suffering), and four main pilgrimage sites (Lumbini, Bodh Gaya, Sarnath, and Kusinara). In Christianity there are four Gospels (Matthew, Mark, Luke, and John), and in Judaism there are four holy cities (Jerusalem, Hebron, Safed, and Tiberius). In Islam there are four: holy cities (Mecca, Medina, Jerusalem, and Damascus), arch angels (Jibraeel [Gabriel], Mikaeel [Michael], Izraeel [Azrael], and Israfil [Raphael]), and holy books (Taurait, Zaboor, Injeel, and Quran). There are four seasons (spring, summer, fall, and winter), four primal elements, four essential musical intervals, four kinds of planetary movements, four faculties (intelligence, reason, perception, and imagination), the man has four ages (childhood, adolescence, youth, and old age), and the Earth has four cardinal points (north, south, east, and west). There are four types of animal (creeping, flying, four-legged, and two-legged), and the four mathematical sciences of the quadrivium. Four points in space give rise to the first three-dimensional solid, the pyramid. Four dimensions of space-time are the framework of the physical Universe, according to relativity. Franklin Delano Roosevelt (1882–1945) the 32nd president of the USA declared four fundamental freedoms (Freedom of Speech, Freedom of Religion, Freedom from Want, and Freedom from Fear). The number 4 is avoided in Japan because the word for 4. shi, sounds similar to the Japanese word for death.

Number 5, was called Pentad, it represents marriage, because it was the union of the first feminine and the first masculine number $2 + 3 = 5$, thus it gives life to the Universe. It also represents reconciliation and concord, and hence sacred to Aphrodite, Goddess of love. Pentad manifested through the five Platonic solids. The five-pointed star (see Fig. 1.1), which contains “triple-interwoven triangle,” was recognized as Pythagorean order. Pythagoras divided the globe into five climatic zones (tropical, dry, mild, continental, polar). The God Shiva has five faces, and his mantra is also called panchakshari (five-worded) mantra. The epic Mahabharata revolves around the battle between Duryodhana and his 99 other brothers and the five pandava princes—Dharma, Arjuna, Bhima, Nakula, and Sahadeva. Bhagavad Gita mentions [also Sikh in their canon] five evils [five thieves] (Kama [desire], Krodha [anger], Lobha [greed], Moha [delusion], and Ahankar [ego or excessive pride]). In Hindu philosophy *Prana* describes the five breaths that are said to govern the vital functions of the human being (prajna, apana, vyana, udana, and samana). There are five gifts of cow (milk, curds, dung, ghi, and urine). In Buddhism there are five virtues (Sat [truthful living], Santokh [contentment], Daya [compassion], Nimrata [humility], and Pyaar [love]). The Lord blessed five loaves and fed 5,000. In Islam, the number 5 is a good omen, and in fact, there are the five Pillars of Islam (Shahadah [declaration of faith], Salat [prayer], Zakat [giving alms], Sawm [fasting during Ramadan], and Hajj making the pilgrimage to Mecca]). Prayers are performed five times every day. Islamic law categorizes human behavior into five classes (Wajib [obligatory], Sunnat or Sunnah [recommended], Mubah [neutral], Makruh [not recommended but not forbidden], and Haram [forbidden]). In Sikhism, five sacred symbols known as *panj kakars* are Kesh (unshorn hair), Kangha (comb), Kara (steel bracelet), Kachhehra (soldier's shorts), and Kirpan (sword). There are five: organs of perception (ears [hearing], skin [touch], eyes [sight], tongue [taste], and nose [smell]), organs of action (speech, hands, legs, anus, and genitals), and five inhabitants of the world (plants, fishes, birds, animals, and humans). Our skin responds to five basic sensations (pressure, touch, cold, heat, and pain). Five is also referred to a hand because of five figures. Five is the atomic number of Boron. In English alphabet, there are five vowels $(a, e, i, o, u).$ Five is the midpoint of the number 10; whatever method we use to add up to 10, 5 will be found to be the arithmetic mean, e.g., $9+1, 8+2, 7+3, 6+4.$ Five is also the only prime number (divisible by 1 and itself only) that is the sum of two consecutive primes, namely 2 and 3, with these indeed being the only possible set of two consecutive primes. The Roman symbol X for ten represents two "V"s, placed apex to apex, which shows the importance of five.

Number 6, was called Hexad, it was the first perfect number, appears from both the addition and the multiplication of its factors: $1+2+3=6$ and $1 \times 2 \times 3 = 6,$ and hence it reflects a state of health and balance. It is seen in the six extensions of geometrical forms, in the six directions of nature (north, south, east, west, up, and down). Like Pentad it arises out of the first odd and even numbers only by multiplication, $2 \times 3,$ rather than by addition, and from this it is associated with androgyny. The Hexad participates in the arithmetic mean—$6, 9, 12,$ the geometric mean—$3, 6, 12,$ and the harmonic mean—$3, 4, 6,$ and seen in music as $6, 8, 12.$ Thus the Pythagoreans used to praise the number six in very vivid eulogies, concluding that the Universe is harmonized by it and, thanks to it, comes by wholeness and permanence as well as perfect health. The ancient Hebrews explained that God chose to create the world in six days instead of one because six is more perfect number (on day 1 light is created, on days 2 and 3 heaven and Earth appear, finally, on days 4, 5, and 6 all living creatures are created). The Romans attributed the number six to the goddess of love, for it is made by the generation of the sexes: from three, which is masculine since it is odd, and from two, which is feminine since it is even. Six similar coins can be arranged around a central coin of the same radius so that each coin makes contact with the central one (and touches both its neighbors without a gap). This makes 6 the answer to the two-dimensional *kissing number problem*. The Earth has six climates zones (tropical, arid, Mediterranean, temperate, continental, and polar). Euouae, a medieval musical term, is the longest English word consisting only of vowels, and the word with the most consecutive vowels. Pythagoreans also realized that like squares, six equilateral triangles (see Fig. 1.3) meeting at a point (add up to 360°) leave no space in tilling a floor.

Number 7, was called Heptad/Hebdomad, it cannot be generated by the operation of any other numbers and hence expresses virginity, symbolized by the virgin Goddess Athena, and has a mystique power. Combined from the second triangular and the second square numbers (see Chap. 7), it is the number of the primary harmony $(3:4),$ the geometric proportion $(1, 2, 4),$ and the sides $(3, 4)$ around the right angle of the archetypal right-angled triangle. Since it cannot be divided by any number other than itself, it represents a fortress or acropolis. Heptad was sacred because it was the number of days of the week named after the Sun, the Moon, and the five planets visible to the naked eye: Mars (French Mardi), Mercury (French Mercredi), Jupiter (French Jeudi), Venus (French Vendredi), and Saturn (English Saturday). The English Tuesday, Wednesday, Thursday, and Friday are named after Teutonic deities that supposedly correspond to the Roman gods after whom the planets were named. Week is in fact the time it takes the Moon to pass from one phase to another, from new Moon to half Moon, from half Moon to full Moon. Heptad was also sacred because Apollo's birthday was celebrated on the seventh day of each month. Pythagorean mathematical pattern in music gives seven an important place, for there are seven distinct notes in the musical scale corresponding roughly to the white notes on a piano. Counting from 1, the eighth note up the scale is the exceedingly harmonious octave (the name for eight).

In Hinduism there are seven sages or Saptarishi, named after seven stars (Atri, Angiras, Kratu, Marichi, Pulaha, Pulastya, and Vashistha), they are said to form the seven stars of the Little Bear which is situated on exactly the same line as the "axis of the world." There are seven ancient scientists *Rishis* (Valmiki, Kashyap, Sukra, Baches, Vyas, Khat, and Kalidas). There are seven sacred rivers of ancient Brahmanism known as Saptasindhava (Ganga, Yamuna, Sarsvati, Satlaj, Parushni, Marurudvridha, and Arjikiya). Our body contains seven chakras (Root, Sacral, Solar plexus, Heart, Throat, Third eye, and Crown), horse with seven heads of the chariot which Surya, the Brahmanic god of the Sun, raced across the sky, there are seven footsteps at the time of marriage (Sapta-padi); in Sumerian religion there are seven gods (An, Enlil, Enki, Ninhursag, Nanna, Utu, and Inanna); in ancient Egypt there were seven paths to heaven and seven heavenly cows; in Greek mythology there are seven wise men; in Christian tradition there are seven deadly sins (pride, greed, lust, envy, gluttony, wrath, and sloth), seven virtues (chastity, temperance, charity, diligence, kindness, patience, and humility), seven spirits of God (spirit of the wisdom, spirit of Lord, spirit of understanding, spirit of counsel, spirit of power, spirit of knowledge, and spirit of the fear of the Lord), seven joys of the Virgin Mary (annunciation, nativity of Jesus, adoration of the Magi, resurrection of Christ, ascension of Christ to Heaven, pentecost or descent of the Holy Spirit upon the Apostles and Mary, and coronation of the Virgin in Heaven), seven devils cast out of Magdalen; for seven days seven priests with seven trumpets invested Jericho (Palestinian city in the West Bank), and on the seventh day they encompassed the city seven times; in the book of Genesis we are told that "God rested from His work on the seventh day"; in Rome there are seven hills (Aventine, Caelian, Capitoline, Esquiline, Palatine, Quirinal, and Viminal); there are seven winds (Sirocco, Aeolian, Gale, Zephyr, Squall, Wuther, and Haboob); and there are seven colors in rainbow (red, orange, yellow, green, blue, indigo, and violet).

In India until very recently grandmother used to give wishes to newlywed bride to have seven children. There are seven seas (Arctic, North Atlantic, South Atlantic, North Pacific, South Pacific, Indian, and Southern oceans); however, according to the Vishnu Purana, the seven seas are (Lavana [salt], Iksu [sugar-cane], Sura [wine], Sarpi [clarified butter or

Ghee], Dadhi [yoghurt], Dugdha [milk], and Jala [water]). In China 7 determines the stages of life of a girl: she gets her milk teeth at seven months, loses them at seven years, reaches puberty at $2 \times 7 = 14$ years, and reaches menopause at $7 \times 7 = 49.$ In general, Chinese do not consider seven as a lucky number. All the offices of the Church are arranged in accordance with number symbolism. The mass itself is composed of 7 parts, or offices. The full episcopal procession is led by 7 acolytes, indicating the 7 gifts of the spirit. There are exactly seven generations from David to the birth of Christ. Then follow the pontiff, 7 subdeacons (7 columns of wisdom), 7 deacons (from apostolic tradition). The Zikkurats, towers of Babel, originally 3 or 4 stories in height but never 5 or 6, were dedicated to the 7 planets and came to consist of 7 steps, faced with glazed bricks of the 7 colors, their angles facing the 4 cardinal points. These 7 steps symbolize the ascent to heaven, and a happy fate is promised the person who ascent to their submit. The tree of life, with 7 branches, each bearing 7 leaves, is perhaps the ancestor of the 7-branched candlestick of the Hebrews. Even the goddesses are called by 7 names and boast of them. In 1966, Manana Ndabezitha Karenga (born 1941, USA) created Kwanzaa the first pan-African holiday which has seven principles of African Heritage described as “a communitarian African philosophy” (Umoja [unity], Kujichagulia [self-determination], Ujima [collective work and responsibility], Ujamaa [cooperative economics], Nia [purpose], Kuumba [creativity], and Imani [faith]). There are seven parts of the body (head, neck, torso, two arms, and two legs). The head has seven orifices (2 nostrils, 2 eyes, 2 ears, and a mouth), the lyre has seven strings, and according to Shakespeare the life of men and women has seven ages (infant, child, adolescent, young adult, adult, elder, and old person).

Herodotus and Callimachus of Cyrene (around 305–240 BC, Greece) made the early list of the seven wonders of the Ancient World (The Great Pyramid at Giza [erected in Egypt around 2600 BC by Khufu, 2589–2566 BC], Colossus of Rhodes [Greece Island], Hanging Gardens of Babylon [Iraq], Lighthouse of Alexandria [Egypt], Mausoleum at Halicarnassus [Turkey], Statue of Zeus at Olympia [Greece], Temple of Artemis [Turkey]), since then the list keeps on changing. A popular nursery rhyme relates: As I was going to St. Ives, I met a man with seven wives, each wife had seven sacks, each sack had seven cats, each cat had seven kits, kits, cats, sacks, and wives, how many were going to St. Ives? The origin of this rhyme seems to be the problem 79 in Rhind mathematical papyrus. Several versions of this riddle have survived in different ages and cultures. The

1937 German fairy tale Snow White has seven dwarfs (Bashful, Doc, Dopey, Grumpy, Happy, Sleepy, and Sneezy).

Number 8, was called Octad/Ogdoad, it was significant because it is the first cube $(2 \times 2 \times 2)$ and is thus associated with safety, steadfastness, and everything in the Universe which is balanced and regulated. It is the source of all the musical ratios and is called "Embracer of Harmonies," it is also known as Eros, as it is a symbol for lasting friendship. The heavens are made up of nine spheres, the eighth of which encompasses the whole, which introduces into the Octad a notion of all-embracing presence. There are eight phases of Moon (new Moon, waxing crescent, first quarter, waxing gibbous, full Moon, waning gibbous, third quarter, and waning crescent). In Hinduism, eight is the number of wealth and abundance, and according to cosmogony there are eight divinities/guardians of the horizons and the points of the compass (Indra, Agni, Yama, Nirriti, Varuna, Kuvera, Vayu, and Ishana). In Buddhism, the branches of the Eightfold Path are embodied by the Eight Great Bodhisattvas (Manjusri, Vajrapani, Avalokitesvara, Maitreya, Ksitigarbha, Nivaranavishkambhi, Akasagarbha, and Samantabhadra). Further, in Buddhism 8 is a lucky number, possibly because of the eight petals of the lotus, a plant associated with luck in India and a favorite Buddhist symbol. In Chinese mythology there are 8 immortals (He Xian Gu, Cao Gou Jiu, Li Tie Guai, Lan Cai, Lü Dongbin, Han Xiang Zi, Zhang Guo Lao, and Zhongli Quan) representing separately male, female, the old, the young, the rich, the noble, the poor, and the humble Chinese. Further, in China, 8 determines the stages of life of a boy: He gets his milk teeth at eight months, loses them at eight years, reaches puberty at $2 \times 8 = 16$, and loses sexual virility at $8 \times 8 = 64.$ The Jews practiced circumcision on the 8th day after birth. At their Feast of Dedication, they kept 8 candles burning, and this Feast lasted 8 days. Eight prophets were descended from Rahab. There were 8 sects of Pharisees. In the eighth century it was pointed out by Alcuin that the second origin of the human race was made from the deficient number eight (see Sect. 4.5), since in Noah's Ark there were eight human animals (Noah [8th in direct descent from Adam], his wife, his three sons, and their wives) from whom the entire human race sprung, this second origin being thus more imperfect than the first, which was made according to the perfect number (see Sect. 4.5) six. The square of any odd number, less one, is always a multiple of $8.$ The eight beatitudes are the teachings of Lord Jesus Christ during his Sermon on the Mount (Beatitudes Mountain) in which he describes the attitudes and actions that should characterize his followers and disciples. In Islam there are seven hells but eight paradises, signifying God's mercy. The atomic number of oxygen is 8.

Number 9, was called Ennead/Nonad, that which brings to fruition, nine months for birth, and the number of the muses that complete dance and movement. Ennead is the number of completion and fulfillment, wisdom and good leadership, and heaven. In Hinduism it is the number of Lord Brahma (the Creator), and represents Navaratna (Dhavantari [pearl], Kshapanaka [ruby], Amarasimah [topaz], Shanku [diamond], Vetalabhatta [emerald], Ghatakarpara [lapis-lazuli], Kalidasa [coral], Varahamihira [sapphire], and Varauchi [not identified to any specific gem]), these Sanskrit names are of courtiers of the legendary King (Vikramaditya, 102 BC-18 AD, India), and gems have immaculate auspicious powers. Navaratri is an annual Hindu festival that spans over nine nights (perhaps these nine days were associated with the nine numerals of the place-value system). It is dedicated to Goddess Devi Durga/Parvati and her nine avatars (Shailaputri, Brahmacharini, Chandraghanta, Kushmanda, Skandamata, Katyayani, Kaalaratri, Mahagauri, and Siddhidatri). In Chinese it is the ninth day of the Chinese New Year which is the birthday of the Jade Emperor/God (the ruler of heaven), the Double Ninth festival is an old Chinese tradition celebrated on the ninth day of the ninth lunar month. In Hebrews it is a symbol of truth, further in some of the Hebrew writings it is taught that God has 9 times descended to this Earth (Garden of Eden, confusion of tongues at Babel, destruction of Sodom and Gomorrah, Moses at Horeb, Sinai when the Ten Commandments were given, Balaam, Elisha, Tabernacle, and Temple at Jerusalem), and it is taught at the 10th coming this Earth will pass away and a new one will be created. In biblical term, the Fruit of the Spirit sums up to nine attributes of a person (love, joy, peace, patience, kindness, goodness, faithfulness, gentleness, and self-control). A group of nine deities in Egyptian mythology worshiped at Heliopolis (the Sun god Atum, his children Shu and Tefnut, their children Geb and Nut, and their children Osiris, Isis, Set, and Nephthys).

Jacques de Longuyon (active 1290–1312, France) in his *Voeux du Paon* introduced Nine Worthies (Pagans [Hector, Alexander the Great, Julius Caesar], Jews [Joshua, David, Judas Maccabeus], and Christians [King Arthur, Charlemagne, Godfrey of Bouillon]). In Catholicism, a novena is the act of saying prayers for nine consecutive days. The Bah$\hat{a}$'s faith/religion was established in the nineteenth century, which teaches the essential worth of all religions and the unity of all people; they consider the number nine a symbol of completeness and fulfillment, as the highest single digit number; their nine-pointed star may symbolize nine great religions of the world; their Houses of Worship have nine sides, nine doors, and nine gardens. Norse cosmology divided the Universe into nine realms with Yggdrasil (the tree of life) in their midst (Asgard [Realm of the Aesir God], Alfheim [Realm of the Light Elves], Jotunheim [Realm of the Giants], Midgard [Realm of the Humans], Muspelheim [Realm of the fire giant or the forces], Nidavellir [Realm of the Dwarves], Niflheim [Realm of the mist world], Svartalfheim [Realm of the Black Elves], and Vanaheim [Realm of the Vanir gods]). Grete Waitz (1953–2011, Norway) won the New York Marathon nine times. Ludwig van Beethoven (1770–1827, Germany) wrote nine symphonies. Lady Jane Grey (1537–1554, England) was Queen of England for nine days (10th of July–19th of July 1553). The ninth President William Henry Harrison (1773–1841, USA) was the first president to die in office. There is a famous saying that a cat has nine lives.

One property of nine known since antiquity known as “casting out nines” or “rule of nine” is that when divided into any power of ten, nine always leaves a remainder of one. Since the days of performing computations on counting boards, nine has been used as a check on the computation. Suppose we multiply $49{,}476$ by $15{,}833$ to obtain $783{,}353{,}508.$ To check the answer, we add the digits

$$4+9+4+7+6 \ = \ 30, \quad 1+5+8+3+3 \ = \ 20, \quad 7+8+3+3+5+3+5+0+8 \ = \ 42$$

and then divide each of these sums by nine, noting only the remainders are $3, 2$ and $6.$ If the computation is correct, the remainder of the numbers being multiplied will produce the remainder of the product. Since $3 \times 2 = 6,$ our multiplication seems to be correct. However, there is always a possibility that digits might have been transposed in the answer, a common error which the check will not catch. The same check can be used for addition and subtraction as well as for division. The sum of the two numbers we multiplied will leave the remainder of five; their difference, a remainder of one. For checking division, we follow the standard rule that the dividend *a* should equal the divisor *b* multiplied by the quotient plus the remainder, or $a = b \times q + r.$ But instead of using the whole number for this check, we cast out nines and use only the remainders

$$49,476 \ = \ 15,833 \times 3 + 1977 \quad \text{gives} \quad 3 \ = \ 2 \times 3 = 6 \quad \text{and hence} \quad 3 = 3.$$

Number 10, was called Decade/Decad, it was considered the greatest and perfect of all because it holds all things through a single form and power. It contains in itself the first four integers—one, two, three, and four $1+2+3+4=10$; it is the smallest integer *n* for which there are just as many primes between 1 and *n* as non-primes, and it gives rise to an equilateral triangle named as tetraktys (see Fig. 2.5). For the Pythagoreans, the tetraktys was the sum of the divine influences that hold the Universe together, or the sum of all the manifest laws of nature. They recognized tetraktys as fate, the Universe, the heaven, and even God and honored it by never gathering in groups larger than ten. Iamblichus states that the tetraktys was so revered by the members of the brotherhood that they shared the following oath, "I swear by him who has transmitted to our minds the holy tetraktys, the roots and source of ever-flowing nature." At its top is the essence of light that illuminates the world of Deity without burning. Its base becomes the square platform of a pyramid, rooted in the world. It is alleged that the Pythagorean musical system was based on the Tetractys as the rows can be read as the ratios of 4:3 (perfect fourth), 3:2 (perfect fifth), 2:1 (octave), forming the basic intervals of the Pythagorean scales. For Plato number ten was the archetypal pattern of the Universe. Theon of Smyrna (around 70–135 AD, Turkey–Greece) believed that Tetraktys is composed of man, family, village, and city. According to Bell [59], "Pythagoras asked a merchant if he could count. On the merchants' replying that he could, Pythagoras told him to go ahead. One, two, three, four…, he began, when Pythagoras shouted Stop! What you name four is really what you would call ten. The fourth number is not four, but decad, our tetractys and inviolable oath by which we swear." The Tetraktys stands like the altar before the bridal couple. In his dialogue *De verbo mirifico* (1494), Reuchlin compared the Pythagorean tetractys to the ineffable divine name YHWH, ascribing each of the four letters of the tetragrammaton a symbolic meaning according to Pythagorean mystical teachings. Andrew Gregory (England) concludes that the tradition linking Pythagoras to the tetractys is probably genuine. The triangular figure of four rows, respectively, represents Monad (a point), Dyad (a line), Triad (a plane), Tetrad (a tetrahedron), adding up to the perfect number Decade, the unity of a higher order.

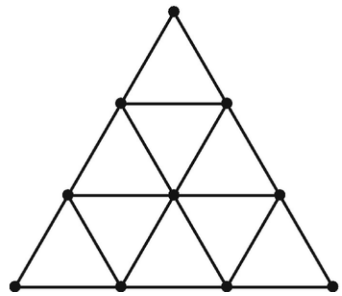


Fig. 2.5 Tetraktys

By connecting the ten dots of the tetractys, nine equilateral triangles are formed. Inadvertently, the tetractys occurs (generally coincidentally) in the following: the arrangement of bowling pins in ten-pin bowling, the arrangement of billiard balls in ten-ball pool, the baryon decuplet, an archbishop's coat of arms, the “Christmas Tree” formation in association football, a Chinese checkers board, and the list continues.

Aleph and other letters of the Hebrew alphabet are shown on a Kabbalistic diagram representing one of the ten emanations of God during creation (Keter [the divine crown], Hokhmah [wisdom], Binah [understanding], Hesed [mercy], Din [justice], Tif'eret [beauty], Nezah [eternity], Hod [glory], Yesod [foundation], and Shekhinah [God's presence in the world]). With the use of decimal system number ten became a significant number. The Rigveda is made up of ten books of hymns celebrating the chief Vedic gods. In Hindus epic Ramayana, the demon king Ravana is 10-headed. There are ten major incarnations of Lord Vishnu (Matsya, Kurma, Varaha, Narasimha, Vamana, Parashu-Rama, Rama, Krishna, Buddha, and Kalki). It is considered as a fortunate number in the sense that one's plans are likely to be carried out according to his/her desires. Jews believe that God gave the Ten Commandments to Moses on two tablets of stone at Mount Sinai (do not have any other gods, do not make or worship idols, do not disrespect or misuse God's name, remember the Sabbath and keep it holy, honor your mother and father, do not commit murder, do not commit adultery, do not steal, do not tell lies, and do not be envious of others), and in Buddhism 10 Commandments are (do not destroy life, do not take what is not given you, do not commit adultery, tell no lies and deceive no one, do not become intoxicated, eat temperately and not at all in the afternoons, do not watch dancing nor listen to singing or plays, wear no garlands nor perfumes or any adornments, sleep not in luxurious beds, and accept no gold or silver). Humans have ten fingers and ten toes. In all Indo-European languages, as well as Semitic, Mongolian, and most primitive languages, the base of numeration (the number of digits or a combination of digits that a system of counting uses to represent numbers) is ten. For the conversion from the base 10 to any base $b \geq 2,$ several algorithms are known, for example, see Krishnamurhty [318].

Number 11, was called Hendecad, it is a master number, because it has its own unique and powerful vibrations. Those influenced by it tend to find inner strength in times of trial, coping well with crisis and chaos. Hendecad often symbolically associated with the God Rudra ($=$ Shiva). In the Babylonian creation myth Enuma Elish Tiamat, the God of chaos, is supported by 11 monsters. The ancient Roman equivalent of a police force comprised 11 men whose job was to hunt down criminals. Apollo 11 was the first manned spacecraft to land on the Moon. The approximate periodicity of a sunspot cycle is 11 years. Several sports involve teams with 11 members (American football, football [soccer], cricket, hockey).

Number 12, was called Dodecad/Duodecade, it represents authority, completeness, magical symbolism, mythology, perfection, and religiousness. The number 12 is strongly associated with the heavens. Most calendar systems—solar or lunar—have twelve months in a year. The Sumerians, Assyrians, and Babylonians used base 12 and its multiples and divisors very widely in their measurements. The ancients recognized 12 main northern stars and 12 main southern stars. Hindu astrology depends on 12 Rasi/Zodiac signs: Aries (March 21–April 19), Taurus (April 20–May 20), Gemini (May 21–June 20), Cancer (June 21–July 22), Leo (July 23–August 22), Virgo (August 23–September 22), Libra (September 23–October 22), Scorpio (October 23–November 21), Sagittarius (November 22–December 21), Capricornus (December 22–January 19), Aquarius (January 20–February 18), and Pisces (February 19–March 20). In Chinese, 12 Chinese Zodiac signs represent animals that have their unique characteristics, and each returns after 12 years: Rat (2020, resourceful, versatile, kind), Ox (2021, dependable, strong, determined), Tiger (2022, brave, confident, competitive), Rabbit (2023, elegant, kind, responsible), Dragon (2024, confident, intelligent, enthusiastic), Snake (2025, enigmatic, intelligent, wise), Horse (2026, animated, active, energetic), Goat (2027, calm, gentle, sympathetic), Monkey (2028, sharp, smart, curiosity), Rooster (2029, observant, hardworking, courageous), Dog (2030, lovely, honest, prudent), and Pig (2031, compassionate, generous, diligent). According to Hindus the Sun God *Surya* has 12 names (Mitra, Ravi, Surya, Bhanu, Kha, Pusha, Hiranyagarbha, Marichin, Aditya, Savitr, Arka, and Bhaskara). There are 12 Gods of Greece (Zeus, Hera, Poseidon, Demeter, Athena, Apollo, Artemis, Ares, Hephaestus, Aphrodite, Hermes, and Hestia).

According to the Hebrew Bible, the twelve lost tribes of Israel (Reuven, Shimon, Levi, Yehuda, Issachar, Zevulun, Dan, Naphtali, Gad, Asher, Joseph, and Benjamin) were said to have descended from the 12 sons of the patriarch Jacob (who was later named Israel) by two wives, Leah and Rachel, and two concubines, Zilpah and Bilhah. There were 12 Roman Gods (Jupiter, Juno, Mars, Mercury, Neptune, Venus, Apollo, Diana, Minerva, Ceres, Vulcan, and Vesta). In Christianity 12 is the number of Christ's disciples (Judas, Thomas, James, Philip, John, Judas Thaddeus, Matthew, Andrew, Peter, Bartholomew, and Simon). King Arthur's Round Table had 12 knights plus King Arthur himself (their names vary). The human body has twelve cranial nerves (Olfactory, Optic, Oculomotor, Trochlear, Trigeminal, Abducens, Facial, Vestibulocochlear, Glossopharyngeal, Vagus, Accessory, and Hypoglossal). Each finger has three articulations (or phalanxes); and if you leave out the thumb (as you have to, since you use it to check off the phalanxes counted), you can get to 12 using only the fingers of one hand. 12 is the number of inches in a foot, ounces in the ancient pound, hours about the clock, and the words dozen and gross are used for counting eggs and oysters as higher units. Twelve is the kissing number in three dimensions, and the atomic number of Magnesium.

Number 13 is considered as an unlucky number because of the Babylonian Code of Hammurabi (Babylonian king, around 1811–1750 BC); there were 13 people present on the Jesus Christ's last supper, and the 13th to sit on the table was the apostle Judas Iscariot, who later betrayed Christ; according to Julian calendar October 13, 1307 was Friday and following the order of King Philip the IV of France the Knights Templar were arrested, tortured, and many were killed; Friday the 13th was a horror film series involving a mass murderer named Jason Voorhees (first appearance Friday the 13th, 1980 and last appearance Friday the 13th, 2009); the end of the Mayan calendar's 13th Baktun was superstitiously feared as a harbinger of the apocalyptic 2012 phenomenon; a year with 13 full moons instead of 12 created troubles for the monks in-charge of the calendars; the baptismal name Adolfus Hitler (1889–1945, Austria–Germany) contains 13 letters; Alfred Joseph Hitchcock (1899–1980, England–USA) could not complete his 13th film in 1922; in 20 years out of 35 Friday on the 13th, 15 times Wall Street recorded losses; 1979's Super Bowl XIII was a huge financial setback for sports Bookies.

There is another cause as to why there is a fear of number 13. The occult symbolism that stood for number 13 was represented by a mystic picture of “A skeleton with a scythe in its bony hands reaping down men,” which different pundits have explained differently. Triskaidekaphobes contemplate Friday the 13 is a cursed/miserable/catastrophic day; they have a great phobia of everything related to the number 13, so they try to avoid everything labeled 13. Many tall buildings do not have a 13th floor, some hospitals avoid the 13th bed, there is no row 13 in planes, the numbers of racing cars as well as triathlon skip from 12 to 14, there are no 13th seats in opera-houses in Italy (however, traditionally in Italy, Friday the 13th was not considered unlucky), it is considered to be unlucky to have thirteen guests on a dinner table (specially in France), and many people avoid getting married or buying a house on a day marked by this dreaded number. In a tarot card deck, XIII is the card of Death, usually picturing the Pale horse with its rider. However, number 13 combines energy of numbers 1 and 3 and hence has a great power, also according to medieval theology $13 = 10 + 3$ (Commandments plus Trinity) and hence the number had some positive aspects. In Hinduism it is considered auspicious to name a baby girl on the 13th day of her life. In Judaism all boys and most reform and conservative girls become bar or bat Mitzvahs at age 13, i.e., a full member of the Jewish faith. According to Rabbinic commentary on the Torah, Lord has 13 attributes (merciful before a person sins, merciful after a person has sinned and repented, all-powerful, compassionate, gracious, slow to anger, abundant in kindness, affluence in truth, keeping mercy for thousands, forgives sins committed willfully, forgives sins committed in defiance of His will, forgives sins committed unwittingly, and cleansing).

In Confucianism there are Thirteen Classics, in Tai Chi there are thirteen postures (consisting of Eight Gates and Five Steps), and in Aztec mythology there are Thirteen Heavens. Sizdah Bedar also known as Nature’s Day is an Iranian festival held annually on the thirteenth day of Farvardin (same as Aries), the first month of the Iranian calendar, a festival dedicated to pranks and spending time outdoors. In ancient cultures, the number 13 represented femininity, because it corresponded to the number of lunar (menstrual) cycles in a year ($13 \times 28 = 364$ days). In some of the ancient writings, it is said, “He who understands the number 13 will be given power and dominion.” In Catholicism the apparitions of the Virgin of Fátima in 1917 were believed to occur in Portugal on the 13th day of six consecutive months, and it is also associated with Saint Anthony of Padua since his feast day falls on June 13. In Islam among Shi'ites, 13 signifies the 13th day of the month of Rajab (the Lunar calendar), which is the birth of Imam Ali. Neo-Pagans, French, and Italians use to consider 13 as a lucky number. In Gurmukhi as well as Hindi the number 13 is called Terah (yours). According to famous Sakhi/story of Guru Nanak Dev Ji, when he was an accountant at a town of Sultanpur Lodhi, he was distributing groceries to people. When he gave groceries to the 13th person, he kept saying, "Yours, yours, yours…" remembering God. People reported to the emperor that Guru Nanak was giving out free food to the people. When treasures were checked, there was more money than before. The Vaisakhi, which commemorates the creation of "Khalsa" or pure Sikh, was celebrated on April 13 for many years. Thirteen is the minimum age of consent in Argentina, Burkina Faso, Japan, two Mexican states, Niger, and in the United States to create an account in compliance with Children's Online Privacy Protection Act. Colgate University was founded in 1819 by 13 men with 13 dollars, 13 prayers, 13 articles, the campus address is 13 Oak Drive in Hamilton, New York, and the male a cappella group is called the Colgate 13.

The Great Seal of the United States consists 13 olive leaves, 13 stars, 13 arrows, and there are thirteen stripes on the American flag representing the thirteen British colonies that declared independence from the Kingdom of Great Britain. The Thirteenth Amendment to the United States Constitution abolished slavery and involuntary servitude (except as a punishment for crime). In 1970 the Apollo 13 NASA Moon mission became famous as "successful failure." It returned to the Earth safely, but exploded; however, all the crew members survived a catastrophic accident. There are 13 Archimedean Solids (cuboctahedron, great rhombicosidodecahedron, great rhombicuboctahedron, icosidodecahedron, small rhombicosidodecahedron, small rhombicuboctahedron, snub cube, snub dodecahedron, truncated cube, truncated dodecahedron, truncated icosahedron, truncated octahedron, and truncated tetrahedron). Thirteen is the first number within the teen numerical range 13–19. Thirteen is the smallest number whose fourth power can be written as a sum of two consecutive square numbers $13^4 = 119^2 + 120^2$, and the sum and the difference of 2 consecutive squares $13 = 2^2 + 3^2 = 7^2 - 6^2$. Thirteen is the sixth prime number, second star number, one of only three known John Wilson primes $5{,}13{,}563$, seventh Fibonacci number, and third centered square number. In the standard 52-card deck of playing cards, there are four suits, each of 13 ranks. In rugby league each side has 13 players on the field at any given time. The birth and death rates of renowned people in every discipline seem to be the same for the number thirteen as of any other number in the range 1–30. Thirteen letter words such as compassionate, confrontation, embarrassment, encouragement, entertainment, understanding, and unfortunately give a mixed reaction. Thus, it all depends on an individual in which way he chooses to perceive the number thirteen and because of the power of thoughts gets the expected results.

Numbers 35 and 36, Plutarch called the number 35 “harmony” because it represents the sum of the first feminine and the first masculine cube $2^3 + 3^3.$ He also showed that 36 is the first number that is both square 6×6 and rectangular $4 \times 9,$ that it is the multiple of the first square numbers, 4 and $9,$ and the sum of the first three cubes, $1^3 + 2^3 + 3^3.$ It is also a parallelogram 3×12 or 4×9 and is named “agreement” because in it the first four odd numbers unite with the first four even $1 + 3 + 5 + 7 = 16, 2 + 4 + 6 + 8 = 20, 16 + 20 = 36.$

Number 40 is found important in many religions such as Hinduism, Sumerian, Buddhism, Judaism, Christianity, and Islam. In Judaism forty is often used for time periods, for example, forty days and forty nights lasted the rain which brought about the great deluge; for forty days and forty nights Moses conferred with Jehovah on Mount Sinai; forty years were the children of Israel wandering in the wilderness; one of the prerequisites for a man to study Kabbalah is that he is forty years old, and the list continues. Six, seven, and forty were the ominous number of the Hebrews.

Number 60 is called sixty. Inheriting from the Sumerian and Akkadian civilizations, the Babylonians and Persians preferred sixty and its multiples. According to Theon of Alexandria (around 335–405, Greece), the Sumerians chose base 60 because it was the “easiest to use” as well as the lowest of “all the numbers that had the greatest number of divisors.” The same argument also cropped up 1,300 years later by Wallis, and again, in a slightly different form, in 1910, when Adolphe Löfler (Switzerland) argued that the system arose “in priestly schools where it was realised that 60 has the property of having all of the first six integers as factors.” In 1789, Vincenzo Formaleoni (1752–1797, Italy) suggested that Sumerian system derived from exclusively “natural” considerations; on this view, the number of days in a year, rounded down to 360, was the reason for the circle being divided into 360 degrees, and the fact that the chord of a sextant (one sixth of a circle) is equal to the radius gave rise to the division of the circle into six equal parts. This would have made 60 a natural unit of counting. This proposition was repeated in 1880 by Moritz Benedikt Cantor (1829–1920, Germany). In 1889, Carl Ferdinand Friedrich Lehmann-Haupt (1861–1938, Germany) believed he had identified the origin of base 60 in the relationship between the Sumerian "hour" (danna), equivalent to two of our current hours, and the visible diameter of the Sun expressed in units of time equivalent to two minutes by current reckoning.

In 1927, Otto Eduard Neugebauer (1899–1990, USA) proposed that the source of base 60 is in terms of systems of weights and measures. Other speculation was made in 1986 by Daniel Joseph Boorstin (1914–2004, USA) Mesopotamians got to 60 by multiplying the number of planets (Mercury, Venus, Mars, Jupiter, and Saturn) by the number of months in the year, i.e., $5 \times 12.$ Anu, the God of heaven, Babylonians attributed to the number 60. Xerxes the Great (around 518–465 BC, Persia) punished the Hellespont with 300 lashes, and Darius ordered the Gyndes to be dug up into 360 ditches, because one of his holy horses had drowned in the river. In Hinduism, the 60th birthday of a man is called *Sashti Poorthi*, which represents a milestone in his life. Sixty occurs several times in the Bible, for example, as the age of Isaac when Jacob and Esau were born, and the number of warriors escorting King Solomon. In time, 60 is the number of seconds in a minute, and the number of minutes in an hour. In geometry, it is the number of seconds in a minute, and the number of minutes in a degree. Out of 13 Archimedean solids four (great rhombicosidodecahedron, snub dodecahedron, truncated icosahedron, and truncated dodecahedron), have 60 vertices. The sixteenth year of marriage is called the diamond wedding anniversary. In many countries a person becomes a senior citizen at $60.$ Since sixty is the smallest number that is divisible by the numbers 1 to 6 and has exactly 12 divisors, it is easy to use in expressing fractions. Sixty is also the product of the side lengths of the smallest whole number right triangle $(3, 4, 5)$ a Pythagorean triple. It is the smallest number that is the sum of two odd primes in six ways $60 = 7 + 53 = 13 + 47 = 17 + 43 = 19 + 41 = 23 + 37 = 29 + 31.$

Number 100, called one hundred, represents perfection and reason for celebration. There are 100: years in a century; centimeters in a meter; yards in an American football field; pennies in one dollar; letter tiles in a Scrabble game; and sebaceous (oil) glands in our one square inch of skin. Hundred is the: square of $10;$ basis of percentages; sum of the first nine prime numbers; sum of the first 10 odd numbers; sum of the cubes of the first four positive integers, i.e., $100 = 1^3 + 2^3 + 3^3 + 4^3$; and square of the sum of the first four positive integers, i.e., $100 = (1 + 2 + 3 + 4)^2$. There are exactly hundred prime numbers whose digits are in strictly ascending order, and the hundredth such prime is 23456789. In many old editions of the Bible, the number $99 = 100 - 1$ appears at the end of a prayer as a substitute for *amen*. A Googol is the number 1 followed by 100 zeroes. On the Celsius scale, 100 degrees is the boiling temperature of pure water at sea level. The United States Senate has 100 Senators. In the game of cricket, scoring 100 runs (a century) is a major feat for a batsman. The record number of points 100 scored in one NBA game by a single player, set by Wilton Norman Chamberlain (1936–1999, USA) of the Philadelphia Warriors on March 2, 1962.

Number 108 has the power of 1 that stands for God or higher truth, 0 that stands for emptiness or completeness in spiritual practice, and 8 that stands for infinity or eternity. It has been considered as a sacred number in mathematics, geometry, astrology, and numerology for thousands of years, mainly in several eastern religions and spiritual traditions. In India more than 5000 years ago, it was known that the distances of the Moon and Sun from the Earth are 108 times the diameters of these heavenly bodies, respectively (the observed values are 110.6 for the Moon and 107.51 for the Sun). Vedic cosmology postulates, 108 is the basis of creation, represents the Universe, and all our existence. In Hinduism the number 108 is very powerful because there are 108: Divya Desams (temples of Lord Vishnu); Mukhya Shivaganas (attendants of Shiva); gopis (followers of Lord Krishna); Upanishads (sacred texts of wisdom from ancient sages), names of each deity, Sanskrit alphabet (54 letters, each has a feminine, or Shakti, and masculine, or Shiva, quality), and for the Holy River Ganga the multiplication of its longitude of 12 degrees (79 to 91) and latitude of 9 degrees (22 to 31). Thus, they use Japa mala (a string of 108 beads made from Tulasi wood) to recite all mantras (or silent repetition of a mantra) with the belief that each unit brings closer to God within. In Ayurveda (traditional Hindu system of medicine), there are 108 marma points (vital points of life forces) known as Shri Yantra in our body. In pranayama (the Yogic practice of regulating breath) it is believed that if an individual can be so calm as to only breathe 108 times in one day, enlightenment will be achieved. As a mark of respect, the numbers 108/1008 are prefixed to the name of a learned Sadhu or Sanyasi.

In Buddhism, there are said to be 108 earthly temptations, 108 lies, and 108 delusions of the mind. Soon after the birth of the Buddha, 108 Brahmans were invited to the name-giving ceremony. Buddha has 108 names, and there are 108 lamps devoted to him. In Kathmandu there are exactly 108 images of Buddha. Some Buddhist temples have 108 steps and 108 columns. The Lankavatara Sutra has a section where the Bodhisattva Mahamati asks Buddha 108 questions. Tibetan Buddhist malas or rosaries are usually 108 beads and have 108 sacred books, and Zen priests wear Juzu (a ring of prayer beads) around their wrists, which consists of 108 beads. In Jainism, the total number of ways 108 of Karma influx (Aasrav) is calculated by multiplying 4 Kashays (anger, pride, conceit, greed), 3 karanas (mind, speech, bodily action), 3 stages of planning (planning, procurement, commencement), and 3 ways of execution (own action, getting it done, supporting or approval of action). Sarsen Circle Stonehenge, whose structure is similar to that of PhNom Bakheng (an ancient Shiva Temple in Cambodia), has a diameter of 108 feet. In Japan, at the end of the year, a bell is chimed 108 times in Buddhist temples to finish the old year and welcome the new one. Each ring represents one of 108 earthly temptations a person must overcome to achieve nirvana. There are several close links between 108 and 9: $1+0+8=9, 108\times 2=216$ and $2+1+6=9, 108\times 3=324$ and $3+2+4=9, 108/2=54$ and $5+4=9,$ also $54/2=27$ and $2+7=9.$ We further have the relation $1^1\times 2^2\times 3^3=108.$ 108 is an important symbolic number in several martial arts and karate styles. 108 degrees Fahrenheit is the internal temperature at which the human body's vital organs begin to fail from overheating.

Number 365, in the Jewish faith there are 365 "negative commandments." The letters of the deity *Abraxas*, in the Greek notation, make up the number 365. This number was subsequently viewed as signifying the levels of heaven. The Bible states that *Enoch* lived for 365 years before entering heaven alive. The number 365 is based on the passage of the Sun through the twelve divisions of the Zodiac, which is the origin of the calculation of the year period that is found in every civilization; thus several solar calendars have a year containing 365 days. It is the product of two prime numbers 5 and 73. It is the smallest number that has more than one expression as a sum of consecutive square numbers $365=10^2+11^2+12^2=13^2+14^2.$

Number 1000, called one thousand, often interpreted in the sense of the multitude or the incalculable, often associated with: the attributes of many Hindu Brahmanic divinities (the Thousand arms, the Thousand rays, or the Thousand of the Brilliant all denote the Sun god Surya; the Thousand names denoted the gods Vishnu and Shiva; the Thousand eyes refer to the gods Vishnu and Indra, etc.); or mythological figures (such as the demon Arjuna who is referred to by the name Thousand arms of Arjuna). This number is also associated with: Mercy; Charity, Sympathy; the Mouths of the Ganga (*jahnavivaktra*); the Arrows of Ravi (= Surya); Ananta (the serpent with thousand heads); the lotus with a thousand petals, etc.

- William John Warner (1866–1936, Ireland), popularly known as Cheiro (the term came from "cheiromancy") as an young adult, went to India for three years in search of knowledge. In his most popular book [124] on *numerology* writes "The ancient Hindus (Brahmins) together with the Chaldeans and Egyptians, were the absolute masters of the occult or hidden meaning of numbers, in their application to time and in their relation to human life." "In India Brahmins kept in their hands from almost prehistoric times studies and practices of an occult nature which they regarded as sacredly as they did their own religious teaching. Among other thing they had theories on the occult significance of numbers and their influence and relation to human life." "The ancient Hindu searches after Nature's laws, it must be remembered, were in former years masters of all such studies, but in transmitting their knowledge to their descendants, they so endeavored to hide their secrets from the common people that in most cases the key to the problem became lost, and the truth that had been discovered became buried in the dust of superstition and charlatanism." He correctly predicted for Arthur James Balfour (1848–1930, U.K.) to be eventually Prime Minister; Russian Revolution and alliance with China; World War II; England's Trade Union strikes of 1926, the time of his own death. In 1925, he predicted the future partition of India, and the sinking of the Titanic, 13 years before it sank. Number mysticism had its precursor in a sort of numerology that to this day persists in otherwise unaccountable omens and superstitions. Numbers in one way or another create and influence our realism that serves as the basis for studying their intrinsic qualities—which we call numerology. Numerologists for an individual consider his name, date of birth, time of birth, place of birth, etc., and somehow covert it into a number between 1 and 9, and based on the property of that number forecast a person's life, successes and misfortunes, and to parents of newborns suggest a good name for their child based on the best prognosis against their birth date. Cheiro's system of numerology is based on the following three steps:

Step1. Connect each letter of the alphabet to a number, as follows:
1 = A, I, J, Q, Y, 2 = B, K, R, 3 = C, G, L, S, 4 = D, M, T,
5 = E, H, N, X, 6 = U, V, W, 7 = O, Z, 8 = F, P.

Step2. Look at each letter of the person's name, assign the numbers given in step 1, add them, and then reduce them through further addition of digits until we reach a single number. For example, for the name RAVI PRAKASH AGARWAL, we have
RAVI = R+A+V+I = 2+1+6+1 = 10 = 1+0 = 1
PRAKASH = P+R+A+K+A+S+H = 8+2+1+2+1+3+5 = 22 = 2+2 =4
AGARWAL = A+G+A+R+W+A+L = 1+3+1+2+6+1+3 = 17 = 1+7 =8.

Now adding these single digit numbers leads to 1+4+8 = 13, which further reduces to as 1+3=4. Thus the Name Number of Ravi Prakash Agarwal is 4.

If we stop at double digit number 13, then from the above number mysticism, we get only mixed reaction. However, for the single digit 4 we can use Cheiro's following classification:

Step3. For numbers 1,2,3,4,5,6,7,8,9 the following list gives names of planets, days, lucky colors, precious stones (which carry potent messages of power, honor, and love), and characteristic nature.

Number 1 – The Sun, Sunday, Golden, Ruby. The brightest star in the sky, without which we would not be here. As such, symbolizes leadership, and the one that everybody looks up to. This number stands for the forefront of creative original capabilities, enduring strength, focus, and positivity. Nothing holds Number 1 back in their quest to rise to the top.
Number 2 – The Moon, Monday, White, Pearl. Like the Sun, stands out from the crowd, but in mastery of the creative planes. As such creativity and artistry are key strengths, but always carried out with restraint. Kindness and tact are positive features of this number. However, number

2 people thrive in positive environments, easily scattered, timid, helpful, and secret keeper.
Number 3 – Jupiter, Tuesday, Yellow, Sapphire. A large and dominant planet, so it reflects good fortune, success, fame, material comfort, and happiness. An authority figure and delegator, is disciplined, and expects the same from others. While a natural born leader, this does not make number 3 a friend to everybody.
Number 4 – Uranus, Wednesday, Blue, Ruby. Definition of a true individual, if not a maverick. Thinking outside of the box and contrary to popular belief and expectation. It represents the law of justice without mercy, tolerance, or sympathy. It is cold, intellectual, and slow in its nature.
Number 5 – Mercury, Thursday, Green, Emerald. Flows and glides effortlessly just like liquid mercury. Excellent with other people, especially other number 5's! Very strong willed, but just as much open to acting on an acute sense of instinct. Dreams up ideas and has the will to make them happen. But far too easy to wind up the wrong way. It seeks present happiness and cares little for cost or consequence. Good in communication, changeability, temperament, writing, and talking.
Number 6 – Venus, Friday, Yellow, Diamond. Attraction is the key concept to number 6, so it represents love, harmony, beauty, art, decoration, sympathy, and cooperation. Just as people are drawn fondly to them, number 6's attach themselves to their dreams and ambitions with long-lasting faithfulness. Appreciate the finer things in life and highly cultured.
Number 7 – Neptune, Saturday, White, Pearl. A healthy independence permeates the number 7. The freedom to move about and travel is craved, and there is a thirst for knowledge about the world. They do not naturally seek out wealth, but still do well for themselves by their originality and application to the task at hand, not least their talent for self-expression. Can be a bit far fetched at times, for better or for worse. It is the number of inventors, musicians, composers, researchers, and shows loneliness and aloofness.
Number 8 – Saturn, green, sapphire. Intense, strong, and an important outlier. Has a crucial arm in overturning systems and making history. It symbolizes justice with mercy, and proponent of robust philosophy and conduct. However is seldom on the same page as others and can get isolated, if not reviled. It deals with the law of Karma.

Number 9 – Mars, red, coral. Gains strength through conflict, learns from battles lost, but in the end always strives toward victory. Likes being in-charge, those beneath a number 9 find them temperamental. Has a lot of potential for success but needs guidance not to fall victim to their own impulsiveness and reckless behavior.

Thus Ravi Prakash Agarwal whose number is 4, must be following justice without mercy, tolerant or sympathetic. Next, we are interested in his date of birth which is 7th October 1948, and it gives $7, 1+0=1, 1+9+4+8=22=2+2=4$, addition of these numbers gives $7+1+4=12=1+2=3$. Thus for Ravi Prakash Agarwal Birth Number is 3, which according to above classification shows he must have good fortune, successful, famous, material comfort, and happiness.

An alternative method to assign each letter of the alphabet to a number is A = 1, B = 2,$\cdots$, Z = 26, and then add all numbers. Thus, for Ravi Prakash Agarwal, we get 187, which further gives $187=1+8+7=16=1+6=7$. Hence, the above classification indicates Ravi Prakash Agarwal likely to be an inventor, a musician, a composer, or a researcher.

There are several shocking reincarnation stories for which science has no explanation. In his book Cheiro rewrites an extraordinary case of St. Louis of France (1215–1270) and King Louis XVI (1754–1793), which was originally published in 1852 in a book entitled *Research into the Efficacy of Dates and Names in the Annals of Nations*. It will certainly add to one more stories who firmly believe in the theory of reincarnation. Clearly, there was an interval of exactly 539 years between the birth of St. Louis and Louis XVI. Birth of Isabel, sister of St. Louis 1225, add interval 539 gives 1764—Birth of Elizabeth, sister of Louis XVI 1764. Death of Louis VIII, father of St. Louis 1226, add interval 539 gives 1765—Death of the Dauphin, father of Louis XVI 1765. Minority of St. Louis commences 1226, add interval 539 gives 1765—Minority of Louis XVI commences 1765. Marriage of St. Louis 1231, add interval 539 gives 1770—Marriage of Louis XVI 1770. Majority of St. Louis (King) 1235, add interval 539 gives 1774—Accession of Louis XVI, King of France 1774. St. Louis concludes a peace with Henry III 1243, add interval 539 gives 1782—Louis XVI concludes a peace with George III 1782. An Eastern prince sends an ambassador to St. Louis desiring to become a Christian 1249, add interval 539 gives 1788—An Eastern prince sends an ambassador to Louis XVI for the same purpose 1788. Captivity of St. Louis 1250, add interval 539 gives 1789—Louis XVI deprived of all power 1789. St. Louis abandoned 1250, add interval 539 gives 1789—

Louis XVI abandoned 1789. Birth of Tristan (sorrow) 1250, add interval 539 gives 1789—Fall of the Bastille and Commencement of the Revolution 1789. Beginning of Pastoral under Jacob 1250, add interval 539 gives 1789—Beginning of the Jacobins in France 1789. Death of Isabel d'Angouleme 1250, add interval 539 gives 1789—Birth of Isabel d'Angouleme in France 1792. Death of Queen Blanche, mother of St. Louis 1253, add interval 539 gives 1792—End of the White Lily of France 1792. St. Louis desires to retire and becomes a Jacobin 1254, add interval 539 gives 1793—Louis XVI quits life at the hands of the Jacobins. St. Louis returns to Madeleine in Provence, add interval 539 gives 1793—Louis XVI interred in the cemetery of the Madeleine in Paris 1973.

After Cheiro's book, several other numerologists have used similar methods to study individuals: pattern of thinking, behavior, profession, finance, and in general his/her comprehensive personality, e.g., see Katakkar [299].

The best known instance of numerology is the "number of the beast," 666, from the biblical Revelation to John (13:18). Curiously, Revelation is the 66th book in the Bible, and the number of the beast occurs in verse 18, which is 6+6+6. But who is the beast? The German Protestant scholar Andreas Helwig (1572–1643) in 1612 added up the Roman numerals in the phrase Vicarius Filii Dei ("Vicar of the Son of God," a title falsely ascribed to the pope) and omitted all the other letters (that is, I = 1, V [and U, which appears as V in Latin inscriptions] = 5, L = 50, C = 100, D = 500) and got 666, proving that the beast is the Roman Catholic Church. This interpretation was taken up by some Seventh-day Adventists in the nineteenth century, but the same method applied to the name Ellen Gould White (1827–1915, USA), a founder of Seventh-day Adventism, also yields 666, provided that the W counts as two V's. Hitler sums to 666 if one uses the code A = 100, B = 101, and so on. Stifel a Protestant, October 3, 1533, and other time used a clever rearrangement of the letters LEO DECIMVS to "prove" that Leo X was 666, whereas Pietro Bongo (died 1601, Italy) a Catholic, unraveled 666 as Martin Luther. In contrast to 666, the number 888 (in Greek) is considered by students of Occultism to be the number of Jesus Christ in His aspect as the Redeemer of the world. Readers of *War and Peace* of Count Lev Nikolayevich Tolstoy (1828–1910, Russia) find that "L'Empereur Napoleon" can also be made equivalent to the number of the beast.

In numerology, *Angel Numbers* are number sequences (usually three or four-digit numbers) that contain repetition (such as 222 or 7777) and/or patterns (such as 4321 or 8686). Angel numbers exhibit the ways in which you are moving through the world. In numerology, 000 or 0000 signifies a fresh start; 111 or 1111 shows extraordinary support; 222 or 2222 indicates a divine collaboration; 333 or 3333 shows opportunity to add your unique talents and abilities to a situation; 444 or 4444 suggests that you are in the process of grounding, rooting, and cultivating an infrastructure that is truly built to last; 555 or 5555 is often associated with change, evolution, love, and abundance; 666 or 6666 although religiously considered as the number of the beast, it can be a gentle, much-needed reminder to treat yourself with kindness and understanding: 777 or 7777 stipulates good fortune—especially finance wise; 888 or 8888 is considered as the most divine numbers; 999 or 9999 gives warning to step outside your comfort zone, expand your horizons, and explore new territory.

In numerology the *Karmic Debt Number* is your unique number that is determined by your life lessons and what you are working on in this lifetime. It is a number that corresponds to the lessons you have yet to learn and the ones that have already been mastered. It is calculated by adding the digits of your birth day and month. There are only four Karmic debt numbers 13, 14, 16, and 19. As an example, Ravi Prakash Agarwal has no Karmic debt number (see his birthday). Each of these four karmic debt numbers holds a significant meaning and indicates a specific set of difficulties that a person would need to work through in this lifetime. Number 13 signifies you will have to work harder and learn more lessons in order to achieve career success; 14 indicates there was an abuse of freedom in a past life so you need to maintain moderation and self-control; 16 relates to past life transgressions when it comes to love so you must be more thoughtful of others and consider how you affect them; 19 is related to the abuse of power in a past life, and hence do not fall victim to ego-based behavior or stubbornness.

Numerology found expression in another form called GEMATRIA. Every number in Hebrew and Greek stood for both sound and number. The *number of the word* was the sum of the numbers represented by each letter in the word, and two words were regarded equivalent if they stood for the same number. Gematria was practiced very extensively in old days, and there are indications of it even in Biblical passages. For example, the word *amen*, which is $\alpha\mu\eta\nu$ in Greek. These letters have the numerical values $M(\alpha)=1,\ M(\mu)=40,\ E(\eta)=8,\ N(\nu)=50$ totaling $99.$ Thus, in many old editions of the Bible, the number 99 appears at the end of a prayer as a substitute for *amen*. An interesting illustration of gematria is also found in the graffiti of Pompeii: “I love her whose number is 545.” It was also used for interpreting the past and foretelling the future.

2.10 Some Interesting Numbers

There are several numbers that have curies properties (see Le Lionnais [348]). In this section, we shall discuss a few of such numbers.

Simple multiplication and addition give

$$\begin{aligned} 1\times 9+2 &= 11 & 1\times 8+1 &= 9\\ 12\times 9+3 &= 111 & 12\times 8+2 &= 98\\ 123\times 9+4 &= 1111 & 123\times 8+3 &= 987\\ 1234\times 9+5 &= 11111 & 1234\times 8+4 &= 9876\\ 12345\times 9+6 &= 111111 & 12345\times 8+5 &= 98765\\ 123456\times 9+7 &= 1111111 & 123456\times 8+6 &= 987654\\ 1234567\times 9+8 &= 11111111 & 1234567\times 8+7 &= 9876543\\ 12345678\times 9+9 &= 111111111 & 12345678\times 8+8 &= 98765432. \end{aligned}$$

Sum of consecutive squares a square

$$\begin{aligned} 1^2+2^2+3^2+\cdots+24^2 &= 70^2\\ 18^2+19^2+20^2+\cdots+28^2 &= 77^2\\ 25^2+26^2+27^2+\cdots+50^2 &= 195^2\\ 38^2+39^2+40^2+\cdots+48^2 &= 143^2\\ 456^2+457^2+458^2+\cdots+466^2 &= 1529^2\\ 854^2+855^2+856^2+\cdots+864^2 &= 2849^2. \end{aligned}$$

Square numbers containing all the ten digits unrepeated

$$\begin{array}{rclcrcl} 32043^2 &=& 1026753849 & & 45624^2 &=& 2081549376 \\ 32286^2 &=& 1042385796 & & 55446^2 &=& 3074258916 \\ 33144^2 &=& 1098524736 & & 68763^2 &=& 4728350169 \\ 35172^2 &=& 1237069584 & & 83919^2 &=& 7042398561 \\ 39147^2 &=& 1532487609 & & 99066^2 &=& 9814072356. \end{array}$$

- Number 45. The sum of all nine and ten digit numbers is 45:
  $123456789 + 123456789 = 246913578,\ 987654321 - 123456789 = 864197532, 123456789 \times 2 = 246913578, 987654321 \times 2 = 1975308642, 1234567890/2 = 617283945, 9876543210/2 = 4938271605,$
  the sum of the quotients, i.e., $617283945 + 4938271605 = 5555555550.$
- Number 153. According to John in the Gospel $21 : 11$, "Simon Peter went up, and drew the net to land full of great fishes, an hundred and fifty and three: and for all there was so many, yet was not the net broken." Mathematically, the number 153 has several obscure properties, for example, $1 + 2 + 3 + \cdots + 17 = (17 \times 18)/2 = 153$ and hence 17th triangular number (see Sect. 7.2), also $1! + 2! + 3! + 4! + 5! = 153.$ Among all the numbers, the following are the only four that can be represented by the cubes of their digits

$$\begin{array}{rclcrcl} 153 &=& 1^3 + 5^3 + 3^3, & & 370 &=& 3^3 + 7^3 + 0^3 \\ 371 &=& 3^3 + 7^3 + 1^3, & & 407 &=& 4^3 + 0^3 + 7^3. \end{array}$$

  If we begin with any integer multiple of $3,$ add up the cubes of its digits, then take the result and sum the cubes of its digits, and so on, we eventually end up with $153.$ For example, consider the number 13701 that is a multiple of $3.$ Now successively we have

$$1^3 + 3^3 + 7^3 + 0^3 + 1^3 = 372, 3^3 + 7^3 + 2^3 = 378,$$
$$3^3 + 7^3 + 8^3 = 882, 8^3 + 8^3 + 2^3 = 1032,$$
$$1^3 + 0^3 + 3^3 + 2^3 = 36, 3^3 + 6^3 = 243, 2^3 + 4^3 + 3^3 = 99, 9^3 + 9^3 = 1458,$$
$$1^3 + 4^3 + 5^3 + 8^3 = 702, 7^3 + 2^3 = 351, 3^3 + 5^3 + 1^3 = 153.$$

  The square root of 153 is 12.369, which is the number of lunar months in a solar year.

- Take any number of three digits, reverse it, subtract the smaller, reverse the result, and add, and you will always have 1089. For example, 287 when reversed is 782, $782 - 287 = 495$ (subtract the smaller), 594 (495 is reversed), $495 + 594 = 1089.$
- 8712 and 9801 are the only 4-digit numbers that are the integral multiples of their reversals, i.e., $8712 = 2178 \times 4$ and $9801 = 1089 \times 9.$

- There are no other numbers below $10,000$ that have the following property $312 \times 221 = 68952$ and $213 \times 122 = 25986.$
- In 1949, Dattatreya Ramchandra Kaprekar (1905–1986, India) discovered an sticking property of the number 6174 which in now known as Kaprekar constant. Take any four-digit number *N* with at least two unequal digits. One can even take a one, two, or three-digit number and consider it as a four-digit number by padding it with zeros on the left. For example, 1 can be considered as 0001. From the four-digit number chosen, form new numbers *A* and *B* by writing the digits of *N* in decreasing and increasing order, respectively. Find $T(N) = A - B.$ This *T* is known as Kaprekar transformation. Kaprekar observed that in at most seven transformations, any four digit gets changed to 6174, and the process stops thereafter, since $T(6174) = 7641 - 1467 = 6174.$ As examples, note that under $T,\ 0001 \to 0999 \to 8991 \to 8082 \to 8532 \to 6174,$ and $2158 \to 7263 \to 5265 \to 3996 \to 6264 \to 4176 \to 6174.$ It is also interesting to note that $18^1 + 18^2 + 18^3 = 18 + 324 + 5832 = 6174.$
- A natural number *N* of *k* digits is called Kaprekar number if it is squared, and then its representation can be partitioned into two *positive* integer parts (the right part with *k* digits and the left part with the *k* or $k-1$ digits) whose sum is equal to the original number. There are infinitely many Kaprekar numbers, and first few of them are 1 (by convention), $9, 45, 55, 99, 297, 703, 999, 2223, 2728, 4879, 4950, 5050, 5292, 7272, 7777,$ 9999. As an example, $2728^2 = 7441984 \to 744 + 1984 = 2728,$ $7272^2 = 52881984 \to 5288 + 1984 = 7272$, and $857143^2 = 734694122449 \to 734694 + 122449 = 857143.$ The restriction that the partitions are positive is essential, in fact, 10 is not a Kaprekar number $10^2 = 100 \to 1 + 00 = 1.$
- In 1955, Kaprekar introduced *Harshad (giving joy) numbers*: A natural number with the property that it is divisible by the sum of their digits is called Harshad number. There are infinite Harshad numbers. In 1977, Ivan Morton Niven (1915–1999, Canada–USA) took interest in these numbers, and in 1994, Helen Giessler Grundman (USA) proved that there is no sequence of more than 20 consecutive Harshad numbers and found the smallest sequence of 20 consecutive Harshad numbers, each member of which has 44363342786 digits. Clearly, $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 36, 40, 60, 100, 108, 153, 6174,$ the biblical number 666, and the taxicab 1729 are Harshad numbers, but $11, 13, 35$

are not Harshad numbers. The number 6804 is called a multiple Harshad number because $6804/18 = 378,\ 378/18 = 21,\ 21/3 = 7,$ and $7/7 = 1.$

- In 1963, Kaprekar defined Devlali (after the town where he lived) or *Self numbers* as integers that cannot be generated by taking some other number and adding its own digits to it. For example, 109 is not a self-number because $109 = 104 + 1 + 0 + 4,$ whereas 108 is a self-number, since it cannot be generated from any other integer. He also gave a test for verifying this property in any number. In the literature these numbers are also referred to as Colombian numbers. First few self-numbers are $1, 3, 5, 7, 9, 20, 31, 42.$ Since $153 = 144 + 1 + 4 + 4, 1729 = 1715 + 1 + 7 + 1 + 5$ and $6156 = 6156 + 6 + 1 + 7 + 4,$ these are not self-numbers.
- If we put after 1 any number of zeros and divide by 7, we get repetitions of the same number 142857 which from time immemorial has been called the "Sacred Number." Now if we add this figure, we get 27 that when added together gives the number 9.
- A *palindromic number* remains the same when its digits are reversed. For example, the first 25 palindromic numbers are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 22, 33, 44, 55, 66, 77, 88, 99, 101, 111, 121, 131, 141, 151. Thus, it has reflectional symmetry across a vertical axis. The term palindromic is derived from palindrome, which refers to a word or sentence (such as racecar or name no one man) whose spelling is unchanged when its letters are reversed. Clearly, there are ten palindromic numbers with one digit, for $n(\geq 2)$ digits the number of palindromic numbers is $9 \times 10^{[(n-1)/2]}$. Thus, there are 9 palindromic numbers with two digits, 90 with three as well as of four digits. The first 15 palindromic square numbers are 0, 1, 4, 9, 121, 484, 676, 10201, 12321, 14641, 40804, 44944, 69696, 94249, 698896. Clearly, there are palindromic numbers whose square is also palindromic, e.g., 11. Also there are palindromic numbers whose square is not palindromic, e.g., 33. The first 10 palindromic cube numbers are 0, 1, 8, 343, 1331, 1030301, 1367631, 1003003001, 10662526601, 1000300030001. It is known that $10662526601 = 2201^3$ is the unique known palindromic cube that has a non-palindromic root number. Methuselah (a biblical patriarch and a figure in Judaism, Christianity, and Islam) is said to have lived 969 years. This number is a palindrome. It is also the 17th tetrahedral number (see Sect. 7.23).

- A Chinese myth says that in about 2200 BC, a divine tortoise emerged from the Yellow River. On his back was a special diagram of numbers from which all of mathematics was derived. The Chinese called this diagram Lo Shu and represented the numbers by knots tied in white and black cords. The Lo Shu diagram

4	3	8
9	5	1
2	7	6

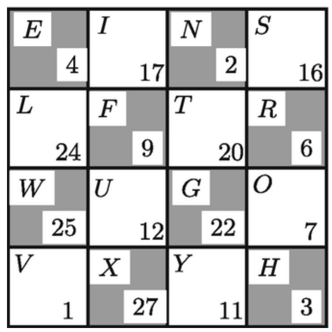
  is the first known *Magic Square*: A magic square is an $n \times n$ square with a whole number written inside each cell, so that the sum of the numbers in every row, in every column, and in each of the main diagonals is equal. This number is called the *magic number*. The magic number of the above 3×3 magic square is 15. Varahamihira examined the *Pandiagonal Magic Squares* (a magic square with the additional property that the broken diagonals, i.e., the diagonals that wrap round at the edges of the square, also add up to the magic constant) of order four. For example, the magic number of the following 4×4 pandiagonal magic square is 34, and note that $8+3+9+14=34=7+13+10+4$

1	12	7	14
8	13	2	11
10	3	16	5
15	6	9	4

  Arab traders brought the Chinese square to Europe during the Middle Ages, when the Black Death was killing millions of people. Magic squares were considered strong talismans against evil, and possession of a magic square was thought to insure health and wealth. Some first rate mathematicians have contributed to the fascinating subject of magic squares. It is of interest to know that even with a few integers an exceedingly large number of magic squares can be formed. In fact, Fermat in 1640 showed that with the first sixty-four natural numbers, the number of different magic squares that can be formed will be more than $1,004,144,995,344!$ One of the most popular and persistent number challenges is the magic square.

- In the following figure a 4×4 magic square known as *Sallow's Magic Square* (named after Lee Cecil Fletcher Sallows, born 1944, England) has been given. Each sub-square has been either shaded or unshaded. Further, each square has a number as well as a letter in it. The magic of this square is as follows: Choose the number from any square. Spell out that number. You will find every letter of that spelling somewhere in the square. If the letter is in an unshaded square, take the number of that square with positive sign, and if the letter is in a shaded square, take the number of that square with negative sign. Find the sum of the numbers corresponding to all the letters in the spelling of the chosen number taking the numbers with proper signs as explained above. You will see that this sum is equal to the number chosen to start with. For example

$$22 \to \text{Twenty two} \to 20 - 25 - 4 - 2 + 20 + 11 + 20 - 25 + 7 = 22.$$



- In the following we have two squares, and the one on the left is a magic square known as *Sallow's Alphamagic Square* with the magic constant 45. This square is related to the one on the right as follows: Spell out each entry in the first square, and place the number of letters in the corresponding position in the second square. Thus the first entry is 5, and since the word "five" has 4 letters, the first entry in the second square is 4. Similarly, for the other members in the square. The interesting fact is that the second square so obtained is also a magic square with the magic constant 21.

5	22	18
28	15	2
12	8	25

$\rightarrow$

Five	Twenty two	Eighteen
Twenty eight	Fifteen	Two
Twelve	Eight	Twenty five

$\rightarrow$

4	9	8
11	7	3
6	5	10

2.11 Complex Numbers

A *complex number* is an expression of the form $a + ib,$ where *a* and $b \in \mathcal{R},$ and *i* is just a symbol. The set of all complex numbers is denoted as $\mathcal{C} = \{a + ib: \ a, \ b \in \mathcal{R}\}.$ For a complex number, $z = a + ib, \ \mathcal{R}e(z) = a$ is the *real part* of $z,$ and $\mathcal{I}m(z) = b$ is the *imaginary part* (or lateral part) of $z.$ If $a = 0,$ then *z* is said to be a *purely imaginary number*. Two complex numbers, *z* and *w*, are equal; i.e., $z = w,$ if and only if, $\mathcal{R}e(z) = \mathcal{R}e(w)$ and $\mathcal{I}m(z) = \mathcal{I}m(w).$ Clearly, $z = 0$ is the only number that is real as well as purely imaginary. The origin of imaginary numbers dates all the way back to Heron of Alexandria (around 75 AD, Egypt) when he attempted to find the volume of a frustum of a pyramid, which required computing the square root of $81 - 144$ (though negative numbers were not conceived in the Hellenistic world). Later, around 850 AD, the Mahavira wrote "as in the nature of things, a negative is not a square, it has not a square root." Cauchy made the same observation a little less than a thousand years later (in 1847).

In 1545, Cardano attempted to solve the following problem: Divide 10 into two parts, one of which multiplied into the other shall produce $40.$ It is evident that this question is impossible, as the product of such parts can be at most $25.$ Cardano called this problem as "manifestly impossible"; however, he considered $x + y = 10$ and $xy = 40,$ or equivalently, the quadratic equation $40 - x(10 - x) = x^2 - 10x + 40 = 0,$ to get the two numbers $5 + \sqrt{-15}$ and $5 - \sqrt{-15},$ and then stated, "putting aside the mental tortures involved, multiply $5 + \sqrt{-15}$ by $5 - \sqrt{-15},$ making $25 - (-15),$ whence the product is 40." He did not pursue the matter but concluded that the result was "as subtle as it is useless." Although eventually rejected, it was the first time that the square root of a negative number was explicitly written. And "the mere writing down of the impossible gave it a symbolic existence." Cardano's solution in *Ars Magna* for the cubic $x^3 = ax + b$ was given as

$$x = \sqrt[3]{\frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 - \left(\frac{a}{3}\right)^3}} + \sqrt[3]{\frac{b}{2} - \sqrt{\left(\frac{b}{2}\right)^2 - \left(\frac{a}{3}\right)^3}}, \qquad (2.6)$$

the so-called Cardano formula. When applied to the historic example $x^3 = 15x + 4,$ the formula yields $x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}.$ Although Cardano claimed that his general formula for the solution of the cubic was inapplicable in this case (because of the appearance of $\sqrt{-121}$), square roots of negative numbers could no longer be so lightly dismissed, whereas for the quadratic (e.g., $x^2 + 1 = 0$) one could say that no solution exists, for the cubic $x^3 = 15x + 4,$ a real solution, namely $x = 4,$ does exist; in fact, the two other solutions, $-2 \pm \sqrt{3},$ are also real. (This work really instigated the whole field and later others worked upon the basis of his solution, including his contemporaries like the Italian mathematicians Scipione del Ferro, 1465–1526, Tartaglia, and Ferrari.) It now remained to reconcile the formal and "meaningless" solution $x = \sqrt[3]{2 + \sqrt{-121}} \; + \; \sqrt[3]{2 - \sqrt{-121}}$ of $x^3 = 15x + 4,$ found by using Cardano's formula, with the solution $x = 4,$ found by inspection. The task was undertaken by Bombelli (he called complex numbers by the alternative name "minus of minus") about 27 years after the publication of Cardano's work. Bombelli had the "wild thought" that since the radicals $2 + \sqrt{-121}$ and $2 - \sqrt{-121}$ differ only in sign, the same might be true of their cube roots. Thus, he let $\sqrt[3]{2 + \sqrt{-121}} = a + \sqrt{-b}$ and $\sqrt[3]{2 - \sqrt{-121}} = a - \sqrt{-b},$ where $a > 0$ and $b > 0$ are to be determined. He proceeded to solve for *a* and *b* by manipulating these expressions according to the established rules for real variables. He deduced that $a = 2$ and $b = 1$ and thereby showed that, indeed,

$$\sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}} \;=\; (2 + \sqrt{-1}) + (2 - \sqrt{-1}) \;=\; 4.$$

Bombelli had thus given meaning to the "meaningless." This event signaled the birth of complex numbers. Breakthrough was achieved by thinking the unthinkable and daring to present it in public. To formalize his discovery, he developed a calculus of arithmetic operations with complex numbers. His rules, in modern symbolism $\sqrt{-1} = i,$ are

$$(\pm 1)i \;=\; \pm\, i, \qquad (+i)(+i) \;=\; -\,1, \qquad (-i)(+i) \;=\; +\,1,$$

$$(\pm 1)(-i) \;=\; \mp\, i, \qquad (+i)(-i) \;=\; +\,1, \qquad (-i)(-i) \;=\; -\,1.$$

He also included examples involving addition and multiplication of complex numbers, such as $8i + (-5i) = +3i,$ and $\left(4 + \sqrt{2i}\right)^{1/3} \left(3 + \sqrt{8i}\right)^{1/3} = (12 + 11\sqrt{2i} + 4i)^{1/3}.$

However, complex numbers were shrouded in mystery, little understood, and often entirely ignored. In fact, for complex numbers, Simon Stevin in 1585 remarked that "there is enough legitimate matter, even infinitely much, to exercise oneself without occupying oneself and wasting time on uncertainties." Napier called complex numbers as the ghosts of real numbers. Wallis in his *Treatise on Algebra* published in 1685 pondered and puzzled over the meaning of imaginary numbers in geometry. He wrote, "These Imaginary Quantities (as they are commonly called) arising from the Supposed Root of a Negative Square (when they happen) are reputed to imply that the Case proposed is Impossible." Leibniz made the following statement in 1702: "The imaginary numbers are a fine and wonderful refuge of the Divine Sprit, almost an amphibian between being and non-being." Christiaan Huygens (1629–1695, The Netherlands), a prominent Dutch mathematician, astronomer, physicist, horologist, and writer of early science fiction, was just as puzzled as Leibniz. In reply to a query, he wrote to Leibniz: "One would never have believed that $\sqrt{1+\sqrt{-3}}+\sqrt{1-\sqrt{-3}}=\sqrt{6}$ and there is something hidden in this which is incomprehensible to us." Euler was candidly astonished by the remarkable fact that expressions such as $\sqrt{-1},\ \sqrt{-2},$ etc., are neither nothing, nor greater than nothing, nor less than nothing, which necessarily constitutes them imaginary or impossible. In fact, he was confused by the absurdity $\sqrt{(-4)(-9)}=\sqrt{36}=6\neq\sqrt{-4}\sqrt{-9}=(2i)(3i)=6i^2=-6.$ Similar doubts concerning the meaning and legitimacy of complex numbers were to Girard and Newton and persisted for two and a half centuries.

Despite of several doubts, during the same period complex numbers were extensively used in both physics (to describe the very real world around us) and pure mathematics, and a considerable amount of theoretical work was done by such distinguished mathematicians as Descartes (who coined the term *imaginary number* in 1637, the terminology, with its aura of the fictional, is perhaps unfortunate but still used in lieu of the word complex, before him these numbers were called *sophisticated*, *impossible*, or *subtle*), and Euler (who was the first in 1777 to designate $\sqrt{-1}$ by $i,$ one of the best-known symbols of mathematics which we use today); De Moivre in 1730 noted that the complicated identities relating trigonometric functions of an integer multiple of an angle to powers of trigonometric functions of that angle could be simply reexpressed by the well-known formula

$$(\cos\theta+i\sin\theta)^n \;=\; \cos n\theta+i\sin n\theta \qquad (2.7)$$

and many others. Complex numbers also found applications in hydrodynamics by d'Alembert and in map projection by Johann Heinrich Lambert (1728–1777, Switzerland).

In 1797, Casper Wessel (1745–1818, Norway) an obscure surveyor presented before the Danish Academy of Sciences a report on the geometrical interpretation of complex quantities in Danish. This report passed unnoticed, and only one hundred years later the Academy issued a French translation, *Essai sur la représentation analytique de la direction*. In the same year, 1797, the twenty-year-Gauss was defending his doctor's thesis on the fundamental theorem of algebra, in which he implicitly used a geometrical representation of the complex domain. In 1806, Jean-Robert Argand (1768–1822) an obscure Parisian bookkeeper Swiss by birth, published an essay on the geometric interpretation of the complex numbers. This again passed unnoticed until about ten years later when it was republished in a prominent mathematical journal. Finally, in 1831 Gauss, in an essay formulated with precision the mathematical equivalence of plane Cartesian geometry with the domain of the complex numbers (which dispelled much of the mystery surrounding complex numbers). He interpreted real numbers as points on the *x*-axis, and the imaginaries as points on the *y*-axis of a rectangular Cartesian system of coordinates, whose intersection point represents the number $0.$ With this perception, the *xy*-plane became complex plane (sometimes it is called as the Argand diagram). In the Gaussian interpretation a rotation through 90^O takes the positive real number axis into the positive imaginary number axis. Gauss did not give a basis for this representation; however, he derived from it the right to operate with imaginary numbers. By means of this interpretation we can also obtain a geometric picture of those numbers which are generated by the addition of an imaginary and a real number, for example, $3+2i$ can be represented as the $x=3, y=2$ on the complex plane. In general, each complex number $z=a+bi$ can be represented as the point (a,b) in the complex plane, and vice versa. This establishes a one-to-one correspondence between the set of all complex numbers and the set of all points in the complex plane.

We can justify the above representation of complex numbers as follows: Let *A* be a point on the real axis such that $OA=a$ (see Fig. 2.6). Since $i\cdot i\,a=i^2\,a=-a,$ we can conclude that twice multiplication of the real number *a* by *i* amounts to the rotation of *OA* through two right angles to the position $OA.''$ Thus, it naturally follows that the multiplication by *i* is equivalent to the rotation of *OA* through one right angle to the position OA'. Hence, if $y'Oy$ is a line perpendicular to the real axis $x'Ox$, then all imaginary numbers are represented by points on $y'Oy$.

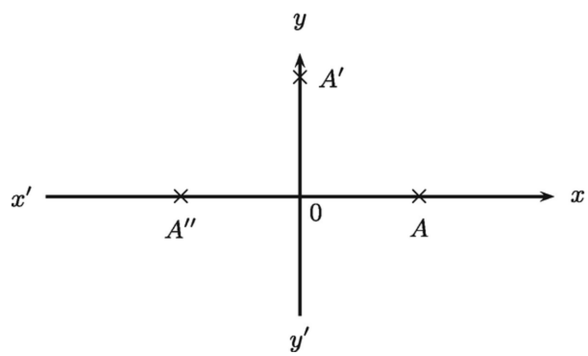


Fig. 2.6 Complex plane

Geometric applications of complex numbers appeared in several memoirs of prominent mathematicians such as August Ferdinand Möbius (1790–1868, Germany), George Peacock (1791–1858, England), Giusto Bellavitis (1803–1880, Italy), De Morgan, and Ernst Eduard Kummer (1810–1893, Germany). In the next three decades, further development took place. Especially, in 1833 William Rowan Hamilton (1805–1865, Ireland) gave an essentially rigorous algebraic definition of complex numbers as ordered pairs of real numbers. However, a lack of confidence in them persisted; for example, the English mathematician and astronomer Sir George Biddell Airy (1801–1892) declared: "I have not the smallest confidence in any result which is essentially obtained by the use of imaginary symbols." The English logician George Boole (1815–1864) in 1854 called $\sqrt{-1}$ an "uninterpretable symbol." Kronecker believed that mathematics should deal only with whole numbers and with a finite number of operations and is credited with saying: "God made the natural numbers; all else is the work of man." He felt that irrational, imaginary, and all other numbers excluding the positive integers were man's work and therefore unreliable. A similar remark was made by Stifel "just as an infinite number is not a true number, so an irrational number is not a true number." However, Hadamard said the shortest path between two truths in the real domain passes through the complex domain, and Edward Charles Titchmarsh (1899–1963, England) said "There are certainly people who regard $\sqrt{2}$ as something perfectly obvious but jib at $\sqrt{-1}$. This is because they think they can visualize the former as something in physical space but not the latter. Actually, $\sqrt{-1}$ is a much simpler concept."

From Gauss's representation we can geometrically visualize those numbers that are generated by the addition and subtraction of two complex numbers. In fact, for the numbers $z = a + ib,\ w = c + id$, we have $z \pm w = (a + ib) \pm (c + id) = (a \pm c) + i(b \pm d)$. Following Wessel the multiplication of *z* and *w* is $zw = (a + ib)(c + id) = (ac - bd) + i(ad + bc)$. The division was introduced as the inverse operation of multiplication. On setting $1/(a + ib) = c + id$, we have $1 = (a + ib) \cdot (c + id)$, which means $1 = (ac - bd) + i(ad + bc)$. On comparing reals with reals, imaginaries with imaginaries, we obtain $c = a/(a^2 + b^2),\ d = -b/(a^2 + b^2)$ and therefore

$$\frac{1}{a + bi} = \frac{a}{a^2 + b^2} - i\frac{b}{a^2 + b^2}, \quad a^2 + b^2 \neq 0.$$

We also note that, for any integer k,

$$i^{4k} = 1, \quad i^{4k+1} = i, \quad i^{4k+2} = -1, \quad i^{4k+3} = -i. \tag{2.8}$$

Thus, for the set $\mathcal{C}$ the usual laws of arithmetic are consistent with those defined in $\mathcal{R}$. However, from the above geometric representation of complex numbers, it is clear that two complex numbers such as $2 + 3i$ and $3 + 2i$ cannot be compared. For example, *i* is neither greater or less than 0. In fact, in either case, $i^2 = -1$ should be greater than zero. Therefore, the concept of betweenness of $\mathcal{R}$ is lost for $\mathcal{C}$. The *absolute value* or *modulus* of the number $z = a + ib$ is denoted by $|z|$ and given by $|z| = \sqrt{a^2 + b^2}$. The term absolute value is due to Weierstrass, whereas modulus is due to Argand. Since $a \leq |a| = \sqrt{a^2} \leq \sqrt{a^2 + b^2}$ and $b \leq |b| = \sqrt{b^2} \leq \sqrt{a^2 + b^2}$, it follows that $\mathcal{R}e(z) \leq |\mathcal{R}e(z)| \leq |z|$ and $\mathcal{I}m(z) \leq |\mathcal{I}m(z)| \leq |z|$. Now, let $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$, and then $|z_1 - z_2| = \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}$. Hence, $|z_1 - z_2|$ is just the Euclidean distance between the points z_1 and z_2, and the absolute value of a complex number is just its distance from the origin. The *complex conjugate* of the number $z = a + bi$ is denoted by $\overline{z}$ and given by $\overline{z} = a - bi$. Geometrically, $\overline{z}$ is the reflection of the point *z* about the real axis. The following relations are immediate:

1. $|z_1 z_2| = |z_1||z_2|, \quad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \quad (z_2 \neq 0).$

2. $|z| \geq 0$, and $|z| = 0$, if and only if $z = 0$.

3. $z = \overline{z}$, if and only if $z \in \mathcal{R}$.

4. $z = -\overline{z}$, if and only if $z = bi$ for some $b \in \mathcal{R}$.

5. $\overline{z_1 \pm z_2} = \overline{z}_1 \pm \overline{z}_2$.

6. $\overline{z_1 z_2} = (\overline{z}_1)(\overline{z}_2)$.

7. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z}_1}{\overline{z}_2}, \quad z_2 \neq 0.$

8. $\mathcal{R}e(z) = \frac{z + \overline{z}}{2}, \ \mathcal{I}m(z) = \frac{z - \overline{z}}{2i}.$

9. $\overline{\overline{z}} = z$.

10. $|z| = |\overline{z}|, \ z\overline{z} = |z|^2$.

11.
$$|z_1 + z_2| \leq |z_1| + |z_2|. \tag{2.9}$$

12.
$$||z_1| - |z_2|| \leq |z_1 - z_2|. \tag{2.10}$$

As an illustration, we shall show only 11 and 12. Each of these is called a *triangle inequality*. For (2.9), we have

$$\begin{aligned}
|z_1 + z_2|^2 &= (z_1 + z_2)\overline{(z_1 + z_2)} = (z_1 + z_2)(\overline{z}_1 + \overline{z}_2) \\
&= z_1\overline{z}_1 + z_1\overline{z}_2 + z_2\overline{z}_1 + z_2\overline{z}_2 \\
&= |z_1|^2 + (z_1\overline{z}_2 + \overline{z_1\overline{z}_2}) + |z_2|^2 \\
&= |z_1|^2 + 2\mathcal{R}e(z_1\overline{z}_2) + |z_2|^2 \\
&\leq |z_1|^2 + 2|z_1 z_2| + |z_2|^2 = (|z_1| + |z_2|)^2.
\end{aligned}$$

It is clear that, in (2.9), equality holds if and only if $\mathcal{R}e(z_1\overline{z}_2) = |z_1 z_2|$; i.e., $z_1\overline{z}_2$ is real and nonnegative. If $z_2 \neq 0$, then since $z_1\overline{z}_2 = z_1|z_2|^2/z_2$, this condition is equivalent to $z_1/z_2 \geq 0$.

For (2.10) we apply the inequality (2.9) to the complex numbers $z_2 - z_1$ and z_1, to get

$$|z_2| \ = \ |z_2 - z_1 + z_1| \ \leq \ |z_2 - z_1| + |z_1|,$$

and hence

$$|z_2| - |z_1| \ \leq \ |z_2 - z_1|. \tag{2.11}$$

Similarly, we have

$$|z_1| - |z_2| \ \leq \ |z_1 - z_2|. \tag{2.12}$$

Combining inequalities (2.11) and (2.12), we obtain (2.10).

The concepts of angle and radius that originated in the work of Hipparchus (around 190–120 BC, Greece) led to *polar coordinates*, also known as (r, θ)-coordinates. In the recent literature, Jacob Bernoulli (1654–1705, Switzerland) has been credited as the inventer of polar coordinates. In 1729, Jacob Hermann (1678–1733, Switzerland) a student of Jacob Bernoulli proclaimed that it was as easy to graph a locus in the polar coordinate system as it was to graph it in the Cartesian coordinate system. To represent a complex number in (r, θ)–coordinates, for the point (a, b) in the Cartesian plane associating $z = a + bi$, we let $r = \sqrt{a^2 + b^2}$ and θ to be its angle measured counterclockwise from the positive *x*-axis to the line joining (a, b) to the origin, i.e., $\theta = \tan^{-1}(b/a)$. Then, *z* can be expressed in *polar* (*trigonometric*) *form* (see Fig. 2.7) as

$$z \ = \ a + ib \ = \ r(\cos\theta + i\sin\theta). \tag{2.13}$$

The number θ is called an *argument* of z, and we write $\theta = \arg z$.

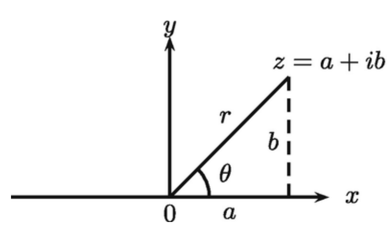


Fig. 2.7 Polar coordinates

To find $\theta,$ we usually compute $\tan^{-1}(b/a)$ and adjust the quadrant problem by adding or subtracting π when appropriate (see Fig. 2.8). Recall that $\tan^{-1}(b/a) \in (-\pi/2, \pi/2)$. The argument of $z = 0$ is not defined.

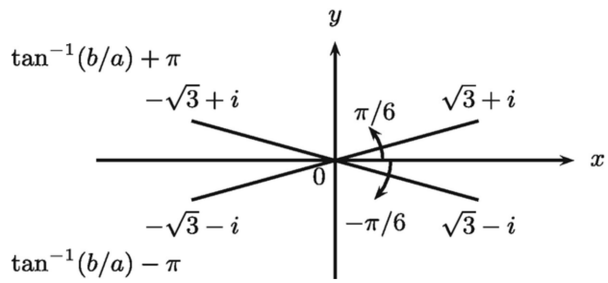


Fig. 2.8 Right quadrant

From (2.13) it is clear that $\arg z$ is not unique, in fact any one of the values $\theta \pm 2n\pi,\ n = 0, 1, \cdots$, can be used. The *principal value* of $\arg z$, denoted by $\operatorname{Arg} z$, is defined as that unique value of $\arg z$ such that $-\pi < \arg z \le \pi$.

If we let $z_1 = a_1 + ib_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = a_2 + ib_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then for the multiplication, we have

$$\begin{aligned} z_1 z_2 = &\ r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \\ = &\ r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]. \end{aligned} \tag{2.14}$$

Thus, $|z_1 z_2| = |z_1||z_2|,\ \arg(z_1 z_2) = \arg z_1 + \arg z_2$. Similarly, for the division, we have

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)], \tag{2.15}$$

and hence, $|z_1/z_2| = |z_1|/|z_2|$ and $\arg\ (z_1/z_2) = \arg z_1 - \arg z_2$. And, $\arg\overline{z} = -\arg z$.

In the fourteenth century, Madhava of Sangamagramma (1340–1425, India) invented the ideas underlying the infinite series expansions of functions, power series, the trigonometric series of sine, cosine, tangent, and arctangent (these series have been credited to James Gregory (1638–1675, England), Brook Taylor (1685–1731, England), and Newton in 1670). We recall for real x,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \tag{2.16}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \tag{2.17}$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{2n-1} + \cdots \tag{2.18}$$

and

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots . \tag{2.19}$$

In (2.19), Euler courageously replaced *x* by ix, used (2.8), separated real and imaginary parts, and then used (2.16) and (2.17), to get

$$\begin{aligned} e^{ix} &= 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \cdots \\ &= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \cdots \\ &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) \\ &= \cos x + i \sin x. \end{aligned}$$

The obtained identity

$$e^{ix} = \cos x + i \sin x \tag{2.20}$$

is one of Euler's famous formula of 1743 which all at once discloses a connection between functions that seem to be of entirely different types. If we let $x = \pi$ in this formula, we have

$$e^{i\pi} = \cos \pi + i \sin \pi = -1 + i \cdot 0 = -1, \tag{2.21}$$

which shows that a unsolvable equation $e^x = -1$ for real *x* has a complex solution $x = \pi i.$ Rewriting (2.21) as

$$e^{i\pi} + 1 = 0, \tag{2.22}$$

we have Euler's most curious formula (credit for its first discovery might belong to Roger Cotes, 1682–1716, England, who published an identity involving the log of both sides of this equation, in 1714), which was considered by some of his metaphysically inclined contemporaries as of mystic significance. Indeed, it contains the most important five symbols of modern mathematics and was regarded as a sort of *mystic union*, in which arithmetic was represented by 0 (the additive identity) and 1 (the multiplicative identity), algebra by the symbol *i* (the imaginary unit), geometry by π (the circular constant), and analysis by the transcendental *e* (the base of the natural logarithms, see Sect. 8.21). However, for many mathematicians, equation (2.22) is the paragon of mathematical beauty, because this extremely simple, compact formula relates all the most important numbers in mathematics in a totally unexpected way.

Now replacing *x* by $-x$ in (2.20), we get

$$e^{-ix} \;=\; \cos x - i\sin x. \tag{2.23}$$

Relations (2.20) and (2.23) immediately give Euler's definitions of cosine and sine in terms of complex exponents

$$\cos x \;=\; \frac{e^{ix}+e^{-ix}}{2} \quad \text{and} \quad \sin x \;=\; \frac{e^{ix}-e^{-ix}}{2i} \tag{2.24}$$

From the exponential representation (2.20) of complex numbers, De Moivre's formula (2.7) follows immediately. In fact, we have

$$(\cos\theta + i\sin\theta)^n \;=\; (e^{i\theta})^n \;=\; e^{i\theta}\cdot e^{i\theta}\cdots e^{i\theta} \;=\; e^{i\theta+i\theta+\cdots+i\theta}$$

$$= \; e^{in\theta} \;=\; \cos n\theta + i\sin n\theta$$

From (2.13) and (2.20), it is clear that any complex number can be written as $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$. By using the multiplicative property of the exponential function, we have

$$z^n \;=\; r^n e^{in\theta} \tag{2.25}$$

for any positive integer n. If $n = -1, -2, \cdots$, we define z^n by $z^n = (z^{-1})^{-n}$. Now since $z^{-1} = e^{-i\theta}/r$, it follows that

$$z^n \;=\; (z^{-1})^{-n} \;=\; \left[\frac{1}{r}e^{i(-\theta)}\right]^{-n} \;=\; \left(\frac{1}{r}\right)^{-n} e^{i(-n)(-\theta)} \;=\; r^n e^{in\theta}.$$

Hence, formula (2.25) is also valid for negative integers n.

Now we shall follow Euler's work of 1751 to see if (2.25) holds for $n = 1/m$. For this, first we shall find the "roots of unity," i.e., solutions of the *cyclotomic equation* $x^m - 1 = 0$. Clearly, 1 has two square roots ± 1, found by solving $x^2 - 1 = 0$. Similarly, unity has three cube roots, which are obtained as the solutions of

$$0 \;=\; x^3 - 1 \;=\; (x-1)(x^2 + x + 1).$$

The first factor gives $x = 1$, and the second, as an application of the quadratic formula, yields $x = \omega = (-1 + i\sqrt{3})/2$, and $x = \omega^2 = (-1 - i\sqrt{3})/2$. By considering $x^4 - 1 = (x^2 - 1)(x^2 + 1) = 0$, Euler showed that the four fourth roots of unity are $1, -1, i$, and $-i$. He also obtained the five fifth roots of unity by solving the equation $x^5 - 1 = 0$ as

$$1, \quad \frac{-1 - \sqrt{5} + \sqrt{-10 + 2\sqrt{5}}}{4}, \qquad \frac{-1 - \sqrt{5} - \sqrt{-10 + 2\sqrt{5}}}{4},$$
$$\frac{-1 + \sqrt{5} + \sqrt{-10 - 2\sqrt{5}}}{4}, \quad \frac{-1 + \sqrt{5} - \sqrt{-10 - 2\sqrt{5}}}{4}. \qquad (2.26)$$

Clearly, the last four of these are imaginary. Then, in view of the formula (2.25), he realized that the numbers

$$x \;=\; e^{(2k\pi i)/m}, \quad k = 0, \pm 1, \pm 2, \cdots$$

are *m*th roots of unity, i.e., satisfy the equation $x^m = 1$. All these roots lie on the unit circle centered at the origin and are equally spaced around the circle every $2\pi/m$ radians (see Fig. 2.9 for $m = 6$).

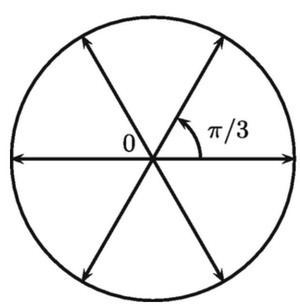


Fig. 2.9 Roots of unity

Hence, all of the distinct *m* roots of unity are obtained by writing

$$\omega_k \;=\; e^{(2k\pi i)/m} \;=\; \cos\frac{2k\pi}{m} + i\sin\frac{2k\pi}{m}, \quad k = 0, 1, \cdots, m-1 \qquad (2.27)$$

From these considerations, it follows that (2.25) holds for $n = 1/m$, and *m* distinct roots of a complex number $z = re^{i\theta}$, i.e., solutions of $\xi^m = z$, are given by

$$\begin{aligned} z^{1/m} &= \sqrt[m]{r}e^{i(\theta+2k\pi)/m} \\ &= \sqrt[m]{r}\left(\cos\frac{\theta+2k\pi}{m} + i\sin\frac{\theta+2k\pi}{m}\right), \quad k = 0, 1, \cdots, m-1. \end{aligned} \tag{2.28}$$

The product of two *m*th roots of unity is itself an *m*th root of unity. Indeed, if $z^m = 1$ and $w^m = 1$, then $(zw)^m = z^m w^m = 1$. Also, the reciprocal of an *m*th root of unity is itself that root. For this, from $zz^{-1} = 1$, it follows that $z^m(z^{-1})^m = 1$, i.e., $(z^{-1})^m = 1$. Generally, any power of the *m*th root of unity is also an *m*th root of unity:

- For the equation $x^3 = 3x + 1$, Cardano's formula (2.6) gives

$$x = \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{1/3} + \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^{1/3}. \tag{2.29}$$

  Now since

$$\frac{1}{2} + i\frac{\sqrt{3}}{2} = e^{i\tan^{-1}\sqrt{3}} = e^{i\pi/3}$$

  from (2.28), we have

$$\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{1/3} = e^{i(\pi/3+2\pi k)/3}, \quad k = 0, 1, 2,$$

  and similarly

$$\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^{1/3} = e^{-i(\pi/3+2\pi k)/3}, \quad k = 0, 1, 2.$$

  Applying these in (2.29) and using (2.24), we get the three solutions of $x^3 = 3x + 1$, respectively, corresponding to $k = 0, 1,$ and $2,$ as

$$x_1 \;=\; e^{20^\circ i} + e^{-20^\circ i} \;=\; 2\cos 20^\circ,$$

$$\begin{aligned} x_2 &= e^{(20^\circ+120^\circ)i} + e^{-(20^\circ+120^\circ)i} \\ &= 2\cos\left(20^\circ + 120^\circ\right) \;=\; -\cos 20^\circ - \sqrt{3}\sin 20^\circ, \end{aligned}$$

$$\begin{aligned} x_3 &= e^{(20^\circ+240^\circ)i} + e^{-(20^\circ+240^\circ)i} \\ &= 2\cos\left(20^\circ + 240^\circ\right) \;=\; -\cos 20^\circ + \sqrt{3}\sin 20^\circ \end{aligned}$$

- For Cardano's equation $x^3 = 15x + 4,$ we shall recover the three solutions from Cardano's solution (2.6). For this, we note that $(2+11i) = (2+i)^3$, which implies that $\arg(2+11i) = \arg(2+i)^3$, i.e., $\tan^{-1} 11/2 = 3\tan^{-1} 1/2.$ Further, since

  $$(2+i) \;=\; \sqrt{5}e^{i\tan^{-1}1/2} \;=\; \sqrt{5}\left(\cos\left(\tan^{-1}1/2\right) + i\sin\left(\tan^{-1}1/2\right)\right),$$

  we have

  $$\sqrt{5}\cos\left(\tan^{-1}1/2\right) \;=\; 2 \quad \text{and} \quad \sqrt{5}\sin\left(\tan^{-1}1/2\right) \;=\; 1.$$

  Now since

  $$(2+11i)^{1/3} \;=\; \sqrt{5}e^{i(\tan^{-1}11/2+2k\pi)/3}, \quad k = 0,1,2$$

  and

  $$(2-11i)^{1/3} \;=\; \sqrt{5}e^{-i(\tan^{-1}11/2+2k\pi)/3}, \quad k = 0,1,2$$

  all three solutions of (3.37) are

  $$\begin{aligned} x_k &= \sqrt{5}\left(e^{i(\tan^{-1}11/2+2k\pi)/3} + e^{-i(\tan^{-1}11/2+2k\pi)/3}\right) \\ &= 2\sqrt{5}\cos\left(\tan^{-1}1/2 + 2k\pi/3\right), \qquad k = 0,1,2. \end{aligned}$$

  Thus corresponding to $k = 0,1,2,$ respectively, we have

$$x_1 = 2\sqrt{5}\cos\left(\tan^{-1} 1/2\right) = 4,$$

$$\begin{aligned} x_2 &= 2\sqrt{5}\cos\left(\tan^{-1} 1/2 + 2\pi/3\right) \\ &= 2\sqrt{5}\left(\cos\left(\tan^{-1} 1/2\right)\cos 2\pi/3 - \sin\left(\tan^{-1} 1/2\right)\sin 2\pi/3\right) \\ &= -2-\sqrt{3}, \end{aligned}$$

$$x_3 = 2\sqrt{5}\cos\left(\tan^{-1} 1/2 + 4\pi/3\right) = -2+\sqrt{3}.$$

The general acceptance of the complex numbers and complex analysis is due to the delegant works of Cauchy, Abel, Pierre Alphonse Laurent (1813–1854, France), Riemann, Karl Gottfried Neumann (1832–1925, Germany), Poincaré, Carl David Tolmé Runge (1856–1927, Germany), Charles Emile Picard (1856–1941, France), Charles Proteus Steinmetz (1865–1923, Poland–USA), and Henri Cartan (1904–2008, France), especially due to Abel who was the first to boldly use complex numbers, with a success that is well-known. Various developments in the nineteenth and twentieth centuries enabled us to gain a deeper insight into the role of complex numbers in mathematics (algebra, analysis, e.g., improper integrals, differential equations, geometry, and the most fundamental work of Dirichlet in number theory); engineering (stresses and strains on beams, resonance phenomena in structures as different as tall buildings and suspension bridges, control theory, signal analysis, quantum mechanics, fluid dynamics, electric circuits, aircraft wings, and electromagnetic waves); and physics (relativity, fractals, e.g., sets due to Benoit Mandelbrot (1924–2010, Poland–France–USA), Gaston Maurice Julia (1893–1978, Algeria–France), and Pierre Joseph Louis Fatou (1878–1929, France) sets, and the equation of Erwin Rudolf Josef Alexander Schrödinger (1887–1961, Austria–Hungary) to describe the quantum theory of atom). By the latter part of the nineteenth century, all vestiges of mystery and distrust of complex numbers could be said to have disappeared (in fact, since early 1930s complex analysis has become one of the required courses for undergraduate as well as graduate students and several text books at all level have been written, e.g., for recent publications, see Agarwal et al. [12] and Pathak et al. [408], and for easy reading Nahin [386]); however, some resistance continued among a few textbook writers well into the twentieth century. Although scholars who employ complex numbers in their work today do not think of them as imaginary/mysterious, these quantities still have an aura for the mathematically naive. For example, the famous twentieth-century French intellectual and psychoanalyst Jacques Lacan (1901–1981) saw a sexual meaning in $\sqrt{-1}.$ On a playful note in the famous novel, *The Da Vinci Code* of 2006, the imaginary number *i* is joked to be the code to the world's secrets.

In recent years complex numbers have been defined in various other (essentially equivalent) ways some of which are as follows:

1. Points or vectors in the plane

2. Ordered pairs of real numbers

3. Operators (i.e., rotations of vectors in the plane)

4. Numbers of the form $a + bi,$ with *a* and *b* real numbers

5. Polynomials with real coefficients modulo $x^2 + 1$

6. Matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix},$ with *a* and *b* real numbers

7. An algebraically closed, complete field (a field is an algebraic structure which has the four operations of arithmetic)

- We conclude this chapter with the following impressive remarks of Gauss which he wrote in 1831: "Our general arithmetic, so far surpassing in extent the geometry of the ancients, is entirely the creation of modern times. Starting originally from the notion of absolute integers it has gradually enlarged its domain. To integers have been added fractions, to rational quantities the irrational, to positive the negative, and to the real the imaginary. This advance, however, had always been made at first with timorous and hesitating steps. The early algebraists called the negative roots of equations false roots, and this is indeed the case, when the problem to which they relate has been stated in such a form that the character of the quantity sought allows of no opposite. But just as in general arithmetic no one would hesitate to admit fractions, although there are so many countable things where a fraction has no meaning, so we ought not deny to negative numbers the rights accorded to positive, simply because innumerable things admit of no opposite. The reality of negative numbers is sufficiently justified since in innumerable other cases they find an adequate interpretation. This has long been admitted, but the imaginary quantities—formerly and occasionally now improperly called impossible, as opposed to real quantities—are still rather tolerated than fully naturalized; they appear more like an empty play upon symbols, to which a thinkable substratum is unhesitatingly denied even by those who would not depreciate the rich contribution which this play upon symbols has made to the treasure of the relations of real quantities."

- For further general reading about numbers, see Adler and Coury [6], Bigelow [70], Burger [107], Conant [136], Crump [143], Dantzig [150], Davenport [153], Davis [156], Dedekind [158], Dickson [164], Dodge [168], Flegg [192], Freidberg [196], Freitag and Freitag [197], Kaplan and Kaplan [297], Georg [211], Hopper [267], Hurford [272], Ifrah [280], Khinchin [306], Landau [333], Menninger [365], Michell [367], Muir [379], Ogilvy and Anderson [399], Ore [402], Schimmel [450], Schröder [453], de Lubicz [456], Shanks [469], Sierpinski [478], Spencer [492], Thurston [509], Upensky and Heasler [516], and Vinogradov [521].

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3. Mathematics, Mathematicians, and Proofs

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3.1 Introduction

Mathematics has often been regarded as a role model for all of science—a paragon of abstraction, logical precision, and objectivity. The origins of mathematics lie in the mists of antiquity. In this chapter, we shall see inspite of several attempts from ancient and modern leading philosophers and mathematicians, the word mathematics is too subtle to enunciate; however, the definitions of a mathematics teacher and a mathematician can be differentiated and defined undoubtedly. We shall detail the importance of the history of mathematics in its further development; in fact, it is essential to know what has already been done before working in a particular field of mathematics, especially, on a particular problem we intend to work. We shall reveal the human nature of mathematicians who are very often believed to be eccentric individuals. We shall explain basic requirements for the deductive mathematics such as a mathematical statement and a mathematical definition. We shall carefully define axioms and list them for geometry, natural numbers, fields, and sets. We shall demonstrate that sometimes eliminating or changing an axiom from the earlier assumed axioms has led to altogether new mathematics, which is as consistent as earlier, and sometimes more useful. Next, we shall define only that part of logic which is required in mathematics and illustrate it through several interesting examples. We shall then be ready to spell out the terms theorem/result, lemma, and corollary, which are the heart of whole mathematics. Any result in mathematics without its proof is incomplete; in fact, there is a famous quote “mathematics is equivalent to proof”; and we shall examine critically the term mathematical proof. We shall also discuss several commonly used methods to prove theorems and illustrate each with elementary, but of paramount interest, examples. Since the invention of electronic computers, several results that were not within the reach of humans have been successfully solved; however, among mathematicians there is a conflict whether to accept computer-based proofs 100%. Certainly, while such proofs provide inside of a result but loses the flavor of classical mathematics. A counterexample to a mathematical statement is an example that satisfies the statement’s condition(s) but does not lead to the statement’s conclusion. While a counterexample is sufficient to disprove a given mathematical statement, often its construction is challenging. We shall provide a few simple examples to clear up this important concept in mathematics.

One of the most pressing questions in mathematics is “can proofs be exact”; we shall take up this problem seriously and show that your today’s proof of a theorem is never permanent, within a few years (sometimes several years) it is modified/simplifies/generalized, and later (often) you and your proof are being criticized. In line with this, we shall mention several proofs that are excessively long for which mathematicians are looking for shorter proofs. In mathematics, an inference formed without proof is called a conjecture. We shall list and explain several conjectures, some of which are challenging from the last several years. A statement for which different approaches lead to different conclusions (namely, true and false) is called a paradox. We shall discuss several paradoxes, some of which are entertaining. We shall also discuss in detail Zeno’s four paradoxes that require the acceptance of infinity. We have made an attempt to answer what is bad, good, and beautiful mathematics. While the answers to these questions are more individualistic, several mathematicians/philosophers have tried to response conclusively. In the last section we shall take up mainly three classical problems of antiquity. We shall show that Euclidean tools are not enough to solve these problems. The most important feature of these problems is that the failure of solving these problems has led to substantial amount of new mathematics.

3.2 Can We Define Mathematics?

The term *mathematics* has its roots from a long-established Greek word mathematikoi (from mathematikos), which means mathematical study, or astronomical, or scientific, or disposed to learn. Its development was contemporaneous with that of philosophy, and it was seen as an unsurpassed exercise in pure thought, a subject at once ideal, immaterial, and eternal. Pythagoras has been credited to coin the label mathematics, and its first part, “math,” comes from an old Indo-European root that is related to the English word “mind.” He grouped arithmetic, astronomy, geometry, and music together and for several centuries’ mathematics referred to only these four subjects (in the Middle Ages it was called as the quadrivium). We owe him the following twofold branches of mathematics, see Fig. 3.1.

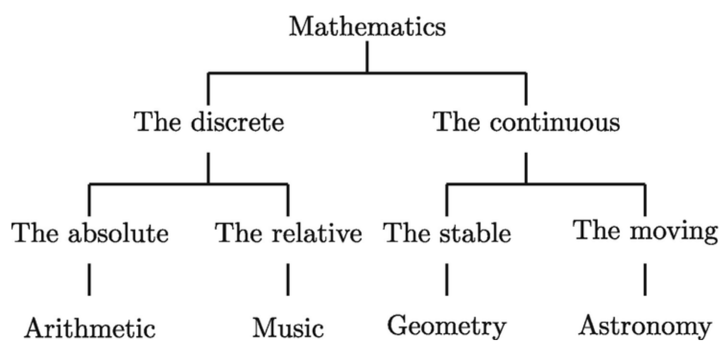

Fig. 3.1 Branches of mathematics

However, as we have noted in the monograph by Agarwal and Sen [14], at least empirical (based on observation, sensory experience, or intuition, which are secrets of scientific power) and heuristic (not regarded as final and strict but merely as provisional and plausible but achieves better understanding) study of arithmetic, astronomy, and geometry began thousands/millions of years before Pythagoras. After Pythagoras several different definitions of mathematics have been proposed. Each one tries to define mathematics with a specific context in mind. Chronologically, some definitions given by prominent philosophers/mathematicians (even politicians) are as follows:

- Mathematics is like draughts (checkers) in being suitable for the young, not too difficult, amusing, and without peril of the state. (Plato)
- Mathematics is the study of “quantity.” (Aristotle)
- The mathematical sciences particularly exhibit order, symmetry, and limitation; and these are the greatest forms of the beautiful. (Aristotle)
- Mathematics consists of objects and of valid statements. (Aristotle)

- Mathematics is the gate and key of the sciences, which the saints discovered at the beginning of the world …and which has always been used by all the saints and sages more than all the sciences. Neglect of mathematics works injury to all knowledge, since he who is ignorant of it cannot know the other sciences or the things of this world. And what is worse, men who are thus ignorant are unable to perceive their own ignorance and so do not seek a remedy. (Roger Bacon)
- Mathematics is the language of science, but important clues about the behavior of various equations can be obtained by observing physical processes. (Galileo)
- Mathematics is the science of order and measure. (Descartes)
- Mathematics-the unshaken foundation of science, and the plentiful fountain of advantage to human affairs. (Isaac Barrow, 1630–1677, England)
- Mathematics is like swine; everything is good. (Lagrange)
- Mathematics is concerned only with the enumeration and comparison of relations. (Gauss)
- Mathematics is persistent intellectual honesty. (Moses Aaron Richardson, 1793–1871, England)
- Mathematics is the science of what is clear by itself. (Carl Guslov Jacob Jacobi, 1804–1851, Prussia now Germany)
- Mathematics is the science which draws necessary conclusions. (Benjamin Peirce, 1809–1880, USA)
- Mathematics seems to endow one with something like a new sense. (Charles Robert Darwin, 1809–1882, England)
- Mathematics is the work of the human mind, which is destined rather to study than to know, to seek the truth rather than to find it. (Evariste Galöis, 1811–1832, France)
- Mathematics is not (as some dictionaries today still assert) merely "the science of measurement and number," but, more broadly, any study consisting of symbols along with precise rules of operation upon those symbols, the rules being subject only to the requirement of inner consistency. (George Boole)
- · · · what is physical is subject to the laws of mathematics, and what is spiritual to the laws of God, and the laws of mathematics are but the expression of the thoughts of God. (Thomas Hill, 1818–1891, USA)
- Mathematics and Poetry are · · · the utterance of the same power of imagination, only that in the one case it is addressed to the head, and in the other, to the heart. (Thomas Hill)

- Mathematics is a branch of logic (an interdisciplinary field which studies truth and reasoning). (Dedekind)
- Mathematics is the science of quantity. (Charles Sanders Peirce, 1839–1914, USA)
- Mathematics is the science of self-evident things. (Felix Christian Klein, 1849–1925, Germany)
- Mathematics is the art of giving the same name to different things. (Poincaré)
- Mathematics in its widest significance is the development "of all types of formal, necessary and deductive reasoning." (Whitehead)
- Mathematics is the most powerful technique for the understanding of pattern, and for the analysis of the relationships of patterns. (Whitehead)
- Mathematics is a game played according to certain rules with meaningless marks on paper. (Hilbert)
- Mathematics knows no races or geographic boundaries; for mathematics, the cultural world is one country. (Hilbert)
- Mathematics is that peculiar science in which the importance of a work can be measured by the number of earlier publications rendered superfluous by it. (Hilbert)
- Mathematics consists of everything which can be formulated and proved by means of the language and axioms. (Ernst Friedrich Ferdinand Zermelo, 1871–1953, Germany, and Abraham Fraenkel)
- Mathematics is a subject identical with logic. (Bertrand Russell)
- Mathematics as the science in which we never know what we are talking about, nor whether what we say is true. (Bertrand Russell)
- The whole of mathematics is nothing more than a refinement of everyday thinking. (Einstein)
- Mathematics is not only the model along the lines of which the exact sciences are striving to design their structure; mathematics is the cement which holds this structure together. A problem, in fact, is not considered solved until the studied phenomenon has been formulated as a mathematical law. Why is it believed that only mathematical processes can lend to observation, experiment, and speculation that precision, that conciseness, that solid certainty which the exact sciences demand? (Dantzig)
- Mathematics is the art of problem solving. (George Polya, 1887–1985, Hungary-USA)

- Mathematical thinking is not purely ‘formal’; it is not concerned only with axioms, definitions, and strict proofs, but many other things belong to it; generalizing from observed cases, inductive arguments, arguments from analog, recognizing a mathematical concept in, or extracting it from, a concrete situation. (Polya)
- In the year 1941, Richard Courant (1888–1972, Germany-USA) and Herbert Robbins (1915–2001, USA) published their book *What Is Mathematics*? [140], which was than highly commended by several distinguished scholars, e.g., Bell reported work as “inspirational collateral reading,” Hermann Klaus Hugo Weyl (1885–1955, Germany) affirmed it as “a work of high perfection,” Harold Calvin Marston Morse (1892–1977, USA) considered it as “a work of art,” and Einstein appreciated it with high praises. In the preface of the second edition of this work mathematics is defined as follows: “mathematics is nothing but a system of conclusions drawn from definitions and postulates that must be consistent but otherwise may be created by the free will of the mathematician.”
- Mathematics is the motley of techniques of proof. (Wittgenstein)
- Mathematics is a tool which ideally permits mediocre minds to solve complicated problems expeditiously. (Floyd Alburn Firestone, 1898–1986, USA)
- Mathematics is a development of thought that had its beginning with the origin of man and culture a million years or so ago. To be sure, little progress was made during hundreds of thousands of years. (Leslie Alvin White, 1900–1975, USA)
- Mathematics is a spirit of rationality. It is this spirit that challenges, simulates, invigorates and drives human minds to exercise themselves to the fullest. It is this spirit that seeks to influence decisively the physical, normal and social life of man, that seeks to answer the problems posed by our very existence, that strives to understand and control nature and that exerts itself to explore and establish the deepest and utmost implications of knowledge already obtained. (Morris Kline)
- Mathematics today is the instrument by which the subtle and new phenomena of nature that we are discovering can be understood and coordinated into a unified whole. In this some of the most advanced and newest branches of mathematics have to be employed and contact with an active school of mathematics is therefore great asset for theoretical physicists. (Homi Jehangir Bhabha, 1909–1966, India)

- Mathematics is like a chest of tools, before studying the tools in detail, a good workman should know the object of each, when it is used and what it is used for. (Walter Warwick Sawyer, 1911–2008, England)
- In mathematics you start with some of the things you want and you try to find postulates to support them. (Richard Wesley Hamming, 1915–1998, USA)
- Mathematics is security, certainty, truth, beauty, insight, structure, architecture. I see mathematics, the part of human knowledge that I call mathematics, as one thing—one great, glorious thing. Whether it is differential topology, or functional analysis, or homological algebra, it is all one thing. …They are intimately interconnected, they are all facets of the same thing. That interconnection, that architecture, is secure truth and is beauty. (Paul Richard Halmos, 1916–2006, Hungary-USA)
- Mathematics is, after all, not a collection of theorems, but a collection of ideas. (Halmos)
- The study of mental objects with reproducible properties is called mathematics. (Philip J. Davis, 1923–2018, USA)
- Mathematics seeks to reduce complexity to a manageable level and also to improve structure where no structure is apparent. (Michael George Aschbacher, born 1944, USA)
- Mathematics is the oldest venue of human intellectual inquiry. (Steven George Krantz, born 1951, USA).
- Mathematics is not about choosing the right set of axioms, but about developing a framework from these starting points. (Richard B Wells, born 1953, USA)
- Mathematics is pure language—the language of science. It is unique among languages in its ability to provide precise expression for every thought or concept that can be formulated in its terms. (Alfred Adler, born 1930, USA)
- Mathematics has also been defined as follows: Mathematics is a natural part of man's cultural heritage; mathematics is the accumulation of human wisdom in an effort to understand and harness the physical, social, and economic worlds; mathematics is a bridge across centuries, civilizations, linguistic barriers, and national frontiers; there is no national prejudice in mathematics; mathematics is a language, and a language can be learned only by continuously using it; mathematics is a tool that ideally permits mediocre minds to solve complicated problems expeditiously; mathematics is the study of numbers, data, quantity, structure, space, models, and change; mathematics is the measurement,

properties, and relationships of quantities and sets using numbers and symbols; mathematics is something that man himself creates, and the type of mathematics he works out is just as much a function of the cultural demands of the time as any of his other adaptive mechanisms; mathematics sheds light on much of the Universe but very little on human psychology; mathematics is the science which uses easy words for hard ideas; mathematics is a living, breathing, changing organism with many facets to its personality; mathematics is a way of thinking which enables to see unifying patterns in diverse context; mathematics allows us to make logical decision based on observations; mathematics is not just a profession, rather it is a cumbersome and tyrannical taskmaster; mathematics is the most problem-solving oriented of all sciences; and, mathematics seeks regularities and pattern in behavior, motion, number, or shape, or even in the substrata of chaos. The portrait of mathematics shows a human face.

A variety of quips and clichés can also define mathematics: "It's an art," "it's a science—in fact, it's the queen and servant of science," "it's just circus artistes," "it's what I use when I balance my checkbook," "a game that we play with rules we're not quite sure of," a certain type of human experience, and the apologetic favorite "something I was never good at," and the list can go on and on.

About mathematics several interesting positive views have also been proposed. Perhaps from these we can find some more definitions of mathematics.

- Mathematics reminds you of the investible form of the soul; she gives life to her own discoveries; she awakens the mind and purifies the intellect; she brings light to our intrinsic ideas, she abolishes the oblivion and ignorance which are ours by birth. (Proclus)
- In mathematics I can report no deficiency, except it be that men do not sufficiently understand the excellent use of the Pure Mathematics. (Francis Bacon)
- If a man's wit be wandering, let him study mathematics. (Francis Bacon)
- …the Universe stands continually open to our gaze, but it cannot be understood unless one first learns to comprehend the language and interpret the characters in which it is written. It is written in the language of mathematics, and its characters are triangles, circles, and other geometric figures, without which it is humanly impossible to

understand a single word of it; without these, one is wandering about in a dark labyrinth. (Galileo)

- If there is anything that can bind the heavenly mind of man to this dreary exile of our earthly home and can reconcile us with our fate so that one can enjoy living-then it is verily the enjoyment of the mathematical sciences and astronomy. (Kepler)
- The chief aim of investigations of the external world should be to discover the rational order and harmony which has been imposed on it by God and which He revealed to us in the language of mathematics. (Kepler)
- Music is a secret mathematical exercise, and he who engage in it is unaware that he is manipulating numbers. (Leibniz)
- There is no philosophy, which is not founded upon knowledge of the phenomena, but to get any profit from this knowledge it is absolutely necessary to be a mathematician. (Daniel Bernoulli, 1700–1782, The Netherlands-Switzerland)
- No employment can be managed without arithmetic, no mechanical invention without geometry, mathematical demonstrations are better than academic logic for training the mind to reason with exactness and distinguish truth from falsity even outside of mathematics. (Benjamin Franklin, 1706–1790, USA)
- What science can there be more noble, more excellent, more useful for men, more admirably high and demonstrative, than mathematics. (Benjamin Franklin)
- In any serious and honest attempt to solve a mathematical problem, there is a faithful look at truth. (Lagrange)
- Trigonometry…is most valuable to every man. There is scarcely a day in which he will not resort to it for some of the purposes of common life. The science of calculation also is indispensable as far as the extraction of the square and cube roots. Algebra as far as the quadratic equation and the use of logarithms are often of value in ordinary cases. But all beyond these is but a luxury, a delicious luxury indeed, but not to be indulged in by one who is to have a profession to follow for his subsistence. In this light I view the conic sections, curves of the higher orders, perhaps even spherical trigonometry, algebraical operations beyond the second dimension, and fluxions. (Thomas Jefferson, 1743–1826, USA)
- The profound study of nature is the most fecund source of mathematical discoveries. The fundamental elements are those which recur in all natural phenomena. (Jean Baptiste Joseph Fourier, 1768–1830, France)

- The advancement and perfection of mathematics are intimately connected with the prosperity of the state. (Napoléon Bonapartet, 1769–1821, France)
- He who does not employ mathematics for himself will some day find it employed against himself. (Johann Friedrich Herbart, 1776–1841, Germany)
- All the measurements in the world are not the equivalent of a single theorem that produces a significant advance in our greatest of science. (Gauss)
- Mathematics is queen of the sciences and number theory the queen of mathematics. (Gauss)
- In mathematics there are no true controversies. (Gauss)
- Life is good for two things, discovering mathematics and teaching mathematics. (Baron Siméon Denis Poisson, 1781–1840, France)
- There is no branch of mathematics, however abstract, which may not someday be applied to the phenomena of the real world. (Nicolai Ivanovich Lobachevsky, 1792–1856, Russia)
- History shows that those heads of empires who have encouraged the cultivation of mathematics, the common nature of all exact sciences, are also those whose reigns have been the most brilliant and whose glory is the most durable. (Michel Floréal Chasles, 1793–1880, France)
- The moving power of mathematical invention is not reasoning but imagination. (De Morgan)
- During the three years which I spent at Cambridge my time was wasted, as far as academical studies were concerned …I attempted mathematics, and even went during the summer of 1828 with a private tutor …but I got on very slowly. The work was repugnant to me, chiefly from my not being able to see any meaning in the early steps in algebra. This impatience was very foolish, and in after years I have deeply regretted that I did not proceed far enough at least to understand something of the great leading principles of mathematics, for men thus endowed seem to have an extra sense. (Charles Darwin wrote in his autobiography)
- The most distinct and beautiful statement of any truth must take at last the mathematical form. (Henry David Thoreau, 1817–1862, USA)
- It may well be doubted whether, in all the range of science, there is any field so fascinating to the explorer-so rich with hidden treasures-so fruitful in delightful surprises-as Pure Mathematics. (Charles Lutwidge Dodgson, 1832–1898, England, pen name: Lewis Carroll)

- The essence of mathematics is in its freedom. (Cantor)
- Mathematics is one of the oldest of sciences; it is also one of the most active; for its strength is the vigor of perpetual youth. (Andrew Russell Forsyth, 1858–1942, England)
- Every kind of science, if it has only reached a certain degree of maturity, automatically becomes a part of mathematics. (Hilbert)
- "Mathematics, even in its present and most abstract estate, is not detached from life. It is just the ideal handling of the problems of life…" (Cassius Jackson Keyser, 1862–1947, USA)
- The true spirit of delight…is to be found in mathematics as surely as in poetry. (Bertrand Russell)
- We have overcome the notion that mathematical truths have an existence independent and apart from our minds. It is even strange to us such a notion could ever have existed. Yet this is what Pythagoras would have thought–and Descartes, along with hundreds of other great mathematicians before the nineteenth century. Today mathematics is unbounded; it has cast off its chains. Whatever its existence, we recognize it to be as free as the mind, as prehensile as the imagination. (Edward Kasner, 1878–1955, USA)
- To all of us who hold the Christian belief that God is truth, anything that is true is a fact about God, and mathematics is a branch of theology… (Hilda Phoebe Hudson, 1881–1965, England)
- Mathematics, as much as music or any other art, is one means by which we rise to a complete self-consciousness. The significance of mathematics resides precisely in the fact that it is an art; by informing us of the nature of our own minds it informs us of much that depends on our minds. It does not enable us to explore some remote region of the eternally existent; it helps to show us how far what exists depends on the way we exist. We are the lawgivers of the Universe; it is even possible that we can experience nothing but what we have created, and that the greatest of our mathematical creations is the Universe itself… Mathematics is of profound significance in the Universe, not because it exhibits principles we obey, but because it exhibits principles that we impose. (John William Navin Sullivan, 1886–1937, England)
- The mathematical phenomenon always develops out of simple arithmetic, so useful in everyday life, out of numbers, those weapons of the gods; the gods are there, behind the wall, at play with numbers. (Charles-Édouard Jeanneret known as Le Corbusier, 1887–1965, Switzerland-France)

- Mathematical study and research are very suggestive of mountaineering. Edward Whymper (1840–1911, England) made several efforts before he successfully climbed the Matterhorn (a mountain of the Alps) in 1865 and even then it cost the life of four of his party. Now, however, any tourist can be hauled up for a small cost, and perhaps does not appreciate the difficulty of the original ascent. So, in mathematics, it may be found hard to realize the great initial difficulty of making a little step which now seems so natural and obvious, and it may not be surprising if such a step has been found and lost again. (Louis Joel Mordell, 1888–1972, USA-England)
- If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is. (John Louis von Neumann, 1903–1957, Hungary-USA)
- If you want to become historian, learn mathematics. If you want to become a doctor, learn mathematics. (Grigore Moisil, 1906–1973, Romania)
- Mathematics has given man miraculous power over nature. (Morris Kline)
- Mathematics is the surest way to immortality. (Erdös)
- Creativity is the heart and soul of mathematics at all levels. The collection of special skills and techniques is only the raw material out of which the subject itself grows. To look at mathematics without the creative side of it, is to look at a black-and-white photograph of a Cezanne; outlines may be there, but everything that matters is missing. (Robert Creighton Buck, 1920–1998, USA)
- Mathematics abounds in bright ideas. No matter how long and hard one pursues her, mathematics never seems to run out of exciting surprises. And by no means are these gems to be found only in difficult work at an advanced level. All kinds of simple notions are full of ingenuity. (Ross Honsberger, 1929–2016, Canada)
- Mathematics is an art, a creative art, that cannot be reduced to logic any more than Shakespeare’s King Lear or Beethoven’s Fifth Can. (Roger Penrose, born 1931, England).
- The apex of mathematical achievement occurs when two or more fields which were thought to be entirely unrelated turn out to be closely intertwined. Mathematicians have never decided whether they should feel excited or upset by such events. (Gian-Carlo Rota, 1932–1999, Italy-USA)

- Mathematics is a natural component of our being. It arises from our body, brain and our daily experiences in the world. (Jean-Pierre Changeux, born 1936, France)
- Mathematics is a part of physics. Physics is an experimental science, a part of natural sciences. Mathematics is the part of physics where experiments are cheap. (Vladimir Igorevich Arnold, 1937–2010, Soviet Union, now Ukraine)
- Mathematics is an art that contains its own justification, in the same way that Michelangelo's sculptures live inside the stone until they are liberated by the chisel. (Enrico Bombieri, born 1940, Italy)
- The great strength of mathematics is its unreal reality! (Ian Nicholas Stewart, born 1945, England)
- …working in mathematics satisfies a personal intellectual curiosity and desire to work on one's own ideas. (Krantz [316]).

We also encounter some negative views about mathematics: Sir William Hamilton (1788–1856, Scotland), the famed philosopher, logician, and meta-physicist, viewed mathematics in a way that may be construed as a cruel attack on mathematics and hence on mathematicians: "Mathematics freeze and parch the mind," "an excessive study of mathematics absolutely incapacitates the mind for those intellectual energies which philosophy and life require," "mathematics cannot conduce to logical habits at all," "in mathematics dullness is thus elevated into talent, and talent degraded into incapacity," and "mathematics may distort, but can never rectify, the mind" (see Bell [59]). Introduction of exactness (in mathematics and logic) is artificial and forced. It is not therefore applicable to this terrestrial life but only imaginary celestial existence (Bertrand Russell). We are talking here about theoretical physics, and therefore of course mathematical rigor is irrelevant and impossible (Edmund Georg Hermann (Yehezkel) Landau, 1877–1938, Germany). Mathematics that is certain does not refer to reality, and mathematics that refers to reality is not certain (Einstein). As the complexity of a system increases, our ability to make precise yet significant statements about its behavior diminishes, until a threshold is reached beyond which precision and significance (or relevance) become mutually exclusive characters (Lotfi Aliasker Zadeh, 1921–2017, Azerbaijan-USA). Mathematics is a human endeavor, and mathematical truths are uncertain like any other truths (Reuben Hersh, 1927–2020, USA, see [261] and its review by Auslander [44]). Mathematics is mired in a language of symbols foreign to most of us and [it] explores regions of the infinitesimally small and the infinitely large that elude words, much less understanding. So specialized is mathematics today…that most mathematical papers appearing in most mathematics journals are indecipherable even to most mathematicians (Robert Kanigel, born 1946, USA). Henry Oldenburg (1618–1677, Germany) pioneered the idea of refereed scientific journals in 1665.

The following parable (which has been widely spread around the world) of "The Blind Men and the Elephant" (a story from the *Buddhist Sutra*) is relevant to our attempt to define mathematics. Several prominent citizens were engaged in a hot argument about God and the different religions and could not come to an agreement. So they approached Buddha to find out what exactly God looks like. Buddha asked one his disciple to get a large majestic elephant and four blind men. He then brought the four blind men to the elephant and told them to find out what the elephant would "look" like. The first blind man touched the elephant's leg and reported that it "looked" like a pillar. The second blind man touched its tummy and said that an elephant was an inverted ceiling. The third blind man touched the elephant's ear and said that it was a piece of cloth. The fourth blind man held on to the tail and described the elephant as a piece of rope. And all of them ran into a hot argument about the "appearance" of an elephant. The Buddha asked the citizens: "Each blind man had touched the elephant but each of them gives a different description of the animal. Which answer is right?" "All of them are right," was the reply. Why? Because everyone can only see part of the elephant. They are not able to see the whole animal. This famous "blind men" episode is not meant to disrespect any mathematician. We state it only to point out that "any definition of mathematics, however elaborate or epigrammatic, will fail to lay bare its fundamental structure and the reasons for its universality" (Herman Weyl).

Similar views were lay down by Mark Kac (1914–1984, Poland-USA) and Ulam: "We shall not undertake to define mathematics, because to do so would be to circumscribe its domain…The structure, however, changes continually and sometimes radically and fundamentally. In view of this, an attempt to define mathematics with any hope of completeness and finality is, in our opinion, doomed to failure." In fact, the characteristic feature of mathematics is that it itself has no meaning, and its function is to connect postulates with observations. Mathematics can be correlated beautifully with a banyan tree which lives for hundreds of years, its roots grow from branches which mature into thick, woody trunks, which can become indistinguishable from the primary trunk with age, and its each part has its own unique medical uses (the bark and seeds are used as a tonic to maintain body temperature and treat diabetes, and roots are used to strengthen teeth and gums by brushing with them). In conclusion, within few years each branch itself is called mathematics, and scientists start searching its applications in all of arts, sciences, engineering, and technology. The primary trunk of the banyan tree can be contemplated as natural numbers and geometry (originated in India, Babylonia, Egypt, and Greece). Thus, we can say that the definition of mathematics continues to change with time and innovation. It also depends on to whom you would ask: for elementary school children mathematics is natural numbers and the basic operations and negative and decimal numbers; for students in high school mathematics is learning rules and formulas to solve equations and elementary geometry and trigonometry (triangle measurement); for undergraduate students mathematics is calculus, differential equations, and linear algebra; and for graduate students and researchers it narrows down to a particular topic/branch they are working.

Nonetheless, mathematics rewards its creator and learner with a strong sense of aesthetic satisfaction. It helps us understand man's place in the Universe and enables us to find order in chaos. Under certain axioms, mathematics is the most absolute, everlasting, precise, significant, and universal (not reserved for any nationality or a group) subject. It is perceived as the highest form of thought in the world of learning. It furnishes a strong sense of rationale to scientific community of their new discoveries (especially for experimental sciences) through well-established mathematics. It also shows how mathematics is woven into the fabric of our lives in diverse means than we perceive. Poincaré noted "Mathematics with one method-mathematical model studies various events of the real world." Most of the branches of science and technology provide mathematics with interesting problems to investigate, and mathematics provides science and technology with powerful tools to use in analyzing data. In fact, mathematics is found in almost every field of human endeavor: Algebra and number theory are used in *cryptography*; computational fluid dynamics in *aircraft and automobile design* and *weather modeling and prediction*; differential equations in *aerodynamics, fluid mechanics, and finance*; discrete mathematics in *communications and information technology*; formal systems and logic in *computer programming*; geometry in *computer-aided engineering and design*; nonlinear control in *operation of mechanical and electrical systems*; numerical analysis in *essentially all applications*; optimization in *asset* *allocation* and *shape and system design*; statistics (the practice of drawing a mathematically precise line from an unwarranted assumption to a foregone conclusion) in *design of experiments* and *analysis of data sets*; stochastic processes in *signal analysis*; and the list goes on and on. Even the most esoteric and abstract parts of mathematics, number theory, and logic, for example, are now used routinely in applications. No doubt, mathematics is one of the greatest creations of mankind—if it is not indeed the greatest. Mathematics will live forever.

Mathematics could be a matter of life and death! During the Russian Revolution, the famous mathematical physicist Igor Yevgenyevich Tamm (1895–1971), suspected of being a communist agitator, was detained by some anticommunist vigilantes. When asked what he did for a living, Tamm said that he was a mathematician. The armed leader of the gang then ordered Tamm to find the remainder when the Brook Taylor series of a certain function was truncated after a certain number of terms. Although quivering, Tamm managed to give the correct answer to the leader and was let go. Tamm went on to win the Nobel Prize in Physics in 1958. He never discovered the identity of the mysterious gang leader, but no doubt spared no effort to impress upon his students the importance of mathematics!

Up to the first half of the twentieth century, there were three main schools of thought on the issue of "mathematical philosophy/reality" (they spent almost 40 years quarreling with each other and then went back to sleep): Platonism/Logistic (modern espousers were Gottlob Frege, Bertrand Russell, Alonzo Church, 1903–1995, USA, Gödel, and Willard Van Orman Quine 1908–2000, USA), Constructivism/Intuitionism (founder Jan Brouwer), and Formalism (originator Hilbert). Generally, the Platonists believe that the whole of mathematics exists eternally, independently of man, and the job of the mathematician is to discover these mathematical truths. For the Platonists (most practicing mathematicians seem to belong to this group), if mathematical objects constitute an ideal nonmaterial world, how does the human mind establish contact with this world? Is there a mental faculty which can directly perceive an ideal reality just as our physical senses perceive physical reality? Is this then a second ideal entity, the counterpart on the subjective level of the ideal mathematical reality on the objective level? For the Platonists, this mental faculty is akin to the "soul" for believers in the hereafter—they know it is there, but no questions can be asked about it. This position makes Platonism a difficult doctrine for any scientifically oriented person to defend. The constructivist asserts that there are certain primitive objects (the natural numbers), and only mathematical objects that can be "constructed" from these primitive ones in a finitistic (finite mathematical objects) way are meaningful. They purport as a universal and unmistakable intuition the notions of the constructive and of the natural numbers, that is, the notion of an operation which can be iterated, which can always be repeated one more time. This dogma, however, is not tenable in the light of historical, pedagogical, or anthropological experience. The natural number system seems an innate intuition only to those who have achieved such a level of sophistication that they cannot remember or conceive of the time before they acquired it.

Finally, the formalist takes the position that mathematics consists merely of formal symbols or expressions that are manipulated according to preassigned rules or agreements. In other words, they believe that mathematics simply provides a language for other disciplines and has no intrinsic meaning of its own. They take the stance that a mathematical formula is just a mathematical formula, and our belief that it has content is an illusion that need not be defended or justified. From this standpoint, a problem of principal concern to mathematicians becomes totally invisible. This is the problem of giving a philosophical account of the actual development of mathematics and the mathematics of the classroom. As most mathematicians would testify, mathematics grows by a process of successive criticism and refinement of theories and the advancement of new and competing theories and not by the deductive pattern of formalized mathematics. There are numerous examples of great mathematicians who could recognize new results before they could supply a formal proof. In the classroom, some of students' favorite questions are "Why do you make this particular definition"? or "Why do you carry out this particular construction"? To which many lecturers, influenced by formalism, like to answer: "Just because it works." Formalism denies completely the utilitarian (utilitarianism is the moral philosophy that right actions are those that bring about more good for the people affected than any alternative) aspect of mathematics and can say nothing about its growth.

Thus, as an example, for Platonists number two is that Platonic "form" or 'idea' in virtue of which things have the property of two-ness; for logicists number two is $\{x \mid \exists_y \exists_z \ (y \neq z \wedge x = \{y, z\})\}$, i.e., two is a set of all unordered pairs; for constructivists number two is the concept which expresses the principle of "two-ity"; and for formalists number two is just a class of expressions manipulated according to certain rules. Hence each of these schools has a different view for number two. What is the true answer? Fortunately, Professor Cuthbert Calculus (French: Professeur Tryphon Tournesol) meaning "Professor Tryphon Sunflower" is a fictional character in The Adventures of Tintin and was able to discover this when attending a conference in France. The number two is a pair of platinum balls kept at room temperature in the second drawer at the Bureau of Standards in Paris.

In a more humanistic approach, which is being advocated more recently, the meaning of mathematics is found in the shared understanding of human beings. In this respect, mathematics is similar to an ideology, a religion, or an art form; it deals with human meanings and is intelligible only within the context of culture. In other words, mathematics is a humanistic study. The special feature of mathematics that distinguishes it from other humanities is its science-like quality. Its conclusions are compelling, like the conclusions of natural science. They are not simply products of opinion and not subject to permanent disagreement like the ideas of literary criticism. Without any doubt, mathematics does the exact thing. And the most important and beautiful fact about Math is it never lies itself. Mathematical knowledge is fallible, corrigible, tentative, and evolving, as is every other kind of human knowledge. However, "…by and large it is uniformly true in mathematics that there is a time lapse between a mathematical discovery and a moment when it is useful; and that this lapse of time can be anything from thirty to a hundred years, in some cases even more; and that the whole system seems to function without any direction, without any reference to usefulness, and without any desire to do the things which are useful" (Lord Kelvin, 1824–1907, England).

3.3 Who Is a Mathematician?

Mathematicians, by their nature and their very way of thinking, abstract the situations presented by simple examples/observations and define an abstract structure. Then, they give a name to such structure. One of the advantages of defining such an abstract structure is that instead of studying a large number of particular examples, separately, we can study their representative abstract structure only once, and all conclusions drawn from such a single study will apply to particular examples. Creating and identifying such structures and studying them are the strength of mathematicians. According to Bell [60], "unless a man adds something new to mathematics, he is not a mathematician. By this standard, the Moslems were not mathematicians in their extremely useful work of translation, and commentary." However, they researched and extended the theoretical and applied science of the Indians, Greeks, and Romans of an earlier era in ways that preserved and strengthened man's knowledge in these important fields. Thus a mathematician must invent at least one new result in the field of his interest, and the scientific community and students must accept its significance. In conclusion, mathematicians create and conserve mathematics. Other quotations about mathematicians are:

- Mathematicians draw tangible figures as an aid to their investigations, "they are not thinking about these figures but of those things which the figures represent: thus it is the square in itself and the diameter in itself which are the matter of their arguments, not that which they draw; similarly when they model or draw objects, which may themselves have images in shadows or in water, they use them in turn as images, endeavoring to see those absolute objects which cannot be seen otherwise than by thought. (Plato)
- Mathematicians are really seeking to behold the things themselves, which can be seen only with the eye of the mind. (Plato)
- Mathematicians are people who do real work as opposed to academic work. (Plato)
- Mathematicians are like Frenchmen; whatever you say to them they translate into their own language, and forthwith it is something entirely different. (Johann Wolfgang von Goethe, 1749–1832, Germany)
- A mathematician is a blind man in a dark room looking for a black hat which isn't there. (Charles Darwin)
- We are servants rather than masters in mathematics. (Charles Hermite, 1822–1901, France)
- "Do you know what a mathematician is" Lord Kelvin once asked a class. He stepped to the board and wrote

  $$\int_{-\infty}^{\infty} e^{-x^2} dx \ = \ \sqrt{\pi}.$$

  Putting his finger on what he had written, he turned to the class. "A mathematician is one to whom that is as obvious as that twice two makes four is to you."
- Mathematicians do not deal in objects, but in relation between objects; thus, they are free to replace some objects by others so long as the relations remain unchanged. Content to them is irrelevant: they are interested in form only. (Poincaré)

- A scientist worthy of the name, above all a mathematician, experiences in his work the same impression as an artist; his pleasure is as great and of the same nature. (Poincaré)
- A mathematician, like a painter or poet, is a maker of patterns. If his patterns are more permanent than a painter's, it is because they are made with ideas. The mathematician's patterns, like the painter's or the poet's, must be beautiful; his ideas, like the colors or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics. (Hardy)
- A mathematician is in the first instance an observer, a man who gazes at a distant range of mountains and notes down his observations. If he sees a peak and wishes someone else to see it, he points to it, either directly or through the chain of summits which leads him to recognize it himself. When his pupil also sees it, the research, the argument, the proof is finished. (Hardy)
- Hardy's [249] view on what mathematics is and what a mathematician does: "I believe that mathematical reality lies outside us, and that our function is to discover or observe it, and that the theorems which we prove, and which we describe grandiloquently as our 'creations' are simply our notes of our observations." Those who persist in their belief that "mathematical reality lies outside us" have at least one unanswerable argument on their side. Invention may be free, they admit, but free only within the law.
- Mathematicians are of two types: the 'we can' men believe (possibly subconsciously) that mathematics is a purely human invention; the 'there exists' men believe that mathematics has an extra-human *existence* of its own, and that 'we' merely come upon the *eternal truths* of mathematics in our journey through life, in much the same way that a man taking a walk in a city comes across a number of streets with whose planning he had nothing whatever to do (Bell).
- The mathematician may be compared to a designer of garments, who is utterly oblivious of the creatures that his garments may fit. To be sure, his art originated in the necessity for clothing such creatures, but this was long ago; to this day a shape will occasionally appear which will fit into the garment as if the garment had been made for it. Then there is no end of surprise and of delight. (Dantzig)
- Mathematicians are the law-givers of the Universe and that the Universe itself is the greatest of our mathematical creations. (Sullivan)

- Summing up the value system of the modern mathematicians: …all the different fields of mathematics are as inseparable as the different parts of a living organism; as a living organism mathematics has to be permanently recreated, each generation must reconstruct it wider, larger and more beautiful. The death of mathematical research would be the death of mathematical thinking which constitutes the structure of scientific language itself and by consequence the death of our scientific civilization. Therefore, we must transmit to our children strength of character, moral values and drive towards an endeavoring life. (Jean Leray, 1906–1998, France)
- Mathematicians may like to rise into the clouds of abstract thought, but they should, and indeed they must, return to earth for nourishing food or else die of mental starvation. (Morris Kline)
- What's the best part of being a mathematician? I'm not a religious man, but it's almost like being in touch with God when you're thinking about mathematics. God is keeping secrets from us, and it's fun to try to learn some of the secrets. (Halmos)
- A mathematician is a machine that turns coffee into theorems. (Alfred Rényi, 1921–1970, Hungary)
- A typical mathematician is a Platonist on working days and a formalist on Sundays. This means that when he pursues his mathematical studies, he is convinced that he deals with an objective reality, the features of which he is trying to figure out. But if he is confronted with the demand to give a philosophical explanation of this reality, he prefers to pretend that he ultimately does not believe in it. (Hersh)
- Much of the mathematics was either initiated in response to external problems or has subsequently found unexpected applications in the real world. This whole linkage between mathematics and science has an appeal to its own, where the criteria must include both the attractiveness of the mathematical theory and the importance of the applications. As the current story of the interaction between geometry and physics shows, the feedback from science to mathematics can be extremely profitable, and this is something I found doubly satisfying. Not only can we mathematicians be useful, but we can create works of art at the same time, partly inspired by the outside world. (Michael Francis Atiyah, 1929–2019, England-Lebanon)
- In the broad light of day mathematicians check their equations and their proofs, leaving no stone unturned in their search for rigor. But, at night, under the full moon, they dream, they float among the stars and wonder

at the miracle of the heavens. They are inspired. Without dreams there is no art, no mathematics, no life. (Michael Atiyah)

- A mathematician finds mathematical truths by way of insight, intuition, and sometimes even leaps of faith. (Penrose)
- Mathematicians are those humans who advance human understanding of mathematics. (William Paul Thurston, 1946–2012, USA)
- A mathematician is someone who: Observes and interprets phenomena, analyzes scientific events and information, formulates concepts, generalizes concepts, performs inductive reasoning, performs analogical reasoning, engages in trial and error (and evaluation), models ideas and phenomena, formulates problems, abstracts from problems, solves problems, uses computation to draw analytical conclusions, makes deductions, makes guesses, proves theorems, and the list continues. (Keith James Devlin, born 1947, England-USA)
- …the difference between a physicist and a mathematician is this: After a physicist has worked on a problem for many years and solved it, he/she is convinced that he is a great historical genius and he/she celebrates. After a mathematician has worked on a problem for many years and solved it, he/she decides that the problem is trivial and that he/she is an idiot. (Krantz [317])
- A mathematician is an individual who proves his ignorance with equations. (Khalid Masood)
- A mathematician is an explorer, and usually an explorer who has no idea what he/she is looking for. It is part of the ordinary course of life to make mistakes, to spend days on calculations that come to no conclusion, to pursue paths that end up being meaningless. But if one continues to analyze and to think critically and to stare mercilessly at all these efforts, then one may draw useful conclusions. In the end, one may formulate a theorem. And then, with an additional huge expenditure of hard work, one may prove that theorem. It is a great adventure, with many pitfalls and missteps. But that is the life of a mathematician. (Anonymous)

However, with the present standards, the above quotations are too restrictive. At the same time, "The mathematical requirements for even the most developed economic structures of antiquity can be satisfied with elementary household arithmetic which no mathematician would call mathematics" (Neugebauer). Today, mathematicians besides developing new theoretical mathematics (pure mathematics) and publishing their research in scientific Journals, professionals (applied mathematicians) use mathematical theory, computational techniques, algorithms, and the computer technology to solve mathematical, physical, economic, environmental, scientific, engineering, military, political, medical technology, and business problems and try to relate mathematics with life and social sciences. Specialists such as statisticians, actuaries, and operations research analysts are also applied mathematicians. Furthermore, mathematicians love mathematics, have passion for mathematics, and are eager to know new results in mathematics and their applications. Although pure and applied mathematics are not distinctly defined and often overlap each other, pure mathematicians develop new principles and recognize previously unknown relationships between existing principles of mathematics, whereas applied mathematicians use theories and techniques to formulate and solve practical problems. While pure mathematics has existed since several millennia, the concept was embellished around the year 1900, especially from the works of Gauss, Hardy, and Ramanujan in number theory. Mathematicians collaborate (one of the best examples is Hardy and Ramanujan) which is the key in order to advance the next generation of mathematics. "It seems plausible that in 100 years we will no longer speak of mathematicians as such but rather of *mathematical scientists*. This will include traditional, pure mathematicians to be sure, but also of others who use mathematics for analytical purposes. It would not be all surprising if the notion of Department of Mathematics at the college and university level gives way to Division of Mathematical Sciences."

Like musicians, mathematicians have created specific vocabulary/symbols and alphabetical identifications (mostly Greek) to make a mathematical statement simple, precise, effective, and understandable. For example, until the sixteenth century algebra was mostly *rhetorical*, that is, each equation was expressed in ordinary language. The next stage of algebra, which was initiated by Brahmagupta and Diophantus and continued by the Arabian mathematicians, has been called *syncopated*, and in it symbols are partially used. Finally, *symbolic* algebra, which we use today, became established through the works of Viéte, Descartes, Wallis, and Leibniz. These three phases of algebra have been recognized by Georg Heinrich Ferdinand Nesselmann (1811–1881, Poland-Germany) in 1842. In the nineteenth century, symbols were used to write most mathematics as formulas. These formulas played a key role in advancing all mathematics, particularly algebra, to great heights. Although Gauss once remarked that mathematical proofs depend on *notions*, not on *notations*, in appreciation of mathematical formulas, Heinrich Hertz (1857–1892, Germany), the discoverer of electromagnetic waves, said "One cannot escape the feeling that these mathematical formulas have an independent existence and an intelligence of their own, that they are wiser than we are, wiser even than their discoverers, that we get more out of them than was originally put into them." Furthermore, according to Morris Kline, "the true value of mathematical formulas lies in the fact that they apply to so many varied situations on heaven and earth."

Archimedes, Newton, and Gauss have been considered *complete mathematicians* due to the fact that they enriched every branch of the subject that they touched. According to Laplace "Newton was a most fortunate man because there is just one universe and Newton had discovered its laws." According to Bell, Poincaré [was] the last man to take practically all mathematics, pure and applied, as his province. Few mathematicians have had the breadth of philosophic vision that Poincaré had and none in his superior in the gift of clear exposition. Unfortunately, despite of unmatched contributions, there is always a tendency to consider mathematicians as an aloof/dumb natured.

3.4 Why History of Mathematics Is Important?

A study of the history of any subject—knowing what and how it has happened earlier—puts the learning of that subject in proper perspective and can help to make progress in the right direction. The history of mathematics is a monumental subject covering more than 10,000 years of human pursuit; it examines the bona fide origin of discoveries in mathematics and of mathematical sciences, also essential part of the world's culture and civilization. It helps in uncovering secrets, and to find the missing pieces, it assists in understanding the reasons and results, it provides a linkage between the definition of a mathematical concept and its applications, it reflects some of the noblest thoughts of countless generations, and most importantly it gives a pride of priority of the nation. The following famous quotations manifest the importance of studying/knowing the history of mathematics:

- In most sciences one generation tears down what another has built, and what one has established, another undoes. In mathematics alone each generation adds a new story (better storey) to the old structure. (Hermann Hankel, 1839–1873, Germany)

- But in science the credit goes to the man who convinces the world, not to the man to whom the idea first occurs. (Francis Darwin, 1848–1925, England)
- To foresee the future of mathematics, the true method is to study its history and its present state. (Poincaré)
- It would be an injustice to pioneers in mathematics to stress modern mathematical ideas with little reference to those who initiated the first and possibly the most difficult steps. Nearly everything useful that was discovered in mathematics before the seventeenth century has either been so greatly simplified that it is now part of every regular school course, or it has long since been absorbed as a detail in some work of greater generality. (Bell)
- Mathematics is a unique aspect of human thought, and its history differs in essence from all other histories. Only in mathematics there is no significant correction-only extensions. Each great mathematician adds to what came previously, but nothing needs to be uprooted. (Isaac Asimov, 1920–1992, Russia-USA)
- Mathematicians scrutinize, memorize, and derive formulas and theorems everyday of their lives, but not many of them realize that the current level of mathematical knowledge has resulted from the strenuous labors of countless generations. In fact, a fair majority of the biggest breakthroughs in mathematics were made possible through the work of people other than those who have been credited in the history books. In many instances, the fault lies with the historians themselves, who inject their own opinion into their texts rather than reporting unbiased fact. One cannot underestimate the influence of every culture, personality, philosophy, region, religion, society, and social status on mathematical development throughout the centuries. (Agarwal and Sen [14])

Thus mathematics can be learnt differently/systematically only through historical notes right from early stages. For this, from the last century several books (at all levels) on the history of mathematics have been written (mostly about the history of European mathematics, specially glorifying the contributions of Greek philosophers, and in general intellectual superiority of Europeans), for example, see Aaboe [1], Carl Benjamin Boyer (1906–1976, USA) [100], Burton [110], Cajori [114], Eves [181], Heath [257,258], Victor Joseph Katz (born 1942, USA) [301], Neugebauer [391], Russell [440], and Scott [457]. Some scientific journals are solely devoted to ancient but unknown mathematical findings, which affects the trends of future research. Furthermore, now an almost inconceivable amount of literature is available on Internet. Inspired by Lakshmikantham et al. [329], in our monographs and articles Agarwal and Sen [14], Sen and Agarwal [463,465], Agarwal et. al. [13], and Agarwal and Hans Agarwal [19], we have recorded the essential discoveries of mathematics in correct chronological order, following the birth of ideas on the basis of prior ideas ad infinitum; these works also examine contemporary events occurring side by side in different countries or cultures, reflecting some of the noblest thoughts of generations; it documents the winding path (e.g., struggle of hundreds of years to develop now casually used concepts such as negative numbers, place value, zero, irrational numbers, and transcendental numbers) of mathematical scholarship throughout history and, most importantly, the thought process of each individual that resulted in the mastery of their subject; it implicitly addresses the nature and character of every mathematician as we try to understand their visible actions; it offers amusing anecdotes and after dinner jokes which reveal the human nature of mathematicians, who are very often believed to be abnormal individuals.

3.5 Are Mathematicians Smart?

The dictionary defines smart person as having or showing a quick-witted intelligence. An extended definition of a smart person is one who responds quickly and effectively, comprehend, having complete logical reasoning skills and successfully applying them to solve political, social, economic, and scientific problems and/or having emotional intelligence. However, smartness is judged differently in different contexts, e.g., a student is smart if he shows quick intelligence or ready mental capability, a businessman is smart if he is shrewd or sharp in dealing with others, a politician is smart if he relates to the state, government, the body politic, public administration, policy-making, etc. In general, a young guy looks smart if he is handsome, dresses nicely, and has muscles to show. According to Plato mathematics strengthens your mind much as physical exercise strengthens your body, helping you negotiate a variety of mental challenges. The 16th president of the United States Abraham Lincoln (1809–1865, USA) believed mastering *Elements* on geometry increased his cognitive capacities, in particular his linguistic and logical abilities. Arthur Ignatius Conan Doyle (1859–1930, England) in 1887 created the characters Sherlock Holmes for his *A Study in Scarlet* in which Dr. Watson (a fictional character) described the deductive powers of Sherlock Homes in these words: “His conclusions were as infallible as so many propositions of Euclid.” Mohandas Karamchand Gandhi (Mahatma Gandhi, 1869–1948, India) in his *The Story of My Experiments with Truth* wrote: When however, with much effort I reached the thirteenth proposition of Euclid the utter simplicity of the subject was suddenly revealed to me. A subject which only required a pure and simple use of one’s reasoning powers could not be difficult. Ever since that time geometry has been both easy and interesting to me. Al-Qifti (around 1172–1248, Egypt) said of Euclid “Nay, there was no one even of later date who did not walk in his footsteps.” Einstein added his own tribute: “If Euclid failed to kindle your youthful enthusiasm, then you were not born to be scientific thinker.” While these observations are phenomenal as *Elements* provides much of the foundation for today’s mathematical thought, in spite of a mathematician thinks mathematical problems logically and/or tries to relate mathematics with almost every real world phenomenon, he is slow in grasping and responding and aloof in his own thoughts. Thus, mathematicians have been labeled as absent-minded, arrogant, eccentric, and egocentric.

3.6 Are Mathematicians Intelligent?

The dictionary defines intelligent person as having ability to acquire and apply knowledge and skills to deal with new or difficult situation/problem. To find the intelligence of a person, psychologists have developed intelligence quotient (IQ) test, as the score goes higher, the person’s intelligence is considered greater. In spite of its broad popularity, this test has been controversial throughout history. Some of the examining factors of the IQ test are language skills, mathematical abilities, memory, processing speed, reasoning abilities, and visual-spatial processing. Since mathematics is one of the factors of the IQ test and mathematics requires critical and analytical thinking, problem-solving, quantitative reasoning, ability to manipulate precise and intricate ideas, construct logical arguments and expose illogical arguments, and communication and most importantly has a universal language (bad English and Greek alphabets), practicing math makes mathematicians intelligent. Although, it is generally perceived that intelligence implies smartness; however, sometimes intelligent people do not use their intelligence in a smart way. For this, it is important to note that intelligence is connected with intellect which is responsible for the faculty of reasoning and understanding objectively, especially with regard to abstract or academic matters; hence if it is not properly trained with the time, hasty decisions and/or overconfidence (dangerous for mathematicians) may end up taking wrong decisions.

The following fable suggests mathematicians are smart and intelligent within their own world: “A psychologist wanted to study the thinking patterns of different types of scientists. So he brought together at one place, many scientists, each of them belonging to a different field of science-one zoologist, one geologist, one mathematician etc. He arranged an experiment for each of them. The experiment consisted of lifting one specified inanimate object (say a duster for definiteness) from the floor of the room and placing it on a table. The psychologist saw to it that each scientist performed the experiment independently, i.e., without observing how it was performed by the other scientists, as he knew that scientists are known to copy other scientists. There were, of course, many observers to watch the scientists performing the experiment. Each scientist came forward one by one, lifted the duster from the floor, and placed it on the table in the normal way. The mathematician’s turn to perform the experiment was at the end. The psychologist and the other observers were eager to see how the mathematician would perform the experiment as mathematicians are generally considered weird and eccentric. The mathematician walked slowly and gracefully, lifted and placed the duster on the table almost exactly the same way as any other scientist did. Here the first part of the experiment was over. The second part of the experiment consisted of lifting the same duster, this time kept on a chair, and placing it on the same table used in the first part of the experiment. Again, each scientist came forward, lifted the duster from the chair and placed it on the table in the normal way. Last came the mathematician, he/she lifted the duster from the chair, placed it on the floor, pointed her finger from the floor to the table and calmly said that this problem was previously solved.”

3.7 What Is a Mathematical Statement?

A statement is a set of words that is complete in itself, typically containing a subject and predicate, question, exclamation, or command and consisting of a main clause and sometimes one or more subordinate clauses. We are born with capability to reason every statement (generally, it improves with age and education). Immanuel Kant (1724–1804, Germany) divided statements into two categories: analytic and synthetic. Analytic statements are those that are true by definition, whereas synthetic statements are possibly true but not necessarily true. A mathematical statement (proposition) is either true or false, i.e., any statement which can be anticipated to be true and false is not a mathematical statement. For example, the sum of two natural numbers is greater than 0 is a true mathematical statement, the sum of two negative integers is greater than 0 is a false mathematical statement, and the sum of any two integers is greater than 0 is not a mathematical statement. Similarly, $x - y = y - x$ is not a mathematical statement, because the symbols are not defined. If $x - y = y - x$ for all $x,\ y$ real numbers, then this is a false proposition; if $x - y = y - x$ for some real numbers, then this is a true proposition. Help me please and your place or mine? are also not mathematical statements. An important aspect of human intellect is by using connecting words such as "and," "or," "if, then," "either, or," etc. two mathematical statements can be combined to make a new mathematical compound statement. For example, consider the statements: squares are rectangles and rectangles have four sides, and then squares are rectangles and rectangles have four sides is a compound statement.

3.8 What Is a Mathematical Definition?

In mathematics, a definition gives an unambiguous meaning to a new term (in terms of certain other terms of which the meaning is already known) or characterizes a concept and occurs repeatedly. A mathematical definition is never self-contradictory and hence cannot be wrong. Definitions are the starting point of serious mathematics and clarify/simplify expressions. To acquire a great wealth of plane geometric facts (Euclidean Geometry), Euclid began his book I of *Elements* with the following 23 definitions (in fact, 22; the last one is a postulate), so the reader would know precisely what his terms meant.

1. A point is that of which there is no part (i.e., it has no width, no length, and no thickness; one of his less illuminating definitions, because some physical space is required for something to exist).

2. And a line is a length without breadth.

3. And the extremities of a line are points.

4. A straight line is (any) one which lies evenly with points on itself.

5. And a surface is that which has length and breadth only.

6. And the extremities of a surface are lines.

7. A plane surface is (any) one which lies evenly with the straight lines on itself.

8. And a plane angle is the inclination of the lines to one another, when two lines in a plane meet one another, and are not lying in a straight line.

9. And when the lines containing the angle are straight then the angle is called rectilinear.

10. And when a straight line stood upon (another) straight line makes adjacent angles (which are) equal to one another, each of the equal angles is a right angle, and the former straight line is called a perpendicular to that upon which it stands.

11. An obtuse angle is one greater than a right angle.

12. And an acute angle (is) one less than a right angle.

13. A boundary is that which is the extremity of something.

14. A figure is that which is contained by some boundary or boundaries.

15. A circle is a plane figure contained by a single line [which is called a circumference], (such that) all of the straight lines radiating towards [the circumference] from one point amongst those lying inside the figure are equal to one another.

16. And the point is called the center of the circle.

17. And a diameter of the circle is any straight line, being drawn through the center, and terminated in each direction by the circumference of the circle. (And) any such (straight line) also cuts the circle in half.

18. And a semi-circle is the figure contained by the diameter and the circumference cuts off by it. And the center of the semi-circle is the same (point) as (the center of) the circle.

19. Rectilinear figures are those (figures) contained by straight lines: trilateral figures being those contained by three straight lines, quadrilateral by four, and multilateral by more than four.

20. And of the trilateral figures: an equilateral triangle is that having three equal sides, an isosceles (triangle) that having only two equal sides, and a scalene (triangle) that having three unequal sides.

21. And further of the trilateral figures: a right-angled triangle is that having a right angle, an obtuse-angled (triangle) that having an obtuse angle, and an acute angled (triangle) that having three acute angles.

22. And of the quadrilateral figures: a square is that which is right-angled and equilateral, a rectangle that which is right-angled but not equilateral, a rhombus that which is equilateral but not right-angled, and a rhomboid that having opposite sides and angles equal to one other which is neither right-angled nor equilateral. And let quadrilateral figures besides these be called trapezia.

23. Parallel lines are straight lines which, being in the same plane, and being produced to infinity in each direction, meet with one another in neither (of these directions).

Euclid began his book VII of *Elements* with the following 22 definitions for positive integer greater than unity, i.e., natural numbers. These definitions remain unchanged from the last almost 2300 years.

1. A unit is that by virtue of which each of the things that exist is called one. (Iamblichus claims the Pythagoreans used a different definition: unity is the boundary between number and parts of a number.)

2. A number is a multitude composed of units.

3. A number is a part of a number, the less of the greater, when it measures the greater. (In other words, a number a is a less part of

another number *b* if there exist a number *n* such that $na = b.$)

4. But parts when it does not measure it.

5. The greater number is a multiple of the less when it is measured by the less.

6. An even number is that which is divisible into two equal parts.

7. An odd number is that which is not divisible into two equal parts, or that which differs by a unit from an even number.

8. An even-times-even number is that which is measured by an even number according to an even number.

9. An even-times-odd number is that which is measured by an even number according to an odd number.

10. An odd-times-odd number is that which is measured by an odd number according to an odd number.

11. A prime number is that which is measured by a unit alone.

12. Numbers relatively prime are those which are measured by a unit alone as a common measure.

13. A composite number is that which is measured by some number.

14. Numbers relatively composite are those which are measured by some number as a common measure.

15. A number is said to *multiply* a number when the latter is added as many times as there are units in the former.

16. And when two numbers having multiplied one another make some number, the number so produced be called plane, and its sides are the numbers which have multiplied one another. (This is the beginning of algebra with geometry.)

17. And when three numbers having multiplied one another make some number, the number so produced be called solid, and its sides are the numbers which have multiplied one another.
18. A square number is equal multiplied by equal, or a number which is contained by two equal numbers.
19. And a cube is equal multiplied by equal and again by equal, or a number which is contained by three equal numbers.
20. Numbers are proportional when the first is the same multiple, or the same part, or the same parts, of the second that the third is of the fourth.
21. Similar plane and solid numbers are those which have their sides proportional.
22. A perfect number is that which is equal to the sum its own parts (factors).

3.9 What Is an Axiom?

According to the New World Encyclopedia, the term axiom comes from the Greek word *axíoma* coined by Aristotle “that which is thought worthy or fit’ or ‘that which commends itself as evident.” In mathematics, an axiom, postulate (mainly in classical geometry, but now meaning is different), or assumption (used synonymously) is a mathematical statement that is regarded as being established, accepted, innocuous, or self-evidently true, to serve as a premise or starting point (building block) for further reasoning and arguments. According to Aristotle “Every demonstrative science must start from indemonstrable principles. Otherwise, the steps of demonstration would be endless.” Thus, axioms are important to get right, because all of mathematics rests on them and should not be contradicting each other. For each particular area of modern mathematics, we start with necessary definitions and require a set of axioms (just like rules before starting any game), e.g., to develop Euclidean Geometry; Euclid in the first book of the *Elements* after definitions of point, line, circle, and other terms stated five postulates (common-sensical geometric facts drawn from our experience) and a list of common notions (very basic, self-evident assertions):

Euclid's postulates (also known as "$4+1$" axioms) in modern language are as follows:

1. A straight line segment can be drawn joining any two points.

2. Any straight line segment can be extended indefinitely in a straight line.

3. Given any straight line segment, a circle can be drawn having the segment as radius and one endpoint as center.

4. All right angles are congruent.

5. If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough. This postulate is known as the *parallel postulate*.

The first three postulates became the common convention for Greek mathematicians in acknowledging geometrical constructions. They do not explicitly say what tools can be used, but in order to abide by these postulates, only an *unmarked straightedge* and a *collapsible compass* are permitted, which are called *Euclidean tools* (instruments of the gods).

Euclid's common assertions are as follows:

(a). Things that are equal to the same thing are also equal to one another, i.e., if $a = b$ and if $a = c$, then $b = c$ (the transitive property).

(b). If equals are added to equals, then the wholes are equal, i.e., if $a = b$ and if $c = d$, then $a + c = b + d$ (Addition property of equality).

(c). If equals are subtracted from equals, then the differences are equal, i.e., if $a = b$ and if $c = d$, then $a - c = b - d$ (subtraction property of equality).

(d). Things that coincide with one another are equal to one another. This is specifically geometrical, where equality means congruence.

(e). The whole is greater than the part, i.e., $a + b > a.$

The following axioms that completely define the natural numbers were formulated by Guiseppe Peano (1858–1932, Italy):

1. 1 is a natural number.
2. For each natural number *n* there is a unique successor $n + 1.$
3. 1 is not the successor of any natural number.
4. Two natural numbers are equal if their successors are equal.
5. Any set of natural numbers which contains 1 and the successor of every natural number *p* whenever it contains *p* is the set $\mathcal{N}.$

A *field*, defined by Eliakim Hastings Moore (1862–1932, USA) in 1893, is a set of scalars, denoted by $F,$ in which two binary operations, addition $(+),$ and multiplication $(\cdot)$, are defined so that the following axioms hold:

1. *Closure Property for Addition and Multiplication*: If $a, b \in F,$ then $a + b \in F$ and $a \cdot b \in F.$
2. *Commutative Property for Addition and Multiplication*: If $a, b \in F,$ then $a + b = b + a$ and $a \cdot b = b \cdot a.$
3. *Associative Property for Addition and Multiplication*: If $a, b, c \in F,$ then $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c).$
4. *Additive and Multiplicative Identity Properties*: There exists a zero element, denoted by $0,$ in *F* such that for all $a \in F,\ a + 0 = 0 + a = a$, and there exists a unit element, denoted by $1,$ in *F* such that for all $a \in F,\ a \cdot 1 = 1 \cdot a = a.$
5. *Additive and Multiplicative Inverses*: For each $a \in F,$ there is a unique element $(-a) \in F$ such that $a + (-a) = (-a) + a = 0$, and for each $a \in F,\ a \neq 0$ there is a unique element $a^{-1} \in F$ such that $a \cdot a^{-1} = a^{-1}a = 1.$

6. *Left Distributivity*: If $a, b, c \in F$, then $a \cdot (b + c) = a \cdot b + a \cdot c$.

7. *Right Distributivity*: If $a, b, c \in F$, then $(a + b) \cdot c = a \cdot c + b \cdot c$.

From the above axioms it immediately follows that the sets $\mathcal{N}$ and $\mathcal{Z}$ are not fields, whereas the set $\mathcal{Q}$, the set of all real numbers $\mathcal{R}$, and the set of complex numbers $\mathcal{C}$ are fields. Clearly, $\mathcal{N} \subset \mathcal{Z} \subset \mathcal{R} \subset \mathcal{C}$. Also, if $a \in \mathcal{R}$, then exactly one of the following statements is true: $a < 0, \ a = 0$, or $a > 0$, i.e., *axiom of order* also known as *trichotomy law* holds. As we have seen earlier, this law does not hold for the set $\mathcal{C}$.

The idea that algebraic operations like addition and multiplication should behave consistently in every number system, especially when developing extensions to established number systems Hankel in 1867 explicitly formulated the *Principle of Permanence*. This principle can be stated as a definition as follows: A collection of symbols infinite in number shall be called a *number field* and each individual element in it a *number*, if:

1. Among the elements of the collection we can identify the sequence of *natural numbers*.

2. We can establish criteria of rank which will permit us to tell of any two elements whether they are equal, or if not equal, which is greater; these criteria reduce to the natural criteria when the two elements are natural numbers.

3. For any two elements of the collection, we can devise a scheme of *addition* and *multiplication* which will have the commutative, associative, and distributive properties of the natural operations bearing these names and which will reduce to these natural operations when the two elements are natural numbers.

The *Axiom of Archimedes* states that for any given real number *x* there exists an integer *n* such that *n* is greater than x.

Let *A* be a non-empty set of real numbers. A number $m \in A$ is called a *least element* of *A* if $x \geq m$ for every $x \in A$. A non-empty set *S* of real numbers is said to be *well-ordered* if every non-empty subset of *S* has a least element. Every non-empty finite set of real numbers is well-ordered. None of the sets $Z, \ Q$, and *R* have a least element (if *x* is a number, then $x - 1$ is another number which is smaller), so none of these sets is well-ordered. Although it may appear evident that the set $\mathcal{N}$ is well-ordered, this statement cannot be shown from the properties of positive integers. Consequently, this statement is accepted as an axiom and stated as follows:

Well-Ordering Principle: The set $\mathcal{N}$ is well-ordered.

The subset $S \subset \mathcal{R}$ is said to be *bounded above* if there exists $M \in \mathcal{R}$ such that $x \in S$ implies $x \leq M.$ The number *M* is called an *upper bound* of $S.$ Similarly, the subset $S \subset \mathcal{R}$ is said to be *bounded below* if there exists $m \in \mathcal{R}$ such that $x \in S$ implies $x \geq m.$ The number *m* is called a *lower bound* of $S.$ The set *S* is said to be *bounded* if it is bounded above and below. If the set *S* has at least one upper bound, then there are infinitely many upper bounds greater than it. If *S* has no upper bound, then *S* is said to be *unbounded above*. If there is a least number among the upper bounds of the set $S,$ then this number is called the *least upper bound* or *supremum* of the set $S.$ Similarly, if the set *S* has at least one lower bound, then there are infinitely many lower bounds smaller than it. If *S* has no lower bound, then *S* is said to be *unbounded below*. If there is a greatest number among the lower bounds of the set $S,$ then this number is called the *greatest lower bound* or *infimum* of the set $S.$ The supremum and/or infimum of the set *S* may not be members of $S,$ e.g., the set $(1, 2)$ (open interval) does not contain either its infimum or its supremum which are 1 and 2, respectively. From these definitions it follows that supremum and infimum of sets, if exist, are unique. The existence of supremum and infimum of non-empty sets bounded above and below, respectively, is the following completeness axiom:

Completeness axiom: If *S* is any non-empty subset of $\mathcal{R}$ that is bounded above (below), then *S* has a supremum (infimum) in $\mathcal{R}.$ The completeness axiom is also known as the *continuity axiom* in $\mathcal{R}.$

Additional popular axioms are those of *axiomatic set theory* due to Zermelo and Abraham Fraenkel known as the Zermelo-Fraenkel (ZF) axioms, which are listed as follows:

1. *The Axiom of Extension*: Two sets are equal (are the same set) if they have the same elements.

2. *The Empty Set Axiom*: A set with no members exists and can be written as $\emptyset.$

3. *The Axiom of Separation*: A subset can be formed from a set, and consist of some elements from the set.

4. *The Pair-Set Axiom*: Given two objects of *a* and $b,$ the set $\{a, b\}$ can be formed.

5. *The Union Axiom*: Two or more sets can be formed into a union of those sets.

6. *The Power Set Axiom*: Given a set, a set of all possible subsets can be formed (power set).

7. *The Axiom of Infinity*: There exists a set with infinitely many elements.

8. *The Axiom of Foundation*: Sets are comprised of simpler sets, so every non-empty set has a minimal member.

9. *The Axiom of Replacement*: A function can be applied to every member of a set, and the answer is still a set.

In addition to these basic axioms, there is a tenth axiom called the *Axiom of Choice* (AC), in short (ZFC). It states "Given infinitely many non-empty sets, one element from each set can be chosen." While this axiom is clear for non-empty finite sets, it does not follow from (ZF) for non-empty infinite sets. Although originally controversial, the axiom of choice is now used without reservation by most mathematicians, and it is included among the nine axioms of set theory. For (ZFC) Bertrand Russell found an analogy: For any (even infinite) collection of pairs of shoes, one can pick out the left shoe from each pair to obtain an appropriate collection (i.e., set) of shoes; this makes it possible to directly define a choice function. For an infinite collection of pairs of socks (assumed to have no distinguishing features), there is no obvious way to make a function that forms a set out of selecting one sock from each pair, without invoking the axiom of choice.

In 1930, Gödel showed that (ZFC) cannot disprove (CH), and in 1963, Paul Cohen showed that (ZFC) cannot prove (CH). That means (CH) can be added to the standard axioms of set theory without creating a contradiction.

After years of struggle, mathematicians have shown that virtually all mathematical concepts and results can be formulated within axiomatic set theory. This has been recognized as one of the greatest achievements of modern mathematics and, as a result, we can now say that "set theory is a unifying theory for mathematics." In fact, all mathematical objects can be defined as sets. Consequently, the results of mathematics can be viewed as statements about sets. Thus it can be said that “mathematics can be embedded in set theory,” or “set theory is woven into the fabric of modern mathematics.”

3.10 Does Abolishing an Axiom Lead to New Mathematics?

We begin with the following excerpt of Bell [61]: “An axiom generally implies a compulsion of rational thought or a restriction of possible action; abolition of the axiom as a necessity invites free invention. In the past, abolition of axioms also invited persecution, today in all sciences except social it merely invites personal abuse, if even that.” In what follows we shall provide some examples where carefully abolishing/challenging an axiom opened altogether a new way of considering a system which turned out to be more beautiful, practical, useful, and often corrected existing theories.

- From Dedekind’s definition of an infinite set, “An infinite set is one which can be placed in one-to-one correspondence with a proper part of itself,” the ancient assertion of mathematics found in *Elements*, that the whole is greater than the part has no place in infinite quantities, it works only for finite quantities (see Euclid’s assertion (e)).
- For almost 2000 years Euclidean geometry stood indisputable as the mathematical model of space. However, Euclid himself was not comfortable with the parallel postulate (Euclid’s Postulate 5) as it was not self-evident in nature. He suspected it might not be necessary. For this postulate several leading mathematicians unsuccessfully tried to prove it from the other four axioms. However, many of them were successful only in proving its equivalent, e.g., Proclus: A line parallel to a given line has a constant distance from it; Hasan Ibn al-Haytham (965–1040, Iraq-Egypt): If a straight line moves so that one end always lies on a second straight line and so that it always remains perpendicular to that line, then the other end of the moving line will trace out a straight line parallel to the second line (his definition characterized parallel lines as lines always equidistant from one another and also introduced the concept of motion into geometry); Omar Khayyám (1048–1131, Iran): Two convergent straight lines intersect, and it is impossible for two convergent (approached one another) straight lines to diverge in the

direction in which they converge; Nasir al-Din al-Tusi (1201–1274, Iran): In 1250 the author wrote detailed critiques of the parallel postulate and later attempted to derive a proof by contradiction (reductio ad absurdum); Wallis: There exist similar (but not equal) triangles, whose angles are equal but whose sides are unequal; Saccheri: There exists at least one rectangle, a quadrangle whose angles are all right angles; John Playfair (1748–1819, Scotland): In a plane, given a line and a point not on it, at most one line parallel to the given line can be drawn through the point (known as Playfair's axiom); Legendre (his books on geometry [339] and number theory [340] are still remembered): A line perpendicular in one arm of an acute angle also intersects the other arm, and also the sum of the angles of a triangle is equal to two right angles, i.e., $180°$; Gauss: There exist triangles of arbitrarily large area; and Eric Wolfgang Weisstein (born 1969, USA): In a right-angled triangle, the square of the hypotenuse equals the sum of the squares of the other two sides (Pythagoras Theorem), and the list goes on.

Finally, the parallel postulate out of three choices, impossible, meaningless, and improperly posed, was avoided by the third possibility. In 1823, Lobachevsky and Janos Bolyai (1802–1860, Hungary-Romania) independently realized that entirely self-consistent "non-Euclidean geometry" could be created in which the parallel postulate did not hold, i.e., the sum of the angles of a triangle is more than or less than $180°$. Nasir al-Tusi considered the cases of non-Euclidean geometry now known as elliptical (also known as Riemannian geometry) and hyperbolic geometry (also known as Bolyai-Lobachevskian geometry), but he ruled out both of them; however, his son Sadr al-Din wrote a book on the subject in 1298, which was later published in Rome in 1594 and was studied by European geometers. Gauss had also discovered but suppressed the existence of non-Euclidean geometry. Upon hearing of Bolyai's results in a letter from Bolyai's father, Farkas Bolyai, Gauss stated: "If I commenced by saying that I am unable to praise this work, you would certainly be surprised for a moment. But I cannot say otherwise. To praise it would be to praise myself. Indeed the whole contents of the work, the path taken by your son, the results to which he is led, coincide almost entirely with my meditations, which have occupied my mind partly for the last thirty or thirty-five years." In the history of science, non-Euclidean geometry in the beginning was treated as mere curiosities invented by mathematicians; however, later it became an example of a scientific revolution, in which mathematicians and scientists changed the way they viewed and taught their subjects. Eugenio Beltrami (1835–1900, Italy) demonstrated that Euclid's geometry and the classical non-Euclidean geometries were either all logical admissible or all logically inadmissible. Non-Euclidean geometry turned out to be a very useful tool for Einstein in conceiving the general theory of relativity. A bibliography of non-Euclidean geometry up to 1911 lists about 4000 titles of books and research papers by about 1350 authors, and since 1911 the subject has expanded enormously. Much of the recent work has disseminated by physics, especially general relativity. Non-Euclidean geometries have taught mathematicians that it is useful to regard postulates as purely formal statements and not as facts based on experience. In conclusion, Euclidean geometry is an approximation to reality, just like Newtonian gravity.

The following well-known story related to non-Euclidean geometry is fascinating: "There were two friends. One was a mathematician, and the other, a politician. They were fast friends throughout even though they belonged to different professions. Once, the politician friend told his mathematician friend: 'We are birds of the same feather. We both talk nonsense. The day before yesterday, I came to meet you in your school. You were teaching in the class. I did not want to disturb you; therefore, I did not call you outside. But your voice being loud, I could hear what you were teaching. You were telling your students that the sum of the angles of a triangle is 180°. We did study something like this, but I do not remember exactly what it was. So I assumed that what you were telling in the class was correct. I came to meet you yesterday too and you were teaching. But this time I heard you telling your students that the sum of the angles of a triangle is greater than 180°. Now how can this be so? In one class, you tell that the sum is 180°, and in the other you tell that the same sum is greater than 180°. Mathematics being an exact subject, only one of these two statements can and must be true-or am I wrong? We politicians are famed for telling one thing on one platform and exactly the opposite on another platform. So I say we both are birds of the same feather, we both talk nonsense.' To this the mathematician replied, Yes, my friend, we both talk nonsense. But there is a difference I talk consistent nonsense, while you talk inconsistent nonsense. My statements though they may look contradictory have to be consistent with respect to the axioms with which I started the subject."

- William Rowan Hamilton in 1833 constructed a rigorous theory of complex numbers based on the idea that a complex number is an ordered pair of real numbers. This work was done independently of Gauss, who had already published the same ideas in 1831, but with emphasis on the interpretation of complex numbers as points in the complex plane. Rowan Hamilton subsequently tried to extend the algebraic structure of the complex numbers, which can be thought of as vectors in a plane, to vectors in three-dimensional space. This project failed; however, the year 1843 marked a triumph for Rowan Hamilton after several years of searching for a way to multiply *his* quaternions (Rowan Hamilton never learned that Gauss had discovered quaternions in 1819 but kept his ideas to himself). A *quaternion* is a kind of hypercomplex number since it represents a force acting in three dimensions rather than in the plane. Quaternions involve the symbols j, k, and i, the imaginary number. Rowan Hamilton could add and subtract quaternions, but he could not find any product that did not abolish the commutative property for multiplication, $ab = ba$. Rowan Hamilton was strolling along the Royal Canal in Dublin when he suddenly realized that a mathematical system could be consistent without obeying the commutative law. This happened at Brougham Bridge, and he carved on it the famous formula $i^2 = j^2 = k^2 = ijk = -1$. This insight freed algebra and paved the way for multiplication of matrices, which is noncommutative. After decades of his death Rowan Hamilton's quaternions found applications in mechanics, geometry, and mathematical physics. Rowan Hamilton's invention of quaternions initiated the study of several other different types of algebras in which one after another postulates of field were discarded (as too restrictive) or modified. As an example we define here group theory which studies algebraic objects called groups and considered a topic in abstract algebra. Starting from Lagrange (also Cauchy and Galöis) group theory has attracted attention of great mathematicians, and it has a wide range of applications in real world problems. A group denoted as $(G, \circ)$ is a set *G* together with a binary operation $\circ$, so that the following axioms hold:

Closure Property: If $a, b \in G$, then $a \circ b \in G$.

Associative property: If $a, b, c \in G$, then $(a \circ b) \circ c = a \circ (b \circ c)$.

Identity Element:

There exists an element $e \in G$ such that for all $a \in G,\ a \circ e = e \circ a = a.$

Inverse Element: For each $a \in G,$ there is a unique element $b \in G$ such that $a \circ b = b \circ a = e.$

If in addition to above axioms

Commutative Property: If $a, b \in G,$ then $a \circ b = b \circ a$

also holds, then $(G, \circ)$ is called abelian group, after Abel.

Clearly, all the above axioms of a group are subset of a field. It follows from the definition that the unit element in any group is unique, and the inverse of any given element in the group is also unique. A simple example of an abelian group is $(\mathcal{Z}, +)$; however $(\mathcal{Z}, \cdot)$ and $(\mathcal{N}, +)$ are not groups.

- *Heliocentric* and *Geocentric* are two feasible interpretations of the arrangement of our Universe, including the solar system. The heliocentric model contemplates that the Sun is the center and all planets circle around the Sun. The geocentric model asserts that the Earth is at the center of the Universe, and every other celestial body rotates around the Earth. The Aitareya Brahmana of the Rigveda, Chap. 3, Verse 44 (Rigveda before 3000 BC) states "The Sun does never set nor rise." Aryabhata claimed that the Earth rotates on its own axis, the Earth moves round the Sun, and the Moon rotates round the Earth; incredibly he believed that the orbits of the planets are ellipses. The Babylonians assumed that the Earth was the center of the Universe and everything revolved around it. Philolaus believed that the world is one and was created from the Central Fire, which is equidistant from top and bottom of the Universe. He supposed that a sphere of the fixed stars, the planets, the Sun, Moon, and Earth all revolved in uniformly round this Central Fire. He presumed the Sun to be a disk of glass that reflects the light of the Universe. Ponticus proposed that the Earth rotates on its axis, from west to east, once every 24 hours. Aristarchus influenced by a concept presented by Philolaus and asserted that the Earth and the other planets (Venus, Mercury, Mars, Jupiter, and Saturn) revolve around the Sun; however, his hypothesis was ignored in favor of Aristotle's strong support of the geocentric theory. Hipparchus thought to be the first to calculate a heliocentric system, but he abandoned his work because the calculations showed the orbits were not perfectly circular.

Ptolemy following the Babylonian and Greek view (lead by Hipparchus) that the Earth is the center of the Universe wondered why at certain times of the year it appears that Mars is moving backward, in *retrograde motion*. Ptolemy postulated a system wherein each body orbiting the Earth spins on a circle of its own, called an *epicycle*. Ptolemy required approximately 80 equations to describe, quite accurately, the locations of all the heavenly bodies of what we now call the solar system. He estimated that the Sun was at an average distance of 1210 Earth radii, while the radius of the sphere of the fixed stars was 20,000 times the radius of the Earth. However, he rejected the idea of a spinning Earth as absurd as he believed it would create huge winds. Ptolemy's geocentric model (superhuman effort of mathematical genius) was believed to be mandatory by the theology of the time. This cosmology was later adopted by the early Christian thinkers and perpetuated for a thousand years after the fall of Rome. This "geocentric theory" was consistent with the Christian belief that the "Son of God" was born at the center of the Universe. Accepting Ptolemaic system and the geocentric model, al-Battani (858–929, Turkey) showed that the distance between the Sun and the Earth varies. However, later several Islamic astronomers including al-Haytham criticized Ptolemaic model and marked it as unworkable. In fact, Nasir al-Tusi developed a heliocentric model. Oresme opposed the theory of a stationary Earth as proposed by Aristotle and advocated the motion of Earth; however, he later rejected his own ideas. Nilakanthan Somayaji (around 1444–1544, India) gave the correct formulation for the equation of the center of the planets and a heliocentric model of the solar system. Leonardo da Vinci wrote "Il sole non si move" (the Sun does not move). Finally, Copernicus challenged the axiom that the Earth is at the center of the solar system and developed a mathematical model in which the Sun is at the center and all planets move around the Sun. This reduced the number of equations describing the motion of the planets from Ptolemy's 80 down to 30. Although Copernicus realized that his theory implied an enormous increase in the size of the Universe, he declined to pronounce it infinite. His book *De revolutionibus orbium coelestium* appeared in 1543, after his death. The main features of his model are:

1. The center of the cosmos (this term was coined by Pythagoras) is the Sun.

2. Around the Sun, Mercury, Venus, Earth, and Moon, Mars, Jupiter, and Saturn move in order in their own orbits, and the stars are fixed in the sky.

3. The motion of the celestial bodies is uniform, eternal, and circular or compounded of several circles.

4. The Earth has three motions: daily rotation, annual revolution, and annual tilting off its axis.

5. The backward motion of the planets is as the Earth's motion.

6. The distance from the Earth to the Sun is small compared to the distance to the stars.

The Vatican ignored Copernicus' book as it only suggested that the mathematical model putting the Sun at the center makes more sense. He did not assert that this is the way things are.

The heliocentric model of Copernicus marked the beginning of the scientific revolution. In fact, as a consequence of this theory, Galileo by the end of 1609 turned his newly invented telescope on the night sky and made remarkable discoveries which convinced him of the truth of Copernicus' heliocentric theory. He had seen mountains on the Moon, rather than a perfect sphere as Aristotle had claimed. Galileo observed the Milky Way, which was previously believed to be a nebula, and found it to be a multitude of stars packed so densely that they appeared to be clouds from the Earth. He saw four small bodies orbiting Jupiter that he named Io, Europa, Callisto, and Ganymede, which later astronomers changed to *the Medicean stars*. In 1612, after a long series of observations, he gave accurate periods for *the Medicean stars*. He also observed the planet Neptune but did not realize that it was a planet and took no particular interest in it. In the same year, opposition arose to the heliocentric theory that Galileo supported. In 1614, from the pulpit of Santa Maria Novella, Father Tommaso Caccini (1574–1648, Italy) denounced Galileo's opinions on the motion of the Earth, judging them to be dangerous and close to heresy. Galileo went to Rome to defend himself against these accusations, but in 1616 Cardinal Roberto Bellarmino (1542–1621, Italy) personally handed Galileo an admonition forbidding him to advocate or teach Copernican astronomy. When three comets appeared, he became involved in a controversy regarding the nature of comets, arguing that they were close to the Earth and caused by optical refraction. This was not acceptable to the Jesuits, and they considered Galileo to be a dangerous opponent. In 1630, he returned to Rome to apply for a license to print the *Dialogue Concerning the Two Chief World Systems* (Ptolemaic and Copernican), which was published in Florence in 1632 with formal authorization from the Inquisition and the Roman Catholic Church. This book takes the form of a dialogue between Salviati, who argues for the Copernican system, and Simplicio, who is an Aristotelian philosopher. Soon after the publication of this book the Inquisition banned its sale and ordered Galileo to appear before the Holy Office in Rome under suspicion of heresy. Because of the illness he was unable to travel to Rome until 1633. Galileo's adherence to experimental results and their honest interpretation led to his rejection of blind allegiance to authority, both philosophical and religious. The Inquisition found Galileo guilty and forced him to recant (publicly withdraw) his support of Copernicus. They forbade further publication of his work and condemned him to life imprisonment, but because of his advanced age allowed him serve his term under house arrest, first with the Archbishop of Siena, and then at his villa in Arcetri outside of Florence. It was a sad end for so great a man to die condemned of heresy. In 1992, 350 years after Galileo's death, Pope John Paul II (1920–2005, Poland-Vatican City) gave an address on behalf of the Catholic Church in which he admitted that errors had been made by the theological advisors in the case of Galileo. He declared Galileo's case closed but did not admit that the Church itself had made mistakes.

Kepler avidly accepted Copernicus' heliocentric theory and went so far as to defend it in public while still a student. During 1617–1621, Kepler formulated his three laws by studying many years' worth of data about the motion of the planets (elliptical orbits) that had been gathered by Tycho Brahe (1546–1601, Sweden-Czechoslovakia). For this, his only strategy was numerous calculations.

1. Every planet describes an ellipse, with the Sun at one focus. The other focus is just a mathematical point at which nothing physical exists.

2. The radius vector from the Sun to a planet sweeps out equal areas in equal intervals of time.

3. The squares of the periodic times of planets are proportional to the cubes of the semimajor axes of the orbits of the planets.

These laws significantly increased accuracy in predicting the position of the planets and placed heliocentrism on a firm theoretical foundation. Descartes in his first cosmological treatise, written between 1629 and 1633 and titled *The World*, included a heliocentric model but later abandoned it in the light of Galileo's treatment. Finally, Newton conceived that Kepler's laws could be derived, using calculus, from his inverse square law of gravitational attraction. In fact, this is one of the main reasons that Newton developed the calculus. The first law made a profound change in the scientific outlook on nature. From ancient times circular motion had reigned supreme, but now the circle was replaced by the ellipse. The second law is an early example of the infinitesimal calculus. The period of the Earth is 1 year; therefore according to the third law, a planet situated twice as far from the Sun would take nearly 3 years to complete its orbit.

- Giambattista Benedetti (1530–1590, Italy) in 1553 challenged the Aristotelian axiom that heavier objects fall faster than lighter objects. Galileo enunciated the correct mathematical law for the acceleration of falling bodies, which states that the distance traveled starting from rest is proportional to the square of time, and the law of time, which states that velocity is proportional to time. There is an apocryphal story that Galileo dropped a cannonball and a musket ball simultaneously from the leaning tower of Pisa to demonstrate that bodies fall at the same rate. He also concluded that objects retain their velocity unless a force acts upon them, contradicting the Aristotelian hypothesis that objects naturally slow down and stop unless a force acts upon them. Although Mo Tzu (around 470–391 BC, China), al-Haytham, and Jean Buridan (around 1300–1358, France) had proposed the same idea centuries earlier, Galileo was the first to express it mathematically. Galileo's *Principle of Inertia* states that "A body moving on a level surface will continue in the same direction at constant speed unless disturbed." This principle was incorporated into Newton's laws of motion as the first law.
- Einstein challenged the axiom that events at different places can be simultaneous: In physics, the relativity of simultaneity is the concept that distant simultaneity—whether two spatially separated events occur at the same time—is not absolute but depends on the observer's reference frame, which gave birth to special theory of relativity.
- Brouwer challenged the axiom that Aristotle's logical law of the excluded middle "There cannot be an intermediate between

contradictions, but of one subject we must either affirm or deny any one predicate" is universally applicable.

Axioms play a key role not only in mathematics but also in other sciences, notably in theoretical physics. In particular, the work of Newton is essentially based on Euclid's postulates, augmented by a postulate on the nonrelation of space-time (space is distinct from body and that time passes uniformly without regard to whether anything happens in the world) and the physics taking place in it at any moment. In 1905, Newton's postulates were replaced by those of Einstein's special relativity (the laws of physics are the same and can be stated in their simplest form in all inertial frames of reference, and the speed of light c is a constant, independent of the relative motion of the source) and later on by those of general relativity (the laws of physics have the same form in all inertial reference frames; light propagates through empty space with a definite speed c independent of the speed of the observer [or source], and in the limit of low speeds, the gravity formalism should agree with Newtonian gravity). Physicists are trying to rewrite the axioms of quantum theory from scratch in an effort to understand what it all means.

3.11 What Is Logic?

In the simplest possible terms logic is a science that deals with the rules and processes used in sound thinking and reasoning. It is not an invention of men but a timeless gift to mankind from the immortal gods. In one form or another, this belief has persisted for well over 2000 years. The main constituent of logic is a premise(s) which is a statement in an argument that provides reason or support for the conclusion. Although, several different types of logic are known and used, the following four are the main. *Informal Logic* which according to Ralph Henry Johnson (born 1940, Canada-USA) is a branch of logic whose task it is to develop non-formal standards, criteria, procedures for the analysis, interpretation, evaluation, criticism, and construction of argumentation in everyday discourse. For example, Premises: Isabella saw a black cat on her way to work. At work, Isabella got fired. Conclusion: Black cats bring bad luck. *Formal Logic* was created by Aristotle and organized by his ancient commentators under the title *Organon*; it deals with abstract propositions, statements (or assertively used sentences), and deductive arguments (drawing truth of a conclusion from something known or assumed). For example, Premises: Socrates is a man. All men are mortal. Conclusion: Therefore, Socrates is mortal. *Symbolic Logic* was advanced by John Venn (1834–1923, England) to develop and represent logical principles by means of a formalized system consisting of primitive symbols and combinations of these symbols, axioms, and rules of inference.

In symbolic logic a single letter is used to denote a statement. For example, letter *p* may be used for the statement eleven is an even number and written as $p : 11$ is an even number. A statement is said to have *truth value T* or *F* according as the statement is true or false. For example, the truth value of $p : 1 + 2 + \cdots + 10 = 55$ is $T,$ whereas for $p : 1^2 + 2^2 + 3^2 = 15$ is $F.$ The knowledge of truth value of a statement enables us to replace it by some other "equivalent" statement. From given statements new statements can be produced by using the following standard logical connectives: If *p* is a statement, then its negation $\sim p$ is the statement not $p,$ e.g., if $p :$ 7 is even number, then $\sim p :$ 7 is not an even number, or 7 is an odd number; if from a statement *p* another statement *q* follows, we say *p* implies *q* and write $p \Rightarrow q,$ e.g., if *n* is an even integer, then $n + 1$ is an odd integer; the statements *p* and *q* are denoted as $p \wedge q$ and called the conjunction of the statements *p* and $q,$ e.g., $p : 4$ is positive and $q : -7$ is negative; the statement *p* or *q* is denoted as $p \vee q$ and is called the disjunction of the statements *p* and $q,$ e.g., if $p :$ Scott is a member of the financial committee and $q :$ Scott is a member of the executive committee, then $p \vee q :$ Scott is a member of the financial committee or of the executive committee; and two statements *p* and *q* are said to be *equivalent* if one implies the other, i.e., $(p \Rightarrow q) \wedge (q \Rightarrow p),$ and we denote this as $p \Leftrightarrow q,$ e.g., *ABC* is an equilateral triangle $\Leftrightarrow$ $AB = BC = CA.$ Propositions that involve the phrases such as if and only if (iff), is equivalent to, or the necessary and sufficient condition are of the type $p \Leftrightarrow q.$ *Mathematical Logic* was systematized by George Boole and De Morgan; it is the study of formal logic within mathematics. For example, $a > b \vee b > c \Rightarrow a > c.$ Mathematical logic and symbolic logic are often used interchangeably. According to Herman Weyl, " logic is the hygiene which the mathematician practices to keep his ideas healthy and strong." Logic also has numerous applications in computer science; in fact, it is used in the design of computer circuits, the construction of computer programs, and the verification of the correctness of programs, and the list continues.

Each type of logic could include deductive reasoning, inductive reasoning (a method of drawing conclusions by going from the specific to the general, e.g., prediction/forecasting, or behavior), or both. With deductive reasoning always correct logical arguments are achieved, whereas inductive reasoning may or may not provide a correct conclusion. The following well-known interesting puzzles illustrate what kind of thinking needed to find their solution:

- An archaeologist found himself trapped inside a chamber of a royal Egyptian tomb. The chamber had one single door with a keyhole, but there was no sign of any key. Inside the chamber, he found three boxes marked I, II, and III with the following inscriptions on their respective covers:

  I. "The key of the door is in this box."

  II. "The key of the door is not in this box."

  III. "The key of the door is not in Box I."

  And on the wall behind the boxes were inscribed the following statements: "At most one of the three statements on the boxes is true. If you open the wrong box or more than one box, the chamber will immediately collapse and bury you alive." Which box should the archaeologist open to get out of the chamber?

  Denote the statements on Boxes I, II, and III by p_1, p_2, and p_3, respectively. Since p_1 and p_3 are negations of each other, one of them must be true. But at most one of the three given statements is true. It follows that p_2 must be false. Therefore, the key is in Box II.
- Four cards are laid in front of you, each of which is explained has a letter on one side and a number on the other. The sides that you see read E, 2, 5, and F. Your task is to turn over only those cards that could decisively prove the truth or falsity of the following rule: "If there is an E on one side, the number on the other side must be a 5." Which ones do you turn over? Clearly, the E should be turned over, since if the other side is not a 5, the rule is untrue. And the only other card that should be flipped is the 2, since an E on the other side would again disprove the rule. Turning over the 5 or the F does not help, since anything on the other side would be consistent with the rule but not prove it to be true.

- This is a puzzle about a man condemned to be hanged. The man was sentenced on Saturday. “The hanging will take place at noon,” said the judge to the prisoner, “on one of the seven days of the next week. But you will not know which day it is until you are informed on the morning of the day of the hanging.” The judge was known to be a man who always kept his word. The prisoner, accompanied by his lawyer, went back to his cell. As soon as the two men were alone the lawyer broke into a grin. “Don’t you see”? he exclaimed. “The judge’s sentence cannot possibly be carried out.” “I don’t see,” said the prisoner. “Let me explain. They obviously can’t hang you next Saturday. Saturday is the last day of the week. On Friday noon you would still be alive and you would know with absolute certainty that the hanging would be on Saturday. You know this before you were told so on Saturday morning. That would violate the judge’s decree.” “True,” said the prisoner. “Saturday, then is positively ruled out,” continued the lawyer. “This leaves Friday as the last day they can hang you. But they can’t hang you on Friday because by Thursday afternoon only two days would remain: Friday and Saturday. Since Saturday is not a possible day, the hanging would have to be on Friday. Your knowledge of that fact would violate the judge’s decree again. So Friday is out. This leaves Thursday as the last possible day. But Thursday is out because if you are alive Wednesday afternoon, you’ll know that Thursday is to be the day.” “I get it,” said the prisoner, who was beginning to feel much better. “In exactly the same way I can rule out Wednesday, Tuesday and Monday. That leaves only tomorrow. But they can’t hang me tomorrow because I know it today!” The prisoner is thus convinced, by what appears to be unimpeachable logic, that he cannot be hanged without contradicting the conditions specified in his sentence. Then, on Thursday morning, the hangman arrives. Clearly, he did not expect him. What is more surprising, the judge’s decree is now seen to be perfectly correct. The question is what is wrong with the reasoning provided by the lawyer? The answer is lawyer believes that there is no alternative to prisoner’s being executed by Friday at the latest, which means that he cannot be executed on Friday.

There are many versions of this puzzle. During World War II, the Swedish Broadcasting Company made the following announcement on the radio: **A civil-defense exercise will be held this week. In order to make sure that the civil-defense units are properly prepared, no one will know in advance on what day the exercise will take place**. Suppose that it was made on a Monday morning. Then the exercise must take place sometime before the following Monday. It cannot happen on Sunday, for by then people will know that it has to take place on Sunday. Since it is to be unexpected, Sunday is ruled out. Also, Saturday is ruled out. Since the exercise cannot take place on Sunday, it also cannot happen on Saturday, for you would know it in advance. If it has not happened during the week so far and if it cannot happen on Sunday, then people would know that it was going to take place on Saturday. So, it cannot. Friday is no good either. With both Saturday and Sunday out, if it has not happened by Friday, everyone will expect it. Similar reasoning applies to Thursday, Wednesday, Tuesday, and Monday itself. Therefore, the civil defense exercise cannot happen at all. But on Wednesday morning, the air-raid sirens wailed and the exercise took place anyway. Irrefutable logic has been refuted by reality.

One of the clearest versions of this puzzle came from Martin Gardner (1914–2010, USA). A loving husband tells his wife that she will receive an unexpected gift for her birthday. It will be a gold watch. Now the wife uses logic. Her husband would not lie to her. Since he has said the gift would be unexpected, it will be unexpected. But she now expects a gold watch. Therefore, it cannot be a gold watch. But, of course, it is. And it is unexpected, for she had used logic to show that it could not be a gold watch.

There is a common feature of the above three puzzles, i.e., two people are required. One (A) says that something will happen and that it will be unexpected. The other person (B) reasons that these conditions are contradictory. Hence the event cannot happen. But it happens anyway.

Following Greek traditions set by Thales and Pythagoras and epitomized by Euclid in his *Elements* from the last 2500 years mathematics continues to be a formal logical deductive system in which hypotheses (axioms and assumptions) lead to conclusions and has been titled as *axiomatic-deductive science* or *axiomatic mathematics*. An important aspect of this passage is that the content of mathematics is no longer defined by quantity or space, rather it could be about anything as long as it exhibits the pattern of assumption-deduction-conclusion. It is vigorously accepted that comparing to empirical axiomatic mathematics is more precise, conceptual, subtle, systematic, and provides mechanism for analyzing deeper extensions. However, whatever ancient Rishis have offered empirically has turned out genuine with axiomatic mathematics also. For example, Hindus discovered what is known as the *precession of the Equinoxes* and their calculation such an occurrence takes place every

25827 years, our modern science after labors of hundreds of years has simply proved them to be correct. How or by what means they were able to arrive at such a calculation has never been discovered. Similarly, the judgment of the Hindus as to the length of what is now known as the cycle of years of the planets has been handed down to us from the most remote ages. Modern science also has proved it to be correct. We should also keep in mind seriously Morris Kline's quote "Logic does not dictate the contents of mathematics, the uses determine the logical structure." However, mathematics students become more skeptical in their reasoning—they begin to think more critically.

In 1931, Gödel answered the fundamental questions: "Can I prove that math is consistent?" and "If I have a true statement, can I prove that it's true?" in his *Incompleteness Theorems*. These theorems state that any logical system either has contradictions or statements that cannot be proved, yet powerful enough to serve as a basis for all of the mathematics that we do. His theorems are considered one of the greatest intellectual achievements of modern times. Most importantly his work showed that mathematics is not a finished object, as had been believed. Furthermore, a computer can never be programmed to answer all mathematical questions. After these theorems philosophers of mathematics lost interest in their three different schools of thought. The present attitude of most mathematicians is best purported by Jean Alexandre Eugéne Dieudonné (1906–1992, France). "In everyday life, we speak as Platonists, treating the objects of our study as real things that exist independently of human thought. If challenged on this, however, we retreat to some sort of formalism, arguing that in fact we are just pushing symbols around without making any metaphysical claims. Most of all, however, we want to do mathematics rather than argue about what it actually is. We're content to leave that to the philosophers."

3.12 What Are Theorem, Lemma, and Corollary?

A *theorem* is a general mathematical statement not self-evident but established/proved by logical deduction, an undoubted truth established by means of accepted truths. In addition to the word theorem, other similar terms are proposition, result, observation, and fact, the choice often depending on the significance or degree of difficulty in its proof. The word theorem has come from Middle French théoréme, from Late Latin the ōrēma, and from Ancient Greek (especially Eudoxus) theorema, “speculation, proposition to be proved.” Thales suggested many early mathematical results and is typically credited with beginning the tradition of a rigorous, logical proof before the general acceptance of a theorem, e.g., he demonstrated that the angles at the bases of any isosceles triangle are equal. However, it was Eudoxus who began the grand tradition of organizing mathematics into theorems. The first major collection of mathematical theorems was developed by Euclid in his *Elements*, which set candidly standards of future mathematics. A mathematician develops a new mathematical statement based on a simple example(s), experience, observation, and/or intuition (empirical) that seems to be true. This original mathematical statement is only given the status of a theorem when it is proven true by logical deduction. Understanding a theorem is not like reading novels or history; one needs to think slowly about every argument and normally reread the same material later several times. A *lemma* is a proven mathematical result, possibly quite important, that is useful in establishing the truth of some other results. The German word for lemma is “hilfsatz,” whose English translation is “helping theorem,” so a lemma is usually considered as a helpful result. Often, complicated proofs are easier to understand when they are established using a series of lemmas and putting them together in a logical way. A *corollary* is a mathematical result that can be deduced from, and is thereby a consequence of, a theorem that has been proved.

3.13 What Is a Mathematical Proof?

In logical terms, a theorem consists of some propositions $H_1, H_2, \cdots, H_n$ called *hypotheses* and a proposition *C* called its *conclusion*. A theorem is true provided $H \equiv H_1 \wedge H_2 \wedge \cdots \wedge H_n \Rightarrow C.$ A *formal proof* of a theorem consists of a sequence of propositions, ending with the conclusion $C,$ that are regarded *valid*. To be valid a proposition may be one of the hypotheses and may be derived or inferred from the propositions known earlier. A formal proof with a valid sequence of propositions is called a *valid proof*. Even if one of the propositions is invalid, then the argument is called a *fallacy*. A great proof not only establishes the truth of the matter, but it also enlightens, it is the heart of mathematics, almost a holy concept, it is what mathematics tick, and the progress of mathematics depends on proving theorems. It is one of the highest intellectual achievements of humankind. Those mathematical theorems whose validity can be demonstrated by only nontrivial proofs are highly regarded for revealing valuable relationships. Thus akin to saying that a writer's job is to construct sentences, a composer's job is to assemble notes, and an artist's job is to draw and color, the job of a mathematician is to establish the truth of a mathematical statement by providing a proof which remains unequivocally true from there on. A mathematical proof is essential to understand, acquire, contribute, and communicate mathematical knowledge to readers. However, in a proof we leave out steps and make small leaps and leave details to the reader—because we want to get the ideas across in the most concise and elegant and effective manner possible. Mathematical proofs are written by humans, studied by humans, judged by humans, and appreciated by humans. While mathematicians take mathematical proofs seriously, not only the layperson but even great scientists fail to appreciate proofs. When young Harish Chandra Mehrotra (1923–1983, India-USA) said to his supervisor Dirac that he was troubled as he could not find proof even though he was sure his answer was correct, Dirac said "I don't care about proofs, I want to know the truth!" It has to be pronounced that while in sciences, and certainly in everyday life, we tend to accept the truth of a principle when experiments repeatedly confirm it, in mathematics any number of cases, though perhaps suggestive, are by no means a proof. We also note that in geometry SSS, the letters stand for "Side-Side-Side," and it means if you have two triangles and you can show that the three pairs of corresponding sides are congruent, then the two triangles are congruent. This is a postulate, not a theorem, meaning that it cannot be proved, but it appears to be true, so everybody accepts it. We begin with the following quotations:

- A proof is a matter not of external discourse but of meditation within the soul. (Aristotle)
- Faith is different from proof; the latter is human, the former is a Gift from God. (Pascal)
- Mathematical proofs, like diamonds, are hard as well as clear, and will be touched with nothing but strict reasoning. (John Locke)
- Gauss used to boast that an architect did not leave up the scaffolding so that people could see how he constructed a building. Just so, a mathematician does not leave clues as to how he constructed or found a proof. (Gauss)
- A proof is a sequence of formulae each of which is either an axiom or follows from earlier formulae by a rule of inference. (Hilbert)

- Mathematicians can never put onto paper the complete process of reasoning, but rather must settle for such an abstract of the proof as is sufficient to convince a properly instructed mind. (Bertrand Russell)
- Proof is an idol before which the mathematician tortures himself. (Eddington)
- Every scientific theory is a system of sentences which are accepted as true and which may be called *laws of asserted statements* or, for short, simply statements. In mathematics, these statements follow one another in a definite order according to certain principles, and they are, as a rule, accompanied by considerations intended to establish their validity. Considerations of this kind are referred to as *Proofs*, and the statements established by them are called *Theorems*. (Alfred Tarski, 1902–1983, Poland-USA)
- Rigor is to the mathematician what morality is to man. (André Weil, 1906–1998, France)
- An elegantly executed proof is a poem in all but the form in which it is written. (Morris Kline)
- A proof tells us where to concentrate our doubts. (Morris Kline)
- The sequence for the understanding mathematics may be: Intuition, trial, error, speculation, conjecture, proof. The mixture and the sequence of these events differ in different domains, but there is general agreement that the product is rigorous proof-which we know and can recognize, without the formal advice of the logicians. (Leslie Saunders Mac Lane, 1909–2005, USA)
- Physics has provided mathematics with many fine suggestions and new initiatives, but mathematics does not need to copy the style of experimental physics. Mathematics rests on proof–and proof is eternal. (Mac Lane)
- Only professional mathematicians learn anything from proofs. Other people learn from explanations. (Ralph Philip Boas Jr. 1912–1992, USA)
- Proofs aren't there to convince you that something is true—they are there to show you why it is true. (Andrew Mattei Gleason, 1921–2008, USA)
- Proof is the glue that holds mathematics together. (Michael Atiyah)
- The overwhelming majority of research papers in mathematics is concerned not with proving, but with reproving; not with axiomatizing, but with reaxiomatizing; not with inventing, but with unifying and streamlining; in short, with what Thomas Kuhn (1922–1996, USA) calls "tidying up." (Rota)

- A good proof is one that makes us wiser. It is just like the solution to Pell's equation

$$\sqrt{1 + 1141 \times (30,693,385,322,765,657,197,397,208)^2}$$
$$= 1,036,782,394,157,\ 223,963,237,125,215$$

it simply does not increase our understanding of the equation $Nx^2 + 1 = y^2$. (Yuri Ivanovich Manin, 1937–2023, Russia)
- Modern mathematics is nearly characterized by the use of rigorous proofs. This practice, the result of literally thousands of years of refinement, has brought to mathematics a clarity and reliability unmatched by any other science. But it also makes mathematics slow and difficult: it is arguably the most disciplined of human intellectual activities. Groups and individuals within the mathematics community have from time to time tried being less compulsive about details arguments. The results have been mixed, and they have occasionally been disastrous. (Arthur Michael Jaffe, born 1937, USA, and Frank Stringfellow Quinn, III, born 1946, Cuba-USA)
- There are no theorems in analysis–only proofs. (John Brady Garnett, born 1940, USA)
- ...it is impossible to write out a very long and complicated argument without error; so, is such a "proof" really a proof? (Aschbacher)
- Indeed, every mathematician knows that a proof has not been 'understood' if one has done nothing more than verifying step by step the correctness of the deductions of which it is composed, and has not tried to gain a clear insight into the ideas which have led to the construction of this particular chain of deductions in preference to every other one. (Bourbaki Group in an article entitled "The architecture of mathematics")
- Proofs remain important in mathematics because they are our signpost for what we can believe in, and what we can depend on. They are timeless and rigid and dependable. They are what hold the subject together, and what make it one of the glories of human thought. (Krantz [317])
- Mathematical proof no longer had a valid role in modern thinking. (John Horgan, born 1953, USA, see [268], also see amendment by Krantz [315])

Polya in his book of 1945 *How to Solve It* suggested the following brilliant four steps when solving a mathematical problem: 1. First, you have to understand the problem. 2. After understanding, make a plan. 3. Carry out the plan. 4. Look back on your work. How could it be better?

Now we shall discuss frequently used methods of proofs and illustrate how these work in practice. It is important to notice that only one type of proof does not work for all results, so out of several different types of known proofs one has to be clever to choose the right method which provides the *cleanest* result.

- **Geometric Proofs**. It is likely that the idea of demonstrating a conclusion first arose in connection with geometry, which originally meant the same as “land measurement.” Like all other types of proofs, geometric proofs are the demonstration of a mathematical statement, true or false, using logic to arrive at a conclusion. In a geometrical proof we draw a figure(s) to clearly visualize what are the given statements and what has to be proved. Mostly we learn geometric proofs in high school, and this later forms the basis of understanding higher mathematics. The following two examples illustrate most of the aspects of geometric proofs.
- We shall prove that the angle subtended by a diameter of a circle at any point in the circumference is a right angle. This result is known as Thales Theorem; however, it was known earlier to Babylonians and Indians. It is a particular case of the inscribed angle theorem mentioned and proved in Euclid’s *Elements* (III:31). To visualize this result, we first draw Fig. 3.2.

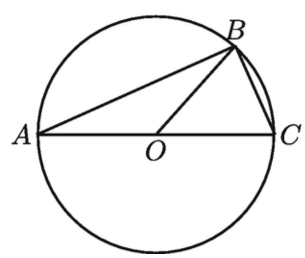

Fig. 3.2 Thales theorem

We are given a circle whose center is at O, diameter is AC, and an arbitrary (but fixed) point *B* is on its circumference. We need to show that $\angle ABC = 90^\circ$. Clearly, $OA = OB = OC$, and hence $\triangle OBA$ and $\triangle OBC$ are isosceles triangles. Thus, $\angle OBC = \angle OCB$ and $\angle OBA = \angle OAB$. Now since the sum of the angles of a triangle is equal to 180°, in the $\triangle ABC$, we have

$\angle OAB + \angle ABC + \angle OCB = \angle OBA + \angle ABC + \angle OBC = 2\angle ABC = 180°$, and hence $\angle ABC = 90°$.

- We shall show that the internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the corresponding sides containing the angle. For this, first we draw Fig. 3.3.

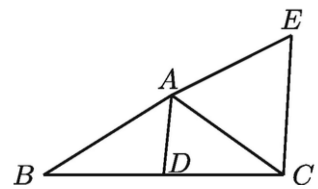

Fig. 3.3 Bisector of an angle

We are given triangle *ABC* and *AD* the internal bisector of $\angle BAC$ which meets *BC* at D. We need to prove that $BD/DC = AB/AC$. We draw a line $CE||DA$ ($||$ stands for parallel) which meets the extended line *BA* at E. Since $CE||DA$ and the line *AC* is the transversal, it follows that $\angle DAC = \angle ACE$. We also have $\angle BAD = \angle AEC$ and $\angle BAD = \angle DAC$. Combining these three relations, we find that $\angle ACE = \angle AEC$. Thus, in view of sides of opposite to equal angles are equal, it follows that $AE = AC$. Now since in $\triangle BCE,\ CE||DA$, it follows that $BD/DC = BA/AE = AB/AC$.

- **Empirical and Heuristic Proofs**. We provide the following examples of empirical and heuristic proofs due to Newton and Euler, which were later demonstrated regressively and became jewels of mathematics.
- In 1665, *Newton developed binomial expansion* (2.1) *empirically*. For this, we recall Pascal's triangle (see Agarwal and Sen [14] for its origin; it has various applications in mathematics and interesting hidden secrets which have caught the interest of numerous mathematicians) which is based on binomial coefficients (2.3).

Going a step further to Wallis, Newton calculated the area of the sequence of curves $y = (1 - x^2)^n,\ n = 0, 1, 2, \cdots$ from 0 to an arbitrary value *x* (compared to the fixed value 1), i.e.,

$$\begin{aligned}
\int_0^x (1-x^2)^0 dx &= x \\
\int_0^x (1-x^2)^1 dx &= x - \tfrac{1}{3}x^3 \\
\int_0^x (1-x^2)^2 dx &= x - \tfrac{2}{3}x^3 + \tfrac{1}{5}x^5 \\
\int_0^x (1-x^2)^3 \qquad &= x - \tfrac{3}{3}x^3 + \tfrac{3}{5}x^5 - \tfrac{1}{7}x^7 \\
\int_0^x (1-x^2)^4 dx &= x - \tfrac{4}{3}x^3 + \tfrac{6}{5}x^5 - \tfrac{4}{7}x^7 + \tfrac{1}{9}x^9
\end{aligned}$$

and then realized that the coefficients of the various powers of *x* satisfy

$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$\cdots$	times
1	1	1	1	1	$\cdots$	x
0	1	2	3	4	$\cdots$	$-x^3/3$
0	0	1	3	6	$\cdots$	$x^5/5$
0	0	0	1	4	$\cdots$	$-x^7/7$
0	0	0	0	1	$\cdots$	$x^9/9,$

which leads to Pascal's triangle (Fig. 3.4). Newton was originally interested in computing the area of a circle, and for this he needed the values in the column corresponding to $n = 1/2$ (see Sect. 8.13). To find these values, he simply interpolated the binomial coefficients (2.3) and the entries in the column $n = 1/2$ he found

$$\binom{\frac{1}{2}}{0} = 1, \quad \binom{\frac{1}{2}}{1} = \frac{1}{2}, \quad \binom{\frac{1}{2}}{2} = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2} = -\frac{1}{8},$$
$$\binom{\frac{1}{2}}{3} = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{6} = \frac{1}{16}, \cdots$$

1

1 1

1 2 **1**

1 **3** **3** 1

1 4 **6** 4 1

1 5 **10** **10** 5 1

1 6 **15** 20 **15** 6 1

1 7 **21** 35 35 **21** 7 1

1 6 **28** 56 70 56 **28** 8 1

Fig. 3.4 Pascal's triangle

Newton could now fill in the table for columns corresponding to $n = k/2$ for any positive integer k. He recognized further that in the original table (Pascal triangle) each entry was the sum of the number to its left and the one above that. If, in his table with extra columns interpolated, he amended that rule marginally to read that each entry should be the sum of the number two columns to its left and the one above that, the new entries found by the binomial coefficient's formula (2.3) satisfied that rule as well. Not only did this give Newton confidence that his interpolation was correct but also convinced him to add columns to the left corresponding to negative values of n. The sum rule made it clear to him that in the column $n = -1$ the first number had to be 1, while the next number had to be -1, since $1 + (-1) = 0$ and 0 was the second entry in the column $n = 0$. Similarly, the third number in the $n = -1$ column was 1, the fourth -1, and so on. Of course, the binomial coefficient formula gave these same alternating values of 1 and -1 as well. Newton's interpolated table for calculating the area under $y = (1 - x^2)^n$ from 0 to *x* was then the following:

$n=-1$	$n=-\frac{1}{2}$	$n=0$	$n=\frac{1}{2}$	$n=1$	$n=\frac{3}{2}$	$n=2$	$n=\frac{5}{2}$	$\cdots$	times
1	1	1	1	1	1	1	1	$\cdots$	x
-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	$\cdots$	$-\frac{x^3}{3}$
1	$\frac{3}{8}$	0	$-\frac{1}{8}$	0	$\frac{3}{8}$	1	$\frac{15}{8}$	$\cdots$	$\frac{x^5}{5}$
-1	$-\frac{5}{16}$	0	$\frac{3}{48}$	0	$-\frac{1}{16}$	0	$\frac{5}{16}$	$\cdots$	$-\frac{x^7}{7}$
1	$\frac{35}{128}$	0	$-\frac{15}{384}$	0	$\frac{3}{128}$	0	$-\frac{5}{128}$	$\cdots$	$\frac{x^9}{9}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$

Newton soon realized that, first, there was no necessity of dealing only with fractions with denominator $2.$ The multiplicative rule for $\binom{n}{k}$ would apply for any fractional value of $n,$ positive or negative. Second, he realized that the terms $(1-x^2)^n$ for *n* integral "could be interpolated in the same way as the areas generated by them; and that noting else was required for this purpose but to omit the denominators $1, 3, 5, 7,$ etc., which are in the terms expressing the areas" (and, of course, reduce the corresponding powers by 1). Finally, there was no reason to limit himself to binomials of the form $1-x^2.$ With appropriate modification, the coefficients of the power series for $(a+bx)^n$ for any value of *n* could be calculated using the formula for the binomial coefficients. Thus, Newton had discovered, although hardly proved, the general binomial theorem. He was, however, completely convinced of its correctness because it provided him in several cases with the same answer that he had derived in other ways. The convergence of Newton's general binomial theorem was proved only in 1826 by Abel.

- In 1740, Philippe Naudé (1684–1745, France) observed the following relation $D(n) = O(n),$ where $D(n)$ is the number of ways of writing *n* as the sum of distinct whole numbers, and $O(n)$ is the number of ways of writing *n* as the sum of odd numbers (not necessarily distinct). To better understand what Philippe wanted to prove, we consider $n = 8,$ so that $D(8) : 8, 7+1, 6+2, 5+3, 5+2+1, 4+3+1,$ i.e., $D(8) = 6$ and hence there are exactly 6 ways of writing 8 as the sum of distinct whole numbers, and
$O(8) : 7+1, 5+3, 5+1+1+1, 3+3+1+1, 3+1+1+1+1+1, 1+1+1+1+1+1+1+1,$
i.e., $O(8) = 6$, and hence there are exactly six ways of writing 8 as the sum of odd numbers. Philippe wrote a letter to Euler inquiring about this problem, and within days, Euler sent back a *heuristic proof* of $D(n) = O(n)$ for all natural numbers $n.$ First Euler wrote

$$\begin{aligned} P(x) &= \prod_{n=1}^{\infty}(1+x^n) = (1+x)(1+x^2)(1+x^3)\cdots \\ &= 1+x+x^2+(x^3+x^{(2+1)})+(x^4+x^{(3+1)})+(x^5+x^{(4+1)}+x^{(3+2)})\cdots \end{aligned}$$

and realized that

$$P(x) = 1 + \sum_{n=1}^{\infty} D(n)x^n.$$

Now recalling the geometric series

$$S_n \;=\; a + ar + ar^2 + \cdots + ar^n \;=\; \frac{a(1 - r^{n+1})}{1 - r}, \tag{3.1}$$

which converges to $a/(1-r)$ as $n \to \infty$ provided $|r| < 1$ (this was shown by Madhava), i.e.,

$$S_\infty \;=\; a + ar + ar^2 + \cdots \;=\; \frac{a}{1-r}, \quad |r| < 1. \tag{3.2}$$

Next Euler introduce $Q(x)$ where

$$\begin{aligned} Q(x) \;&=\; \frac{1}{(1-x)} \cdot \frac{1}{(1-x^3)} \cdot \frac{1}{(1-x^5)} \cdots \\ &=\; (1 + x + x^2 + x^3 + \cdots)(1 + x^3 + x^6 + x^9 + \cdots) \\ &\quad\; (1 + x^5 + x^{10} + x^{15} + \cdots) \cdots \end{aligned}$$

and noticed that

$$\begin{aligned} Q(x) \;&=\; (1 + x^1 + x^{(1+1)} + x^{(1+1+1)} + \cdots)(1 + x^3 + x^{(3+3)} + x^{(3+3+3)} + \cdots) \\ &\qquad \times (1 + x^5 + x^{(5+5)} + x^{(5+5+5)} + \cdots) \cdots, \end{aligned}$$

which is the same as

$$\begin{aligned} Q(x) \;&=\; 1 + x^1 + x^{(1+1)} + (x^3 + x^{(1+1+1)}) + (x^{(3+1)} + x^{(1+1+1+1)}) \\ &\qquad + (x^5 + x^{(3+1+1)} + x^{(1+1+1+1+1)}) + \cdots. \end{aligned}$$

Euler then recognized that

$$Q(x) \;=\; 1 + \sum_{n=1}^{\infty} O(n)x^n.$$

Finally, to prove that $D(n) = O(n)$ for all $n,$ we need to only show that $P(x) = Q(x),$ i.e.,

$$P(x) \;=\; Q(x) \quad \to \quad 1 + \sum_{n=1}^{\infty} D(n)x^n \;=\; 1 + \sum_{n=1}^{\infty} O(n)x^n.$$

For this, we multiply and divide $P(x) = (1+x)(1+x^2)(1+x^3)\cdots$ by $(1-x)(1-x^2)(1-x^3)\cdots$, to obtain

$$P(x) \;=\; \frac{(1+x)(1-x)(1+x^2)(1-x^2)(1+x^3)(1-x^3)\cdots}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)\cdots},$$

which is the same as

$$P(x) \;=\; \frac{(1-x^2)(1-x^4)(1-x^6)\cdots}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)\cdots},$$

and hence

$$P(x) \;=\; \frac{1}{(1-x)(1-x^3)(1-x^5)\cdots} \;=\; Q(x).$$

- The following example is also of *heuristic proof appeared in a* 1747 *memoir of Euler*. It offered a “most extraordinary law of the number concerning the sum of their divisors.” For ease of exposition, we shall adopt today’s notation $\sigma(n) =$ sum of all divisors of *n* (including 1 and n). For instance, $\sigma(6) = 1+2+3+6 = 12$ and $\sigma(n) = 1+n$ iff *n* is a prime. Euler devised a table of $\sigma(n)$ for *n* in the range $1 \le n \le 99$. It looks pretty erratic.

n	0	1	2	3	4	5	6	7	8	9
0		1	3	4	7	6	12	8	15	13
10	18	12	28	14	24	24	31	18	39	20
20	42	32	36	24	60	31	42	40	56	30
30	72	32	63	48	54	48	91	38	60	56
40	90	42	96	44	84	78	72	48	124	57
50	93	72	98	54	120	72	120	80	90	60
60	168	62	96	104	127	84	144	68	126	96
70	144	72	195	74	114	124	140	96	168	80
80	186	121	126	84	224	108	132	120	180	90
90	234	112	168	128	144	120	252	98	171	156

(The table is self-explanatory. For instance, the entry in the row labeled 40 and column labeled 7 is $\sigma(47) = 48$.)

He then gave the rule, the recurrence relation

$$\begin{aligned}\sigma(n) \;=\;& \sigma(n-1)+\sigma(n-2)-\sigma(n-5)-\sigma(n-7)\\ &+\sigma(n-12)+\sigma(n-15)-\sigma(n-22)-\sigma(n-26)\\ &+\sigma(n-35)+\sigma(n-40)-\sigma(n-51)-\sigma(n-57)+\cdots,\end{aligned} \tag{3.3}$$

where (i) the signs $+$ and $-$ each arise in succession, (ii) the sequences continue as long as the number under the sign σ is nonnegative (so sequence stops somewhere), (iii) if $\sigma(0)$ turns up, it is to be interpreted as $n,$ (iv) the sequence $1, 2, 5, 7, 12, 15, 22, 26, 35, 40, 51, 57, \cdots$ follows the pattern in which differences between consecutive terms are $1, 3, 2, 5, 3, 7, 4, 9, 5, 11, 6, \cdots$. As illustration, Euler computed a few examples to convince the reader of the validity of his rule. He then said "The examples that I have just developed will undoubtedly dispel any qualms which we might have had about the truth of my formula." He continued "I confess I did not hit on this discovery by mere chance, but another proposition opened the path to this beautiful property—another proposition of the same nature which must be accepted as true although I am unable to prove it." What Euler referred to is his investigation of the infinite product

$$\prod_{n=1}^{\infty}(1-x^n) \;=\; (1-x)(1-x^2)(1-x^3)\cdots$$

in 1741. This investigation was motivated by the problem of Naudé. By actually computing the product, Euler observed that the pattern came out as

$$1-x-x^2+x^5+x^7-x^{12}-x^{15}+x^{22}+x^{26}-x^{35}-x^{40}+x^{51}+\cdots.$$

Euler noticed that alternate exponents formed two sequences

$$\begin{aligned}&1, 5, 12, 22, 35, 51, \cdots\\ &2, 7, 15, 26, 40, 57, \cdots.\end{aligned}$$

The first sequence is that of pentagonal numbers of the general form $n(3n-1)/2$ (see Sect. 7.5). The second sequences is obtained from the first by adding, respectively, $1, 2, 3, 4, \cdots$, i.e., with *n*th term being $n(3n+1)/2.$ Thus, Euler observed that the following remarkable formula might hold:

$$\prod_{n=1}^{\infty}(1-x^n) = 1+\sum_{n=1}^{\infty}(-1)^n x^{n(3n+1)/2}+\sum_{n=1}^{\infty}(-1)^n x^{n(3n-1)/2}$$

$$= \sum_{n=-\infty}^{\infty}(-1)^n x^{n(3n+1)/2}.$$

According to Euler,"this is quite certain, although I cannot prove it." However, he did prove it 10 years later. He could not possibly guess that both series and product were part of the theory of elliptic modular functions developed by Jacobi 80 years later! Euler in his 1747 memoir further said "As we have thus discovered that those two infinite expressions are equal even though it has not been possible to demonstrate their equality, all the conclusions which may be deduced from it will be of the same nature, that is, true but not demonstrated. Or, if one of these conclusions could be demonstrated, one could reciprocally obtain a clue to the demonstration of that equation; and it was with this purpose in mind that I maneuvered those two expressions in many ways."

Euler applied calculus to "explain" his proposed rule (3.3). He assumed that the observation about the equality of the series and product was correct, i.e.,

$$s = (1-x)(1-x^2)(1-x^3)\cdots = 1-x-x^2+x^5+x^7-x^{12}-x^{15}+\cdots.$$

Then

$$\ln s = \ln(1-x)+\ln(1-x^2)+\ln(1-x^3)+\cdots$$

from the product, and hence

$$\frac{1}{s}\frac{ds}{dx} = -\frac{1}{1-x}-\frac{2x}{1-x^2}-\frac{3x^2}{1-x^3}-\cdots,$$

which is the same as

$$-\frac{x}{s}\frac{ds}{dx} = \frac{x}{1-x}+\frac{2x^2}{1-x^2}+\frac{3x^2}{1-x^3}+\cdots. \tag{3.4}$$

Also,

$$\frac{ds}{dx} = -1-2x+5x^4+7x^6-12x^{11}-15x^{14}+\cdots$$

from the series,

$$-\frac{x}{s}\frac{ds}{dx} = \frac{x + 2x^2 - 5x^5 - 7x^7 + 12x^{12} + 15x^{15} - \cdots}{1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \cdots}. \qquad (3.5)$$

Putting $t = -(x/s)ds/dx$, he obtained from (3.4) by expanding each term as a geometric series

$$\begin{array}{lllllllllllllllllll} t & = & x & + & x^2 & + & x^3 & + & x^4 & + & x^5 & + & x^6 & + & x^7 & + & x^8 & + & \cdots \\ & & & + & 2x^2 & & & + & 2x^4 & & & + & 2x^6 & & & + & 2x^8 & + & \cdots \\ & & & & & + & 3x^3 & & & & & + & 3x^6 & & & & & + & \cdots \\ & & & & & & & + & 4x^4 & & & & & & & + & 4x^8 & + & \cdots \\ & & & & & & & & & + & 5x^5 & & & & & & & + & \cdots \\ & & & & & & & & & & & + & 6x^6 & & & & & + & \cdots \\ & & & & & & & & & & & & & & & & & + & \cdots \end{array}$$

Each power of *x* arises as many times as its exponent has divisors, and each divisor arises as a coefficient of the same power of x. (For example, terms involving x^6 yield $x^6 + 2x^6 + 3x^6 + 6x^6$ with $1, 2, 3, 6$ being all the divisors of 6.) Hence, $t = \sigma(1)x + \sigma(2)x^2 + \sigma(3)x^3 + \sigma(4)x^4 + \cdots$. From (3.5) he obtained

$$t(1-x-x^2+x^5 + x^7-x^{12} - x^{15} + \cdots) - x - 2x^2 + 5x^5 + 7x^7-12x^{12}-15x^{15} + \cdots = 0.$$

Substituting the new expression for t, he obtained finally

$$\begin{array}{lllllllllllllll} 0 & = & \sigma(1)x & + & \sigma(2)x^2 & + & \sigma(3)x^3 & + & \sigma(4)x^4 & + & \sigma(5)x^5 & + & \sigma(6)x^6 & + & \cdots \\ & & -x & - & \sigma(1)x^2 & - & \sigma(2)x^3 & - & \sigma(3)x^4 & - & \sigma(4)x^5 & - & \sigma(5)x^6 & - & \cdots \\ & & & - & 2x^2 & - & \sigma(1)x^3 & - & \sigma(2)x^4 & - & \sigma(3)x^5 & - & \sigma(4)x^6 & - & \cdots \\ & & & & & & & & & + & 5x^5 & + & \sigma(1)x^6 & + & \cdots . \end{array}$$

The coefficient of x^n is

$$\sigma(n) - \sigma(n-1) - \sigma(n-2) + \sigma(n-5) + \sigma(n-7) - \sigma(n-12) - \sigma(n-15) + \cdots ,$$

continued as long as the number under the sign σ is nonnegative, and if $\sigma(0)$ arises, substituted by n, which is the same as (3.3). He then said "This reasoning, although still very far from perfect demonstration, will certainly lifts some doubts about the most extraordinary law that I explained here."

- The following example shows reliance on formal manipulation and careless handling of limits and convergence of Euler in 1748. To find the familiar series expansion of $e,$ he begins with a number $a > 1$ and first writes $a^w = 1 + kw,$ where *w* is taken to be an infinitely small number and *k* is a constant depending only on $a.$ For any real number $x,$ he puts $j = x/w,$ so that

$$a^x \;=\; a^{jw} \;=\; (1+kw)^j \;=\; \left(1+\frac{kx}{j}\right)^j,$$

which can be expanded by Newton's binomial expansion (2.1) into

$$a^x \;=\; 1+\frac{j}{1!}\left(\frac{kx}{j}\right)+\frac{j(j-1)}{2!}\left(\frac{kx}{j}\right)^2+\frac{j(j-1)(j-2)}{3!}\left(\frac{kx}{j}\right)^3+\cdots.$$

Because *w* is infinitely small, *j* will be infinitely large; this allowed Euler of passing to the limit, to assume that

$$1=\frac{j-1}{j}=\frac{j-2}{j}=\frac{j-3}{j}=\cdots.$$

Thus, he concluded that

$$a^x \;=\; 1+\frac{kx}{1!}+\frac{(kx)^2}{2!}+\frac{(kx)^3}{3!}+\cdots,$$

so that, when $x = 1,$

$$a \;=\; 1+\frac{k}{1!}+\frac{k^2}{2!}+\frac{k^3}{3!}+\cdots.$$

Because the number *e* is the value of *a* when $k = 1$ (Euler identified *k* with $\lim_{w\to 0}\frac{a^w-1}{w},$ which is being equal to $\log_e a$), it follows that

$$e \;=\; \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \;=\; 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots. \qquad (3.6)$$

To Euler this series well suited for computing $e,$ in fact, in 1748, he obtain a numerical value to 23 decimal places:

$$e \;=\; 2.71828182845904523536028\cdots.$$

He also used formal manipulations to derive the infinite series expansions of the sine and cosine functions. Euler began with the relation

(2.7) with $\pm$ signs, i.e., $(\cos z \pm i \sin z)^n = \cos nz \pm i \sin nz.$ The two choices of signs allowed him to solve for $\cos nz$ as

$$\cos nz = (\cos z)^n - \frac{n(n-1)}{1\cdot 2}(\cos z)^{n-2}(\sin z)^2 + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}(\cos z)^{n-4}(\sin z)^4 - \cdots .$$

Euler then assumed *z* to be an infinitely small number and *n* to be an infinitely large one, so that $\cos z = 1,\ \ \sin z = z$, and

$$n(n-1) = n^2, \quad n(n-1)(n-2)(n-3) = n^4, \cdots .$$

Substituting and simplifying, this gave him the series for $\cos nz$ as

$$\cos nz = 1 - \frac{n^2 z^2}{2!} + \frac{n^4 z^4}{4!} - \cdots .$$

Finally, in view of the sizes of *z* and $n,$ he concluded that *nz* will be a finite quantity, say $nz = x.$ This transforms the series to the well-known form (2.16). Reasoning along similar lines, he presented the infinite series for $\sin x$ as (2.17).

- **Principle of Mathematical Induction.** According to Polya, "induction is the process of discovering general laws by the observation and combination of particular instances. It is used in all of science, even in mathematics. Mathematical induction is used in mathematics alone to prove theorems of a certain kind." In fact, mathematical induction is not a method of discovery but a technique of proving rigorously what has already been discovered. It is very likely that al-Karkhi (953–1029, Iraq) was the first to use mathematical induction to validate his discoveries of the general binomial theorem and Pascal's triangle. This was followed by Francesco Maurolico (1494–1575, Italy), who used the method of mathematical induction to establish a variety of properties of integers. In 1838 De Morgan presented the first clear explanation of the method of mathematical induction, a term that he coined. Mathematical induction is used to establish the truth of an infinite list of propositions which depend on natural numbers. This method can be described as follows: All the propositions $p(m), p(m+1), \cdots,$ where $m \geq 0$ is an integer, are true provided (i) $p(m)$ is true, and (ii) $p(n)$ implies $p(n+1)$ for all $n \geq m \geq 1.$ The first step (i) is called the *basis of induction*, whereas the second step (ii) is called the *inductive step*. The principle of mathematical induction is also called the "Domino Principle." In fact, we push the first domino (the initial step). This falling domino then pushes the next one, which in turn pushes the next one, which in turn pushes the next one, "ad infinitum" (the induction step). We shall illustrate this method in the following examples.

- For all integers $n \geq 3$, the following inequality holds:

$$n \; > \; \left(1+\frac{1}{n}\right)^n . \tag{3.7}$$

Let $p(n) : n > (1+1/n)^n \,, \; n \geq 3.$ Clearly, $p(3)$ is true (basis of induction). Now suppose that $p(n)$ is true for a given natural number $n \geq 3.$ Then, from (3.7), we have

$$n+1 \; > \; \left(1+\frac{1}{n}\right)^n + 1 \; > \; \left(1+\frac{1}{n+1}\right)^n + 1 \tag{3.8}$$

and

$$n+1 \; > \; n \; > \; \left(1+\frac{1}{n+1}\right)^n ,$$

which implies

$$1 \; > \; \frac{1}{n+1}\left(1+\frac{1}{n+1}\right)^n . \tag{3.9}$$

Combining (3.8) and (3.9), we get

$$n+1 \; > \; \left(1+\frac{1}{n+1}\right)^n + \frac{1}{n+1}\left(1+\frac{1}{n+1}\right)^n \; = \; \left(1+\frac{1}{n+1}\right)^{n+1} .$$

Hence, $p(n+1)$ holds (inductive step) whenever $p(n)$ holds. By the principle of mathematical induction, we conclude that $p(n)$ is true for all $n \geq 3.$

- We shall show that the *binomial theorem* (2.4) holds. For this, we let $p(n) : (a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k, \; n \geq 1.$ Clearly, $p(1)$ is true (basis of induction). To show the inductive step, we need Pascal's identity in the form

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}, \quad 0 < k \le r, \tag{3.10}$$

which follows immediately from (2.3). Now we have

$$\begin{aligned}
(a+b)^{n+1} &= (a+b)\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k = a\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k + b\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k \\
&= \sum_{k=0}^{n}\binom{n}{k}a^{n+1-k}b^k + \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^{k+1} \\
&= a^{n+1} + \sum_{k=1}^{n}\left[\binom{n}{k} + \binom{n}{k-1}\right]a^{n+1-k}b^k + b^{n+1} \\
&= \textstyle\sum_{k=0}^{n+1}\binom{n+1}{k}a^{n+1-k}b^k \; = \; p(n+1).
\end{aligned}$$

- In relation to harmonic series, the number defined by

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n},$$

where $n \ge 1$, is called a *Harmonic Number*, and it arises frequently in the analysis of computer algorithms. We shall show that for all $n \ge 1$, $p(n): \;\; \sum_{k=1}^{n} H_k = (n+1)H_n - n$. Clearly, $p(1)$ is $H_1 = 2H_1 - 1$. Since $H_1 = 1, \; p(1)$ (the basis of induction) is true. For the inductive step, we assume that for $n \ge 1, \; p(n)$ is true, that is, $H_1 + H_2 + \cdots + H_n = (n+1)H_n - n.$ Now $p(n+1)$ has the left side $(H_1 + H_2 + \cdots + H_n) + H_{n+1}$, and because $p(n)$ is true, in view of the definition of H_n, we have

$$\begin{aligned}
(n+1)H_n - n + H_{n+1} &= (n+1)\left(H_{n+1} - \frac{1}{n+1}\right) - n + H_{n+1} \\
&= (n+1)H_{n+1} - 1 - n + H_{n+1} \\
&= (n+2)H_{n+1} - (n+1).
\end{aligned}$$

Hence $p(n+1)$ is true whenever $p(n)$ is true.

- We shall prove that well-ordering principle follows by mathematical induction (in fact both are equivalent). Precisely we shall show that for any fixed integer n_0, any non-empty set of integers $\ge n_0$ has a least element. For this, let *S* denote the set of integers $\geq n_0$ with no least member and let $p(n)$: Every element of *S* is greater than or equal to n_0. By definition of $S,\ p(n_0)$ is true. Suppose that $p(n)$ is true for some $n \geq n_0$. We have to prove that $p(n+1)$ is true. Assume the contrary. If $p(n+1)$ is false, then some number $\leq (n+1)$ is in S. But $p(n)$ is true. So no number $\leq n$ is in S. Hence $(n+1)$, the only integer $\leq (n+1)$ and not $\leq n$, will be in *S* and will be the least element of S. This is impossible because $n \geq n_0$. Thus if $p(n)$ is true, so is $p(n+1)$. By induction, $p(n)$ is true for all n, and hence *S* is empty. So there cannot be a set *S* with integers $\geq n_0$ having no least element. This establishes the truth of the well-ordering principle.

- **Direct Proofs**. A direct proof is a way of showing the truth or falsehood of a given statement by a straightforward combination of established facts, usually axioms, existing lemmas, and theorems, without making any further assumptions. For example, we shall show that for every natural number *n* with two or more digits, the product of the digits of *n* is less than n. For this, in view of (2.5), *n* can be written as $n = \sum_{p=0}^{m} a_{m-p} \cdot 10^{m-p},\ m \leq n-1$. Thus, the product of the digits of *n* is $a_m(a_{m-1}\cdots a_2a_1a_0)$. Now since each $0 \leq a_i \leq 9$, it follows that $a_m(a_{m-1}\cdots a_2a_1a_0) < a_m10^m \leq n$. As another example, we shall demonstrate that there exists a power of 7 that has exactly 100 digits. For this, first we note that $7^k,\ k = 0, 1, 2, \cdots$ leads to the sequence of digits $S = \{1, 1, 2, 3, 4, 5, 6, 6, \cdots\}$. Now since, $7^{200} = 49^{100} > 10^{101}$ it follows that 7^{200} has more than 100 digits. As a final step, we need to show that the sequence *S* does not skip any natural number. We assume that 7^k has *s* digits; then since $7^k < 7^{k+1} < 10 \times 7^k$ and 10×7^k has $s+1$ digits, 7^{k+1} has either *s* digits or $s+1$ digits. In conclusion, there exists at least one *k* so that 7^k has exactly 100 digits.

Given two integers $a \neq 0$ and b, we say *a divides b* or *a* is a *divisor* of *b* (written as $a|b$) if there is an integer *c* such that $b = ac$. If *a* does not divide b, then we write $a \not| b$. For example, $13|182, -7|63$, but $8 \not| 60$. The divisibility of numbers is encountered constantly in practice and also plays an important role in some questions of mathematical analysis. In the following result we establish some basic properties of divisibility.

Theorem 3.1 *Let $a, b,$ and c be integers with $a \neq 0$.*

(i). *If $a|b$, then $a|bc$.*

(ii). *If $a|b$ and $b|c$, where $b \neq 0$, then $a|c$.*

(iii). *If $a|b$ and $a|c$, then $a|(bx+cy)$ for all integers x and y.*

(iv). *If $a|b$ and $b|a$, where $b \neq 0$, then $a = \pm b$, i.e., $|a| = |b|$.*

(v). *If $a|b$, where $b \neq 0$, then $|a| \leq |b|$.*

(vi). *If $a|b$, then $(b/a)\,|b$. The integer b/a is called the conjugate divisor of a.*

Proof For part (i), let $a|b$. Then there exists an integer *q* such that $b = aq$. Therefore, $bc = a(qc)$. Since *qc* is an integer, $a|bc$.

For part (ii), let $a|b$ and $b|c$. Then there exist integers q_1 and q_2 such that $b = aq_1$ and $c = bq_2$. Consequently, $c = bq_2 = (aq_1)q_2 = a(q_1q_2)$. Since q_1q_2 is an integer, $a|c$.

For part (iii), let $a|b$ and $a|c$. Then there exist integers q_1 and q_2 such that $b = aq_1$ and $c = aq_2$. Hence, for integers *x* and y, $bx + cy = (aq_1)x + (aq_2)y = a(q_1x + q_2y)$. Since $q_1x + q_2y$ is an integer, $a|(bx+cy)$.

For part (iv), let $a|b$ and $b|a$. Then it follows that $b = aq_1$ and $a = bq_2$ for some integers q_1 and q_2. Therefore, $a = bq_2 = (aq_1)q_2 = a(q_1q_2)$. Dividing by a, we obtain $1 = q_1q_2$. Hence $q_1 = q_2 = 1$ or $q_1 = q_2 = -1$. Therefore $a = \pm b$.

For part (v), let $a|b$. Then it follows that $b = aq$ for some integer q. Furthermore, $q \neq 0$ since $b \neq 0$. So $|q| \geq 1$. Hence, $|b| = |aq| = |a||q| \geq |a| \cdot 1 = |a|$.

For part (vi), it suffices to note that $b = (b/a)a$. ■

Theorem 3.2 (Euclidean Division) *If a and b are integers such that $a \geq b > 0$, then there exist unique integers q (called quotient) and r (called remainder) such that $a = qb + r$ and $0 \leq r < b$.*

Proof If $b|a,$ then we have $q = a/b$ and $r = 0.$ If $a > b$ and $b \not| a,$ we let *S* to be the set of natural numbers of the form $s_k = a - kb, \ k = 1, 2, \cdots .$ Clearly, $0 < a - b = s_1 \in S,$ and hence *S* is non-empty. Thus by the well-ordering principle, *S* has a least element, say, $s_q = a - qb = r.$ If $r = b,$ then $a = (q+1)b,$ but this implies that $b|a.$ If $r > b,$ then $s_{q+1} = a - (q+1)b > 0,$ which implies that $s_{q+1} \in S,$ and $s_{q+1} < s_q.$ However, this contradicts our definition of $s_q.$ Therefore, it is necessary that $0 < r < b.$ To show uniqueness, suppose that $a = q_1 b + r_1 = q_2 b + r_2,$ where $r_1 > r_2.$ But this implies that $(q_1 - q_2)b = (r_1 - r_2),$ i.e., $b|(r_1 - r_2),$ and hence $b \leq r_1 - r_2 < r_1,$ which is impossible. Thus, *r* is unique, and this in turn implies that *q* is also unique. ■

The proof of Theorem 3.2 can be easily modified to prove the following corollary.

Corollary 3.1 *If a and b are integers such that* $b \neq 0,$ *then there exist unique integers q and r such that* $a = qb + r$ *and* $0 \leq |r| < |b|.$

As examples, for $a = 182, b = 13$ we have $182 = 14 \cdot 13 + 0,$ and hence $q = 14, r = 0;$ for $a = 54, b = 12$ we have $54 = 4 \cdot 12 + 6,$ and hence $q = 4, r = 6;$ and for $a = 13, b = 182$ we have $13 = 1 \cdot 182 + (-169),$ and hence $q = 1, r = -169.$

We say that *d* is the *greatest common divisor* of integers *a* and *b* (written as $d = \gcd(a, b)$) iff (i) *d* is a common divisor of *a* and $b,$ i.e., $d|a$ and $d|b,$ and (ii) *d* is the greatest such divisor, i.e., if $c|a$ and $c|b,$ then $c \leq d.$ For example, $\gcd(12, 54) = 6, \ \ \gcd(13, 51) = 1.$ It is clear that if *a* and *b* are not both zero, then the set of common divisors of *a* and *b* is a set of integers that is bounded above by the largest of $|a|$ and $|b|.$ Hence, by the well-ordering principle for the integers, the set has a largest element, so the $d = \gcd(a, b)$ exists, unique, and $\gcd(a/d, b/d) = 1.$ Furthermore, if *a* and *b* are nonzero integers, then from the definition it follows that

$$\gcd(a, b) = \gcd(-a, b) = \gcd(a, -b) = \gcd(-a, -b) = \gcd(|a|, |b|), \quad (3.11)$$

also for a nonzero integer $a, \ \ \gcd(a, 0) = |a|.$ Hence, we can assume that both *a* and *b* are positive. The greatest common divisor of arbitrary number of integers $a_1, a_2, \cdots, a_n$ denoted as $\gcd(a_1, a_2, \cdots, a_n)$ is defined analogously.

Lemma 3.1 *If $a = bq + r$, then $\gcd(a, b) = \gcd(b, r)$.*

Proof Let $d = \gcd(a, b)$. Since $d|a$ and $d|b$, it follows from $a = bq + r$ that $d|r$. Thus *d* is a common divisor of *b* and r. Now let *c* be any common divisor of *b* and r, i.e., $c|b$ and $c|r$, but then from $a = bq + r$ it follows that $c|a$. Thus *c* is also a common divisor of *a* and b, and hence $c \leq d$. Now from the definition it follows that $d = \gcd(b, r)$. ■

Theorem 3.3 (Euclidean Algorithm) *If a and b are integers such that $a \geq b > 0$ and*

$$\begin{array}{lcll} a & = & q_1 b + r_1 & 0 < r_1 < b \\ b & = & q_2 r_1 + r_2 & 0 < r_2 < r_1 \\ r_1 & = & q_3 r_2 + r_3 & 0 < r_3 < r_2 \\ & \vdots & & \\ r_{n-2} & = & q_n r_{n-1} + r_n & 0 < r_n < r_{n-1}, \end{array}$$

then for n large enough, say $n = N$, we have

$$r_{N-1} = q_{N+1} r_N + 0$$

and $\gcd(a, b) = r_N$.

Proof The strictly decreasing sequence $b > r_1 > r_2 > \cdots \geq 0$ cannot contain more than *b* integers. This means that one of the remainders must be zero. Suppose that it is r_{N+1}; then $r_{N-1} = q_{N+1} r_N$. Now from a successive application of Lemma 3.1, we have

$$\gcd(a, b) = \gcd(b, r_1) = \gcd(r_1, r_2) = \cdots = \gcd(r_{N-1}, r_N) = \gcd(r_N, 0) = r_N.$$

■

For integers *a* and b, an integer of the form $ax + by$, where $x, y \in Z$, is called a *linear combination* of *a* and b. The following result is due to Étienne Bézout (1730–1783, France).

Theorem 3.4 *If a and b are integers and* $d = \gcd(a, b)$, *then there exist integers x and y such that* $d = ax + by$. *In fact,*
$\gcd(a, b) = \min\{ax + by : x, y \in \mathcal{Z} \text{ and } ax + by > 0\}$.

Proof To prove this result, we shall use Euclidean algorithm backward. In fact, successively, we have

$$\begin{aligned} d = \gcd(a, b) = r_N &= r_{N-2} - q_N r_{N-1} \\ &= r_{N-2} - q_N(r_{N-3} - q_{N-1} r_{N-2}) \\ &= (q_N q_{N-1} + 1) r_{N-2} - q_N r_{N-3}. \end{aligned}$$

This represents *d* as a linear combination of r_{N-2} and r_{N-3}. Next, we eliminate r_{N-2} by using $r_{N-4} = q_{N-2} r_{N-3} + r_{N-2}$, to get

$$d = (\text{integer}) r_{N-3} + (\text{integer}) r_{N-4}.$$

We continue eliminating the remainders $r_{N-3}, r_{N-4}, \cdots, r_1$ until we get integers *x* and *y* such that $d = ax + by$. ■

In Theorem 3.4 the integers *x* and *y* are not unique. Indeed, if $\gcd(a, b) = d$ and $d = ax + by$, then $d = a(x + b) + b(y - a)$ is as well. Furthermore, from Theorem 3.4 the following corollaries follow.

Corollary 3.2 *Let a and b be two integers, not both* 0. *Then* $d = \gcd(a, b)$, *and if c is any common divisor of a and* b, *then* $c|d$.

Proof Since $d = \gcd(a, b)$, there exist integers x_0 and y_0 such that $d = ax_0 + by_0$. Since $c|a$ and $c|b$, it follows by Theorem 3.1(iii) that *c* divides $ax_0 + by_0 = d$. Therefore, $c|d$. ■

Corollary 3.3 *Let a and b be integers, not both* 0. *Then a and b are relatively prime (coprime), i.e.,* $\gcd(a, b) = 1$ *iff there exist integers x and y such that* $1 = ax + by$.

- As examples to Corollary 3.3, we have $\gcd(2n + 1, 3n + 2) = 1$ because $(-3)(2n + 1) + 2(3n + 2) = 1$, and for every positive integer *n* and every integer k, $\gcd(nk + (n - 1), nk + (2n - 1)) = 1$ because $[nk + (n - 1)][(n - 1)k + (2n - 3)] + [nk + (2n - 1)][-(n - 1)k - (n - 2)] = 1$.

- Let $\{a_n\}$ be a sequence of positive integers for which $a_0 = 1,\ a_{2n+1} = a_n,\ n \geq 0$, and $a_{2n+2} = a_n + a_{n+1},\ n \geq 0$. Then a_n and a_{n+1} are relatively prime for every nonnegative integer n. For this, first we note that if $\gcd(a,b) = d$, then $\gcd(a+b,b) = \gcd(a,a+b) = d$. Indeed, if $\gcd(a+b,b) = m > d$, then since $m|b$ and $m|a+b$, it follows that $m|(a+b)-b$, or $m|a$, which is a contradiction since $\gcd(a,b) = d$. Now using mathematical induction, we shall show that $\gcd(a_n, a_{n+1}) = 1,\ n \geq 0$. Since $a_0 = 1, a_1 = 1, a_2 = 2, a_3 = 1$, the initial step is clear. For the inductive step, we assume that $\gcd(a_k, a_{k+1}) = 1$, then $\gcd(a_{2k+1}, a_{2k+2}) = \gcd(a_k, a_k + a_{k+1}) = 1$, and if $\gcd(a_{k-1}, a_k) = 1$, then $\gcd(a_{2k}, a_{2k+1}) = \gcd(a_{k-1} + a_k, a_k) = 1$. For this sequence it also follows that $a_{2^n-1} = 1$ for all *n* by repeatedly applying $a_{2n+1} = a_n$.
- Erdös asked Lajos Pósa (born 1947, Hungary) to prove that if you choose $n+1$ integers from 1 to $2n$, at least two of them are relatively prime. In an instant, Pósa realized that when you choose more than half of the numbers from 1 to $2n$, two of them must be consecutive, and consecutive numbers are always relatively prime.
- As examples of the Euclidean algorithm, we have

$$\begin{array}{rcl} 45692 &=& 46\cdot 987 + 290 \\ 987 &=& 3\cdot 290 + 117 \\ 290 &=& 2\cdot 117 + 56 \\ 117 &=& 2\cdot 56 + 5 \\ 56 &=& 11\cdot 5 + 1 \\ 5 &=& 5\cdot 1 + 0 \end{array} \qquad \begin{array}{rcl} 1234569 &=& 501\cdot 2463 + 606 \\ 2463 &=& 4\cdot 606 + 39 \\ 606 &=& 15\cdot 39 + 21 \\ 39 &=& 1\cdot 21 + 18 \\ 21 &=& 1\cdot 18 + 3 \\ 18 &=& 6\cdot 3 + 0, \end{array}$$

  and hence $\gcd(45692, 987) = 1$ and $\gcd(1234569, 2463) = 3$. From the procedure given in Theorem 3.4, it also follows that

$$1 = 45692\cdot 194 - -987\cdot 8981 \quad \text{and} \quad 3 = 1234569\cdot 126 - -2463\cdot 63157.$$

We say that *m* is the *least common multiple* of positive integers *a* and *b* (written as $m = \text{lcm}(a,b)$) if it is the smallest positive integer that is divisible by both *a* and b. For example, $\text{lcm}(12,54) = 108,\ \text{lcm}(13,51) = 663$. The relation between gcd and lcm of two positive integers *a* and *b* is $ab = \gcd(a,b) \times \text{lcm}(a,b)$. Indeed, clearly if $\gcd(a,b) = 1$, then $\text{lcm}(a,b) = ab$, and since if $d = \gcd(a,b)$, then $\gcd(a/d, b/d) = 1$, and hence $\text{lcm}(a/d, b/d) = a/d \cdot b/d$, which in view of $\text{lcm}(a/d, b/d) = \text{lcm}(a, b)/d$ impels that $\text{lcm}(a, b) = ab/d$.

Gauss in his 1798 influential treatise *Disquisitiones Arithmeticae* (systematizing the then-completely unsystematized theory of numbers and making out a path for others to follow gratefully), which was first published in 1801, introduced the concept of *congruence*, which later became a powerful technique. He explained that he was induced to adopt the symbol $\equiv$ because of the close analogy with algebraic equality. According to Gauss, "if a number *n* measures the difference between two numbers *a* and b, then *a* and *b* are said to be congruent with respect to n; if not, incongruent." In the form of a mathematical definition, this becomes the following: Let *n* be a fixed positive integer. Two integers *a* and *b* are said to be congruent modulo n, symbolized by $a \equiv b(\text{mod } n)$ if *n* divides the difference $a - b$, that is, provided that $a - b = kn$ for some integer k. This relation is read as "*a* is congruent to *b* modulo *n*." For example, $5 \equiv 1(\text{mod } 2),\ 84 \equiv 0(\text{mod } 6),\ 173 \equiv 8(\text{mod } 11),\ \ -51 \equiv 5(\text{mod } 7)$. It is to be noted that any two integers are congruent modulo 1, whereas two integers are congruent modulo 2 when they are both even or both odd. It is also clear that if $a = nq + r$ and $0 \leq r < n$ for some integers q, r and n, then $a \equiv r(\text{mod } n)$. Thus, if $n \in \mathcal{N}$, then each integer *a* is congruent, modulo n, to precisely one of the integers $0, 1, 2, \cdots, n-1$. Here *r* is called a *least positive residue* of *a* mod n; if $-n/2 < r \leq n/2$, then *r* is called a *least or absolutely least residue* of *a* mod n. Hence, in particular, if *n* is an odd integer, then $n = 4k + 1$ or $n = 4k + 3$ (primes of the form $4k + 1$ have been named as *Pythagorean primes*, whereas of the form $4k + 3$ as *Gaussian primes*) for some integer k. The following result is a useful characterization of congruence modulo *n* in terms of remainders on division by n.

Theorem 3.5 *For a and b arbitrary integers,* $a \equiv b(\text{mod } n)$ *iff a and b leave the same nonnegative remainder when divided by* n.

Proof If $a \equiv b(\text{mod } n)$, then $a = b + hn$ for some integer h. On division by n, we find that *b* leaves a certain remainder r, i.e., $b = kn + r$, where $0 \leq r < n$. Therefore, $a = b + hn = (kn + r) + hn = (k + h)n + r$, which shows that *a* has the same remainder as b. Conversely, suppose we can write $a = pn + r$ and $b = qn + r$, with the same remainder

$r(0 \le r < n)$. Then, $a - b = (pn + r) - (qn + r) = (p - q)n$, whence $n|(a - b)$, i.e., $a \equiv b(\mathrm{mod}\ n)$. ■

Congruence can be viewed as a generalized form of equality, in the sense that its behavior with respect to addition and multiplication is reminiscent of ordinary equality. Some of the elementary properties of equality that carry over to congruences appear in the next theorem.

Theorem 3.6 *Let* $n > 0$ *be fixed and* a, b, c, d *be arbitrary integers. Then the following properties hold:*

(1). $a \equiv a(\mathrm{mod}\ n)$ *(reflexive property).*

(2). *If* $a \equiv b(\mathrm{mod}\ n)$, *then* $b \equiv a(\mathrm{mod}\ n)$ *(symmetric property).*

(3). *If* $a \equiv b(\mathrm{mod}\ n)$ *and* $b \equiv c(\mathrm{mod}\ n)$, *then* $a \equiv c(\mathrm{mod}\ n)$ *(transitive property).*

(4). *If* $a \equiv b(\mathrm{mod}\ n)$ *and* $c \equiv d(\mathrm{mod}\ n)$, *then* $a + c \equiv b + d(\mathrm{mod}\ n)$ *and* $ac \equiv bd(\mathrm{mod}\ n)$.

(5). *If* $a \equiv b(\mathrm{mod}\ n)$, *then* $a + c \equiv b + c(\mathrm{mod}\ n)$ *and* $ac \equiv bc(\mathrm{mod}\ n)$.

(6). *If* $a \equiv b(\mathrm{mod}\ n)$, *then* $a^k \equiv b^k(\mathrm{mod}\ n)$ *for any positive integer* k.

Proof For any integer a, we have $a - a = 0 \cdot n$, so that $a \equiv a(\mathrm{mod}\ n)$, and hence property (1) follows. Now if $a \equiv b(\mathrm{mod}\ n)$, then $a - b = kn$ for some integer k. Hence, $b - a = -(kn) = (-k)n$, and since $-k$ is an integer, this yields property (2). To show property (3), note that there exist integers *p* and *q* such that $a - b = np,\ b - c = nq$, and hence $(a - b) + (b - c) = np + nq$, which implies $a - c = n(p + q)$, i.e., $n|(a - c)$, which is the same as $a \equiv c(\mathrm{mod}\ n)$. Similarly, if $a \equiv b(\mathrm{mod}\ n)$ and $c \equiv d(\mathrm{mod}\ n)$, then there exist integers *r* and *s* such that $a - b = nr,\ c - d = ns$, and hence $(a + c) - (b + d) = (a - b) + (c - d) = nr + ns \ = \ (r + s)n$; thus $a + c \equiv b + d(\mathrm{mod}\ n)$. Similarly, we have

$ac = (b + nr)(d + ns) = bd + (bs + dr + rsn)n$, i.e., $ac \equiv bd(\text{mod } n)$. This completes the proof of property (4). Property (5) is a special case of properties (4) and (1). The proof of property (6) is by induction. The statement obviously holds for $k = 1$; now we assume it is true for some fixed k. From property (4), it follows that $a \equiv b(\text{mod } n)$ and $a^k \equiv b^k(\text{mod } n)$ together imply that $aa^k \equiv bb^k(\text{mod } n)$, or equivalently $a^{k+1} \equiv b^{k+1}(\text{mod } n)$. ■

- Congruence reduces considerably the labor in certain types of computations. Properties (1)–(3) in Theorem 3.6 confirm that congruence is an *equivalence relation*. The relation $ac \equiv bc(\text{mod } n)$ does not imply $a \equiv b(\text{mod } n)$, for example, $5 \cdot 4 \equiv 7 \cdot 4(\text{mod } 4)$, but $5 \not\equiv 7(\text{mod } 4)$; however, later we shall show that it holds if $\gcd(c, n) = 1$, and in general, if $d = \gcd(c, n)$, then $a \equiv b(\text{mod } n/d)$.
- The general congruence upon which the ancient method of checking a computation by "casting out nines" rests is $10^n \equiv 1(\text{mod } 9)$. If we look at all numbers for the same modulus, we find that they fall into nine different groups:

$$\begin{array}{lcll}
\text{numbers} & \equiv & 0(\text{mod } 9), & \text{such as } 0, 9, 18, 27, 36, \cdots \\
'' & \equiv & 1(\text{mod } 9), & \text{such as } 1, 10, 19, 28, 37, \cdots \\
'' & \equiv & 2(\text{mod } 9), & \text{such as } 2, 11, 20, 29, 38, \cdots \\
'' & \equiv & 3(\text{mod } 9), & \text{such as } 3, 12, 21, 30, 39, \cdots \\
'' & \equiv & 4(\text{mod } 9), & \text{such as } 4, 13, 22, 31, 40, \cdots \\
'' & \equiv & 5(\text{mod } 9), & \text{such as } 5, 14, 23, 32, 41, \cdots \\
'' & \equiv & 6(\text{mod } 9), & \text{such as } 6, 15, 24, 33, 42, \cdots \\
'' & \equiv & 7(\text{mod } 9), & \text{such as } 7, 16, 25, 34, 43, \cdots \\
'' & \equiv & 8(\text{mod } 9), & \text{such as } 8, 17, 26, 35, 44, \cdots
\end{array}$$

  Every number falls in one of these nine groups and no number in more than one group.
- We shall show that every palindromic integer with an even number of digits is divisible by 11. For this, let a palindromic integer *m* have 2*n* digits and can be written as $m = a_1a_2 \cdots a_{n-1}a_na_na_{n-1} \cdots a_2a_1$. We expand this as

$$\begin{aligned} m &= a_1 + a_2(10) + \cdots + a_n(10)^{n-1} + a_n(10)^n + a_{n-1}(10)^{n+1} \\ &\quad + \cdots + a_2(10)^{2n-2} + a_1(10)^{2n-1}. \end{aligned}$$

Since $10 \equiv -1 (\bmod\ 11)$, in view of Theorem 3.6(4) and (6), the above expression on taking (mod 11) leads to

$$\begin{aligned} & a_1 + a_2(-1) + \cdots + a_n(-1)^{n-1} + a_n(-1)^n + a_{n-1}(-1)^{n+1} \\ & \quad + \cdots + a_2(-1)^{2n-2} + a_1(-1)^{2n-1}, \end{aligned}$$

which is the same as

$$a_1 - a_2 + \cdots + a_n(-1)^{n-1} + a_n(-1)^n + \cdots + a_2 - a_1 = 0,$$

and hence $m \equiv 0 (\bmod\ 11)$.

- Let $\{a_n\}$ be a sequence of positive integers generated by the recurrence relation $a_1 = 2,\ a_{n+1} = a_n^2 - a_n + 1$. Then, a_i and $a_j,\ i \neq j$ are relatively prime. For this by induction, it follows that $a_{n+1} = a_1 a_2 \cdots a_n + 1$. Clearly, it suffices to show that for a given $n \in \mathcal{N}$ for each $1 \leq k < n,\ a_k$ and a_n are relatively prime. We define $a_{n,k} = a_1 a_2 \cdots a_n / a_k$. From Theorem 3.4, it is known that

$$\gcd(a_n, a_k) = \min\{a_n x + a_k y : x, y \in \mathcal{Z} \text{ and } a_n x + a_k y > 0\}.$$

Now since $1 \leq k < n$, we have

$$1 = a_n - a_{n-1,k} a_k \in \{a_n x + a_k y : x, y \in \mathcal{Z} \text{ and } a_n x + a_k y\},$$

and hence, from Corollary 3.3, $\gcd(a_n, a_k) = 1$.

For an alternative verification of the above result, we let *p* be a prime that divides at least one a_k. Let *q* be the smallest *k* such that *p* divides a_k. We shall show by induction on *n* that $a_{q+n} \equiv 1 (\bmod\ p)$. Thus, *p* does not divide a_{q+n}. So any prime divides at most one of the a_i, which implies that a_i's are pairwise relatively prime. For the initial step, we have $a_{q+1} = a_q^2 - a_q + 1 \equiv 0^2 - 0 + 1 (\bmod\ p) \equiv 1 (\bmod\ p)$. For the inductive step, we let $a_{q+n} \equiv 1 (\bmod\ p)$; then $a_{q+n+1} \equiv 1^2 - 1 + 1 (\bmod\ p) \equiv 1 (\bmod\ p)$.

- Congruence is used to describe cycles in the world of the integers. For example, the day of the week is a cyclic phenomenon in that the day of the week repeats every 7 days. In fact, it can be utilized very accurately with the help of astronomers' concept of the Julian day. To avoid the confusion which results from months and years of unequal lengths, they number the days consecutively from January 1, 4713 BC, the beginning of the Julian era. January 1, 1930, which fell on Wednesday, was by this numbering Julian day $2,425,978.$ With this information and the congruence relationship based on the modulus 7, we can compute

January 1,1930 $=$ J.D. 2,425,978 $\equiv$ 2(mod 7) = Wednesday
January 1,1960 $=$ J.D. 2,436,935 $\equiv$ 4(mod 7) = Friday

- For the Gregorian and Julian calendar, Julius Christian Johannes Zeller (1822–1899, Germany) in 1883 formulated *Zeller's congruences*: For determining the day of the week for any given date, his formula is

$$\text{weekday} \;=\; \left(D + \left[\frac{13M-1}{5}\right] + Y + \left[\frac{X}{4}\right] + \left[\frac{Y}{4}\right] - 2X\right) \pmod 7,$$

where *D* is the day of the month, *X* is the first two digits of the year, *Y* is the last two digits of the year, $[\cdot]$ is the usual greatest integer function, and *M* is the month according to the numbering, March$=1$, April$=2$, May$=3$, June$=4$, July$=5$, August$=6$, September$=7$, October$=8$, November$=9$, December$=10$, January$=11$, and February$=12$. Because of this wacky numbering, it is important to subtract 1 from the year when dealing with a date in January and February. For example, the date of January 7, 1999 gives us $M=11,\;D=7,\;X=19,$ and after subtracting $1,\;Y=98.$ Plugging this into the formula gives us

$$\begin{aligned}\text{weekday} \;&=\; \left(7 + \left[\frac{13\times 11-1}{5}\right] + 98 + \left[\frac{19}{4}\right] + \left[\frac{98}{4}\right] - 2\times 19\right) \bmod 7\\ &=\; (7+28+98+4+24-38) \pmod 7\\ &=\; 123 \pmod 7 \;=\; 4.\end{aligned}$$

From the code Sunday$=0$, Monday$=1$, Tuesday$=2$, Wednesday$=3$, Thursday$=4$, Friday$=5$, Saturday$=6,$ we see that January 7, 1999 was on a Thursday.

- This problem is from Brahmagupta's work on congruences. Given that the Sun makes 30 revolutions through the ecliptic in $10,960$ days, how many days have elapsed (since the Sun was at a given starting point) if the Sun has made an integral number of revolutions plus $8080/10,960$ of a revolution, that is, "when the remainder of solar revolutions is $8080.$" If *y* is the number of days sought and *x* is the number of revolutions, then, because 30 revolutions take $10,960$ days, *x* revolutions take $(1096/3)x$ days. Therefore, $y=(x+808/1096)(1096/3)$, or $1096x+808=3y$. Thus, we need to solve $N\equiv$ 808(mod 1096) and $N\equiv 0$ (mod 3). ($x=2,\ y=1000,\ N=3000$)

If a,b, and *n* are integers such that $a\not\equiv 0(\text{mod } n)$ and *x* is unknown, then the *linear congruence* is defined as $ax\equiv b(\text{mod } n)$, and by a solution of such an equation we mean an integer x_0 for which $ax_0\equiv b(\text{mod } n)$. It is clear that the linear congruence has a solution iff there are integers *x* and *k* such that $ax=b+kn$. The following result provides necessary and sufficient condition when a linear congruence has a solution.

Theorem 3.7 *Let* $d=\gcd(a,n)$. *Then the linear congruence* $ax\equiv b(\text{mod } n)$ *has at least one solution iff* $d|b$. *Furthermore, if* $\gcd(a,n)=1$, *then the solution is unique modulo* n.

Proof We need to show that there exists an integer *k* such that $ax-kn=b$. Since $d|a$ and $d|n$, it follows that $d|b$, and hence $b=qd$ for some q. Now from Theorem 3.4 there exist integers *r* and *s* such that $d=ar+ns$; thus $b=q(ar+ns)=a(qr)+n(qs)$. Hence, $a(qr)-b$ is a multiple of n, or $a(qr)\equiv b(\text{mod } n)$, which on letting $qr=x$ is the same as $ax\equiv b(\text{mod } n)$. Now assume that $\gcd(a,n)=1$, and the linear congruence has two solutions, say, *w* and z, i.e., $aw\equiv b(\text{mod } n)$ and $az\equiv b(\text{mod } n)$, and then $aw\equiv az(\text{mod } n)$. But then in view of $\gcd(a,n)=1$, it follows that $w\equiv z(\text{mod } n)$. ■

Theorem 3.8 *Let* $d=\gcd(a,n)$. *If* $d|b$, *then the linear congruence* $ax\equiv b(\text{mod } n)$ *has exactly d distinct solutions modulo* n.

Proof Let $a=da_1, b=db_1, n=dn_1$. Then, it follows that $a_1x\equiv b_1(\text{mod } n_1)$ and $\gcd(a_1,n_1)=1$. Thus in view of Theorem 3.7, we find that this linear congruence has a unique solution, say, h. Now we shall show that $h+in_1,\ 0\le i\le d-1$, are the only distinct solutions of the linear congruence $ax\equiv b(\text{mod } n)$. For this, in view of $a_1h\equiv b_1(\text{mod } n_1)$ and $n_1d=n$, we note that

$$a(h+in_1)=a_1dh+a_1din_1=a_1hd+a_1i(n_1d)\equiv b_1d+a_1in\ \equiv\ b_1d\ \equiv b(\text{mod } n),$$

and hence $a(h + in_1) \equiv b(\text{mod } n)$. We also note that

$$0 \ \le \ h + in_1 \ \le \ h + (d-1)n_1 \ < \ n_1 + (d-1)n_1 \ = \ dn_1 \ = \ n.$$

This shows that $ax \equiv b(\text{mod } n)$ has *d* distinct solutions. To show these are the only solutions, for fixed $0 \le i \le d-1$, let $h + in_1 = u$ be one solution and *v* be any solution (other than obtained *d* solutions) of $ax \equiv b(\text{mod } n)$. Then, we have $au \equiv av \equiv b(\text{mod } n)$, which implies that $u \equiv v(\text{mod } n_1)$, or $u - v = pn_1$, or $u = v + pn_1$ for some p. Finally, since *u* is the least residue $(\text{mod } n)$ and all least residues $(\text{mod } n)$ are of the form $h + in_1, \ 0 \le i \le d-1$, *v* must be one of these. ■

- We have seen above that $d = \gcd(45692, 987) = 1$, and hence $45692x \equiv 290(\text{mod } 987)$ has only one solution $x = 1$ modulo 987; similarly the only solution of $45692x \equiv 56(\text{mod } 987)$ is $x = 7$. Furthermore, since $d = \gcd(1234569, 2463) = 3$, the linear congruence $1234569x \equiv 606(\text{mod } 2463)$ has three solutions modulo 2463. To find these solutions, we note that $a = 1234569 = 3 \cdot 411523 = da_1, b = 606 = 3 \cdot 202 = db_1, n = 2463 = 3 \cdot 821 = dn_1$, and the only solution of $411523x \equiv 202(\text{mod } 821)$ is $h = 1$. Thus, all the three required solutions are $x_1 = 1, x_2 = 1 + 821 = 822$, and $x_3 = 1 + 2 \times 821 = 1643$.

Related with linear congruence there are linear equations of Diophantine type $ax + by = n$; however, linear equations were not a part of Gauss study. In fact, such equations were first examined in Sulbasutras, and Aryabhata used the Kuttaka (pulverize) method to solve these equations. His method is essentially the same as the Euclidean algorithm, Theorem 3.3. We state the following result for such equations.

Theorem 3.9 *Let $d = \gcd(a, b)$; then the equation $ax + by = n$ has a solution iff $d|n$. Furthermore, if x_0, y_0 is the solution of this equation, then it has an infinite number of solutions given by* $x = x_0 + (b/d)s, y = y_0 - (a/d)s, \ s = 0, 1, 2, \cdots$.

The following two examples follow the traditions of Chinese starting as early as the first century.

- Consider the following problem: If horses cost 8 coins and cows cost 6 coins, how many of each animal can be purchased for 106 coins. For this, let *x* be the number of horses and *y* be cows; then we are given $8x + 6y = 106.$ Since $\gcd(8, 6) = 2|106$ and from the inspection $x_0 = 2, y_0 = 15$ is a solution, and from Theorem 3.9, all solutions of this equation can be written as $x = 2 + 3s, y = 15 - 4s.$ Now since *x* and *y* have to be nonnegative integers, we must have $2 + 3s \geq 0$ and $15 - 4s \geq 0$, which gives $-\frac{2}{3} \leq s \leq \frac{15}{4}$, and hence the integer values of *s* are $s = 0, 1, 2, 3.$ Therefore, in total there are four solutions, $x = 2, y = 15;\ x = 5, y = 11;\ x = 8, y = 7;$ and $x = 11, y = 3.$
- Cockerels cost 5 *qian* each, hens 3 *qian* each, and three chickens 1 *qian*. If 100 fowls are bought for 100 *qian*, how many cockerels, hens, and chickens are there? Let $x, y,$ and *z* be the numbers of cockerels, hens, and chickens, respectively. Then, we are given the equations

$$5x + 3y + \frac{1}{3}z = 100, \quad x + y + z \ = \ 100.$$

Eliminating the unknown *z* from these equations, we find $7x + 4y = 100.$ Comparing this equation with $ax + by = n$, we have $a = 7, b = 4, n = 100.$ Now since

$$\begin{aligned} 7 &= 1 \cdot 4 + 3 & 1 &= 4 - 1 \cdot 3 \\ 4 &= 1 \cdot 3 + 1 & 1 &= 4 - 1 \cdot (7 - 1 \cdot 4) \\ 3 &= 3 \cdot 1 + 0 & 1 &= -1 \cdot 7 + 2 \cdot 4 \\ \gcd(7, 4) &= 1 & 100 &= -100 \cdot 7 + 200 \cdot 4, \end{aligned}$$

it follows that $x_0 = -100, y_0 = 200$ is a solution of $7x + 4y = 100.$ Thus, from Theorem 3.9 all solutions of fowls problem can be written as

$$x \ = \ -100 + 4s, \quad y = 200 - 7s, \quad z = 100 - x - y \ = \ 3s.$$

Next, since the number of fowls must be nonnegative, we must have $-100 + 4s \geq 0$ and $200 - 7s \geq 0$, which gives $25 \leq s \leq 28\frac{4}{7}$. Thus, the only choices for an integer *s* are $s = 25, 26, 27, 28,$ and these choices give the following solutions:
$x = 0, y = 25, z = 75; x = 4, y = 18, z = 78; x = 8, y = 11, z = 81;$ and $x = 12, y = 4, z = 84.$ Zhang Qiujian without explaining the method (perhaps by trial and error) gives only the last three (positive) solutions.

Now we shall prove a result whose origin is third century AD. The result is extensively used for computing with large integers, as it allows replacing a computation for which one knows a bound on the size of the result by several similar computations on small integers. In recent years this result has found applications in cryptography.

Theorem 3.10 (Chinese Remainder Theorem) *The system of linear congruences*

$$x \equiv a_i (\text{mod } n_i), \quad i = 1, 2, \cdots, m, \tag{3.12}$$

where $\gcd(n_i, n_j) = 1$ *if* $1 \leq i \neq j \leq m$, *has a unique least solution modulo* $N = n_1 n_2 \cdots n_m$.

Proof Let $N_i = N/n_i,\ 1 \leq i \leq m$. Since $\gcd(n_i, n_j) = 1$ if $1 \leq i \neq j \leq m$, it follows that $\gcd(N_i, n_i) = 1,\ 1 \leq i \leq m$. Thus from Theorem 3.7, it follows that there exists a unique x_i such that $N_i x_i \equiv 1 (\text{mod } n_i)$. We claim that $x \equiv \sum_{i=1}^{m} N_i x_i a_i\ (\text{mod } N)$ is the required solution. For this, since $N_k \equiv 0 (\text{mod } n_i)$ for all $1 \leq k \neq i \leq m$, it follows that $x \equiv N_i x_i a_i \equiv a_i (\text{mod } n_i)$ for all $1 \leq i \leq m$. Now to show the uniqueness, let *y* be also a solution. Then, $x \equiv y \equiv a_i (\text{mod } n_i),\ 1 \leq i \leq m$. Thus, $n_i | (x - y),\ 1 \leq i \leq m$. This means $(x - y)$ is the common multiple of all $n_i,\ 1 \leq i \leq m$. But since $\gcd(n_i, n_j) = 1$ if $1 \leq i \neq j \leq m$, this implies that $N | (x - y)$, and hence $x \equiv y (\text{mod } N)$. ■

- To illustrate Theorem 3.10, we consider the system

  $$x \equiv 6 (\text{mod } 11), \quad x \equiv 13 (\text{mod } 16), \quad x \equiv 9 (\text{mod } 21), \quad x \equiv 19 (\text{mod } 25).$$

  Clearly, $n_1 = 11, n_2 = 16, n_3 = 21, n_4 = 25$ satisfy $\gcd(n_i, n_j) = 1$ if $1 \leq i \neq j \leq 4$. Since $N = 92400, N_1 = 8400, N_2 = 5775, N_3 = 4400, N_4 = 3696$, it follows that

$$
\begin{aligned}
N_1x_1 &\equiv 1(\text{mod } n_1) \to 8400x_1 \equiv 1(\text{mod}11) \to 7x_1 \equiv 1(\text{mod}11)\\
&\to x_1 \equiv 8(\text{mod}11)\\
N_2x_2 &\equiv 1(\text{mod } n_2) \ \to \ 5775x_2 \equiv 1(\text{mod } 16) \ \to \ 15x_2 \equiv 1(\text{mod } 16)\\
&\to \ x_2 \equiv 15(\text{mod } 16)\\
N_3x_3 &\equiv 1(\text{mod } n_3) \ \to \ 4400x_3 \equiv 1(\text{mod } 21) \ \to \ 11x_3 \equiv 1(\text{mod } 21)\\
&\to \ x_3 \equiv 2(\text{mod } 21)\\
N_4x_4 &\equiv 1(\text{mod } n_4) \ \to \ 3696x_4 \equiv 1(\text{mod } 25) \ \to \ 21x_4 \equiv 1(\text{mod } 25)\\
&\to \ x_4 \equiv 6(\text{mod } 25).
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
x \ \equiv \ \sum_{i=1}^{4} N_i x_i a_i &\equiv [8400 \cdot 8 \cdot 6 + 5775 \cdot 15 \cdot 13 + 4400 \cdot 2 \cdot 9\\
&+ 3696 \cdot 6 \cdot 19](\text{mod } 92400)\\
&\equiv 2029869(\text{mod } 92400) \ \equiv \ 89469(\text{mod } 92400).
\end{aligned}
$$

We can verify this solution as

$89469 = 8133(11) + 6 = 5591(16) + 13 = 4260(21) + 9 = 3578(25) + 19.$

- The following problem is from ancient Chinese origin: A band of 17 pirates stole a sack of gold coins. When they tried to divide the fortune into equal portions, 3 coins remained. In the ensuing brawl over who should get the extra coins, one pirate was killed. The wealth was redistributed, but this time an equal division left 10 coins. Again, an argument developed in which another pirate was killed. But now the total fortune was evenly distributed among the survivors. What was the least number of coins that could have been stolen? This problem leads to the system

$$
x \equiv 3(\text{mod } 17), \quad x \equiv 10(\text{mod } 16), \quad x \equiv 0(\text{mod } 15).
$$

Clearly, $n_1 = 17, n_2 = 16, n_3 = 15$ satisfy $\text{gcd}(n_i, n_j) = 1$ if $1 \le i \ne j \le 3$. Since $N = 4080, N_1 = 240, N_2 = 255, N_3 = 272$, it follows that

$$\begin{aligned}
N_1 x_1 \equiv 1 (\text{mod } n_1) \ &\rightarrow \ 240 x_1 \equiv 1 (\text{mod } 17) \ \rightarrow \ 2x_1 \equiv 1 (\text{mod } 17) \\
&\rightarrow \ x_1 \equiv 9 (\text{mod } 17) \\
N_2 x_2 \equiv 1 (\text{mod } n_2) \ &\rightarrow \ 255 x_2 \equiv 1 (\text{mod } 16) \ \rightarrow \ 15x_2 \equiv 1 (\text{mod } 16) \\
&\rightarrow \ x_2 \equiv 15 (\text{mod } 16) \\
N_3 x_3 \equiv 1 (\text{mod } n_3) \ &\rightarrow \ 272 x_3 \equiv 1 (\text{mod } 15) \ \rightarrow \ 2x_3 \equiv 1 (\text{mod } 15) \\
&\rightarrow \ x_3 \equiv 8 (\text{mod } 15).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
x \ \equiv \ \sum_{i=1}^{3} N_i x_i a_i \ &\equiv \ [240 \cdot 9 \cdot 3 + 255 \cdot 15 \cdot 10] (\text{mod } 4080) \\
&\equiv \ 44730 (\text{mod } 4080) \ \equiv \ 3930 (\text{mod } 4080).
\end{aligned}$$

We can verify this solution as

$$3930 = (231)(17) + 3 = (245)(16) + 10 = (262)(15) + 0.$$

- **Constructive Proofs**. A technique for proving that there is an object with a certain property such that something happens. To do so, construct, guess, produce, or devise an algorithm to produce the desired object. Then show that the object you constructed has the certain property and satisfies the something that happens. Until the end of nineteenth century, all mathematical proofs were essentially constructive. We shall lay out this method in the following examples.
- If $a, b, c, d, e,$ and *f* are real numbers such that $ad - bc \neq 0,$ then the two equations $ax + by = e$ and $cx + dy = f$ can be solved for real numbers *x* and y. For this, we multiply the equation $ax + by = e$ by $d,$ and the equation $cx + dy = f$ by $b,$ and then subtract the two equations one obtains $(ad - bc)x = (de - bf).$ From the hypothesis, $ad - bc \neq 0,$ and so dividing by $ad - bc$ yields $x = (de - bf)/(ad - bc).$ A similar argument shows that $y = (af - ce)/(ad - bc).$
- Consider the quadratic equation

$$ax^2 + bx + c \ = \ 0, \tag{3.13}$$

where for simplicity we assume that $a > 0$ (if $a = 0,$ (3.13) reduces to a linear equation, and if $a < 0$, we multiply the equation by -1) and b, c are given real numbers. To find the solutions of (3.13), we complete the square to rewrite it as

$$\left(\sqrt{a}x + \frac{b}{2\sqrt{a}}\right)^2 = \frac{b^2}{4a} - c,$$

which gives

$$\sqrt{a}x + \frac{b}{2\sqrt{a}} = \pm\sqrt{\frac{b^2}{4a} - c},$$

and hence the solutions of (3.13) are given by the *quadratic formula*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{3.14}$$

Using first the $+$ sign and then the $-$ sign, we get two different solutions, except when $b^2 - 4ac = 0$ in which case (3.13) has two repeated solutions. If $b^2 - 4ac \geq 0$ solutions are real and if $b^2 - 4ac < 0$, the solutions are complex, which are complex conjugates of each other.

- Theorems 3.2, 3.3, and 3.4 are excellent examples of finite algorithms and hence provide constructive methods. The following systematic steps for finding the maximum (largest) value in a finite sequence of integers is another significant example of a finite algorithm: 1. Set the temporary maximum equal to the first integer in the sequence, 2. compare the next integer in the sequence to the temporary maximum, and if it is larger than the temporary maximum, set the temporary maximum equal to this integer, 3. repeat the previous step if there are more integers in the sequence, and 4. stop when there are no integers left in the sequence. The temporary maximum at this point is the largest integer in the sequence. On repeating this algorithm we can also arrange all numbers of the sequence in decreasing order.
- A number x^* real or complex is said to be a *root of multiplicity* $m \geq 1$ of the equation $f(x) = 0,$ if $f(x^*) = f'(x^*) = \cdots = f^{(m-1)}(x^*) = 0,\ f^{(m)}(x^*) \neq 0.$ If $m = 1,$ then x^* is said to be a *simple root*. *Bolzano's bisection method/algorithm* for finding a root of the equation $f(x) = 0$ can be described as follows: Let $f(x),\ x \in I = [a, b]$ be a continuous function, and let $I_0 = [a_0, b_0] \subseteq I$ be such that $f(a_0)f(b_0) < 0$ (this ensures that the equation $f(x) = 0$ has a root in I_0). Now let $x_0 = (a_0 + b_0)/2$ be the midpoint of the interval I_0. If $f(x_0) = 0,$ a zero of $f(x)$ has been found, and the process

terminates here. Otherwise, $f(x_0) \neq 0$ and either $f(a_0)f(x_0) < 0$ or $f(x_0)f(b_0) < 0.$ We then select a new interval $I_1 = [a_1, b_1]$ by taking either $a_1 = a_0$ and $b_1 = x_0$ if $f(a_0)f(x_0) < 0$ or $a_1 = x_0$ and $b_1 = b_0$ if $f(x_0)f(b_0) < 0.$ We continue this process, which either will terminate after a finite number of steps or will lead to intervals
$I_n = [a_n, b_n], \ n = 0, 1, 2, \cdots$ such that

$$f(a_n)f(b_n) \ < \ 0, \tag{3.15}$$

$$x_n \ = \ \frac{1}{2}(a_n + b_n), \tag{3.16}$$

$$a_{n+1} = a_n \quad \text{and} \quad b_{n+1} = x_n \quad \text{if} \quad f(a_n)f(x_n) < 0,$$

or

$$a_{n+1} = x_n \quad \text{and} \quad b_{n+1} = b_n \quad \text{if} \quad f(x_n)f(b_n) < 0.$$

By this construction the intervals $I_n, \ n = 0, 1, 2, \cdots$, satisfy

$$I_{n+1} \subseteq I_n, \quad \text{i.e.,} \quad a_n \leq a_{n+1} \leq b_{n+1} \leq b_n, \quad n = 0, 1, 2, \cdots \tag{3.17}$$

and

$$b_n - a_n \ = \ \frac{1}{2^n}(b_0 - a_0), \quad n = 0, 1, 2, \cdots . \tag{3.18}$$

From (3.17) and (3.18) it is clear that the sequences $\{a_n\}, \ \{b_n\}$ are monotone, and

$$a_n \ \rightarrow \ x^*, \quad b_n \ \rightarrow \ x^*, \quad x_n \ \rightarrow \ x^*.$$

Furthermore, from the continuity of $f(x)$ and (3.15), it follows that

$$f^2(x^*) \ = \ \lim_{n\to\infty} f(a_n)f(b_n) \ \leq \ 0, \quad \text{i.e.,} \quad f(x^*) = 0.$$

Hence, x^* is a zero of $f(x)$, which lies in each interval
$I_n, \ n = 0, 1, 2, \cdots$. Thus, from (3.18), we have

$$|x_n - x^*| \ \leq \ \frac{1}{2}(b_n - a_n) \ = \ \frac{1}{2^{n+1}}(b_0 - a_0). \tag{3.19}$$

Inequality (3.19) is an a priori error estimate. In particular, if we let $b_0 - a_0 = 1$, then since $2^{-10} < 10^{-3}$, we have $|x_9 - x^*| < 10^{-3},\ |x_{19} - x^*| < 10^{-6},\ |x_{29} - x^*| < 10^{-9}$.

- **Proofs by Disjunction Elimination**. This type of proofs is also known as *proofs by cases* or simply as *proofs by elimination*. Here we allow one to eliminate a disjunctive statement (a compound statement formed by joining two or more statements with the connector "or" $\vee$) one by one from a logical proof.
- We shall show that if *a* is a negative real number, then $y = -b/(2a)$ is a maximum of the function $ax^2 + bx + c$. For this, we let *x* to be a real number. It suffices to show that $ay^2 + by + c \geq ax^2 + bx + c$, or equivalently, $(y - x)[a(y + x) + b] \geq 0$. This is clearly true if $y - x = 0$, so we assume that $y - x \neq 0$. Then either $y - x > 0$ or $y - x < 0$. Assume that $y - x > 0$. Because $a < 0$ and $y = -b/(2a)$, it follows that $a(y + x) + b > a(y + y) + b = 0$. The proof for the case $y - x < 0$ is similar.
- We shall prove that if *n* is a positive integer, then either *n* is prime or *n* is a square, or *n* divides $(n - 1)!$ If $n = 1$, then $n = 1^2$ is a square and the proposition is true. Similarly, if $n = 2$, then *n* is prime and again the proposition is true. So suppose that $n > 2$ is neither prime nor a square. Because $n > 2$ is not prime, there are integers *a* and *b* with $1 < a < n$ and $1 < b < n$ such that $n = ab$. Also, because *n* is not square, $a \neq b$. This means that *a* and *b* are integers with $2 \leq a \neq b \leq n - 1$. That is, *a* and *b* are two different terms of $(n - 1)(n - 2) \cdots 1$. Thus, $ab = n$ divides $(n - 1)!$.
- **Proofs by Contrapositive**. In this technique instead of proving $H \Rightarrow C$, we prove its equivalent form $\sim C \Rightarrow \sim H$, i.e., NOT C implies NOT H. This means we interchange the hypothesis and the conclusion with the negative statements.
- Suppose that *S* and *T* are sets of real numbers with $S \subseteq T$. We shall show that if *S* is not bounded, then *T* is not bounded. For this, we shall assume that *T* is bounded and show that *S* is bounded. Hence, there is a real number $M > 0$ such that, for all $x \in T$, $|x| < M$. Now let $x' \in S$. Because $S \subseteq T$, it follows that $x' \in T$. But then $|x'| < M$ and so *S* is bounded.

- To prove the statement that "if *n* is a positive integer such that $n^2 \equiv 2(\mathrm{mod}\ 4)$ or $n^2 \equiv 3(\mathrm{mod}\ 4)$, then *n* is not a perfect square," we shall use contrapositive argument to show that "if *n* is a perfect square then $n^2 \equiv 0(\mathrm{mod}\ 4)$ or $n^2 \equiv 1(\mathrm{mod}\ 4)$." For this, it suffices to note that if *n* is even, i.e., $n = 2k$, then $n^2 = 4k^2$, and hence $n^2 \equiv 0(\mathrm{mod}\ 4)$, and if *n* is odd, i.e., $n = 2k + 1$, then $n^2 = 4k(k + 1) + 1$, and therefore $n^2 \equiv 1(\mathrm{mod}\ 4)$. We further note that since $k(k + 1)$ is even, if *n* is odd, then $n^2 \equiv 1(\mathrm{mod}\ 8)$. For example, $171^2 = 8(3655) + 1$.
- *Generalized Pigeonhole Principle* states that "if *N* objects are placed into *k* boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects." This principle is also called *Dirichlet Drawer Principle* after Dirichlet who stated this in 1834. To prove this principle, we shall use contrapositive form. We assume that none of the boxes contains more than $\lceil N/k \rceil - 1$ objects, but then using $\lceil N/k \rceil < (N/k) + 1$ the total number of objects must be

$$k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) \ < \ k\left(\left(\frac{N}{k} + 1\right) - 1\right) = N.$$

  Thus the total number of objects is less than N. As an example, in a party of 200 people, at least $\lceil 200/12 \rceil = 17$ people were born in the same month.
- **Contradiction Method**. According to Hardy, "this method is a wonderful logical tactic. By assuming the opposite of what we intend to show, we seem to be putting the eventual goal in jeopardy. Yet, in the end, calamity is averted. He described proof by contradiction as "one of mathematicians' finest weapons. It is far finer gambit than any chess gambit; a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers the game." Hippocrates invented the proof by contradiction. Mathematically, for proving that the statement *H* implies the statement *C*, we work forward from the assumption that *H* and not *C* are true to reach a contradiction to some statement that you know is true. We shall illustrate this method through following examples which demonstrate certain numbers and certain geometric constructions cannot exist. We shall also show how it helps in proving the uniqueness of the solutions and finding solutions of some puzzles.
- Let *x* and *y* be real numbers. We shall show that the indeterminate equation $5x + 25y = 331$ has no integer solution. For this, we assume

that *x* and *y* are integer solutions of $5x + 25y = 331.$ But then, $5(x + 5y) = 331$ implies that $5|331,$ which is not true. This contradiction establishes the result.

- We shall show that if $x, y \in \mathcal{Z},$ then $x^2 - 4y \neq 2.$ For contradiction we assume that there exist $x, y \in \mathcal{Z}$ so that $x^2 - 4y = 2.$ But then, $x^2 = 2(2y + 1),$ i.e., x^2 is even. But this implies that *x* is even, and hence there exist $a \in \mathcal{Z}$ such that $x = 2a.$ Putting this in our equation leads to $(2a)^2 - 4y = 2,$ which is the same as $2(a^2 - y) = 1.$ Since $a^2 - y$ is an integer, this means $2|1,$ which is not true.
- By construction we have seen that the equations $ax + by = e, \ cx + dy = f$ have a solution provided $ad - bc \neq 0.$ We shall show that these equations have a unique solution (to prove this we do not need the existence of a solution). For contradiction, we assume that (x_1, y_1) and (x_2, y_2) are two different solutions of the equations (a generally used tactics), so that

$$ax_1 + by_1 = e \tag{3.20}$$

$$cx_1 + dy_1 = f \tag{3.21}$$

$$ax_2 + by_2 = e \tag{3.22}$$

$$cx_2 + dy_2 = f. \tag{3.23}$$

Subtracting (3.22) from (3.20) and (3.23) from (3.21) yields

$$a(x_1 - x_2) + b(y_1 - y_2) = 0 \tag{3.24}$$

$$c(x_1 - x_2) + d(y_1 - y_2) = 0. \tag{3.25}$$

Multiplying (3.24) by *d* and (3.25) by *b* and then subtracting (3.25) from (3.24), it follows that $(ad - bc)(x_1 - x_2) = 0.$ Because, by hypothesis, $ad - bc \neq 0,$ one has $x_1 - x_2 = 0,$ and hence $x_1 = x_2.$ A similar sequence of algebraic manipulations establishes that $y_1 = y_2$, and thus the uniqueness is proved. Thus, the obtained solution of these equations is the only solution.

- We shall show that if *a* is a positive real number, then there is a unique real number *x* such that $x^3 = a.$ For contradiction, we assume that *x* and *y* are two different real numbers for which $x^3 = a$ and $y^3 = a.$ Hence it follows that $0 = x^3 - y^3 = (x - y)(x^2 + xy + y^2).$ Because $x \neq y,$ it must be that $x^2 + xy + y^2 = 0.$ By the quadratic formula,

$$x = \frac{-y \pm \sqrt{y^2 - 4y^2}}{2} = \frac{-y \pm \sqrt{-3y^2}}{2}.$$

  Now *x* is real, so it must be that $y = 0.$ But then $a = y^3 = 0,$ and this contradicts the hypothesis that $a > 0.$
- We shall use contradiction to show the nonexistence of a quadrilateral with sides of length $2, 3, 4,$ and $10.$ We assume the contrary by assuming that there exists such a quadrilateral. We depict this assumed quadrilateral in Fig. 3.5.

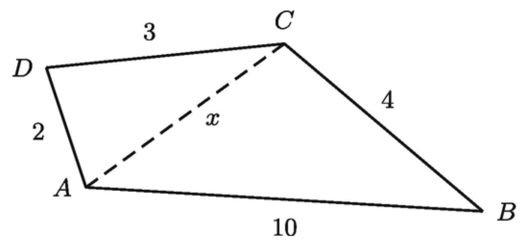

Fig. 3.5 Assumed quadrilateral

We drew the dotted diagonal which splits the quadrilateral into two triangles and assumed *x* be the diagonal's length. As any side of a triangle is shorter than the sum of the other two, in $\triangle\ ABC,$ we know $10 < 4 + x.$ The same principle applied to $\triangle ADC$ yields $x < 2 + 3.$ Combining these inequalities gives $10 < 4 + x < 4 + (2 + 3) = 9,$ which certainly contradicts our assumption. In our demonstration, we considered the sides in the order $10, 2, 3,$ and $4.$ Clearly, there are other configurations, such as $10, 3, 4, 2,$ but similar reasoning still leads to a contradiction.

- Three men (let us name them A,B, and C) have been sentenced to long terms in prison, but due to over crowed conditions, one man must be released. The warden devises a scheme to determine which man is to be released. He tells the men that he will blindfold them and then paint a red dot or blue dot on each forehead. After he paints the dots, he will remove the blindfolds, and a man should raise his hand if he sees a red dot on one of the other two prisoners. The first man to identify the color of the dot on his own forehead will be released. Of course, the men gladly agree to this. The warden blindfolds the men, as promised, and then paints a red dot on the foreheads of all three men. He removes the blindfolds and, since each man sees a red dot (in fact two red dots), each prisoner raises his hand. Some time passes when A exclaims, "I know what color my dot is! It's red!" A is then released. A's conclusion is based on contradicting argument. He assumes, to the contrary, that his dot is blue. Then, B knows A has blue dot and C has red dot (B raised his hand). Similarly, C knows A has blue dot and B has red dot. As a consequence, if A has blue dot, both B and C know they have red dot. But time has passed, and they (B and C) have not determined the color of their dots; A's dot must be red.

In the late nineteenth and early twentieth century, there was controversy in the mathematical world as to whether a theorem is really proved if it is only proved by contradiction. For example, Brouwer's existence proof of his fixed-point theorem was followed by wholesale rejection by contradiction. There was a feeling that a proof is stronger and more convincing if it is not by contradiction. With the rise of computer science and interest in computability, this became a serious issue in certain circles. However, after the work of Alan Mathison Turing (1912–1954, England) who cracked the Enigma code by applying ideas of proof by contradiction in the context of computing machines, proofs by contradiction are accepted as valid by all but a tiny number of mathematicians and computer scientists.

- **Proofs Require More Than One Method**. We present some examples where to prove a statement more than one method is needed.
- We shall show that if $p, q \in \mathcal{Z},$ then *pq* is even iff *p* or *q* is even. If *p* or *q* is even, then by direct argument *pq* is even. To show *pq* even implies *p* or *q* is even, we shall apply contrapositive argument. We assume that both *p* and *q* are odd, i.e., $p = 2s + 1$ and $q = 2t + 1,$ but then $pq = (2s + 1)(2t + 1) = 2(2st + s + t) + 1.$ Since $2st + s + t$ is an integer, *pq* is odd. As a corollary, if $p = q = x,$ we have x^2 even iff *x* is even.
- Let $x \in \mathcal{Z}.$ We shall show that for every integer $n \geq 2,$ x^n is even iff *x* is even. First we shall use direct argument to show that if *x* is even, then x^n is even. We let $x = 2a,$ $a \in \mathcal{Z},$ and then $x^n = x \times x^{n-1} = 2(ax^{n-1}).$ Since ax^{n-1} is an integer, x^n is even. To

prove the converse, i.e., if x^n is even, then *x* is even, we shall employ mathematical induction. For this the intimal step that x^2 even implies *x* is even we have already proved in the previous example. For the inductive step we assume that $p = x^{n-1}$ even implies $q = x$ is even, now $x^n = (x^{n-1})x = pq$ is even, but then again from the previous example either *x* is even (then the result is proved), or x^{n-1} is even (then by assumption *x* is even).

- Let $x, y \in \mathcal{Z}$. We shall use proof by cases and contrapositive argument to show that *x* and *y* are of the same parity iff $x + y$ is even. If *x* and *y* are of the same parity, then both are either even or odd. Thus, there exist $a, b \in \mathcal{Z}$ such that $x = 2a, y = 2b$ or $x = 2a + 1, y = 2b + 1$. But then $x + y = 2a + 2b = 2(a + b)$ or $x + y = 2(a + b + 1)$. However, since $a + b$ and $a + b + 1$ are integers, in both the cases $x + y$ is even. Conversely, we shall show that if $x + y$ is odd, then *x* and *y* are of different parity. For this, there exists $c \in \mathcal{Z}$ such that $x + y = 2c + 1$, and hence $x = 2c + 1 - y$. Now if *y* is odd, i.e., $y = 2d + 1,\ d \in \mathcal{Z}$, then $x = 2c + 1 - (2d + 1) = 2(c - d)$. But, since $c - d$ is an integer, *x* must be even. Also, if *y* is even, i.e., $y = 2s,\ s \in \mathcal{Z}$, then $x = 2c + 1 - 2s = 2(c - s) + 1$, and since $c - s$ is an integer, *x* must be odd.
- **Nonconstructive Proofs.** In this form of proof, we establish the existence of a solution but give no indication as to how it might be found.
- We follow Dov Jarden (1911–1986, Israel) work of 1953 to show that there exist irrational numbers *a* and *b* such that a^b is rational. Consider the irrational numbers $a = b = \sqrt{2}$. If the number $a^b = \sqrt{2}^{\sqrt{2}}$ is rational, we are done. If $\sqrt{2}^{\sqrt{2}}$ is irrational, we consider the numbers

$$a = \sqrt{2}^{\sqrt{2}} \text{ and } b = \sqrt{2} \text{ so that } a^b = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\sqrt{2}} = \sqrt{2}^2 = 2$$

is rational. Note that in this proof we could not find irrational numbers *a* and *b* such that a^b is rational.
- In calculus there are several results that allow us to give nonconstructive proofs, e.g., Extreme Value Theorem, Intermediate Value Theorem, Rolle's Theorem, Mean Value Theorem, Cauchy's Mean Value Theorem, Darboux Theorem, and many others, see Agarwal et al. [16]. We shall show that every polynomial with real coefficients $P_n(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ of odd degree (*n* is odd)

has at least one real zero. For definiteness, we assume that $a_0 > 0.$ Clearly, P_n is continuous on $\mathcal{R}.$ Also, $\lim_{x\to\infty} P_n(x) = \infty,$ and $\lim_{x\to-\infty} P_n(x) = -\infty.$ Thus, there exist real numbers *a* and *b* such that $P_n(a) < 0 < P_n(b).$ This ensures that there exists an $x_0 \in (a, b)$ such that $P_n(x_0) = 0.$ In particular, since for the polynomial $P_5(x) = x^5 + x^2 + 2x + 3 = 0,$ we have $P_5(-2) = -29 < 0$ and $P_5(-1) = 1 > 0,$ there exists an $x_0 \in (-2, -1)$ such that $P_n(x_0) = 0.$

- **Statistical Proofs.** This type of proofs demonstrates the validity of propositions only to a certain extent. Often, the conclusions are based on the experimental data, the facts, and the tests. Statistical proofs are normally used to convince the public about the current status and predict the future estimations in physical and social sciences, business, humanities, government, and manufacturing. While statisticians derive their conclusions using mathematics heavily, their proofs usually are not mathematical, rather fall within the branch known as mathematical statistics. However, in recent years statistical proofs have been employed even in pure mathematics such as cryptography, chaotic series, and analytic number theory (uses complex analysis, its foundations were laid by Euler). Similarly, in probabilistic proofs (e.g., based on opinion survey), examples are shown to exist only to certain degree of confidence, by using methods of probability theory. As an example, in 2004, Pascu [407] has given a probabilistic proof of the fundamental theorem of algebra. It is expected that probabilistic proofs might become a useful tool for verifying mathematical propositions and large computations.

3.14 What Is a Computer-Based Proof?

In Agarwal and Sen [14] we have provided sizable history of computational devices beginning from *abacus* (probably invented in Babylon in 2400 BC) to *electronic computers* (first invented by John Vincent Atanasoff, 1903–1995, USA) and beginning of *supercomputers* (unveiled by Seymour Roger Cray, 1925–1996, USA, in 1976, the CRAY-1). These devises were invented to do from basic arithmetic operations to encompass mathematical research in areas of science where computing was the sole alternative and was beyond human capabilities, as it has zillions of possibilities. In mathematical subjects such as numerical linear algebra, numerical solutions of ordinary and partial differential equations, discrete mathematics (e.g., in its search for mathematical structures like groups), number theory (primarily testing and factorization), cryptography, and computational algebraic topology, these devices are immensely used. From late the 1950s programs have been developed so that computer can rediscover several results of geometry and propose thousands of conjectures in *Graph Theory* (a branch of mathematics), which mathematicians gladly accepted. Mathematicians (including computational scientists) also accepted without any doubt algorithms that were mathematical and computer-assisted rather than computer-based, for example, see AKS in Sect. 4.3. However, during the 1970s and 1980s, an unsettling image entered the mathematical consciousness. It is the image of computers, with their lightning speed and virtual infallibility, taking over the job of proving theorems. This compelled mathematicians to reconsider the very nature of proof, in fact, evoked controversy because it reflected a continuing desire for human understanding of a proof, in addition to knowledge that the theorem is true. Specially, Thurston suggested that traditional proofs may be set aside in favor of experimentation, that is, testing of thousands or millions of examples, on the computer. However, supplying compelling evidence of a fact is not a proof (see Mertens Conjecture in Sect. 3.17), and hence mathematical proofs will never be obsolete. In what follows we provide a few examples where proofs required enormous calculations by computers.

- *Four Color Conjecture/Theorem.* The four color conjecture was first proposed in 1840 during his lecture by Möbius; it states that given any separation of a plane into contiguous regions, producing a figure called a map, at most four colors are needed to color the regions of the map so that no two adjacent regions have the same color. This problem belongs to graph theory. In 1852, the same conjecture occurred to a student Francis Guthrie (1831–1899, England-South Africa) when he was coloring a map of the counties of England. He communicated this conjecture to his mentor De Morgan. Then within a few years, some of the best known mathematicians of the time became aware of this conjecture. For example, in the following map we must make regions $A, B,$ and *C* of different colors because they share pairwise boundaries, but then it would be impossible to color region *D* unless a fourth color was introduced.

It is clear that anyone wishing to resolve this conjecture had one of the two options. Either come up with a specific counterexample—that is, a particular map—that cannot be colored with four colors or else devise a general proof that any map can be so colored. For mathematicians a counterexample proved elusive. Every map they created, no matter how intricate and convoluted, could be colored with four colors. Then, in 1879, a London lawyer Alfred Bray Kempe (1849–1922), a former student of Arthur Cayley (1821–1895, England), published a proof of the four color conjecture. Others also published alternative proofs, e.g., in 1880, Peter Guthrie Tait (1831–1901, Scotland). Kempe was honored. However, after 11 years, in 1890, Percy John Heawood (1861–1955, England) identified a subtle mistake in Kempe's proof. Other proofs were demolished more easily, e.g., Tait's proof was shown incorrect by Julius Peter Christian Petersen (1839–1910, Denmark) in 1891. Thus the problem returned to the status of a conjecture. Nevertheless, Heawood was able to use Kempe's work to provide an elementary proof of five color theorem (five colors suffice to color a map). On the other hand, there are maps, e.g., Fig. 3.6, for which three colors are insufficient. So five colors are plenty, but three are too few, and hence it all comes down to 4.

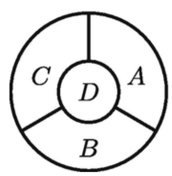

Fig. 3.6 Four color conjecture

To prove four color theorem, in 1976, Kenneth Ira Appel (1932–2013, USA) with his colleague Wolfgang Haken (1928–2022, Germany-USA) of the University of Illinois used 1200 hours of a supercomputer time to perform an extremely complex analysis involving a set of 1936 maps. Their demonstration also required hundreds of pages of manual analysis, see [34–37]. However, their computer proof of the four color conjecture busted dramatically upon the mathematical scene with the resolution. In fact, their proof was not accepted by all mathematicians because the computer-assisted proof was infeasible/intractable for a human to check by hand. Although the proof has gained wider acceptance, doubts continue to persist, because it can be checked only by another machine. As mathematician Ronald Lewis Graham (1935–2020, USA) asked when mulling over this complicated issue, "The real question is this: if no human being can ever hope to check a proof, is it really a proof?" Till today to this point the question has no definite answer, although as computer proofs become more common mathematicians will probably feel more comfortable with them. It is fair to say, however, that most mathematicians would breathe a sigh of relief if the Four Color Theorem were established with a two-page proof—short, ingenious, and elegant—rather than with the brute-force machinations of a computer. Traditionalists long for the good old days of mathematics unplugged. The shortest known proof of the Four Color Theorem of 2011 still has over 600 cases, and each case has to be checked by a computer program.

- *Party Problem (Ramsey's Theorem).* In 1920s, Frank Plumpton Ramsey (1903–1930, England) proposed a theory which deals with the distribution of subsets of elements of sets. His theory has been phrased as a question about the relationship between people at a party. What is the minimum number of guests that must be invited to guarantee that at least *m* people will all know each other (friends) or at least *n* people will all not know each other (enemies)? This number is called *Ramsey number* and denoted as $R(m,n),$ where the integers *m* and *n* are greater than 2. This problem also belongs to graph theory. From interchanging the concept of friends and enemies, it follows that $R(m,n)=R(n,m),$ i.e., Ramsey numbers are symmetric. Now, we shall show that $R(2,n)=n.$ For this, among a group of *n* people either there is a pair of friends, and if not, then every pair of people are enemies. Furthermore, if we have a group of $n-1$ people all of whom are enemies of each other, then there is neither a pair of friends nor a subset of *n* of them all of whom are mutual enemies. In 1955, Robert E. Greenwood (1881–1999, USA) and Gleason computed $R(3,3)=6, R(3,4)=9, R(3,5)=14, R(4,4)=18.$ Some other known Ramsey numbers are $R(3,6)=18, R(3,7)=23, R(3,8)=28, R(3,9)=36.$ In 1993, Stanislaw P. Radziszowski (born 1953, Poland-USA) and Brendan Damien McKay (born 1951, Australia) employed computer-assisted proof to show that $R(4,5)=25,$ which was published in 1995. Radziszowski and Brendan McKay estimated that their proof consumed the equivalent of 11 years of computation by a standard desktop machine. That may be a record, Radziszowski said, for a problem in pure mathematics. While mathematicians have accepted their result, the question of computation of $R(m,n)$ for large *m* and *n* is of social and moral issue. For Ramsey numbers some lower and upper bounds in terms of numbers m,n are known, e.g., $R(m,n)\le\binom{m+n-2}{n-1}$ and

$R(m,m) > 2^{m/2},\ m \geq 3.$ For several particular cases of m, n, much sharper bounds are known, e.g., $908 \leq R(7,19) \leq 134595,\ 17885 \leq R(19,19) \leq 9075135299.$

- In the winter of 1988, a team at Concordia University in Canada, led by Clement Wing Hong Lam (Canada), proved the nonexistence of finite projective planes of order 10. Using a CRAY–1A supercomputer from the US Institute of Defense Analysis (IDA) and VAX machines at Concordia, they spent 3 years and a total of over 2000 hours of computer time to complete the proof. No one could guarantee that no mistake had been made, and, if a mistake was indeed made, it would be difficult to pinpoint whether it was a machine fault or a mathematical error.
- The *Travelling Salesman Problem* (in short TSP) was mathematically formulated by Rowan Hamilton as follows: Given *n* cities and the distances between each pair of cities, find the shortest path that passes through each city only once and returns to the city of origin. For this problem many heuristics and exact algorithms are known, so that some instances with tens of thousands of cities can be solved completely; further problems with millions of cities can be computationally approximated within a small error. Several variations of this problem also have been studied.

For mathematical proofs several philosophical questions have been extensively addressed: Does every theorem have a proof? For the same theorem, why are several proofs useful? Is it the case that proof is mathematics or mathematics is proof? What is the difference between a beautiful proof and an ugly proof? What is the role of proof in understanding the mathematics? Does the concept of a valid proof vary from culture to culture, as well as from age to age? This list continues. Earlier we have succinctly answered some of these questions, and we shall take up some more in Sects. 3.16 and 3.19. We conclude this section with the remark that like mathematics, a proof cannot be defined. We have seen over the course of 50 years of teaching that there is a constant decline in the appreciation of proofs; in fact, students try to avoid proofs, and they are usually happy with numerical or computational examples. If this trend continues, in the near future the definition of proof will be very different. However, professional mathematicians will definitely continue seeking traditional pure mathematical proofs.

3.15 What Is a Counterexample in Mathematics?

To prove a general statement requires a series of logical arguments; to disprove it requires but a single specific instance in which the statement fails. The latter is called a *counterexample*, and a good counterexample is priceless. However, finding counterexamples is not as easy as it might seem; in fact, certain propositions had to wait for many years, while others have refuted all efforts of great mathematicians.

- Consider the statement if *a* and *b* are positive numbers; then $\sqrt{a^2+b^2}=a+b.$ Over the years hundreds of thousands of students have invoked this very formula, as any mathematics teacher will confirm. But it is fallacious, and to show this we need a counterexample. For this, if $a=3$ and $b=4,$ then $\sqrt{a^2+b^2}=\sqrt{3^2+4^2}=\sqrt{25}=5,$ whereas $a+b=3+4=7.$ This lone counterexample is enough to dispose off the statement. Similarly, to show for positive integers $a,b,c,\ \ a^{b^c}\neq(a^b)^c$, it suffices to consider $a=b=2$ and $c=3.$ Indeed, then we have $a^{b^c}=2^{2^3}=2^8=256,$ whereas $(a^b)^c=(2^2)^3=4^3=64.$
- If *n* is an odd prime number, then the only possibilities for *n* are $3,5,$ and $6k+1,6k+5,\ \ k=1,2,\cdots$. We consider the statement that for every natural number $k,\ \ 6k+1,6k+5$ are primes. For such type of statement which involves natural numbers, a useful first step is to substitute specific numbers *k* and determine whether the statement is true. If the statement is false, a counterexample is found quickly, allowing us to reject the statement. For example, in our statement for $k=1,2,3,4,5,\ \ 6k+1$, and $6k+5,$ respectively, take the values $7,13,19,25,31$, and $11,17,23,29,35.$ Thus, $6k+1$ is not prime for $k=4$ and $6k+5$ is not prime for $k=5.$ Hence, our statement is not true for all natural numbers $k.$
- Sophie Germain, *living in an era of male chauvinism*, wrote to Gauss, in her letter dated February 20, 1807: "If the sum of the *n*th powers of any two numbers is of the form $a^2+nb^2,$ then the sum of the two numbers is also of that form." Gauss in his reply, dated April 30, 1807, gave a single counterexample to disprove her proposition. He showed that $15^{11}+8^{11}=8649755859375+8589934592=8658345793967$ is of the form $a^2+11b^2=1595826^2+11\cdot745391^2,$ but $15+8$ is not of the form

$c^2 + 11d^2$. Another similar example is $13^{11} + 10^{11} = (661539)^2 + 11(363634)^2$.

- Jacques Ozanam (1640–1718, France) stated "The sum of all the divisors of 2^{4n} is a prime." G.W. Kraft gave a counterexample of $n = 2$, showing that the sum $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256$ of all the divisors of 2^8, with sum 511 which is not a prime, since $511 = 7 \times 73$. Thus Kraft's single counterexample smashed Ozanam's statement.

3.16 Can Proofs Be Exact?

According to Agarwal and Sen [14], "a **proof** is an explanatory note that the human senses can perceive and that leads the mind and intellect to establish the validity of a fact or argument. Proofs express subjective truth, not universal truth. Since humans have limited perceptive power, there are innumerable subtle assumptions that must be made during the process of proof which are beyond human intellect; therefore any proof can never be exact. This is why we accept whatever truth can be established within our human limitations. Yet the great Sages and Rishis of history possessed extraordinarily subtle insight as well as unique experiences, which led them to establish certain truths. These are known as empirical theories, and are simply well-founded empirical generalizations or laws about the properties and behaviors of objects that are obtained by examining a large number of instances and seeing that they conform without exception to a single general pattern. These theories could never be questioned and have withstood the test of time. One accepts such theories because of his total faith in the wisdom of these sages, or because he believes this to be the primary (or the sole) path to knowledge."

It should be noted that proof has different shades of meaning depending upon the field in which it is considered. For example, within the legal system, the burden of proof in a civil court is "preponderance of evidence," whereas in a criminal court it is "beyond a reasonable doubt." Furthermore, the proof of one man's guilt in court does not mean that all men are guilty, but in geometry we must be able to prove that what is true of one triangle, for instance, is true of all triangles. In a private or public workplace, policies are established by the owner or by a set of elected or appointed individuals, and these policies become the accepted "truth" of the matter. As a result, every employee is expected to adhere to these policies, and when an employee is alleged to have violated these policies, it becomes the responsibility of the owner or the corporation to prove that such alleged violations of the policies have taken place. In this situation, the proof may involve altered data, inaccurate document, or lack of acceptable performance, and so on. In the scientific world, truth is established by experiments. After hypotheses are proposed, experiments are conducted, and based on the data gathered and analysis performed, conclusions are obtained in the form of either statements or mathematical laws. They are published in scientific Journals for anyone interested to verify. As soon as a general acceptance takes place in the scientific community, it serves as the proof, and these conclusions become the accepted "truth" of that particular scientific idea or concept. In the business world, statistical analyses of data are performed, and conclusions are derived. These are published, and after a general acceptance these conclusions become the accepted "truth" of the matter. In political or international matters, organizations or group of individuals are entrusted with finding the "truth" of a given issue, and they come up with their conclusion based on classified intelligence reports, site visits, interviews, and other such activities.

Another way in which a proof can come is in the form of existing buildings or through archeological discoveries. For example, the fact that very knowledgeable people existed hundreds of years ago can be inferred by the existence of the Great Pyramid at Gizeh, whom the Greeks called Cheops (flourished around 2680 BC), is one of the most massive buildings ever erected. It has at least twice the volume and 30 times the mass (the resistance an object offers to a change in its speed or direction of motion) of the Empire State Building in New York and is built from individual stones weighing up to 70 tons each. The slope of the face to the base (or the angle of inclination) of the Great Pyramid at Gizeh is $51^0 50' 35.''$ The same angle also appears in the ancient Hindu *Srichakra*. According to Herodotus four groups of a hundred thousand men labored 3 months each over 20 years to build this pyramid; however, calculations show that not more than 36,000 men could have worked on the pyramid at one time without bumping into one another. Another example is the existence of *Brihadeeswarar Temple* in the city of Thanjavur, State of Tamil Nadu, India. This temple is a part of the United Nations Educational, Scientific, and Cultural Organization's (UNESCO) World Heritage Sites. This temple turned 1,000 years old in 2010! However, what was used as a proof, and therefore accepted, may change with time either due to better understanding or a new discovery. Furthermore, in scientific measurements, due to uncertainty, the absolute truth cannot be found although we may get extremely close to doing so. These facts are supported by the following a few examples:

Iraq and Weapons of Mass Destruction (WMD): On March 19, 2003, the United States and Britain with other allies launched Operation Iraqi Freedom that brought about the end of Saddam Hussein's regime and ultimately resulted in his capture. As U.S. forces moved through Iraq, there were initial reports that chemical and biological weapons might have been uncovered, but closer examinations produced negative results. Although several classified information was presented, data reviewed and analyzed, very serious discussions and debates took place at the United Nations and across the globe, no major WMD have yet been found. On March 19, 2013, on the tenth anniversary of the war, National Security Archive released the briefing book of declassified documents which indicates that the US invasion of Iraq turned out to be a textbook case of flawed assumptions, wrong-headed intelligence, propaganda manipulation, and administrative ad hockery.

Atomic Theory: In 1913, Niels Henrik David Bohr (1885–1962, Denmark) developed a model for the atom. He proposed that electrons are revolving in concentric circles around the nucleus just as planets are revolving around the Sun. Thus, using this "planetary model," he successfully explained the hydrogen spectra. However, this model could not be extended to other atoms with more than one electron. In 1926, Schrödinger took Bohr's model one step further. Using complex mathematical analysis, he obtained equations (functions) that give the probability of finding an electron at a given distance away from the nucleus of an atom. Thus, Bohr (orbit) model became the Quantum mechanical (orbital) model of the atom.

Cold Fusion: Cold Fusion is a hypothetical type of nuclear reaction that would occur at, or near, room temperature. Hot Fusion takes place naturally within stars under immense pressure and at very high temperatures. In 1989, Martin Fleischmann (1927–2012, England) and Bobby Stanley Pons (born 1943, USA) reported, based on their electrolysis experiments, that they have observed cold fusion. They also reported that they measured small amounts of nuclear byproducts, including neutrons and tritium. These reported results received wide media attention and raised hopes of a cheap and abundant source of energy. Many scientists tried to replicate the experiment and hopes faded due to the large number of negative replications, the withdrawal of many positive replications, the discovery of flaws and sources of experimental error in the original experiment, and finally the discovery that Fleischmann and Pons had not actually detected nuclear reaction byproducts. By late 1989, most scientists considered cold fusion claims to be dead.

Radiocarbon Measurements: Radiocarbon measurements have a range of uses, from analyzing archaeological finds to detecting fraudulent works of art, to identifying illegal ivory trading, and to assessing the regeneration of brain cells in neurological patients. Radiocarbon dating works by measuring how much the fraction of carbon 14 versus nonradioactive carbon in an object has changed and therefore how long the object has been around. Fossil fuel emissions could soon make it impossible for radiocarbon dating to distinguish new materials from artifacts that are hundreds of years old. Carbon released by burning fossil fuels is diluting radioactive carbon 14 and artificially raising the radiocarbon "age" of the atmosphere, according to a paper published in the journal Proceedings of the National Academy of Sciences (PNAS). Using Marine records, research is on-going to recalibrate carbon 14 dating work that has been done previously. Although this may shift the ages by only a few hundred years, it could help narrow the window of key events and therefore our current understanding of human history.

Astronomy: Recently NASA announced that it found water on Mars in solid or gaseous form. How was that possible? Based on scientific experiments, analysis of photographs, etc. This conclusion may be revised, augmented, or completely negated at a later date. At present, it is not known how this discovery may change. For example, at one point, Pluto was considered to be a planet. However, on August 24, 2006, members of the International Astronomical Union voted on a new definition of a planet, and as a result, Pluto lost its status as a planet.

Hindu Mythology: In the original version of the *Valmiki Ramayana*, a vivid description of a total solar eclipse is given in the first 15 *slokas* of the 23rd *sarga* of the *Aranyakandam*, but there is indeed a mention of *Rahu* as the cause. It is suggested in the *Mahabharata* that during the 18-day long war between the *Pandavas* and *Kauravas*, there were only 13 days between a Full Moon and a possible total solar eclipse presumed to have been seen over the battle field of *Kuruksetra*. Also, solar eclipse is mentioned in the *Mahabharata*, where Lord Krishna skillfully used his knowledge of eclipse predictions to save the life of *Arjuna*, the great warrior. A sizable portion of the 35th chapter of the *Bhagvata Purana* is devoted to narrating a fable as to why *Rahu* is responsible for all the solar and lunar eclipses by "swallowing" the Sun or the Moon. The *Surya Siddhanta* gives detailed methods for making ecliptic calculations. It was around this time that *Rahu* and *Ketu* were astronomically defined to be the ascending and the descending nodes of the lunar orbit, intersecting the plane of the Earth's orbit. The astronomical significance of *Rahu* and *Ketu* was reduced to mere imaginary points of intersection between the lunar orbit and the plane of the ecliptic. Clearly, whatever the reason that was considered a long time ago has changed!

Religion: In a religious sense, proof may be difficult to even describe. For example, Sri Chandrasekharendra Saraswati Mahaswamiji (1894–1994, India), or the Sage of Kanchi, was the 68th Jagadguru of the Kanchi Kamakoti Peetham. He was usually referred to as Paramacharya, Mahaswami, or Maha Periyavar. He was widely considered as one of the greatest Indian sages of recent times. By all accounts, he was considered as a "living God" by thousands and thousands of individuals. What "Proof" did they have? None other than PERSONAL EXPERIENCE. How can this be defined and explained? In 1920, while on his death bed, Ramanujan cryptically wrote down functions that he said were revealed to him by the goddess Namagiri. Ramanujan believed that 17 new functions that he discovered were "mock modular forms" that looked like theta functions when written out as an infinite sum but were not super-symmetric. Unfortunately, he could not prove them. After more than 90 years, in 2012, it was proved that these functions indeed mimicked modular forms but do not share their defining characteristics, such as super-symmetry. How can the experience of Ramanujan be proved?

In all of these cases mentioned above, we considered examples from fields other than mathematics. In what follows we shall show that the history of mathematics is full of erroneous proofs and why any proof can never be exact.

- In the following list each result is wrongly stated by a leading mathematician(s), but a modified version was later rigorously proved.

  Perimeter of an Ellipse: Archimedes proved that the area formula for an ellipse: $A = \pi ab,$ where *a* and *b* are the semi-major and semi-minor axes, respectively. If $a = b = r,$ this reduces to the well-known formula for the area of a circle, namely, $A = \pi r^2$. Now, consider the square whose sides are tangential to the circle. The ratio of the area of the circle to that of its tangential square is $\pi : 4$, and this ratio happens to coincide with that of the perimeters of the circle and the square. So, by analogy, it seems perfectly reasonable to guess that, as the ratio of the area of an ellipse to that of its tangential rectangle is $\pi : 4$, it should also be equal to the ratio between their perimeters. In this case, since the perimeter of the rectangle is $4(a+b)$, the perimeter of the ellipse would be $p = (\pi/4) \times 4(a+b) = \pi(a+b)$. Note that when $a = b = r$, this reduces to the correct formula (for the perimeter of a circle). Indeed, Fibonacci did propose this formula, which of course we now know is wrong. As a matter of fact, the perimeter of an ellipse is not simple to compute. Ramanujan proposed the following two approximate formulae $p \simeq \pi[3(a+b) - \sqrt{(3a+b)(a+3b)}]$, and

$$p \simeq \pi(a+b)\left[1 + \frac{3h}{10 + \sqrt{4-3h}}\right], \quad h = \frac{(a-b)^2}{(a+b)^2}.$$

Exactly *p* is also known in terms of infinite series and integral

$$p = 4a\int_0^{\pi/2} \sqrt{(1 - e^2 \sin^2\theta)}\, d\theta, \quad e = \frac{\sqrt{a^2 - b^2}}{a} \quad \text{(eccentricity)}.$$

Rational Cubes: In 1798, Legendre claimed that 6 is not the sum of two rational cubes, which as Gabriel Lamé (1795–1870, France) pointed out in 1865 is false as $6 = (37/21)^3 + (17/21)^3$.

Uniform Convergence: In his Cours d'analyse of 1821, Cauchy "proved" that if a sum of continuous functions converges pointwise, then its limit is also continuous. However, Abel observed 3 years later that this is not the case. For the conclusion to hold, "pointwise convergence" must be replaced with "uniform convergence." As a counterexample, a Fourier series of sine and cosine functions, all continuous, may converge pointwise to a discontinuous function such as a step function.

Matrix Multiplication: Cayley asserted that if the product of two nonzero square matrices, *A* and B, is zero, then at least one of the factors must be singular (non-inevitable). Cayley was correct, but surprisingly he overlooked an important point, namely, that if $AB = 0$, then *A* and *B* must both be singular.

Intersection Theory: In 1848, Jakob Steiner (1796–1863, Switzerland) claimed that the number of conics tangent to 5 given conics is $7776 = 6^5,$ but later realized this was wrong. The correct number 3264 was found by Ernest Jean Philippe Fauque de Jonquiéres (1820–1901, France) around 1859 and Michel Chasles in 1864. However, these results, like many others in classical intersection theory, do not seem to have been given complete proofs until the work of William Edgar Fulton (born 1939, USA) and Robert Duncan MacPherson (born 1944, USA) in 1978.

Dirichlet's Principle: The principle of Dirichlet (the assumption that the minimizer of a certain energy functional is a solution to Poisson's equation) was used by Riemann in 1851 in the study of complex analytic functions, but Weierstrass found a counterexample to one version of this principle in 1870, and Hilbert stated and proved a correct version in 1900.

Kronecker-Weber Theorem: The proofs of this theorem by Kronecker in 1853 and Heinrich Martin Weber (1842–1913, Germany) in 1886 both had gaps. The first complete proof was given by Hilbert in 1896.

Wronskians: In 1887, Paul Mansion (1844–1919, Belgium) claimed in his textbook that if a Wronskian (after Józef Maria Hoëné-Wroński, 1776–1853, Poland) of some functions vanishes everywhere, then the functions are linearly dependent. In 1889, Peano pointed out the counterexample x^2 and $x|x|.$ The result is correct if the functions are analytic.

Groups of order 64: In 1930, George Abram Miller (1863–1951, USA) published a paper claiming that there are 294 groups of order 64. Marshall Hall, Jr. (1910–1990, USA) and James Kuhn Senior (1935–2020, USA) in 1964 that the correct number is 267.

Grunwald-Wang Theorem: Wilhelm Grunwald (1909–1989, Germany) published an incorrect proof in 1933 of an incorrect theorem, and George William Whaples (1914–1981, USA) later published another incorrect proof. Shianghao Wang (1915–1993, China) found a counterexample in 1948 and published a corrected version of the theorem in 1950.

Rokhlin Invariant: Vladimir Abramovich Rokhlin (1919–1984, Russia) in 1951 incorrectly claimed that the third stable stem of the homotopy groups of spheres is of order 12. In 1952 he discovered his error: It is in fact cyclic of order 24. The difference is crucial as it results in the existence of the Rokhlin invariant, a fundamental tool in the theory of 3- and 4-dimensional manifolds.

Nielsen Realization Problem: Saul Kravetz (127–1974, USA) claimed to solve Jakob Nielsen (1890–1959, Denmark) realization problem in 1959 by first showing that Teichmüller space is negatively curved, but in 1974 Howard Masur (USA) showed that it is not negatively curved. Finally, the problem was solved in 1980 by Steven Paul Kerckhoff (born 1952, USA).

Classification of N-Groups: The original statement of the classification of *N*-groups by John Griggs Thompson (born 1932, USA) in 1968 accidentally omitted the Tits group, though he soon fixed this.

- The following is the list of proofs which were originally shown to be wrong.

In 1803, Giovanni Francesco Giuseppe Malfatti (1731–1807, Italy) claimed to prove that a certain arrangement of three circles would cover the maximum possible area inside a right triangle. However, to do so he made certain unwarranted assumptions about the configuration of the circles. It was shown in 1930 that circles in a different configuration could cover a greater area, and in 1967 that Malfatti's configuration was never optimal.
In 1806, André Marie Ampére (1775–1836, France) claimed to prove that a continuous function is differentiable at most points, but in 1872 Weierstrass surprised the mathematical community by giving the following function which is continuous everywhere, but nowhere differentiable

$$W(x) \;=\; \sum_{k=0}^{\infty} a^k \cos\left(b^k \pi x\right),$$

where *a* is a real number with $0 < a < 1,$ b is an odd integer, and $ab > 1 + 3\pi/2.$ We also have nowhere continuous functions, e.g., in 1829 Dirichlet defined the function $f : \mathcal{R} \to \mathcal{R}$ by

$$f(x) \;=\; \begin{cases} 1 & \text{if } x \in \mathcal{Q} \\ 0 & \text{if } x \notin \mathcal{Q}. \end{cases}$$

In fact, for this function $\lim_{x\to x_0} f(x)$ does not exist at any $x_0 \in \mathcal{R}$.

In 1878, Cayley incorrectly claimed that there are three different groups of order 6. This mistake is strange because in an earlier 1854 paper he correctly stated that there are just two such groups.
In 1891, Karl Theodor Vahlen (1869–1945, Austria) published a purported example of an algebraic curve in three-dimensional projective space that could not be defined as the zeros of three polynomials, but in 1941 Oskar Perron (1880–1975, Germany) found three equations defining Vahlen's curve. In 1961, Martin Kneser (1928–2004, Germany) showed that any algebraic curve in projective 3-space can be given as the zeros of three polynomials.
In 1898, George Abram Miller published a paper incorrectly claiming to prove that the Mathieu group M_{24} (after Émile Léonard Mathieu, 1835–1890, France) does not exist, though in 1900 he pointed out that his proof was wrong.
In 1900, Charles Newton Little (1858–1923, USA) claimed that the writhe of a reduced knot diagram is an invariant. However, in 1974, Kenneth Perko (USA) discovered a counterexample called the Perko pair, a pair of knots listed as distinct in tables for many years that are in fact the same.
In 1905, Henri Léon Lebesgue (1875–1941, France) tried to prove the (correct) result that a function implicitly defined by a René-Louis Baire (1874–1932, France) function is Baire, but his proof incorrectly assumed that the projection of a Félix Édouard Justin Émile Borel (1871–1956, France) set is Borel. Mikhail Yakovlevich Suslin (1894–1919, Russia) pointed out the error and was inspired by it to define analytic sets as continuous images of Borel sets.
In 1908, Josip Plemelj (1873–1967, Slovenia) claimed to have shown the existence of a Fuchsian differential equations with any given monodromy group (Hilbert's 21st Problem), but in 1989 Andrei Andreevich Bolibrukh (1950–2003, Russia) discovered a counterexample.
In 1911, Hardy and Littlewood announced their first joint work at the meeting of the London Mathematical Society. The result was never published because they later discovered that their proof was incorrect.
In 1929, Lazar Aronovich Lyusternik (1899–1981, Poland-Russia) and Lev Genrikhovich Schnirelmann (1905–1938, Russia) published a proof of the theorem of the three geodesics, which was later found to

be flawed. The proof was completed by Hans Werner Ballmann (born 1951, Germany) about 50 years later.
In 1934, Francesco Severi (1879–1961, Italy) claimed that the space of rational equivalence classes of cycles on an algebraic surface is finite-dimensional, but in 1968 David Bryant Mumford (born 1937, USA) showed that this is false for surfaces of positive geometric genus.
In 1961, Jan-Erik Ingvar Roos (1935–2017, Sweden) published an incorrect theorem about the vanishing of the first derived functor of the inverse limit functor under certain general conditions. However, over 40 years later, Amnon Neeman (born 1957, Israel-Australia) constructed a counterexample.
In 1994 and 1999, Gaoyong Zhang (China-USA) published two papers in the Annals of Mathematics. In the first paper he proved that the 1956 *Busemann-Petty problem* in R^4 (after Herbert Busemann, 1905–1994, Germany-USA, and Clinton Myers Petty, 1923–2021, USA) has a negative solution, and in the second paper he proved that it has a positive solution.

- The following is the list of proofs for which the status is not clear.

Hilbert's 16th problem consists of two similar problems in different branches of mathematics: An investigation of the relative positions of the branches of real algebraic curves of degree *n* (and similarly for algebraic surfaces) and the determination of the upper bound for the number of limit cycles in two-dimensional polynomial vector fields of degree *n* and an investigation of their relative positions. For the question whether there exists a finite upper bound for the number of limit cycles of planar polynomial vector fields of degree $n,$ Evgenii Mikhailovich Landis (1921–1997, Russia) and Ivan Georgievich Petrovsky (1901–1973, Russia) claimed a solution in the 1950s, but it was shown wrong in the early 1960s. In 1991/1992, Yulij Sergeevich Ilyashenko (born 1943, Russia) and Jean Écalle (born 1947, France) showed that every polynomial vector field in the plane has only finitely many limit cycles; however, in an article published in 1823, Henri Claudius Rosarius Dulac (1870–1955, France) had already claimed that a proof of this statement contains a gap.
Italian school of algebraic geometry. Most gaps in proofs are caused either by a subtle technical oversight or before the twentieth century by a lack of precise definitions. A major exception to this is the Italian school of algebraic geometry in the first half of the twentieth century,

where lower standards of rigor gradually became acceptable. The result was that there are many papers in this area where the proofs are incomplete or the theorems are not stated precisely. This list contains a few representative examples, where the result was not just incompletely proved but also hopelessly wrong.
In 1933, George David Birkhoff (1884–1944, USA) and Waldemar Joseph Trjitzinsky (1901–1973, Russia-USA) published a very general theorem on the asymptotics of sequences satisfying linear recurrences. The theorem was popularized by Jet Wimp (born 1934, England-USA) and Doron Zeilberger (born 1950, Israel) in 1985. However, while the result is probably true, Birkhoff and Trjitzinsky's proof is not generally accepted by experts, and the theorem is proved only in special cases.
In 1978, Wilhelm Paul Albert Klingenberg (1924–2010, Germany) published a proof that smooth compact manifolds without boundary have infinitely many *Closed Geodesics*. His proof was controversial, and there is no consensus on whether his proof is complete.
In 2003, Daniel Kálmán Biss (born 1977, USA) published a paper in the Annals of Mathematics claiming to show that *Matroid Bundles* are equivalent to real vector bundles, but in 2009 published a correction pointing out a serious gap in the proof.

- Proofs often extend to hundreds of pages or more and are so complicated that years of intensive study may pass before they are confirmed by the handful experts in the field. Can we assume 100% validity of such proofs? Marianne Freiberger (born 1972, Germany) writes "these days mathematics contains proofs so long and complex that a few people are able to check and understand them in full, yet once a result has made it through the peer review process and into a journal, its truth is taken as read." We hope over time the essence of such proofs becomes clearer, and more concise and enlightening versions are written. The following is a small list of excessively long mathematical proofs.

In 1799, Paolo Ruffini (1765–1822, Italy) almost proved there is no solution in radicals to general polynomial equations of degree 5 or higher with arbitrary coefficients. His 500-pages proof was largely ignored, but then in 1824, Abel published a proof that required just six pages. In the literature this result is known as Abel-Ruffini theorem.

In 1890, Wilhelm Karl Joseph Killing (1847–1923, Germany) classified the complex finite-dimensional simple Lie algebras and discovered the exceptional Lie algebras. He published his 180-pages work in four research papers.
In 1894, Johann Gustav Hermes (1846–1912, Germany) gave the ruler-and-compass construction of a polygon of 65537 sides in over 200 pages.
In 1905, Emanuel Lasker (1868–1941, Germany) proved in 98 pages a special case of the result: Every ideal can be decomposed as an intersection, called primary decomposition, of finitely many primary ideals. The general result now known as Lasker-Noether theorem was proved by Amalie Emmy Noether (1882–1935, Germany-USA) in 1921. Present-day less than one-page proofs are known of this important theorem.
In 1966, Shreeram Shankar Abhyankar (1930–2012, India-USA) proved the resolution of singularities for three-fold in characteristic greater than 6 which covered about 500 pages (in the form of several research papers). In 2009, Steven Dale Cutkosky (USA) simplified this to 69 pages.
In 1966, Harish Chandra provided the discrete series representations of Lie groups in a long series of papers totaling around 500 pages. His later work on the Plancherel theorem (after Michel Plancherel, 1885–1967, Switzerland) for semisimple groups added another 150 pages.
In 1974, Thompson gave the classification of *N*-groups in six papers totaling about 400 pages. He also needed his earlier results, which brought to total length up to more than 700 pages.
In 1974, Pierre René, Viscount Deligne (born 1944, Belgium) proved Ramanujan conjecture (about tau function) and Weil conjectures (about zeta functions of varieties over finite fields) in only 30 pages; however, he used results from algebraic geometry and étale cohomology that estimated to be about 2000 pages long.
In 1980s, the classification of finite, simple groups, which originated around 1890, was completed. The demonstration consists of 500 articles totaling nearly 15000 pages and written by more than 100 workers. It has been said that the only person who grasped the entire proof was its general contractor, Daniel Gorenstein (1923–1992, USA).
In 1983, Dennis Arnold Hejhal (born 1948, USA) proved a general form of the Atle Selberg (1917–2007, Norway) trace formula which consists of two volumes with a total length of 1322 pages.

In 2000, Frederick Justin Almgren (1933–1997, USA) gave a proof of regularity theorem (the singular set of an *m*-dimensional mass-minimizing surface has dimension at most $m-2$) in 955 pages.
In 2004, Aschbacher and Stephen Smith (born 1948, USA) published the classification of the simple quasithin groups in 1221 pages, which is one of the longest single papers ever written.

- Recent research on ancient Indian and Chinese mathematics shows that there were other methods of investigating mathematical truths and of making new and useful applications other than commonly accepted proof methods as described by Euclid. Unfortunately, in all exiting literature these methods have not either been mentioned or completely condemned. For example, according to Morris Kline (1972), "as our survey indicates, the Hindus were interested in and contributed to the arithmetical and computational activities of mathematics rather than to the deductive patterns. Their name for mathematics was Ganita, which means *the science of calculation*. There is much good procedure and technical facility, but no evidence that they considered proof at all. They had rules, but apparently no logical scruples. Moreover, no general methods or new viewpoints were arrived at in any area of mathematics." Hindu mathematicians for proof used the word *upapattis* (a common word in Buddhism, Pali, Hinduism, Sanskrit, Jainism, Prakrit, Marathi, and Hindi). Some of the important features of upapattis in Indian mathematics are as follows (see Srinivas [494] and Sudhakaran [501]):

  1. The Indian mathematicians are clear that results in mathematics, even those enunciated in authoritative texts, cannot be accepted as valid unless they are supported by yukti or upapatti. It is not enough that one has merely observed the validity of a result in a large number of instances.

  2. Several commentaries written on major texts of Indian mathematics and astronomy present upapattis for the results and procedures enunciated in the text.

  3. The upapattis are presented in a sequence proceeding systematically from known or established results to finally arrive at the result to be established.

  4. In the Indian mathematical tradition the upapattis mainly serve to remove doubts and obtain consent for the result among the

community of mathematicians.

5. The upapattis may involve observation or experimentation. They also depend on the prevailing understanding of the nature of the mathematical objects involved.

6. The method of tarka or "proof by contradiction" is used occasionally. But there are no upapattis which purport to establish the existence of any mathematical object merely on the basis of tarka alone. In this sense the Indian mathematical tradition takes a "constructivist" approach to the existence of mathematical objects. However, in India over the centuries there were several scholars well-versed in Tarka Sastras starting from Adi Shankara (788–820 CE).

7. The Indian mathematical tradition did not subscribe to the ideal that upapattis should seek to provide irrefutable demonstrations establishing the absolute truth of mathematical results.

8. There was no attempt made in Indian mathematical tradition to present the upapattis in an axiomatic framework based on a set of self-evident (or arbitrarily postulated) axioms which are fixed at the outset.

9. While Indian mathematicians made great strides in the invention and manipulation of symbols in representing mathematical results and in facilitating mathematical processes, there was no attempt at formalization of mathematics.

- Karl Raimund Popper (1902–1994, Austria-England) believed that nothing can ever be known with absolute certainty. He had a concept of "truth up to falsifiability." There is a widespread notion that once something has been proved mathematically, then it is, as it were, set in stone that we have a mathematical proof that remains "true for all time" (as an official certification). "Not so," says James R. Meyer (Ireland). He writes that most mathematical proofs that anyone will encounter fall a long way short of idealistic concepts. In conclusion, the standards of mathematical proof also keep on changeling according to time. Thus, Greeks who conceived of the world as being made in a mathematical mold—a conception that is still held by most people, seems to be falling apart.

3.17 What Is a Conjecture in Mathematics?

In 1963, Popper pioneered the use of the term conjecture in scientific philosophy. A conjecture in mathematics is an unproved intuitive and shrewd guess or opinion, preferably based on some experience or other source of wisdom such as patterns, made by a person with skilled mathematical insight. Conjectures in mathematics have been a source of constant challenge to the human brain. Some of them have been proved and received the status of theorems (Conjecture$+$Proof $=$ Theorem), while many remain unsolved from centuries but believed to be correct, and others have been disproven through counterexamples. The quality of a conjecture is judged by the simplicity, and the time gap between it is proposed and the day it is proved or disproved. It is a glorious day for mathematics and the mathematician when a certain long-standing conjecture is proved or disproved. The following are some examples:

- Kepler Conjecture: In 1611, Kepler posed the problem "what is the most efficient way to pack balls of the same size into space"? As a mathematical result it states that no arrangement of equally sized spheres filling Euclidean three-dimensional space has a greater average density than that of the cubic close packing (face-centered cubic) and hexagonal close packing arrangements. The density of these arrangements is around $\pi/\sqrt{18} \simeq 0.74.$ In 1990, Wu-Yi Hsiang (born 1937, China-USA) claimed to have proven the Kepler conjecture by using spherical trigonometry; however, the current consensus is that Hsiang's almost 100-page proof is incomplete. In 2003, Thomas Callister Hales (born 1958, USA) used a long computation on a computer to verify conjecture, which was accepted for publication in 2005 (published in 2008 [243]), but with 99% correctness of the proof; however, in 2014 it was verified to be correct. In 2017, the formal proof of Hales has been accepted, so Kepler conjecture is now Kepler theorem.
- Graeco-Latin Squares: Following Ozanam consider the 16 picture cards from the usual pack of playing cards, namely, four aces, four kings, four queens, and four jacks. These 16 cards can be classified into two different ways: (i) according to their denominations, ace, king, queen, and jack, and (ii) according to their suit, club, diamond, heart, and spade. Now suppose that we want to arrange these 16 picture cards in a 4×4 square (i.e., a square having four rows and four columns) in such a way that every denomination and every suit must occur once and only once

in each row and in each column. Now let the Latin (or Roman) letters A, B, C, D stand for the denomination namely ace, king, queen, jack, respectively, and let the Greek letters $\alpha, \beta, \chi,$ and δ denote the four suits namely club, diamond, heart, and spade, respectively. Thus $A\alpha$ indicates the club ace, and similarly $A\beta, B\alpha$, etc. Thus we have the 16 picture cards represented by $A\alpha, A\beta, A\chi, A\delta, B\alpha, \cdots, D\delta$. Figure 3.7 then gives one of the arrangements that satisfies the requirements laid down above by us. Observe that each Latin letter and each Greek letter occur once and only once in each row and each column of the 4×4 square. Also observe that all the 16 possible representations or combinations $A\alpha, \cdots, D\delta$ have found places in the cells of the square.

$A\alpha$	$B\beta$	$C\chi$	$D\delta$
$B\chi$	$A\delta$	$D\alpha$	$C\beta$
$C\delta$	$D\chi$	$A\beta$	$B\alpha$
$D\beta$	$C\alpha$	$B\delta$	$A\chi$

Fig. 3.7 4×4 Graeco-Latin square

Such a square configuration is called a Graeco-Latin square of order $4.$ The name is so given because the Greek and Latin letters are used in its construction. Graeco-Latin square of order 4, given in Fig. 3.7, can be obtained by super-imposing on each other the two squares given in Fig. 3.8, and four different symbols are used; in one, symbols used are $A, B, C, D,$, while in the other they are $\alpha, \beta, \chi, \delta.$ Actual symbols used are immaterial; one could use, for example, 1,2,3,4 if one likes. What is important, however, is that each symbol used must occur once and only once in each row and in each column of the square. One can verify that this is so in the case of each of the squares of Fig. 3.8. A square satisfying this property is called a Latin square irrespective of the nature of the symbols used in constructing the square (maybe, the first time such a square was constructed, Latin letters were used). Thus each of the two squares in Fig. 3.8 is a Latin square.

A	B	C	D
B	A	D	C
C	D	A	B
D	C	B	A

α	β	χ	δ
χ	δ	α	β
δ	χ	β	α
β	α	δ	χ

Fig. 3.8 4-Graeco-Latin squares

Two Latin squares, which when superimposed gives a Graeco-Latin square, are called orthogonal Latin squares. The two Latin squares of Fig. 3.8 are thus orthogonal Latin squares (orthogonal to each other). One can also notice that the second Latin square of Fig. 3.8 is obtainable from the first by establishing a one-one correspondence between the Latin and Greek letters used in them and by suitably rearranging the rows of the first square. If we want to construct all possible Graeco-Latin squares by first systematically enumerating all possible Latin squares and superimposing two using every possible pair out of them and verifying whether what we get is a Graeco-Latin square or not, we should first try the exercise with Latin squares of order 3. This will give us the feeling, idea, and insight of the amount of effort involved in the construction of Latin and Graeco-Latin squares of higher orders. We can prove that there are at least $[n!(n-1)!(n-2)!\cdots 1!]$ Latin squares of order $n.$ Now we can verify that there are only 12 Latin squares of order 3. The number of possible Latin squares increases rapidly as *n* increases. There are 576 Latin squares of order 4 and 161,280 of order 5. Furthermore, there are never more than $n-1$ mutually orthogonal Latin squares of order $n.$

Now the question is can we have a Graeco-Latin square of order *n* for any positive integer n? In 1782, Euler made the conjecture: “A Graeco-Latin square of order $4n+2,\ n=0,1,2,\cdots$ does not exist.” Thus, according to Euler’s conjecture, Graeco-Latin squares of order $2,6,10,\cdots$ do not exist. One can easily verify that a Graeco-Latin square of order 2 cannot exist. In 1901, Gaston Tarry (1843–1913, France) used mechanical aids to verify that Graeco-Latin squares of order 6 do not exist. In 1959, Sharadchandra Shankar Shrikhande (1917–2020, India) along with Raj Chandra Bose (1901–1987, India) and Ernest Tilden Parker (1926–1991, USA) disproved Euler’s conjecture; in fact, they showed any order $4n+2,\ (n\geq 2)$ Graeco-Latin square exists, i.e., Graeco-Latin squares exist for all orders $n>1$ except $n=2,6.$ An example of a 10×10 Graeco-

Latin square is given in Fig. 3.9. Here digits have been used instead of Greek and Roman letters. It is interesting to note that the 3×3 square at the right-hand bottom corner of this 10×10 Graeco-Latin squares is itself a Graeco-Latin square. Bose, Shirkhande, and Parker constructed many different Graeco-Latin squares of order $10,$ and each of them contained such a Graeco-Latin square of order $3.$ Graeco-Latin squares find applications in designing experiments in biology, agriculture, medicine, sociology, marketing, etc.

- Waring Conjecture: In 1770, Edward Waring (1736–1798, England) emitted the conjecture that every integer $n > 0$ is the sum of a fixed least number $g(s)$ (Little Gee) of *s*th powers of integers $\geq 0.$ For $s = 2,$ this is the result proved by Lagrange in 1770 and Euler in 1773 that every positive integer > 0 is a sum of four integer squares $\geq 0,$ (known as *Four-Square Theorem*, see Sect. 4.20), i.e., $g(2) = 4.$ For example, $141 = 1^2 + 2^2 + 6^2 + 10^2$ and $293 = 2^2 + 8^2 + 9^2 + 12^2.$ Next, $g(3) = 9$ was established from 1909 to 1912 by Arthur Josef Alwin Wieferich (1884–1954, Germany) and Aubrey John Kempner (1880–1973, England-USA). In 1939, Leonard Eugene Dickson (1874–1954, USA) showed that the only integers requiring nine cubes are $23 = 2^3 + 2^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3 + 1^3$ and $239 = 4^3 + 4^3 + 3^3 + 3^3 + 3^3 + 3^3 + 1^3 + 1^3 + 1^3.$ Wieferich also proved that only 15 integers require eight cubes: 15,22,50,114,167,175,186,212,231,238,303,364,420,428, and 454. For example, $454 = 7^3 + 4 \times 3^3 + 3 \times 1^3.$ It is also known that every other integer larger than 454 can be represented as the sum of at most seven positive cubes. The largest number known requiring seven cubes is 8042.

00	47	18	76	29	93	85	34	61	52
86	11	57	28	70	39	94	45	02	63
95	80	22	67	38	71	49	56	13	04
59	96	81	33	07	48	72	60	24	15
73	69	90	82	44	17	58	01	35	26
68	74	09	91	83	55	27	12	46	30
37	08	75	19	92	84	66	23	50	41
14	25	36	40	51	62	03	77	88	99
21	32	43	54	65	06	10	89	97	78
42	53	64	05	16	20	31	98	79	87

Fig. 3.9 10×10 Graeco-Latin square

Waring himself proved no single case of his problem, nor did he offer any suggestion for its solution. For all that he or anyone else in the eighteenth century knew $g(s)$ might not exist. Hilbert's proof of Waring's conjecture in 1909 established the existence of $g(s)$ for every $s,$ but did not determine its numerical value for any $s.$ Hardy and Littlewood during 1920–1928 invented an analytic method (the spirit of which is applicable to many other extremely difficult questions in arithmetic) for Waring's problem. The problem affiliated with $g(s)$ is that of finding $G(s)$ (Big Gee), defined as the least integer *n* such that every positive integer beyond a certain finite value is the sum of *n* the *s*th powers of integers $\geq 0.$ Specifically, Hardy and Littlewood showed that $g(s) \leq (1/2\ s - 1)2^{s-1} + 3$ for $s = 3,$ or $s \geq 5$ and $G(s) \leq (s-2)2^{s-1} + 5$ for all $s.$ Their work remained the standard till Ivan Matveevich Vinogradov (1891–1983, Russia), who developed his own more penetrating technique in the 1930s. The best value of $g(4)$ up to 1933 was $g(4) \leq 35,$ and in contrast with Hardy and Littlewood's $G(4) \leq 19,$ it was shown in 1836 that $G(4)$ is either 16 or 17. In 1986, Ramachandran Balasubramanian (born 1951, India), François Dress

(France), and Jean-Marc Deshouillers (born 1946, France) finally established that $g(4) = 19.$ There are only seven integers which cannot be expressed as the sum of fewer than 19 fourth powers $79, 159, 239, 319, 399, 479, 559.$ For example, $79 = 4 \times 2^4 + 15 \times 1^4$ and $559 = 6 \times 3^4 + 4 \times 2^4 + 9 \times 1^4.$ After that every integer can be represented as the sum of at most 18 fourth powers. It has also been proven that there are infinitely many positive integers which require at least 16 fourth powers. In 1964 and 1940, respectively, it was proved that $g(5) = 37$ and $g(6) = 73$ by Jingrun Chen (1933–1996, China) and Pillai. For $s > 6,$ Dickson and Pillai in 1936 proved almost simultaneously an explicit formula for $g(s) = 2^s + [(3/2)^s] - 2$ valid for all $s > 6,$ except possibly for certain doubtful cases (this formula was earlier conjectured by Euler in 1772). However, these doubtful cases were disposed of by Niven in 1943. Thus, after 173 years Waring's guess was finally proved.

- Riemann Hypothesis/Conjecture: The well-known *Riemann Zeta Function* is defined as

$$\zeta(s) \;=\; \sum_{n=1}^{\infty} \frac{1}{n^s} \;=\; 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots , \qquad s = \sigma + it, \quad \sigma > 1 \tag{3.26}$$

  (for real *s* this series was considered by Euler in 1737). In 1859 memoir *On the Number of Primes Less Than a Given Magnitude*, Riemann made six conjectures regarding the zeta function. By almost 1920, five of them were proved right. The sixth is now known as the Riemann hypothesis, which asserts that on the Argand diagram for *s* the nontrivial zeros of the zeta function lie on the line $\sigma = 1/2$ (trivial zeros are at $-2, -4, -6 \cdots$). Many efforts and achievements have been made toward proving this celebrated hypothesis, but it is still an open problem. Its importance and fascination can be understood by the following anecdotes: Hilbert, once said that, were he awakened after having slept for a thousand years, his first question would be, has the Riemann hypothesis been proved? Hardy once took a risky boat and wrote a post card to Bohr, "I have proved Riemann's hypothesis." His argument was that if the boat sank and he drowned, everybody would believe that Hardy had proved the hypothesis, but since God would not afford Hardy such a great honor, he could not allow the boat to sink. In 2000, during the centennial of Hilbert's 23 problems, Bombieri considered it to be the most important unsolved problem in pure mathematics. In the year 2000, Clay Mathematics Institute selected seven well-known problems known as The Millennium Prize Problems and pledged US dollar 1 million prize for the correct solution of any of them. Riemann hypothesis is one of those seven problems. Around 1914, Hardy proved something short of the hypothesis. He proved that an infinite number of solutions of the zeta functions lie on $\sigma = 1/2.$ But that does not amount to saying that all did as claimed by the conjecture. In 1919, Polya stated that "most" (i.e., 50% or more) of the natural numbers less than any given number have an odd number of prime factors. He showed that the truth of this statement would imply the Riemann hypothesis. Unfortunately, in 1958, Colin Brian Haselgrove (1926–1964, England) proved that Polya's statement is wrong. In 2004, Riemann's conjecture was verified on computer for the first 10^{13} zeros. Few number theorists doubt that the Riemann hypothesis is true; in fact Selberg was once a sceptic, whereas Littlewood always was.

The following sensational news was reported in a 1945 issue of Time magazine: The German-American mathematician Hans Adolph Rademacher (1892–1969) had announced a solution to one of the most famous of all mathematical problems—the Riemann Hypothesis. However, he later had to withdraw the claim because Carl Ludwig Siegel (1896–1981, Germany) found an error. In 2018, Michael Atiyah at the Heidelberg Laureate Forum gave a lecture in which he claimed to have proved the hypothesis. But unfortunately, his proof also turned out to be another failed attempt. In anticipation of Riemann's hypothesis (and other unproved conjectures of a similar character), numerous profound theorems specially in number theory have been proved, which require the most refined analysis of the twentieth century. These are called *conditional proofs*: The conjectures assumed appear in the hypotheses of the theorem, for the time being. These "proofs," however, would fall apart if it turned out that the hypothesis was false, so there is considerable interest in verifying the truth or falsity of conjectures of this type. The Riemann hypothesis has been repeatedly generalized, and the more far-reaching the generalizations, the more central they appear to be to the structure of modern number theory. It also has potential implications in physics.

- Mertens Conjecture: An *n*th root of unity (see Sect. 2.11) is said to be *primitive* if it is not an $1 \le m \le (n-1)$th root of unity. If *n* is a prime number, then all *n*th roots of unity, except 1, are primitive. The *Möbius function* $\mu(n)$ is defined in 1831 as the sum of the primitive *n*th roots of unity. It has values in $\{-1, 0, 1\}$ depending on the factorization of *n* into prime factors: $\mu(n) = +1$ if *n* is a square-free positive integer with an even number of prime factors, e.g., $6 = 2 \times 3,$ $\mu(n) = -1$ if *n* is a square-free positive integer with an odd number of prime factors, e.g., $30 = 5 \times 2 \times 3,$ and $\mu(n) = 0$ if *n* has a squared prime factor, e.g., $9 = 3^2.$ Möbius found that the probability of $\mu(n)$ having its value -1 and 1 is the same to $3/\pi^2.$ The *Mertens function* (after Franz Mertens, 1840–1927, Poland) is defined as $M(n) = \sum_{k=1}^{n} \mu(k).$ In 1897, Mertens made the conjecture that for all $n > 1,$ $|M(n)| < \sqrt{n}.$ However, it was earlier conjectured in 1885 by Thomas Joannes Stieltjes (1856–1894, The Netherlands) in a letter to Hermite. It has been shown that Mertens conjecture implies the Riemann hypothesis. By the early 1980s computers had shown that Mertens conjecture indeed holds for at least the first 10 billion integers; however, in 1985, Andrew Michael Odlyzko (born 1949, Poland-USA) and Hermanus Johannes Joseph Te Riele (born 1947, The Netherlands) proved the Mertens conjecture is false. In 1987, János Pintz (born 1950, Hungary) showed that the first counterexample appears below $e^{3.21\times10^{64}} \approx 10^{1.39\times10^{64}}$, whereas in 2016, Greg Hurst (USA) has shown the example must be above 10^{16}. In 2006, Tadej Kotnik (Slovenia) and Te Riele have lowered the upper bound to $e^{1.59\times10^{40}} \approx 10^{6.91\times10^{39}}$. However, no explicit counterexample is known. It is a striking example of a mathematical conjecture proven false despite a large amount of computational evidence in its favor.

- Poincaré Conjecture: In 1904, Poincaré thought he had proved that "every simply connected, closed 3-manifold is homeomorphic to the 3-sphere," but later an error was discovered. This conjecture has fascinated mathematicians for 100 years, because it has important implications for the geometry of our universe and is of central interest to mathematicians and cosmologists alike. In the spring of 1986, the New York Times reported with quite a bit of fanfare that the English mathematician Colin Rourke (born 1943, England) and his graduate student Eduardo Rêgo from Portugal had solved the famous problem—the Poincaré Conjecture. However, a fatal error was found in the proof. In 2002 and 2003, Grigori Yakovlevich Perelman (born 1966, Russia) gave sketchy proofs of the Poincaré conjecture (and the more powerful geometrization conjecture of Thurston: Each of certain three-dimensional topological spaces has a unique geometric structure). On

March 18, 2010, it was announced that Perelman had met the criteria to receive the first Clay Millennium Prize for resolution of the Poincaré conjecture, but on July 01, 2010, he rejected the prize of 1 million dollars, by saying I am not interested in money or fame; I do not want to be on display like an animal in a zoo. After his work several mathematicians have published proofs with the details filled in, which come to several hundred pages. In 1961, Stephen Smale (born 1930, USA) had already proved the Poincaré conjecture for all dimensions greater than or equal to 5, and for dimension 4 it was settled in 1982 by Michael Hartley Freedman (born 1951, USA).

- Carmichael's Totient Function Conjecture: In 1760, Euler introduced the totient function $\phi(n)$: The number of positive integers less than *n* and relatively prime to $n,$ so that if *n* is prime, $\phi(n) = n-1.$ As an example, since each number $1, 5, 7, 11, 13, 17, 19, 23, 25, 29, 31, 35$ is relatively prime to $36,\ \phi(36) = 12.$ In 1907, Carmichael claimed to have proved the conjecture that for every *n* there is at least one other integer $m \neq n$ such that $\phi(m) = \phi(n),$ but in 1922 he retracted his claim and stated the conjecture as an open problem. Using computational techniques Kevin Barry Ford (born 1967, USA) in 1998 has shown that any counterexample to this conjecture must be at least $10^{10^{10}}.$
- Bieberbach Conjecture: Let $\mathcal{S}$ be the class of functions which are analytic and one-to-one in the unit disk $B(0,1)$ and are normalized by the conditions $f(0) = 0$ and $f'(0) = 1.$ The class $\mathcal{S}$ has many interesting properties. A function $f \in S$ in terms of Maclaurin's series (after Colin Maclaurin, 1698–1746, Scotland) can be expressed as

$$f(z) \;=\; z + a_2 z^2 + a_3 z^3 + \cdots . \tag{3.27}$$

In 1916, Ludwig Bieberbach (1886–1982), a German mathematician (remembered as a notorious uniform-wearing Nazi and vicious anti-Semite, who sought to eliminate Jews from the profession of German mathematics) conjectured that in (3.27) the coefficients $|a_n| \leq n,\ n \geq 2.$ This conjecture attracted the attention of several distinguished mathematicians. The proof for the case $n = 2$ was known to Bieberbach. In the year 1923, Charles Loewner (1893–1968, Czechoslovakia-USA) used a differential equation to treat the case $n = 3,$ whereas in 1925, Littlewood proved that $|a_n| \leq en$ for all $n,$ showing that the Bieberbach conjecture is true up to a factor of $e.$

Several authors later reduced the constant in the inequality below e. Variational methods were employed in 1930s, which led to the conjecture established for $n = 4$ in 1955 by Paul Roesel Garabedian (1927–2010, USA) and Menahem Max Schiffer (1911–1997, Germany-USA) and for $n = 6$ by Roger Pederson (1930–1996, USA) in 1968 and Mitsuru Ozawa (Japan) in 1969. The case $n = 5$ was settled by Pederson and Schiffer in 1972. From time to time, proofs of other special cases were announced, but they have not been substantiated. Finally, 12 years later Louis de Branges de Bourcia (born 1932, France-USA) in the year 1984 proved the general case. As expected, his proof is not simple—it ran to over 350 pages. At one point, even computer was used to validate the work; however, the proof itself does not rely on a machine. In 1985, Richard Allen Askey (1933–2019, USA) and George Gasper (born 1939, USA) proved Bieberbach conjecture (also now known as de Branges's theorem) in the traditional manner which also shortened Branges's proof considerably.

- Ramanujan's Conjecture About the Partition Function: Integer 4 can be obtained from positive integers, using only addition, in five different ways: $4, 3+1, 2+2, 2+1+1, 1+1+1+1.$ These are called partitions of 4, and the fact is denoted as $P(4) = 5.$ In general, $P(n),$ an arithmetical function (originates in trying to learn certain properties of numbers), stands for the number of partitions of positive integer $n.$ Actual computation shows that the partition function $P(n)$ increases very rapidly with $n;$ in fact $P(0) = 1, P(1) = 1, P(2) = 2, P(3) = 3, P(4) = 5, P(5) = 7, \cdots,$ $P(200)$ has the enormous value $P(200) = 3,972,999,029,388.$ Ramanujan's conjecture regarding this partition function is "If $(24n-1) = 0(\text{mod } m^{\alpha}),$ where *m* is $5, 7$ or $11,$ and $\alpha \geq 1,$ then $P(n) = O(\text{mod } m^{\alpha}).$" Sarvadaman Chowla (1907–1995, England-India-USA) showed that Ramanujan's conjecture failed for $n = 243.$ For his proof, he used the table of partitions prepared by Hansraj Gupta (1902–1988, India) in 1967. Later it was also found that Ramanujan's conjecture failed for $n = 586.$ In 1918, Hardy and Ramanujan proved what is considered one of the masterpieces in number theory: Namely, that for large *n* the partition function satisfies the relation

$$P(n) \;\simeq\; \frac{e^{c\sqrt{n}}}{4n\sqrt{3}},$$

where the constant $c = \pi(2/3)^{1/2}$. For $n = 200$, the right-hand side of the relation is approximately $4 \cdot 10^{12}$, which is remarkably close to the actual value of $P(200)$. In 1937, Rademacher refined their work into the first known formula for $P(n)$. It is interesting to note that Euler in 1753 gave the following *generating function* (for a given sequence $\{a_n\}$, the corresponding generating function $f(x)$ is defined by the series $\sum_{n=0}^{\infty} a_n x^n$) for the sequence $\{P(n)\}$

$$\begin{aligned}\prod_{n=1}^{\infty}\frac{1}{(1-x^n)} &= (1+x+x^2+x^3+x^4+x^5+x^6+\cdots) \\ &\quad \times(1+x^2+x^4+x^6+x^8+\cdots) \\ &\quad \times(1+x^3+x^6+x^9+x^{12}+\cdots)\times(1+x^4+x^8+x^{12}+\cdots) \\ &\quad \times(1+x^5+x^{10}+x^{15}+\cdots)\times\cdots \\ &= 1+x+2x^2+3x^3+5x^4+7x^5+11x^6+\cdots \\ &= \sum_{n=0}^{\infty} p(n)x^n, \quad \text{where} \quad p(0)=1.\end{aligned}$$

A typical case related to the above partition is to find *r*th partition of the positive integer *n* denoted as $P^r(n)$. For example, $P^1(4) = 1, P^2(4) = 2, P^3(4) = 1,$ and $P^4(4) = 1$. The earliest studies of *r*-partitions occur in gambling and games of chance, where several dice are thrown simultaneously. In 1699, Leibniz wrote to Johann Bernoulli "Have you ever considered the number of partitions or divulsions of a given number, namely, the number of ways it may be broken up into two, three, etc., pieces? It seems to me that this is not an easy problem, and yet it would be worth knowing." Leibniz work was studied systematically by Euler in 1741. The final result is known as mu1tinomial theorem.

- Ramanujan's Hypothesis or Tau Conjecture: Let $\sigma_s(n)$ denote the sum of the *s*th powers of the divisors of n. If $n = 4$, its divisors are $1, 2, 4$. If $s = 2,\ \sigma_2(4) = 1^2 + 2^2 + 4^2 = 21$. In order to calculate $\sigma_s(n)$ generally, Ramanujan introduced the arithmetical tau function, $\tau(n)$. In 1616, Ramanujan conjectured that $\tau(n)$ is of the form $O(n^{(11/2)+\epsilon})$. Hardy called it Ramanujan's hypothesis. It is more generally known as the tau conjecture. It meant that the value of $\tau(n) \leq kn^{(11/2)+\epsilon}$ for some constant $k,$ for all $n > n_0$ (say). Ramanujan himself had proved that $\tau(n) = O(n^7).$ But the power 7 here, of $n,$ is much more than $11/2$ and very much short of proving the conjecture. Two years later, Hardy came little closer to the proof when he proved $\tau(n) = O(n^6).$ Hendrik Douwe Kloosterman (1900–1968, The Netherlands) got closer in 1927 and Harold Davenport (1907–1969, England) and Hans Oscar Emil Salié (1902–1978, Germany) still closer in 1933. Robert Alexander Rankin (1915–2001, Scotland), Hardy's student, proved in 1939 that $\tau(n) = O(n^{29/5}).$ Finally, Viscount Deligne proved the conjecture in 1974 using algebraic geometry. This is considered as one of the sparkling achievements of the twentieth century. For this great achievement Deligne was awarded the Fields Medal, which is considered as the Nobel prize for mathematics.

- Collatz $3n+1$ Conjecture: Hailstone sequence $\{a_i\}$ of integers (because numbers go up and down like hailstones) with $a_0 = n \in \mathcal{N}$ is defined as

$$a_i \;=\; \begin{cases} n/2 & \text{if } n \text{ is even} \\ 3n+1 & \text{if } n \text{ is odd.} \end{cases}$$

  For example, with $a_0 = 7,$ the hailstone sequence is 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1,$\cdots$. In 1937, Lothar Collatz (1910–1990, Germany) conjectured that for any $a_0 = n \in \mathcal{N}$ the hailstone sequence eventually reaches to 1 and then continues indefinitely to 1, 4, 2, 1,$\cdots$. In the literature it is also known as $3x+1$ Ulam conjecture, Kakutani's problem (after Shizuo Kakutani, 1911–2004, Japan-USA), the Thwaites conjecture (after Bryan Thwaites, born 1923, England), Hasse's algorithm (after Helmut Hasse, 1898–1979, Germany), and the Syracuse problem. For Collatz's conjecture Erdös said (confirmed by Collatz himself in 1988) "Mathematics may not be ready for such problems." In 1985, Jeffrey Clark Lagarias (born 1949, USA) gave a heuristic probabilistic argument to support the conjecture; however, in 2010, he added Collatz conjecture is an extraordinarily difficult problem, completely out of reach of present-day mathematics. As of 2020, the conjecture has been checked by computer for all starting values up to 2^{68}.

On September 8, 2019 Terence Chi-Shen Tao (born 1975, Australia-USA) posted a proof (Almost All Orbits of the Collatz Map Attain Almost

Bounded Values. https://arxiv.org/pdf/1909.03562.pdf), showing that—at the very least—the Collatz conjecture is "almost" true for "almost" all numbers. While his result is not a full proof of the conjecture, it is a major advance on a problem that does not give up its secrets easily. Modestly he said "I wasn't expecting to solve this problem completely, but what I did was more than I expected." In 2019, Journal of Mathematics (Article ID 6129836), in 2021, International Journal of Mathematics Trends and Technology (67, 178–182), and in January and February 2022 issues of the Journal Advances in Pure Mathematics (www.scirp.org/journal/apm) articles have appeared claiming the proofs of Collatz's conjecture. However, at this point only one can hope that experts will confirm the legitimacy of at least one of these proofs.

- Jacobian Conjecture: In 1939, Eduard Ott-Heinrich Keller (1906–1990, Germany) claimed that if a polynomial function from an *n*-dimensional space to itself has Jacobian determinant (after Jacobi) which is a nonzero constant, then the function has a polynomial inverse. This conjecture was widely publicized by Abhyankar. For this conjecture over the period of 60 years several incomplete proofs have been announced. This list includes three proofs of Beniamino Segre (1903–1977, Italy).
- Erdös-Straus Conjecture: The table of decompositions of fractions $2/(2k+1)$ as a sum of two, three, or four unit fractions (numerator equal to one and a positive integer as its denominator), e.g., $2/9 = 1/6 + 1/18,\ 2/41 = 1/24 + 1/246 + 1/328,\ 2/43 = 1/42 + 1/86 + 1/129 + 1/301$ found in the Rhind mathematical papyrus has been the matter of wonder and stirred controversy for quite some time between the historians. Fibonacci, in 1202, published an algorithm for expressing any rational number between 0 and 1 as a sum of distinct unit fractions; this was rediscovered and more intensely investigated by Sylvester in 1880. In 1884, he proved that any proper fraction a/b can be written as a sum of distinct unit fractions. This is certainly true when the numerator $a = 1.$ For $a > 1,$ Sylvester assumed $1/q$ to be the largest unit fraction less than $a/b.$ Then

$$\frac{1}{q} < \frac{a}{b} < \frac{1}{q-1},$$

which implies that $0 < aq - b < a.$ Since

$$\frac{a}{b} = \frac{1}{q} + \frac{aq-b}{bq}$$

by induction, $(aq-b)/bq$ is the sum of distinct unit fractions. Furthermore, none of them is $1/q$, since

$$\frac{1}{q} > \frac{aq-b}{bq}.$$

This completes the proof by induction and gives a procedure to find a distinct unit fraction sum equal to a given proper fraction. To apply Sylvester method for $2/41$, we note that $q = 21$ and $aq - b/bq = 1/861$, and hence $2/41 = 1/21 + 1/861$. For $2/41$ this is the only sum of two distinct unit fractions. In fact, if *p* is prime, then $2/p$ can be expressed as a sum of two distinct unit fractions in exactly one way. For this, if $2/p = 1/a + 1/b$, then $(2a-p)(2b-p) = p^2$. Now p^2 can be factored into distinct factors in only one way: $p^2 \times 1$. Hence if $a < b$, we have $a = (1+p)/2$ and $b = p(1+p)/2$. Clearly, this is different from the three unit fractions given above from the Rhind mathematical papyrus. Thus for a given proper fraction a/b, there may be more than one unit fraction. Similarly, we can check that $3/7 = 1/3 + 1/11 + 1/231 = 1/4 + 1/7 + 1/28$ and $4/2009 = 1/504 + 1/144648 = 1/574 + 1/4018 = 1/588 + 1/3444$ (these are the only sums of two unit fractions as claimed by Ionascu and Wilson, see [281]); we also have $4/2009 = 1/670 + 1/2009 + 1/1346030$. Furthermore, Sylvester's method for the same fraction gives

$$\frac{4}{2009} = \frac{1}{503} + \frac{1}{336843} + \frac{1}{170194473131} + \frac{1}{57932317368507167413191}.$$

In 1948, Erdös and Ernst Gabor Straus (1922–1983, Germany-USA) conjectured that for every $n \in \mathcal{N},\ n \geq 2$, there exist a, b, c natural numbers, not necessarily distinct and not necessarily unique, so that $4/n = 1/a + 1/b + 1/c$, i.e., $4/n$ can be written as a sum of three positive unit fractions. For example, we have

$4/2 = 1/1 + 1/2 + 1/2,\ 4/3 = 1/2 + 1/2 + 1/3,\ 4/4 = 1/2 + 1/3 + 1/6,\ 4/5 = 1/2 + 1/4 + 1/20 = 1/2 + 1/5 + 1/10,$

$4/6 = 1/6 + 1/6 + 1/3 = 1/2 + 1/8 + 1/24 = 1/3 + 1/4 + 1/12,$

$4/7 = 1/2 + 1/28 + 1/28 = 1/2 + 1/21 + 1/42 = 1/3 + 1/6 + 1/14,$

$4/14 = 1/4 + 1/56 + 1/56 = 1/4 + 1/42 + 1/84 = 1/5 + 1/14 + 1/70 = 1/6 + 1/14 + 1/21 = 1/7 + 1/8 + 1/56; 1/6 + 1/12 + 1/28.$

When $n = pq,$ with *p* prime and *q* any other integer, solutions for *n* can be simply derived from those for *p* by multiplying those denominators by $q.$ For example,

$$\frac{4}{75} = \frac{4}{5 \times 15} = \frac{1}{2 \times 15} + \frac{1}{4 \times 15} + \frac{1}{20 \times 15} = \frac{1}{30} + \frac{1}{60} + \frac{1}{300}.$$

Thus in view of fundamental theorem of arithmetic (Theorem 4.1) to prove Erdös-Straus conjecture, it suffices to show that for every prime $n > 4,$ $4/n$ is a sum of three unit fractions. This problem has not yet been solved, although several partial results, generalizations for several other fractions $m/n,$ and number of possible representations have been studied extensively. We refer here to an excellent survey paper of Graham [223] and among several known identities list the following:

$$\frac{4}{4n} = \frac{1}{2n} + \frac{1}{3n} + \frac{1}{6n}$$

$$\frac{4}{4n+2} = \frac{1}{n+1} + \frac{1}{(n+1)(2n+1)}$$

$$\frac{4}{4n+3} = \frac{1}{n+2} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(4n+3)}$$

$$\frac{4}{8n+5} = \frac{1}{2(n+1)} + \frac{1}{2(n+1)(3n+2)} + \frac{1}{2(3n+2)(8n+5)}$$

$$\frac{4}{3n+2} = \frac{1}{n+1} + \frac{1}{3n+2} + \frac{1}{(n+1)(3n+2)}.$$

- Palindrome Number Conjecture: Take any non-palindromic natural number with two or more digits, add its inverse ordinal number, continue to use the inverted number of sum plus sum; repeating this process continuously leads to a palindromic number. In 1979, Gardner claimed that of the first 10000 numbers, only 251 do not produce a palindromic number in ≤ 23 steps. However, his claim is wrong; numbers 89 and 98 need the following 24 steps (89 or 98) $\to$ 187 $\to$ 968 $\to$ 1837 $\to$ 9218 $\to$ 17347 $\to$ 91718 $\to$ 173437 $\to$ 907808 $\to$

1716517 → 8872688 → 17735476 → 85189247 → 159487405 → 664272356 → 1317544822 → 3602001953 → 7193004016 → 13297007933 → 47267087164 → 93445163438 → 176881317877 → 955594506548 → 1801200002107 → 8813200023188. Fred Gruenberger (1918–1998, USA) in 1984 showed that among the first 100000 numbers, 5996 numbers apparently never generate a palindromic number. The first few such numbers are 196, 887, 1675, 7436, 13783, 52514, 94039, 187088, 1067869, 10755470. Recently in [528] Wang et. al. have shown that the palindrome number conjecture is not true (their proof is under scrutiny). They also claimed that 196, 295, 394, 493, 592, 691, 790, 1495, 1585, 1675, 1765, 1855, 1945, 227386 are the counterexamples. Thus, we can conclude that for two digits numbers palindrome number conjecture is true.

3.18 What Is a Paradox?

The ancient Greek word parádoxos (derived from two Greek words; para: meaning faulty, disordered, false, contrary to, or abnormal; and doxa: meaning opinion) was first time used in English as paradox in 1540, and it meant for a statement that is seemingly self-contradictory yet not illogical or obviously untrue, perhaps a vicious circle. A paradox, in De Morgan's special sense of the word, was any curious tale about science or scientists that he had come across in his extensive reading, any piece of gossip, choice examples of lunacy, assorted riddles, and puns. In fact, paradoxes are most important to human logic because it exposes weakness in our reasoning; however, some claim paradoxes are compact energy sources, talismans. Paradoxes can be found abundantly throughout: mathematics, physics, logic, philosophy, economics, biology, and mechanics (in hydrodynamics it is used in the sense of an inconsistency between experimental facts and conclusions based on plausible arguments). The very existence of a paradox can be used to drive some interesting facts about the relationship between mind and the Universe. In mathematics, the logicists considered paradoxes to be the common errors, caused by errant mathematicians and not by a faulty mathematics. The intuitionists, on the other hand, considered paradoxes to be clear indications that classical mathematics itself is far from perfect. They felt that mathematics had to be rebuilt from the bottom up. Indeed from time to time challenging paradoxes have been contemplated which have shaken the basic foundation of a particular field and thus forced mathematicians to rethink a fresh from the beginning. We list some of the famous paradoxes that are counterintuitive and so astonishing.

- Epimenides of Crete (around sixth century BC, Greece) is remembered for the *Liar Paradox*: A Cretan says "All Cretans always do lie." If that Cretan is speaking the truth, in that case he is lying, and if he is lying, then he is telling the truth. There are several variants of this paradox, e.g., popular in medieval Europe and often quoted in literature is Plato: What Socrates is about to say is false. Socrates: Plato has spoken correctly. According to Eubulides of Miletus (fourth century BC, Greece), and also Bertrand Russell, the statement "I am lying" is true only if it is false and false if it is true. Chrysippus of Soli (around 279–206 BC, Turkey-Greece) wrote: If someone says: "All people lie" is that person telling the truth or lying? Bhartrhari (fifth century AD, India) formulated it as "everything I am saying is false." Philip Edward Bertrand Jourdain (1879–1919, England) in 1913 had an interesting calling card. On the one side was written "The statement on the other side of this card is true." On the other side was written "The statement on the other side of this card is false." The simplest versions of the Liar Paradox are the statements "This sentence is false," "I am a compulsive liar," and Pinocchio says "My nose will grow now."
- Eubulides is also known for the *Small Heap Paradox*: Suppose we have a small heap (sorites in Greek) of stones. If we remove a stone from a heap, then of course it will be smaller. But even if we add a stone to the small heap, it will remain small. Now let us start from the "heap" of a single stone which is definitely small. Since addition of a stone does not affect the smallness of the heap, one can go on adding a stone at a time and still continue to get a "small" heap. This leads to the paradoxical situation where every heap is small. Thus, smallness is a fuzzy concept.

In 1926, Bertrand Russell proposed a dual of the small heap paradox known as *Bald Man's Paradox*. We have a bald man. Can you specify the day from when he would have been declared bald? Of course, he was not bald when he was a young man of 25. But thereafter he started losing hair. When he lost his first hair, nobody called him bald, nor after losing the second, third, and so on. But today he is called bald, so there must have been a hair such that before its fall, he was not bald, and after its fall, he was declared bald. Thus, the difference between baldness and non-baldness is just one hair!

- Zeno put forth four paradoxes which confounded thinkers for centuries (until Cantor's development of the theory of infinite sets) the accepted notions of space and time and the relation of the discrete to the continuous. These paradoxes arose because he was attempting to rationally understand the notions of infinity for the first time. He tried to express his intuitions that a line/space cannot be conceived of as a set of points (see Fig. 2.4), and time cannot be discrete/instantaneous. Thus, he must be credited in setting a stage for the very foundation of mathematics and physics. Bertrand Russell described Zeno's arguments, in some form, have "afforded grounds for almost all the theories of space and time and infinity which have been constructed from his day to our own."

Achilles Cannot Overtake the Tortoise. Suppose there is a race between Achilles (the ancient Greek hero famed for his strength and being several times faster than the tortoise) and a tortoise (the slowest creature on the Earth). Achilles offers a small concession to the poor creature. Achilles stands a few steps behind the tortoise when the race starts (Fig. 3.10). Zeno claims that because of this, Achilles cannot win the race. To understand this suppose that Achilles is at point *A* and the tortoise is at point $T,$ when the race starts. If Achilles is to overtake the tortoise, then he first has to reach the tortoise. For this to happen, both Achilles and the tortoise will have to be at the same point, say $P,$ at the same time. In other words, Achilles will have to cover the length *AP* in the same amount of time that the tortoise requires to cover the length $TP.$

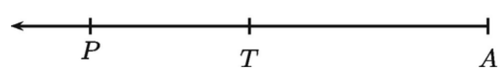

Fig. 3.10 Race between Achilles and a tortoise

Since the line segment *TP* is a part of the line segment $AP,$ and since the whole is always greater than its part (Euclid's assertion (e)), the segment *AP* has more points in it than the segment $TP?$ Thus Achilles can win the race, only if he crosses more points than the tortoise in the same time interval. Is this possible? Zeno's argument is an emphatic "no," based on the following "self-evident" facts.

I. An instant is the last indestructible part of time. There is no fractional instant.

II. A point is the last indestructible part of a line. There is no fraction of a point.

If Achilles is faster than the tortoise, he will cover more points per instant than the tortoise does. Does he cover (say) two points while the tortoise covers just one, in an instant? If so, then the next question arises; how much time does he take to cross a point? Half an instant? Since a half-instant does not exist, Achilles must cover only a point in an instant. Similarly, the assumption that the tortoise, being slow, may take two instants to cross a point is absurd (the word absurd comes from the Latin absurdus meaning unmelodious or discordant). Thus both Achilles and the tortoise cover just a point per instant. Clearly, Achilles cannot win over the tortoise, as he is required to cover "more points" in the same "amount" of instants. In fact, as late as the nineteenth century, some European scholars were arguing that Zeno was correct, and Achilles had lost the race!

Grégoire de Saint-Vincent (1584–1667, Belgium) was the first to use the method of infinite series to argue against Zeno's paradox, as follows: Suppose Achilles is ten times as fast as the tortoise, e.g., suppose Achilles runs 10 meters/sec, and the tortoise crawls 1 meter/sec. If the tortoise has a 10 meter head start, then at time $t = 0$, Achilles' position at $t = 0$ is $A(0) = 0$, and the tortoise's position at $t = 0$ is $T(0) =$10 m. Achilles' position at time $t = 1$ sec is $A(1) =$10 m, and the tortoise's position at time $t = 1$ sec is $T(1) =$11 m. Continuing on with this argument,

$$A(1 + 1/10) \;=\; 11\,\mathrm{m} \quad \text{and} \quad T(1 + 1/10) \;=\; 11.1\,\mathrm{m}.$$

At "stage *n*,"

$$A(1 + 1/10 + 1/10^2 + \cdots + 1/10^n) \;=\; 10 + 1 + 1/10 + \cdots + 1/10^{n-1}$$

and

$$T(1 + 1/10 + 1/10^2 + \cdots + 1/10^n) \;=\; 10 + 1 + 1/10 + \cdots + 1/10^n.$$

Now recalling the geometric series (3.2) (Zeno's time infinite number of steps, which was not acceptable as the problem had to be resolved in a finite number of steps), it follows that

$$A(10/9) \;=\; T(10/9) \;=\; 100/9,$$

so that Achilles catches up with the tortoise after 10/9 seconds, at the 11- and 1/9- meter mark. Thus, mathematically the major takeaway from this paradox was that an infinite series is also summable.

Horses Cannot Meet. If two horses approach each other, crossing a point per instant, then effectively they are crossing two points per instant. As examined above, this is impossible. Hence, the two horses running toward each other cannot meet.

No One Can Cross a Field. In order to cross a field, you will first have to cross half of it. But in order to cross the half, you will have to cross half of the half, i.e., $1/4$ of the distance. Repeating the argument, you will have to cover infinitely many distances $1/2, 1/4, 1/8, \ldots, 1/2^n, \ldots.$ To cover any positive distance, you will need some positive amount of time, however small. Hence to cover infinitely many distances, you will need infinite time. Clearly this is not possible for a mortal.

An Arrow in Motion Is at Rest. Suppose an arrow leaves a bow and hits the target at 300 meters in 10 seconds. If we assume (for simplicity) that the velocity is uniform, it is obvious that it will be at the distance of 150 meters at the end of 5 seconds. More generally, given an instant of time, we will be able to state precisely the point at which the arrow will be found. Clearly, if we are able to locate the point precisely at each instant of time, the arrow must be static at that point and that instant. Thus, an arrow in flight is actually at rest.

- One place where probability contradicted “common sense” occurred in the *St. Petersburg Paradox* worked out between Daniel Bernoulli and his bother Nicholas Bernoulli (1695–1726, Switzerland) some time before Nicholas death from a cold caught after plunging into the freezing Neva River. The St. Petersburg Paradox first appeared in 1713 in a problem posed by Nicholas Bernoulli to Pierre Remond de Montmort (1678–1719, France); Nicholas and Daniel began discussing the problem somewhat later. The problem revolves around the following game between two players: Suppose that Peter and Paul agree to play a game based on the toss of a coin. If a head is thrown on the first toss, Paul will give Peter one crown; if the first toss is tail, but a head appears on the second toss, Paul will give Peter two crowns; if a head appears for the first time on the third toss, Paul will give Peter four crowns, and so on, the amount to be paid if head appears for the first time on the *n*th toss being 2^{n-1} crowns. What should Peter pay Paul for the privilege of playing the game? Peter's mathematical expectation (the sum of the probabilities for each possible outcome of the game multiplied by the payoff for each outcome), given by

$$\frac{1}{2}\cdot 1+\frac{1}{2^2}\cdot 2+\frac{1}{2^3}\cdot 2^2+\cdots+\frac{1}{2^n}\cdot 2^{n-1}+\cdots ,$$

evidently is infinite, yet common sense suggests a very modest finite sum. When Georges Louis Leclerc, Comte de Buffon (1707–1788, France) made an empirical test of the matter, he found that in 2084 games Paul would have paid Peter 10,057 crowns. This indicates that for any game Paul's expectation, instead of being infinite, is actually something less than 5 crowns. The paradox raised in the Petersburg problem was widely discussed during the eighteenth century, with differing explanations being given. Daniel Bernoulli sought to resolve it through his principle of moral expectation, in accordance with which he replaced the amounts $1, 2, 2^2, 2^3, \cdots, 2^{n-1}, \cdots$ by $1^{1/2}, 2^{1/4}, 4^{1/8}, 8^{1/16}, \cdots$. Others preferred, as a solution of the paradox, to point out that the problem is inherently impossible in view of the fact that Paul's fortune is necessarily finite; hence he could not pay the unlimited sums that might be required in the case of a long delay in the appearance of a head.

For many years, because of paradoxes such as St. Petersburg Paradox, the subject of probability theory was in ill repute. It was not until the invention of a branch of mathematics called *measure theory* pioneered by Lebesgue in 1901 that the tools became available to put probability theory on a rigorous footing. The celebrated Russian mathematician Andrey Nikolaevich Kolmogorov (1903–1987) is credited with this development.

- The *Bertrand Paradox* is a problem within the classical interpretation of probability theory, which was discovered by François Bertrand in 1889. The resolution of this paradox came in the late 1930s. Fix a circle of radius 1. Draw the inscribed equilateral triangle as shown in Fig. 3.11. We let ℓ denote the length of a side of this triangle. Suppose that a chord *d* (with length *m*) of the circle is chosen "at random." What is the probability that the length *m* of *d* exceeds the length ℓ of a side of the inscribed triangle?

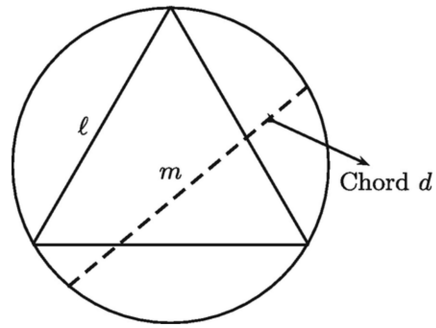

Fig. 3.11 Bertrand's Paradox (a)

The "paradox" is that this problem has three different but equally valid solutions. We now present these apparently contradictory solutions in sequence. At the end we shall explain why it is possible for a problem like this to have three distinct solutions.

Solution 1. Examine Fig. 3.12. It shows a shaded, open disk whose boundary circle is internally tangent to the inscribed equilateral triangle. If the center of the random chord *d* lies inside that shaded disk, then $m > \ell.$ If the center of the random chord *d* lies outside that shaded disk, then $m \leq \ell.$ Thus the probability that the length *d* is greater than the length ℓ is (area of shaded disk)/(area of unit disk). But an analysis of the equilateral triangle (Fig. 3.13) shows that the shaded disk has radius $1/2$ and hence area $\pi/4.$ The larger unit has area $\pi.$ The ratio of these areas is $1/4.$ We conclude that the probability that the length of the randomly chosen exceeds ℓ is $1/4.$

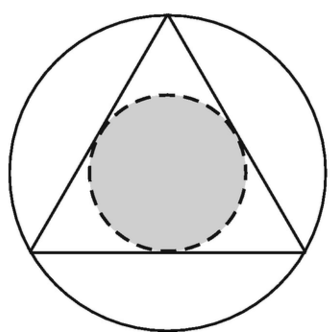

Fig. 3.12 Bertrand's Paradox (b)

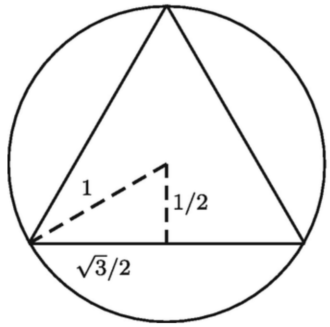

Fig. 3.13 Bertrand's Paradox (c)

Solution 2. Examine Fig. 3.14. We may also assume that our randomly chosen chord is horizontal (the equilateral triangle and the chord can both be rotated so that the chord is horizontal and one side of the triangle is horizontal). Notice that if the height, from the base of the triangle, of the chord *d* is less than or equal to $1/2$, then $m \leq \ell,$ while if the height is greater than $1/2$ (and not more than 1), then $m > \ell.$ We thus see that there is probability $1/2$ that the length *m* of *d* exceeds the length ℓ of a side of the equilateral triangle.

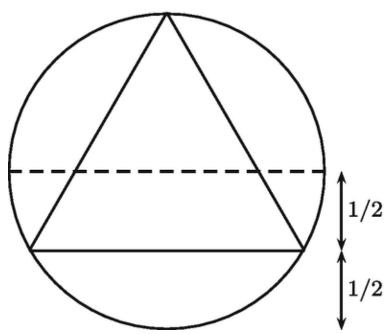

Fig. 3.14 Bertrand's Paradox (d)

Solution 3. Examine Fig. 3.15. We may also assume that one vertex of our randomly chosen chord occurs at the lower left vertex *A* of the inscribed triangle (by rotating the triangle we may always arrange this to be the case). Now look at the angle θ that the chord subtends with the tangent line to the circle at the vertex *A* (shown in Fig. 3.16). If that angle is between $0°$ and $60°$ inclusive, then the chord is shorter than or equal to $\ell.$ If the angle is strictly between $60°$ and $120°$, then the chord is longer than $\ell.$ Finally, if the angle is between $120°$ and $180°$ inclusive, then the chord is shorter than $\ell.$ In sum we see that the probability is $60/180 = 1/3$ that the randomly chosen chord has length exceeding ℓ.

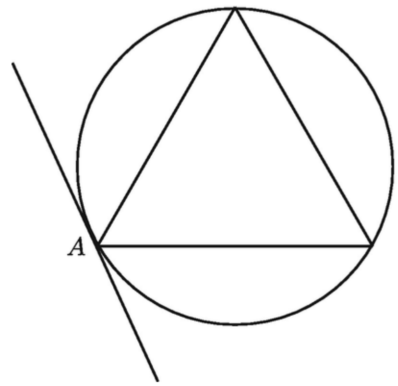

Fig. 3.15 Bertrand's Paradox (e)

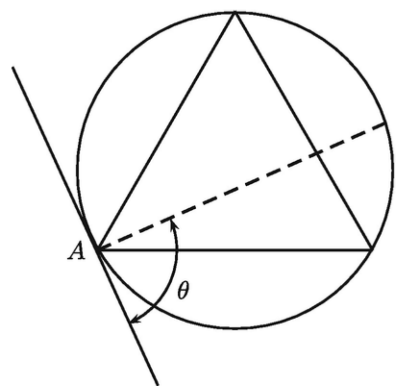

Fig. 3.16 Bertrand's Paradox (f)

We have seen then three solutions to our problem. And they are different: We have found valid answers to be $1/4, 1/2,$ and $1/3.$ How can a perfectly reasonable problem have three distinct solutions? And be assured that each of these solutions is correct! The answer is that when one is dealing with a probability space having infinitely many elements (i.e., a problem in which there are infinitely many outcomes—in this case there are infinitely many positions for the random chord); then there are infinitely many different ways to fairly assign probabilities to those different outcomes. Our three distinct solutions arise from three distinct ways to assign probabilities: Notice that the one of these is based on area, one is based on height, and one is based on angle.

- From 1897, Cesare Burali-Forti (1861–1931, Italy), Bertrand Russell, Julius König (1849–1914, Hungary), and others found several paradoxes within Cantor's set theory. This is because Cantor created set theory "naively," meaning non-axiomatically, and he also abandoned his own theory. Among these paradoxes *Russell's Paradox* of 1903 is most popular which shook the very core of the foundation of mathematics: Most of the sets are not elements of themselves. For instance, the set of all integers is not an integer, and the set of all horses is not a horse. However, some sets are elements of themselves. For instance, the set of all abstract ideas is an abstract idea. Let *S* be the set of all sets that are not elements of themselves, i.e.,

$$S = \{A : A \text{ is a set and } A \notin A\}.$$

Is *S* an element of itself? The answer is neither yes or no. For if $S \in S$, then *S* satisfies the defining property for S, and hence $S \notin S$. But if $S \notin S$, then *S* is a set such that $S \notin S$ and so *S* satisfies the defining property for S, which implies that $S \in S$. Thus neither is $S \in S$ nor is $S \notin S$, which is a contradiction. This paradox was a big blow to Gottlob Frege's work as during this period the second volume of his definite work *The Basic Laws of Arithmetic* was with the printer.

To help explain his paradox to lay people, Bertrand Russell devised a puzzle, *The Barber Paradox*, whose solution exhibits the same logic as his paradox. (1). The men in a village are of two types: men who do not shave themselves and men who do. (2). The village barber shaves all men who do not shave themselves, and he shaves only those men. But who shaves the barber? The barber cannot shave himself. If he did, he would fall into the category of men who shave themselves. However, (2) states that the barber does not shave such men. So barber does not shave himself. But then he falls into the category of men who do not shave themselves. According to (2), the barber shaves all of these men; hence, the barber shaves himself, too. We find that the barber cannot shave himself, yet the barber does shave himself—a paradox.

Some other popularizations of Russell's Paradox are as follows: Every municipality of a certain country must have a mayor, and no two municipalities may have the same mayor. Some mayors do not reside in the municipalities they govern. A law is passed compelling nonresident mayors to reside by themselves in a certain special area A. There are so many nonresident mayors that A is proclaimed a municipality. Where shall the mayor of A reside? An adjective in the English language is said to be *autological* if it applies to itself; otherwise the adjective is said to be *heterological*. Thus the adjectives "short," "English," and "polysyllabic" all apply to themselves and hence are autological, whereas the adjectives "long," "French," and "monosyllabic" do not apply to themselves and hence are heterological. Now is the adjective "heterological" autological or heterological? Suppose a librarian complies, for inclusion in his library, a bibliography of all those bibliographies in his library that do not list themselves.

For Russell's Paradox several different ways were invented to define the basic concepts of set theory so as to avoid his contradiction. One of the simple way is, except for the power set whose existence is guaranteed, the prerequisite must also be made that the set is a subset of a known set. This assumption does not allow us to have "the set of all sets that are not elements of themselves." We can only say "the set of all sets that are subsets of some known set and that are not elements of themselves." When this restriction is made, Russell's Paradox immediately ceases. Then the following holds: Let *U* be a set of sets, and let $S = \{A \in U : A \notin A\}$. Is $S \in S?$ The answer is no. For if $S \in S,$ then *S* satisfies the defining property for $S,$ and hence $S \notin S.$

- Bertrand Russell's *Paradox of Tristram Shandy* of 1903. Tristram Shandy is a hero of a novel of the same name by the eighteenth century novelist, Laurence Sterne (1713–1768, Ireland-England). Shandy intends to write an autobiography. But he encounters a peculiar problem. His life is too eventful to complete the work. It takes him 1 year to cover the events of a day. Obviously, however hard he may try, he cannot finish the work. But Shandy is immortal. He may complete the first day's account during the first year of his life. Second day's in the second year, …, *n*th day's in the *n*th year. Does this mean that there are as many days as there are years in the life of Shandy.
- In 1904, Poincaré in his book *La Science et I'Hypothése and La Valeur de la Science* emphasized that the physical continuum is not transitive. *Poincaré's Paradox* begins as follows: Suppose we are comparing various pieces of metals, say, $m_1, m_2, \cdots, m_n$ in a balance for their relative weight. We keep m_1 in one pan and m_2 in the other to find that $m_1 = m_2.$ We replace m_1 by m_3 to observe that $m_2 = m_3,$ and continue in this manner to find $m_1 = m_2,\ m_2 = m_3,\ m_3 = m_4, \cdots, m_{n-1} = m_n.$ But at each instant of weighing, there will be a small difference between the two weights, which even the most sensitive machine will fail to detect. And then a stage may come when the machine will show that $m_1 \neq m_n.$ In describing denseness of real points (always there is another real

number between any two real numbers) on an infinite continuum, he observed that adjacent points on the continuum are so indistinguishable from each other that they almost overlap. He describes this situation by the equation $a = b,\ b = c,$ but $a \neq c.$

- In 1908, Bertrand Russell attributed the paradox "The least integer not nameable in fewer than nineteen syllables" to G.G. Berry (1867–1928, England), known as *Berry's Paradox*. This paradox is itself a name consisting of 18 syllables; hence the least integer not nameable in fewer than 19 syllables can be named in 18 syllables, which is a contradiction.
- In 1924, Stefan Banach (1892–1945, Poland-Ukraine) and Tarski published a joint paper proving the remarkable result that a solid ball could be decomposed into sets, which could then be reassembled to form two identical balls, each equal to the first, which contradicts fundamental principles of physics. The result, now known as the *Banach-Tarski Paradox*, which implied that volume was a meaningless concept. Banach-Tarski Paradox has even more dramatic formulations. It is actually possible to take a solid ball, break it up into finitely many pieces, and reassemble those pieces into a full-sized replica of the Empire State Building. Their proof relied on the axioms of choice and the measure theory of Lebesgue. The paradox caused many mathematicians to reconsider the validity of the axiom of choice.
- In 1939, Richard von Mises (1883–1953, Ukraine-USA) raised the question of how many people must be within a room, so that the probability of two or more people having their birthday on the same day is more than 50%, which has become popular as *The Birthday Paradox*. He proved that with 365 days per year only 23 people are needed to raise the probability of two or more of them having their birthday on the same day to 50%. With 57 people the probability is already is 99%. For a probability of 100%, however, 366 people must be within the room. Von Mises developed the following formula for calculating the probability with which greater than or equal to celebrate their birthday on the same day $1 - [365!/[365^n(365 - n)!]$.
- In 1954, James Francis Thomson (1921–1984, England) posed a puzzle which has become popular as *Thomson Lamp Paradox*. Its original version is due to Aristotle which can be elaborated as follows: Assume that there is a lamp that comes on at time $t = 1 - 1/2^n,$ if *n* is even, but goes off at time $t = 1 - 1/2^n,$ if *n* is odd. If, indeed, we can divide an interval of time into an actually infinite number of instances, then this

lamp is theoretically possible, and, theoretically, it would turn on and off an actually infinite number of times in the time interval $t = 0$ to $t = 1$. However, at time $t = 1$ the lamp would be neither on nor off—because the infinite is neither even nor odd. But this is impossible. Hence, we cannot divide an interval of time into an actually infinite number of instances.

- In 1959, Gardner introduced a distinction between “dull” and “interesting” numbers. Interesting numbers, according to Gardner, are those that have some peculiar pattern or property that separates them from all other numbers. Dull numbers, on the other hand, are all those numbers that are not interesting. Gardner then went on to show the paradoxical nature of this dichotomy, by showing that dull numbers cannot possibly exist. His argument was to first list the integers in order, letting *D* stand for the first dull number on the list. But the very fact that *D* is the first dull number makes it interesting! Therefore, there can be no dull numbers.

The axiomatic set theory (ZF) avoids paradoxes such as those of Russell’s and Berry’s. The fuzzy set theory resolves the paradoxes such as small heap and Poincaré. According to Quine, there are three classifications of paradoxes: veridical paradoxes, falsidical paradoxes, and antinomies. A fourth type of paradox was later recognized as dialetheia. A veridical paradox is one in which even though the result seems ridiculous, it is still true. Hilbert’s paradox is an example of a veridical paradox because despite the outcome being laughable, it is correct. A falsidical paradox is one that is just outright incorrect. Zeno’s paradox is false in that it assumes something traveling at a greater speed can never catch something going at a slower pace. An antinomy is a phrase, or statement, that has parts within in that contradict and do not agree with other. Liar’s paradox is an antinomy because in order for one to be true the other must be false and vice versa. Dialetheia is a special case in which a paradox is both true and false simultaneously. For example, “that glass is half-full” is both true and false because it indeed is filled halfway with water, but it is also false because it can be said that it is half-empty rather than half-full.

3.19 What Is Bad, Good, and Beautiful Mathematics?

There are several possible ways of bad mathematics. The most common is an innocent error (and hence bad mathematics) in calculations and/or in algebraic rules, which can be corrected swiftly. For example, forgetting the rule that the square root of a positive number is equal to both a negative and a positive number, which can lead to catastrophic situation such as following: Let *n* be a positive integer, since $(n+1)^2 = n^2 + 2n + 1$ or $(n+1)^2 - (2n+1) = n^2$, subtracting $n(2n+1)$ from both sides and factoring, we get $(n+1)^2 - (n+1)(2n+1) = n^2 - n(2n+1)$, now adding $(1/4)(2n+1)^2$ to both sides yields $(n+1)^2 - (n+1)(2n+1) + \frac{1}{4}(2n+1)^2 \ = \ n^2 - n(2n+1) + \frac{1}{4}(2n+1)^2$, which may be written as $[(n+1) - (1/2)(2n+1)]^2 = [n - (1/2)(2n+1)]^2$, and hence taking square roots on both sides gives $n + 1 - (1/2)(2n+1) = n - (1/2)(2n+1)$; therefore, $n = n + 1.$ This leads to an absurd conclusion that all integers are the same as 1. A wrong application of mathematical induction also gives the same answer. For this, let $P[n]$: All numbers in a set of *n* numbers are equal to one another. Clearly, $P[1]$ is true. Suppose *k* is a natural number for which $P[k]$ is true. Let $a_1, a_2, \cdots, a_k, a_{k+1}$ be any set of $k+1$ numbers. Then, by the supposition, $a_1 = a_2 = \cdots = a_k$ and $a_2 = a_3 = \cdots = a_k = a_{k+1}$. Therefore $a_1 = a_2 = \cdots = a_k = a_{k+1}$, and $P[k+1]$ is true. Thus, it follows that $P[n]$ is true for all natural numbers n.

Invalid reasoning also leads to bad mathematical conclusions. For example, since x^2 is positive, *x* is positive; since $\sqrt{a} \times \sqrt{b} = \sqrt{a \times b}$, we have $\sqrt{-1} \times \sqrt{-1} = \sqrt{(-1) \times (-1)}$, and hence $\sqrt{(-1)^2} = \sqrt{1}$, i.e., $-1 = 1$; since $(-x)^2 = (x)^2$, we have $\ln(-x)^2 = \ln(x)^2$, whence $2\ln(-x) = 2\ln(x)$, and hence $\ln(-x) = \ln(x)$. Let *e* denote the eccentricity of the ellipse $x^2/a^2 + y^2/b^2 = 1$. It is well known that the length *r* of the radius vector drawn from the left-hand focus of the ellipse to any point $P(x, y)$ on the curve is given by $r = a + ex$. Now $dr/dx = e$. Since there are no values of *x* for which dr/dx vanishes, it follows that *e* has no maximum or minimum. But the only closed curve for which the radius vectors have no maximum or minimum is a circle. It follows that every ellipse is a circle; consider the isosceles triangle *ABC* as in Fig. 3.17, in which base $AB = 12$ and altitude $CD = 3$.

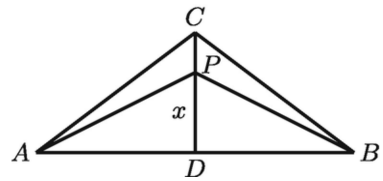

Fig. 3.17 Invalid reasoning

Clearly there is a point *P* on *CD* such that $S = PC + PA + PB$ is a minimum. We denote *DP* by x. Then $PC = 3 - x$ and $PA = PB = (x^2 + 36)^{1/2}$. Hence,

$$S = 3 - x + 2(x^2 + 36)^{1/2}$$

and

$$\frac{dS}{dx} = -1 + 2x(x^2 + 36)^{-1/2}.$$

Setting $dS/dx = 0$, we find $x = 2\sqrt{3} > 3$, and *P* lies outside the triangle on *DC* produced. Hence there is no point on the segment *CD* for which *S* is a minimum.

Although mathematicians are extraordinarily careful, sometimes bad logic leads to bad mathematics. For example, the *Principia Mathematica* is a three-volume work on the foundations of mathematics written by Whitehead and Bertrand Russell and published in 1910, 1912, and 1913. This collection takes 360 pages to prove $1 + 1 = 2$, and gets it wrong. According to a critical article by Adrian Richard David Mathias (born 1944, England), Robert Martin Solovay (born 1938, USA) showed that Bourbaki's definition of the number 1, written out using the formalism in the 1970 edition of *Théorie des Ensembles*, requires $2,409,875,496,393,137,472,149,767,527,877,436,912,979,508,338,752,$ $092,897 \simeq 2.4 \times 10^{54}$ symbols and $871,880,233,733,949,069,946,182,804,910,$ $912,227,472,430,953,034,182,177 \simeq 8.7 \times 10^{53}$ connective links used in their treatment of bound variables. Mathias notes that at 80 symbols per line, 50 lines per page, and 1,000 pages per book, this definition would fill up 6×10^{47} books. (If each book weighed a kilogram, these books would be about 200,000 times the mass of the Milky Way.) It appears that Bourbaki's proof of $1 + 1 = 2$, written on paper would not fit inside the observable Universe.

One of the most important tasks of mathematics is the examination of assumptions. Mathematics enables us to see which assumptions are necessary and which are sufficient, and if any of the assumptions contradicts one of the others. For example, for integers greater than 2, being odd is necessary (but not sufficient) to being prime, since 2 is the only whole number that is both even and prime. A number being divisible by 4 is sufficient (but not necessary) for it to be even, but being divisible by 2 is both sufficient and necessary for it to be even. For the convergence of the series $\sum_{n=1}^{\infty} a_n$, it is necessary that $a_n \to 0,$ but not sufficient. For example, for the harmonic series $\sum_{n=1}^{\infty} 1/n, \ a_n = 1/n \to 0,$ but the series diverges. Suppose $a_n > 0, \ n \in \mathcal{N}$, the condition $\lim_{n\to\infty} a_{n+1}/a_n < 1$ is sufficient for the series to converge, but not necessary. For example, *p* series $\sum_{n=1}^{\infty} 1/n^p, \ p > 1$ converges; however, $a_{n+1}/a_n = n^p/(n+1)^p = 1/(1+1/n)^p \to 1.$ Corollary 3.3 gives necessary and sufficient conditions for two positive integers to be relatively prime.

Several other criteria that can be classified as bad mathematics are wrong assumptions, e.g., every bounded sequence is convergent, and in fact, the sequence $a_n = 0,$ when *n* is odd, and $a_n = 1,$ when *n* is even, is bounded but has no limit; enormous formulas, e.g., Ptolemy's 80 equations to describe the solar system, 15000 pages for the classification of finite, simple groups, in theoretical physics the addition of general gauge-fixing term to second variation of fourth-order action in quantum gravity involves an equation with 5000 terms; derivations constructed to produce a correct result in spite of incorrect logic or operations, e.g., $16/64$ canceling 6 gives $1/4;$ proving a result with an example, e.g., to show there is no integer *a* such that $a \equiv 7 (\mathrm{mod}\ 14)$ and $a \equiv 5 (\mathrm{mod}\ 21),$ we just let $a = 14$ and confirm the truth; not verifying all the conditions of a result, e.g., Leibniz criteria for the convergence of an alternating series $\sum_{n=0}^{\infty}(-1)^n a_n,$ are 1. a_n are positive, 2. $\lim_{n\to\infty} a_n = 0,$ 3. a_n decreases monotonically; we just show one or two of these and leave the other; use the terms such as "clearly$\cdots$," "it is self-evident that$\cdots$," "it can be easily shown that$\cdots$," "$\cdots$ does not warrant a proof," "the proof is left as an exercise for the reader," and the list continues. Finally, mathematics is really considered bad if your proof is correct but hard to follow for anyone because there are too many gaps for the reader to fill. In 1657, Pascal realized this fact and wrote "I have made this letter longer because I lack the time to make it short.” Similarly, Samuel Langhorne Clemens (1835–1910, USA), best known by his pen name Mark Twain, understood this phenomenon when he apologized to a correspondent,“I didn’t have time to write you a short letter so I wrote you a long instead.” Although most of the solutions discovered for Hilbert’s 20 problems which he proposed as a challenge to the International Congress of Mathematicians, held in Paris in 1900, have been quite complex and intellectually demanding, he valued on simplicity and intelligibility. His famous quote is “A mathematical theory is not to be considered complete until you have made it so clear that you can explain it to the first man whom you meet on the street.” Einstein made it more relevant by saying “Everything should be made as simple as possible, but not simpler.” However, the current mathematics has reached to the extent that often-understanding others’ work on the same problem you are working for several years takes weeks.

In 2007, Terence Tao [506] cautiously made an attempt to list the desirable requirements for good mathematics: problem-solving (e.g., a major breakthrough on an important mathematical problem), technique (e.g., a masterful use of existing methods or the development of new tools), theory (e.g., a conceptual framework or the choice of notation which systematically unifies and generalizes an existing body of results), insight (e.g., a major conceptual simplification or the realization of a unifying principle, heuristic, analogy, or theme), discovery (e.g., the revelation of an unexpected and intriguing new mathematical phenomenon, connection, or counterexample), application (e.g., to important problems in physics, engineering, computer science, statistics, etc. or from one field of mathematics to another), exposition (e.g., a detailed and informative survey on a timely mathematical topic or a clear and well-motivated argument), pedagogy (e.g., a lecture or writing style which enables others to learn and do mathematics more effectively or contributions to mathematical education), vision (e.g., a long-range and fruitful program or set of conjectures), taste (e.g., a research goal which is inherently interesting and impacts important topics, themes, or questions), relations (e.g., an effective showcasing of a mathematical achievement to nonmathematicians or from one field of mathematics to another), meta-mathematics (e.g., advances in the foundations, philosophy, history, scholarship, or practice of mathematics), rigorous (all details correctly and carefully given in full), beautiful (e.g., the amazing identities of Ramanujan; results that are easy (and pretty) to state but not to prove), elegant (e.g., pleasant, graceful, stylish, and succinct), achieving a difficult result with a minimum of effort), creative (e.g., a radically new and original technique, viewpoint, or species of result), useful (e.g., a lemma or method which will be used repeatedly in future work on the subject), strong (e.g., a sharp result that matches the known counterexamples or a result which deduces an unexpectedly strong conclusion from a seemingly weak hypothesis), deep (e.g., a result which is manifestly nontrivial, for instance, by capturing a subtle phenomenon beyond the reach of more elementary tools), intuitive (e.g., an argument which is natural and easily visualizable), and definitive (e.g., a classification of all objects of a certain type; the final word on a mathematical topic). However, a single work can hardly incorporate them all; in fact, some are mutually incompatible.

We all know what we like in music, painting, or poetry, but it is much harder to explain why we like it. The same is true in mathematics; in fact, Cayley said "As with everything else, so with a mathematical theory; beauty can be perceived, but not explained." Mathematical beauty is not linked to simplicity, complexity, or formal rigor. Bertrand Russell eloquently observed: Mathematics, rightly viewed, possesses not only truth but also supreme beauty—a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of paintings or music, yet sublimely pure and capable of a stern perfection such as only the greatest art can show. He also said: The true spirit of delight, the exaltation, the sense of being more than Man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as poetry. According to Poincaré, the mathematician does not study pure mathematics because it is useful; [s]he studies it because [s]he delights in it and [s]he delights in it because it is beautiful. Hardy in his book [249] compares mathematics with painting and poetry. For Hardy, the most beautiful mathematics was that which had no practical applications in the outside world (pure mathematics) and, in particular, his own special field of number theory. Hardy contends that if useful knowledge is defined as knowledge which is likely to contribute to the material comfort of mankind in the near future (if not right now), so that mere intellectual satisfaction is irrelevant, then the great bulk of higher mathematics is useless. He justifies the pursuit of pure mathematics with the argument that its very "uselessness" on the whole meant that it could not be misused to cause harm. On the other hand, Hardy denigrates much of the applied mathematics as either being "trivial," "ugly," or "dull" and contrasts it with "real mathematics," which is how he ranks the higher, pure mathematics. However, much of mathematics was either initiated in response to external problems or has subsequently found unexpected applications in the real world. This whole linkage between mathematics and science has an appeal of its own, where the criteria must include both the attractiveness of the mathematical theory and the importance of the applications.

As reported by Dirac, there are two strategies for studying nature: *experimental method*, which, beginning with observed facts, looks for the relationship that exists between them, and the *mathematical reasoning*, which only involves the search for mathematical beauty, the physical significance of which is investigated only later. For him the second method was more profitable because nature manifests itself in terms of beautiful mathematical equations. According to Erdös God has a transfinite book "The Book" with all the theorems and their best proofs. You do not have to believe in God, but you have to believe in "The Book." The best proofs, according to Erdös, are the simplest and most elegant, though he admits that "in some cases it's not really clearly defined." Fortunately, a handful of proofs that belong in "The Book" can be understood by anyone with a vague memory of high school algebra and what Erdös would call a brain that is "open." Understanding such a proof is a little like viewing one of those three-dimensional pictures that appear at first glance to be nothing more than a sheet of marbled paper. You relax your eyes, open your mind, drop your prejudices, and concentrate. In a little while the surface of the paper seems to dissolve, revealing a three-dimensional image of a dolphin or a dinosaur. That moment is a revelation, magic, from emerging from formlessness. Doing mathematics can feel like that. For example, in Euclid's *Elements* there are several results which are in "The Book," e.g., Euclidean division and algorithm (Theorems 3.2 and 3.3), fundamental theorem of arithmetic (Theorem 4.1), infinitely many prime numbers (Theorem 4.2), characterization of perfect numbers (Theorem 4.3), Pythagorean theorem (Chap. 5), and the list continues. We are sure Chinese remainder theorem (Theorem 3.10), Fermat's little theorem (Theorem 4.5), and Lagrange's four-square theorem (Theorem 4.19) also belong to "The Book." Martin Aigner (born 1942, Austria) and Günter Matthias Ziegler (born 1963, Germany) have included several such theorems in their book [23]. In conclusion, the essence of mathematics lies in its beauty and its intellectual challenge.

Erdös once said, having himself written clumsy proofs. Over time the essence of such proofs becomes clearer, and more concise and enlightening versions are written. Here we illustrate two examples that completely support his statements. However, we do not consider the initial proof as clumsy, rather it is the first most important step.

In 1960, Zdzislaw Opial (1930–1974, Poland) proved the following interesting integral inequality: Let $x(t) \in C^{(1)}[0, h]$ be such that $x(0) = x(h) = 0,$ and $x(t) > 0$ in $(0, h).$ Then, the following inequality holds:

$$\int_0^h |x(t)x'(t)|dt \ \leq \ \frac{h}{4}\int_0^h (x'(t))^2 dt.$$

In this inequality the constant $h/4$ is the best possible. Opial's proof is nearly 3-pages long. In 1960 itself, Czeslaw Olech (1931–2015, Poland) showed that in Opial's inequality the positivity requirement of $x(t)$ is unnecessary, and the inequality holds even for the functions $x(t)$ which are only absolutely continuous in $[0, h].$ Furthermore, Olech's proof is simpler than that of Opial. Two years later Paul Richard Beesack (Canada) gave an even simpler proof; his proof paved the way for some nontrivial generalizations. In 1964, Norman Levinson (1912–1975, USA) extended Opial's inequality to complex-valued functions. A year later C.L. Mallows (USA) conjectured that ultimate simplicity is attained in his proof, which is only few lines. Yet, in the same year 1965, an equally simpler proof was offered independently by L.K. Hua (China) and R.N. Pederson (USA). Since then hundreds of generalizations of Opial's inequality have been published and new applications have been provided. In 1995, Agarwal and Pang in their book of 393 pages [9] collected most of the known results published in the past 35 years on Opial's type inequalities. The work published after 1995 is only in research papers. In 1981, during the General Inequalities, three meeting at Oberwolfach (Germany) Agarwal, during questions and answers session, proved the following: If $u(x,y) \in C^{(1,1)}([0,a]\times[0,b]),\ u(0,y) = u(x,0) = 0,$ then

$$\int_0^a\int_0^b |u(x,y)u_{xy}(x,y)|dx\,dy \ \leq \ Cab\int_0^a\int_0^b |u_{xy}|^2 dx\,dy,$$

where the constant $C = 1/2\sqrt{2},$ and asked if *C* can be replaced by $1/4.$ However, Raymond Moos Redheffer (1921–2005, USA) gave an example to show that $C \geq (3+\sqrt{13})/24.$ The problem of obtaining the optimal constant *C* still remains unsolved.

In 1892, Aleksandr Mikhaĭlovich Lyapunov (also Romanized as Liapunov, Liapounoff, Ljapunov, or Ljapunow), 1857–1918, Russia, proved the following result: Let *q* be a real-valued and continuous function defined on the interval $[a, b]$. If the so-called Hill differential equation (after George William Hill, 1838–1914, USA)

$$x''(t) + q(t)x(t) \ = \ 0, \quad t \in (a, b)$$

has a nontrivial solution that vanishes at two points of $[a, b]$, then *q* satisfies the inequality

$$\int_a^b |q(t)| dt \ > \ \frac{4}{b-a}.$$

This remarkable inequality is sharp, in the sense that the constant 4 cannot be replaced by a larger number. Moreover, one may show that if *q* is a real-valued function such that the second-order Hill differential equation has a nontrivial solution having two distinct zeros on $[a, b]$, then the nonnegative part $q^+ = \max\{0, q\}$ must satisfy the so-called Lyapunov inequality

$$\int_a^b q^+(t) dt \ > \ \frac{4}{b-a}.$$

Since then and until the publication of our book in 2021 (see Agarwal et. al. [20]) of 607 pages, research on Lyapunov's inequality and many of its generalizations has far exceeded 5000 publications, and numerous applications to many branches of mathematics have been given. One of the improvements of Lyapunov's inequality is the so-called Hartman inequality (after Philip Hartman, 1915–2015, USA)

$$\int_a^b (b-t)(t-a)q^+(t) dt \ > \ b-a.$$

has also received considerable attention.

3.20 Do Classical Problems from Antiquity Lead to New Mathematics?

Despite ancient Greek geometers innovative work, they were unable to settle the following three questions whose origin is only various speculations:

Problem1. How to construct the side of a cube whose volume shall be twice of a given cube?

Problem2. How to trisect any triangle?

Problem3. How to construct a square equal in area to any given circle (also known as quadrature of a circle)?

Following strictly Euclidean tools to find solutions of these three problems left conundrums to mathematical ingenuity for over 2000 years, but their unsuccessful attempts influenced modern mathematics and helped the world to realize that it is not about calculations but understanding and testing the reality of the world. In particular, these problems established strong and profound links between geometry, algebra, and arithmetic and strengthened the foundation of mathematics. Details about these problems have been reported in short treatises of Claude Comiers (died 1693, France) in 1677 [134], Felix Klein in 1897 [308], and Wilbur Richard Knorr (1945–1997, USA) in 1986 [310]. In recent years, these problems have been discussed on hundreds of websites. In what follows, we shall provide a brief account of these problems.

- For the first problem Heath in [257,258] has quoted Eratosthenes comment: When the God Apollo proclaimed to the Delians by the oracle that, to get rid of a plague they should construct an altar double of the existing one, and their craftsmen fell into great perplexity in their efforts to discover how solid could be made double of a (similar) solid; they therefore went to ask Plato about it, and he replied that the oracle meant, not that the God wanted an altar of double the size, but that he wished, in setting the task, to shame the Greeks for their neglect of mathematics and their contempt for geometry (believing geometry was the root of all reality). It is estimated that the plaque accounted for between 75 and 100000 deaths, almost 25 percent of the Athenian population. Another version of problem's origin also comes from Greek mythology which relates to King Minos (son of Zeus and Europa). This question of doubling a cube came after the passing of his late son Glaucus. Apparently, Minos was very upset with the construction of his son's tomb and did not believe it was fitted for the son of a king. He immediately thereafter ordered builders to double the size of it. The doubling of the cube was also needed to solve the following problem: Given a catapult, construct a second catapult that is capable of firing a projectile twice as heavy as the projectile of the first catapult.

The first decisive step in the problem was taken by Hippocrates: Using analytical arguments if for a given line segment of length *a*, it is required to find *x* such that $x^3 = 2a^3,$ line segments of lengths *x* and $y,$ respectively, may be searched such that $a : x = x : y = y : 2a$, but then $a^3/x^3 = (a/x)^3 = (a/x)(x/y)(y/2a) = a/2a = 1/2,$ or $x^3 = 2a^3.$ Thus, letting $a = 1,$ the problem is equivalent to determining a geometrical construction, by means of Euclidean tools, for the real root of $x^3 - 2 = 0,$ that is, $x = 2^{1/3}.$ Later many mathematicians/philosophers used alternative tools to solve the problem, e.g., Archytas used three surfaces of revolution right cone, cylinder, and torus (this work is considered as one of the finest mathematical achievements of antiquity); Plato, in spite of his disgust to mechanical solutions "The good of geometry is set aside and destroyed, for we again reduce it to the world of sense, instead of elevating and imbuing it with the eternal and incorporeal images of thought, even as it is employed by God, for which reason He always is God," provided a mechanical solution; Eudoxus projected the intersection of the cone and torus onto *xy*-plane, for this he discovered the curve known as *Kampyle of Eudoxus* which in Cartesian and polar coordinates can be written as $x^4 = a^2(x^2 + y^2),\ r = a\sec^2\theta;$ Dinostratus (390–320 BC, Greece) employed Hippias of Elis's (around 460 BC, Greece) curve *quadratrix* which in Cartesian coordinate appears as $y = x\cot(\pi x/2a);$ Menaechmus of Proconnesus (around 375–325 BC, Greece) used parabolas and rectangular hyperbola $x^2 = ay$ and $xy = ab,$ or $y^2 = bx$ and $xy = ab,$ or $x^2 = ay$ and $y^2 = bx$ (historians even say Menaechmus developed conic sections, circles, hyperbola, parabola, and ellipse, while decoding this problem); Philo of Byzantium's (around 280–220 BC, Greece) line finds the point of intersection of a rectangular hyperbola and a circle; Nicomedes (280–210 BC, Greece) employed his curve *conchoid* which in Cartesian coordinates may be written as $(x-a)^2(x^2+y^2) = b^2x^2;$ Eratosthenes used two parallel horizontal bars, two movable triangle plates, and a fixed triangle; Apollonius solution is similar to that of Philo; Diocles (around 240–180 BC, Greece) used his curve called *cissoid* which in Cartesian coordinate appears as $(x^2+y^2)x = 2ay^2;$ Hero of Alexandria (around 75

AD, Egypt-Greece) solution is similar to to that of Philo; Sporus of Nicaea (around 240–300, Greece) used approximations which are early examples of integration; Pappus developed an approximate method; Pandrosion of Alexandria (around 300–360, Greece) established a numerically accurate approximate method; and Newton also suggested a geometric method.

The frustration of failure of not able to solve duplication problem by Euclidean tools compounded when prominent mathematicians including Descartes, Gauss, and Abel convinced that the problem is unsolvable. In his treatise *Disquisitiones Arithmeticae* Gauss went so far as to claim he had proved the impossibility result. But he then proceeded to announce that in the interest of brevity he was omitting the proof. Finally, in 1837 Pierre Wantzel (1814–1848, France) obtained necessary and sufficient conditions for the solution of an algebraic equation with rational coefficients to be geometrically constructible in the manner specified. His proof was published in the second volume of the *Journal de Mathématiques pures et appliquées* founded in 1836 by Joseph Liouville (1809–1882, France) and was the culmination of the work started by Gauss. Wantzel's conditions were not satisfied for the cubic equation $x^3 - 2 = 0,$ and this demonstrated the impossibility. An improved proof of unattainability was later given by Jacques Charles Francois Sturm (1803–1855, Switzerland), but he did not publish it. In 2003, Abe Hisashi in his publication *Amazing Origami*, (in Japanese), ISBN 4-535-78409-4 has demonstrated that Origami can be used for doubling the cube. Menaechmus's conic sections have far-reaching applications throughout mathematics: Omar Khayyám used the intersection of conic sections to solve cubic equations (a problem closely related to duplicating the cube), the paths of the planets around the Sun are ellipses with the Sun at one focus, parabolic mirrors are used to converge light beams at the focus of the parabola, parabolic microphones perform a similar function with sound waves, solar ovens use parabolic mirrors to converge light beams to use for heating, the parabola is used in the design of car headlights and in spotlights because it aids in concentrating the light beam, the trajectory of objects thrown or shot near the Earth's surface follow a parabolic path, hyperbolas are used in a navigation system known as LORAN (long range navigation), and hyperbolic and parabolic mirrors and lenses are used in systems of telescopes.

- Mark Kac and Ulam asserted that "The unique and peculiar character of mathematical reasoning is best exhibited in proofs of impossibility.

When it is asserted that doubling the cube (i.e. in constructing the cube root of two) is impossible, the statement does not merely refer to a temporary limitation of human ability to perform this feat. It goes far beyond this, for it proclaims that never, no matter what will anybody ever be able to construct the cube root of two…if the only instruments at his disposal are a straightedge and a compass. No other science, or for that matter no other discipline of human endeavor, can even contemplate anything of such finality.”

- For the second problem we begin with the note that some angles, e.g., $45, 72, 90, 180$ degrees can be divided into three equal parts by using only Euclidean tools. Thus, the trisection of an angle here means to partition an arbitrary angle into three equivalent angles. The origin of this problem is not known; however, a traditional problem in ancient Greek geometry was the construction of regular polygons and of regular solids. Greek mathematicians knew how to construct regular polygons with $3, 4,$ and 5 sides and of an even number of sides, however failed to construct with seven sides (heptagon). As we have seen earlier in Chap. 1, they could also construct five regular polyhedra. Perhaps this leads to the natural question: is it possible to construct a regular polygon with an arbitrary number of sides with compass and straightedge? If not, which regular polygons are constructible and which are not? To find a solution of this problem, it became necessary to divide a given arbitrary angle to an arbitrary number of times. Clearly, the trisection of an arbitrary angle is just a particular case of this general problem. An excellent survey of known methods for the trisection of an arbitrary angle has been documented in [540] by Robert Carl Yates (1904–1963, USA) in 1942. Like for the first problem, trisection of an angle avoiding Euclidean tools has a long history. Hippocrates seems to be the first who did not use Euclidean tools to find a relatively simple method to trisect any given angle (see Fig. 3.18): Given an angle *CAB*, draw *CD* perpendicular to *AB* to cut it at $D.$ Complete the rectangle $CDAF.$ Extend *FC* to *E*, and let *AE* be drawn to cut *CD* at $H.$ The point *E* is such that $HE = 2AC.$ The angle *EAB* is $1/3$ of the $CAB.$

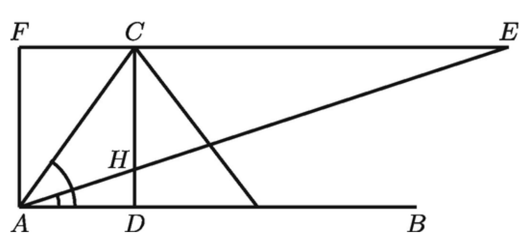

Fig. 3.18 Trisection of an angle

Then, Hippias used his *quadratrix*; Pappus used hyperbola $y^2+(1-e^2)x^2-2kx+k^2=0$; Archimedes used his *spiral* $y=x\tan\sqrt{x^2+y^2}$; Nicomedes used his *conchoid*; Nicholas of Cusa and later Willebrord Snell (Snellius) (1580–1626, The Netherlands) showed that for an arbitrary angle ψ, the following approximate formula holds:

$$\frac{\psi}{3} \simeq \tan^{-1}\left(\frac{\sin\psi}{2+\cos\psi}\right)$$

whose right side can be calculated geometrically, and this formula for $30°, 60°,$ and $90°$, respectively, gives $9.896090638°, 19.10660535°,$ and $26.56505118°$; Albrecht Dürer (1471–1528, Germany) in 1525 demonstrated that for an arbitrary angle ψ, the following approximate formula holds:

$$\frac{\psi}{3} \simeq 2\sin^{-1}\left[\frac{1}{9}\sin\frac{\psi}{2}+\sqrt{\frac{2}{27}}\sqrt{2+\cos^2\frac{\psi}{2}-\cos\frac{\psi}{2}\sqrt{8+\cos^2\frac{\psi}{2}}}\right]$$

whose right side can be obtained geometrically, and this formula for $60°, 120°,$ and $180°$, respectively, gives $19.999974804°, 39.96789639°,$ and $59.47291622°$; Viéte showed that the problems of trisection of an angle and the duplication of a cube both depend upon the solution of cubic equations; Descartes used intersection of a circle $x^2+y^2-2ax-4y=0$ and a parabola $y=x^2$ to obtain trisection equation $x^3-3x-2a=0,$ where $x=2\cos\theta$ and $a=\cos 3\theta,$ which follows immediately from the equality $\cos 3\theta = 4\cos^3\theta-3\cos\theta$; Pascal used his *limaçon* $(x^2+y^2-ax)^2=b^2(x^2+y^2)$; Tommaso Ceva (1648–1737, Italy) used his *cycloidum anomalarum* (cycloid of Ceva) $(x^2+y^2)^3=(3x^2-y^2)^2$; in 1695 he also offered a mechanical trisecting tool called *Ceva's pantograph*; and Maclaurin used his *trisectrix* $2x(x^2+y^2)=a(3x^2-y^2)$.

Gauss in 1796 was able to construct a regular polygon with 17 sides using Euclidean tools, and in his treatise *Disquisitiones Arithmeticae* proved that a regular polygon with *n* sides is constructible if *n* is the product of a power of 2 and distinct prime numbers (Fermat's primes, see Sect. 4.6) of the form $F_k=2^{2^k}+1$; in particular, when $k=0,1,2,3,4$ we see that each of the corresponding numbers $F_k=3,5,17,257,65537$ is prime, so regular polygons with these numbers of sides are constructible. Thus the geometric question of the possibility of construction of a regular *n*-polygon with ruler and compass turns out to depend on the arithmetic nature of the number $n.$ According to one story, Gauss approached his colleague Abraham Gotthelf Kästner (1719–1800, Germany) with this discovery; however, he told Gauss that the discovery was useless since approximate constructions were “well known,” furthermore that the construction was impossible, so Gauss’ proof had to be flawed, and finally that Gauss’ method was something that he (Kästner) already knew about, so Gauss’ discovery was unimportant. Despite Kästner’s discouraging remarks, Gauss was so proud of this discovery that he requested that a 17 regular polygon be inscribed upon his tombstone (but, it was not, as grave smith thought it would simply look like a circle). Gauss later toasted Kästner, who was an amateur poet, as the best poet among mathematicians and the best mathematician among poets. Gauss did not prove that regular *n*-gons, where $n=7,11,13,19,$ and so on cannot be constructed. This gap was filled in 1837 by Wantzel.

In 1822, Magnus Georg Paucker (1787–1855, Estonia-Latvia), and in 1832, Friedrich Julius Richelot (1808–1875, Germany) gave the construction of a regular polygon of 257 sides. As we have noted earlier the construction for a regular 65537 sides was given by Hermes. In 1895, James Pierpont (1866–1938, USA) proved that a regular polygon of *n* sides (*n* prime) can be constructed using conic sections iff $n-1$ contains no prime factors other than 2 and $3.$ For example, a regular 7-gon can be constructed using conics, while a regular 11-gon cannot. Similar to cube doubling problem, Gauss in his treatise *Disquisitiones Arithmeticae* also claimed he had proved the impossibility of trisection, but later withdrew. John Francis Lagarrigue in 1831 proposed compass of proportions to accomplish trisection of angles and arches; Wantzel showed that his same result of 1837 is also applicable for the cubic equation $x^3-3x-2a=0,$ and this displayed the unfeasibility; in conclusion, a regular polygon with *n* sides is constructible iff the cosine of its common angle is a constructible number, that is, can be written in terms of the four basic arithmetic operations and the extraction of square roots; and Nicolaus Fialkowski (Poland) in 1860, having in mind that an arbitrary angle ψ can be bisected into equal parts, used the geometric series (3.1), which for $a=1/2, r=-1/2$ gives the following approximations of $\psi/3$:

$$\theta_n \; = \; \frac{1}{3}\left(1-\left(-\frac{1}{2}\right)^{n+1}\right)\psi, \quad n \ge 0 \quad (\theta_{2n+1} \le \psi/3 \le \theta_{2n}).$$

This expression for $\psi = 60^\circ$ with $n = 10$ gives $\theta_{10} \simeq 20.000976562^\circ$, whereas for $a = r = 1/2^2$, (3.1) leads to the approximations

$$\phi_n \; = \; \frac{1}{3}\left(1-\frac{1}{2^{2n+2}}\right)\psi, \quad n \ge 0 \quad (\phi_n \le \phi_{n+1} \le \psi/3),$$

which again in particular, for $\psi = 60^\circ$ with $n = 10,$ gives $\phi_{10} \simeq 19.99999523^\circ.$

Sylvester in 1875 announced a linkage trisector that he called *A Lady's fan*, and his idea was used in an optical apparatus to keep moving prisms equally inclined to each other which was called *Sylvester's isoklinostat*; Charles-Ange Laisant (1841–1920, France) in 1875 provided two mechanical instruments known as *Laisant's links*; Pierre René Jean Baptiste Henri Brocard (1845–1922, France) in 1887 (seems to be the first) used the tool *tomahawk* whose inventor is unknown; Kempe in 1875 designed a mechanical instrument known as *Kempe's trisector*; Andrew Doyle in 1881 showed that trisection of any rectilineal angle by elementary geometry and solutions of other problems is impossible except by aid of the higher geometry; Amadori (Italy) in 1883 following Ceva suggested a mechanical instrument known as *Amadori's link*; T.W. Nicalson (England) in 1883 used a right-angled square with parallel edges whose legs have the same width, and his procedure is named as *carpenter's square*; Philbert Maurice d'Ocagne (1862–1938, France) in 1934, then Tadeusz Wazewski (1896–1972, Poland) in 1945, G. Peterson in 1983, and Wladyslaw Hugo Dionizy Steinhaus (1887–1972, Poland) in 1999 approximated $\psi/3$ by the following formula:

$$\frac{\psi}{3} \; \simeq \; \tan^{-1}\left(\frac{2\sin(\psi/2)}{1+2\cos(\psi/2)}\right)$$

whose right side can be obtained geometrically, and this formula for $30^\circ, 60^\circ,$ and 90°, respectively, gives $10.01276527^\circ, 20.10390936^\circ,$ and $30.3611934^\circ.$

August Adler (1863–1923, Czech Republic-Austria) in 1906 showed that two celluloid triangles, each having a right angle, can be drafted to trisect any angle and obtained the same trisection equation as Descartes, and his method is known as *drafter's triangles*; A. Aubry (France) in 1909 constructed a right circular cone whose slant height equal to three times the base radius, and his method is called *the cone trisector*; Karajordanoff in 1928 estimated $\psi/3$ by the following approximate formula:

$$\frac{\psi}{3} \simeq \sin^{-1}\left[\frac{(1-\cos\psi)\sin(\psi/2)}{2(\cos(\psi/2)-\cos\psi)}\right]$$

whose right side can be obtained geometrically, and this formula for $30°, 60°,$ and $90°$, respectively, gives $9.994212497°, 19.9686757°,$ and $30°;$ Dobri Naidenoff Petkoff in 1941 refreshed trisection problem; Robert Lee Durham (1870–1949, USA) in 1944 furnished a geometric construction to approximate $\psi/3$ by the following formula:

$$\frac{\psi}{3} \simeq \frac{1}{2}\psi - \tan^{-1}\left[\frac{4}{3}\sin\left(\frac{1}{4}\psi - \sin^{-1}\left(\frac{1}{2}\sin\frac{\psi}{4}\right)\right)\right];$$

this approximation for $40°, 80°,$ and $120°$, respectively, gives $13.34652558°, 26.7729122°,$ and $40.36247719°;$ Free Jamison (USA) in 1954 provided a geometric construction (originally due to C.R. Lindberg) to approximate $\psi/3$ by the following formula:

$$\frac{\psi}{3} \simeq \frac{1}{4}\psi + \tan^{-1}\frac{\sin(\psi/4)}{2+\cos(\psi/4)},$$

which coincides with Avni Pllana's approximation of 2003

$$\frac{\psi}{3} \simeq \tan^{-1}\frac{\sin(\psi/2)+2\sin(\psi/4)}{\cos(\psi/2)+2\cos(\psi/4)};$$

these approximation for $40°, 80°,$ and $120°$, respectively, give $13.32956306°, 26.63627259°,$ and $39.89609064°;$ Orman Quine in 1990 used elementary algebra to show that some angles cannot be trisected by ruler and compass; in 2003, Abe Hisashi in his publication *Amazing Origami* also showed that Origami can be used for trisecting an angle; and George Goodwin polished Doyle's work in 2018 to provide trisection of any rectilineal angle.

- The problem of squaring a circle is one of the oldest and most famous/studied/ unlimited/praised/intriguing problems in whole of mathematics. All circles have the same shape and traditionally represent the infinite, immeasurable, and even spiritual world. Some circles may be large and some small, but their “circleness,” their perfect roundness, is immediately evident. Mathematicians say that all circles are *similar*. Behind this unexciting observation, however, lies a profound fact of mathematics: that the *ratio* of circumference to diameter is the same for one circle as for another. Whether the circle is gigantic, with large circumference and large diameter, or minute, with tiny circumference and tiny diameter, the *relative* size of circumference to diameter will be exactly the same. In fact, the ratio of the circumference to the diameter of a circle produces the most ubiquitous/external/mysterious mathematical number known to the human race. It is written as Pi, or symbolically mathematicians chose the 16th letter of the Greek alphabet π and defined as (see Fig. 3.19)

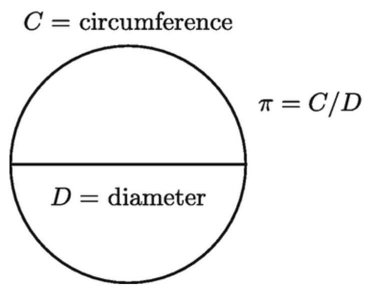

Fig. 3.19 Pi

$$\text{Pi} = \frac{\text{distance around a circle}}{\text{distance across and through the center of the circle}} = \frac{C}{D} = \pi.$$

Since the exact date of birth of π is unknown, one could imagine that π existed before the Universe came into being and will exist after the Universe is gone. Its appearance in the disks of the Moon and the Sun, makes it as one of the most ancient numbers known to humanity. It keeps on popping up inside and outside the scientific community, for example, in many formulas in geometry and trigonometry, physics, complex analysis, cosmology, number theory, general relativity, navigation, genetic engineering, statistics, fractals, thermodynamics, mechanics, and electromagnetism. Pi hides in the rainbow and sits in the pupil of the eye, and when a raindrop falls into water π emerges in the spreading rings. Pi can be found in waves and ripples and spectra of all kinds, and therefore π occurs in colors and music. The double helix of DNA revolves around π. Pi has lately turned up in super-strings, the hypothetical loops of energy vibrating inside subatomic particles. Pi has been used as a symbol for mathematical societies and mathematics in general, e.g., $\pi(x)$ is used in number theory and built into calculators and programming languages. Pi is represented in the mosaic outside the mathematics building at the Technische Universität Berlin. Pi is also engraved on a mosaic at Delft University. Even a movie has been named after it.

Pi is the secret code in Alfred Hitchcock's Torn Curtain and in The Net starring Sandra Bullock. Pi day is celebrated on March 14 (which was chosen because it resembles 3.14). The official celebration begins at 1:59 p.m., to make an appropriate 3.14159 when combined with the date. In 2009, the United States House of Representatives supported the designation of Pi Day. Einstein was born on Pi Day (3/14/1879). The first 144 digits of π add up to 666; since there are 360 degrees in a circle, some mathematicians were delighted to discover that the number 360 is at the 359th digit position of $\pi.$ A mysterious 2008 crop circle in Britain shows a coded image representing the first ten digits of $\pi.$ The website "The Pi-Search Page" finds a person's birthday and other well-known numbers in the digits of $\pi.$ Several people have endeavored to memorize the value of π with increasing precision, leading to records of over 100,000 digits.

Throughout the history of $\pi,$ which according to Beckmann [57] "is a quaint little mirror of the history of man," and James Glaisher "has engaged the attention of many mathematicians and calculators from the time of Archimedes to the present day, and has been computed from so many different formula, that a complete account of its calculation would almost amount to a history of mathematics," one of the enduring challenges for mathematicians has been to understand the nature of the number π and to find its exact/approximate value. The quest in fact started during prehistoric era and continues to the present day of supercomputers. The constant search by many including greatest Mathematical thinkers that the world produced continues for new formulas/bounds based on geometry/algebra/analysis, relationship among them, and relationship with other numbers. Right from the beginning till modern times, attempts were made to exactly fix the value of π but always failed, although hundreds constructed circle squares and claimed the success. At one point, the quest to square the circle became such an obsession that De Morgan, coined the term *morbus cyclometricus*, the circle-squaring disease. Roger Webster (England) called the problem squaring the circle as "Mount Everest of mathematics." Stories of these contributors are amusing and at times almost unbelievable. In Chap. 8 we shall show that like other two problems by employing only Euclidean tools squaring a circle is also impossible. Furthermore, in this chapter we shall show that like other two problems on ignoring Euclidean tools, squaring the circle is possible but requires more advanced techniques and mechanical instruments. In addition, we shall provide computational details of π from the beginning till very recent approximation to 100 trillion decimal places, and we find its nature (rational/irrational/transcendental). We believe that the study and discoveries of π will never end; there will be books, research articles, new record-setting calculations of the digits, clubs, and computer programs dedicated to π.

In mathematics there are several other innocent looking problems whose rigorous proofs or impossibility kept outstanding mathematicians engaged for an extended period. We list some of these problems.

Königsberg Bridge Problem: This problem asks if the seven bridges of the city of Königsberg, formerly in Germany but now known as Kaliningrad a part of Russia, over the river Preger can all be traversed in a single trip without doubling back, with the additional requirement that the trip ends in the same place it began. In 1736, Euler proved the impossibility (negative result) of a solution of this problem, which represented the beginning of graph theory.

Euler's Polyhedral Formula of 1752: For any polyhedron (a solid figure with many plane faces, typically more than 6), $V - E + F = 2,$ where V is the number of vertexes, E is the number of edges, and F is the number of faces. For this theorem at least 21 different proofs are known (these different proofs serve not merely to convince but also to enlighten as they require different tools of mathematics which helps in progressing mathematics; failed attempts also spawn new areas of mathematical research). For all five regular polyhedra, the result is immediate (see Fig. 1.4).

The *Isoperimetric Problem* is to determine a plane figure of the largest possible area whose boundary has a specified length. For three dimensions it states that the shape enclosing the maximum volume for its surface area is the sphere. It was formulated by Archimedes but not proved rigorously until 1884 by Karl Hermann Amandus Schwarz (1843–1921, Germany).

The *Fundamental Theorem of Algebra* states that every polynomial equation of degree n with complex number coefficients has n roots, or solutions, in the complex numbers. Many incomplete or incorrect attempts were made at proving this theorem in the eighteenth century, including by d'Alembert in 1746, Euler in 1749, François Daviet de Foncenex (1734–1799, France) in 1759, Lagrange in 1772, Laplace in 1795, James Wood (1760–1839, England), and Gauss in 1799. The first rigorous proof was published by Argand in 1806 and then by Gauss 1815, 1816, and 1848. There have been claims that now more than 200 different proofs are known.

Marie Ennemond Camille Jordan (1838–1922, France): *The Jordan Curve Theorem*. This is the result that a closed non-self-intersecting curve in the plane divides the plane into two regions—a bounded and an unbounded one. Intuitively obvious, this theorem incredibly difficult to prove in the traditional sense of the word. Jordan's original proof of this theorem in 1887 contains gaps. Its first valid proof was given by Oswald Veblen (1880–1960, USA) in 1905.

- For further reading about the topics discussed in this chapter see, Adler [5], Berggren [66], Bittinger [72], Bloch [74], Byers [113], Carlson [115], Coolidge [139], Cupillari [144], D'Angelo and West [149], Dedron and Itard [159], Gerstein [212], Granier and Taylor [224], Hall [246], Hanna [247], Hardy [248], Karpinski [298], Kaye [302], Khan [305], Kulkarni [321], Kushyar [322], Lakshmikantham [328], Lamport [332], Lewis [344], Mikami [371], Murthy [380], Needham [389], Pandit [404], Paramesvaran [405], Plofker [411], Poincaré [414], Polya [415], Rao [428], Reid [429], Rota [437], Sarasvati [446], Sen and Bag [467], Snapper [487], Srinivas [494], Szabo [503], Velleman [519], Viéte [520], and Yehuda [541].

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4. Prime Numbers

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4.1 Introduction

It would be preposterous to write a history of mathematics without acknowledging prime numbers, and it would not be justified to give an account of prime numbers without providing the history of mathematics. Prime numbers occur in almost every branch of mathematics; they are as fundamental as they are ubiquitous, see Loweke [351]. Prime numbers are indivisible. The traditional count of Holy Quran has 6348 verses, and in these verses the word "Allah" (with variation bismillah) appears 2699 times; 2699 is a prime which means it is indivisible just like Allah is indivisible. Skeptics claim that this is just a coincidence. It turned out that prime numbers are one of the characteristics of the entire Holy Quran. A few prime numbers were known as early as 22,000 years ago; the Rhind mathematical Papyrus has Egyptian fraction expansions of different forms for prime and composite numbers; Hindus had adequate knowledge of prime, perfect, and amicable numbers, much before the days of Pythagoreans; the sieve of Eratosthenes is still used to construct lists of small primes. The first systematic treatment of number theory was given by Euclid in Books VII, VIII, and IX of his *Elements*. His "fundamental theorem of arithmetic," "infinitude of primes" (which makes the study of primes fascinating), and "construction of all even perfect numbers" are three painite crystals of numbers theory. In the 1600s, Fermat took the initiative, and through his educated guessed problems (with solutions to only a few) in number theory, especially for prime numbers, captivated/hypnotized the best minds then, and paved the road for generations to follow his footsteps. In fact, over the period of 400 years, Mersenne, Euler, Lagrange, Gauss, Dirichlet, Pafnuty Lvovich Chebyshev (1821–1894, Russia), Riemann, Dedekind, Hadamard, Poussin, Hardy, Littlewood, Ramanujan, Vinogradov, Pillai, Erdös, Selberg, Jingrun Chen, Terence Tao, and the list continues, greatly intensified the theory of primes; however, still there are more unsolved problems than solutions. In this chapter, we shall detail some of their works (leaving behind complicated technicalities) and discuss the present status of several known conjectures.

4.2 Prime and Composite Numbers

An integer *p* greater than 1 is called a *prime number* or simply *prime* if it has no integer factors (or divisors) other than 1 and itself, e.g., the first ten primes are $2, 3, 5, 7, 11, 13, 17, 19, 23,$ and $29.$ In some contexts, negative counterparts of primes, $-2, -3, -5,$ etc., are also regarded as primes. An integer *n* greater than 1 is called a *composite number* or just *composite* if it has divisors between 1 and $n,$ e.g., the first ten composite numbers are $4, 6, 8, 9, 10, 12, 14, 15, 16,$ and $18.$ Except 2 all the prime numbers are odd, since all even numbers after 2 are divisible by the prime 2 and hence composite. Numbers 0 and 1 are neither prime nor composite. Two numbers *a* and *b* are called *relatively prime* if $\gcd(a, b) = 1.$ It is clear that two consecutive odd positive integers are relatively prime. The numbers $a_1, a_2, \cdots, a_n$ are called *pairwise relatively prime* if $\gcd(a_i, a_j) = 1$ whenever $1 \leq i < j \leq n.$ Odd composite numbers are Pythagoreans called *effeminate*. Primes and composites play a very precious role among all numbers. Further, they constitute a mine of beautiful interesting and useful behavior patterns.

The *Ishango bone*, unearthed in 1960 at the Lake Edward in Zaire, close the Ugandan border. It is estimated to be 22,000 years old. It is one of the oldest known remarkable objects that shows evidence of an understanding of numbers beyond counting. It contains three columns of marks: the middle column contains the sequence of numbers $3, 6, 4, 8, 10, 5, 5, 7,$ which add up to $48;$ the numbers in the left and right columns are all odd numbers, the left column lists the prime numbers between 10 and $20,$ i.e., $11, 13, 17, 19,$ and the right column consists of the numbers $11, 21, 19, 9;$ both side columns add up to 60. These numbers cannot be random and suggest an understanding of multiplication and division by 2 (and perhaps familiarity with prime numbers and the lunar cycle). So far, nobody really knows what these prime numbers were used for 22,000 years ago. Ahmes Papyrus contains Egyptian fraction expansions of different forms for prime and composite numbers. The first definition of a prime numbers appeared in the *Elements* (book VII) of Euclid.

Eratosthenes method/algorithm known as sieve of Eratosthenes was apparently the first methodical attempt to separate the primes from the composite numbers, and all subsequent tables of primes and of prime factors have been based on extensions of it. His method is based on a simple observation: if *n* is composite, then it has a divisor *m* such that $1 < m < n^{1/2}$. For this, since *n* is composite, there are integers m_1 and m_2 such that $m_1 m_2 = n$ and $1 < m_1 < n,\ 1 < m_2 < n.$ If m_1 and m_2 are both larger than $n^{1/2}$, then $n = m_1 m_2 > n^{1/2} n^{1/2} = n,$ which is impossible. From this, we can immediately conclude that if *n* is composite, then it has a prime divisor less than equal to $n^{1/2}$. In conclusion, if an integer $n > 1$ is not divisible by a prime $p \leq \sqrt{n},$ then *n* itself is necessarily a prime. To see how his method works to find all primes under $100,$ we list all numbers 0 to 100 in an array, and we cross numbers 0 and 1 as they are not prime. Since 2 is a prime, we cross all even integers from our listing, except 2 itself. The first of the remaining integers is $3,$ which must be a prime. We keep $3,$ but cross out all higher multiples of $3,$ so that $6, 9, 12, \cdots$ are now removed. The smallest integer after 3 not yet crossed is $5.$ It is not divisible by either 2 or 3 (otherwise it would have been canceled), hence is also a prime. Because all proper multiples of 5 are composite numbers, we next cross $10, 15, 20, \cdots,$ retaining 5 itself. The first surviving integer 7 is a prime, for it is not divisible by $2, 3,$ or $5,$ the only primes that precede it. After the proper multiples of $7,$ the largest prime less than $\sqrt{100} = 10,$ have been eliminated, all composite integers under 100 have been crossed out through the sieve. The positive integers that remain,
$2,\ 3,\ 5,\ 7,\ 11,\ 13,\ 17,\ 19,\ 23,\ 29,\ 31,\ 37,\ 41,\ 43,\ 47, 53, 59, 61, 67, 71, 73, 79,$
$83, 89, 97$, are all the primes less than 100.

×	×	2	3	×	5	×	7	×	×
×	11	×	13	×	×	×	17	×	19
×	×	×	23	×	×	×	×	×	29
×	31	×	×	×	×	×	37	×	×
×	41	×	43	×	×	×	47	×	×
×	×	×	53	×	×	×	×	×	59
×	61	×	×	×	×	×	67	×	×
×	71	×	73	×	×	×	×	×	79
×	×	×	83	×	×	×	×	×	89
×	×	×	×	×	×	×	97	×	×

The compilation of such a table involves a fantastic amount of work, which is not always rewarded. The first fair-sized table, one giving the least prime factors of all numbers not divisible by 2 or 5, up to $24{,}000,$ was published by Johann Rahn (1622–1676, Switzerland) as an appendix to his *Teusche Algebra* of 1659. In 1668, Pell extended this table to include numbers up to $100{,}000.$ The table constructed by schoolmaster Anton Felkel (1740–1800, Austria) gave factors of numbers not divisible by 2, 3, or 5 up to 408,000, was published in 1776 by the Imperial Treasury of Austria; when only a few subscriptions were received the remaining copies were confiscated, their pages used as paper in the manufacture of cartridges for a war against the Turks. Jakob Philipp Kulik (1793–1863, Austria) devoted 20 years of his life preparing, unassisted and alone, the factors of numbers up to $100{,}000{,}000.$ He deposited the table in the library of the Vienna Royal Academy in 1867. It was never published; moreover, they lost the part that included the integers from 12,642,600 to 22,852,800.

Boethius reproduced the sieve of Eratosthenes. From the sieve of Eratosthenes, a cumbersome formula can be obtained that will determine the number of primes below *n* when the primes below $\sqrt{n}$ are known. This formula was considerably improved in 1870 by Daniel Friedrich Ernst Meissel (1826–1895, Germany), who showed that the number of primes below 10^8 is $5{,}761{,}455.$ Niels Peder Bertelsen (1869–1938, Denmark) continued these computations, and in 1893, he announced that the number of primes below 10^9 is $50{,}847{,}478$ (the correct last three digits are 534). In 1959, Derrick Henry Lehmer (1905–1991, USA) showed that Bertelsen's result is incorrect, and it should read $50{,}847{,}534;$ he also showed that the number of primes below 10^{10} is $455{,}052{,}511.$

- We observe that $24, 25, 26, 27$, and 28 is a string of five consecutive composites. Now since $6! = 1 \times 2 \times 3 \times 4 \times 5 \times 6$, it is clear that $6! + 2 = 722,\ 6! + 3 = 723,\ 6! + 4 = 724,\ 6! + 5 = 725,$ and $6! + 6 = 726$ is also a string of five consecutive composites. Similarly, we have $702! + 2, 702! + 3, \cdots, 702! + 702$ as a string of 701 consecutive composites. From these two examples, it follows that a string of *n* consecutive composites can be written as

$$(n+1)! + 2,\ (n+1)! + 3, \cdots, (n+1)! + (n+1).$$

  Thus, it is possible to construct a string of arbitrarily long length (in the theory of numbers, this term is known as "as many as we please") of consecutive composites. This also shows that consecutive primes can be far apart.
- Let p_n denote the *n*th prime and c_n the *n*th composite number. Thus, $p_1 = 2$ and $p_2 = 3;$ while $c_1 = 4$ and $c_2 = 6.$ Of course, $p_n \neq c_n,$ for all integers $n.$ To find all integers $n,$ such that $|p_n - c_n| = 1,$ we note that $(p_1, c_1) = (2, 4),\ (p_2, c_2) = (3, 6),\ (p_3, c_3) = (5, 8),\ (p_4, c_4) = (7, 9),\ (p_5, c_5) = (11, 10),\ (p_6, c_6) = (13, 12)$, and $(p_7, c_7) = (17, 14).$ Now since for all integers $k \geq 0,\ p_{7+k} \geq 17 + 2k$ and $c_{7+k} \leq 14 + 2k$, it follows that $|p_{7+k} - c_{7+k}| \geq 3$ for all $k \geq 0.$ Therefore, 5 and 6 are the only positive integers *n* such that $|p_n - c_n| = 1.$
- In 1915, Ramanujan called a composite number as *highly composite number* if the number of its divisors is more than those of any other smaller composite number. For example, composite number 24 has eight divisors, namely, $1, 2, 3, 4, 6, 8, 12,$ and 24 and this number 8 is more than the number of divisors of any of the composite numbers $4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21,$ and $22,$ each of them being smaller than $24.$ Therefore, 24 is a highly composite number. From the definition, it also follows that 45 is not highly composite number because it has only six divisors $1, 3, 5, 9, 15,$ and $45.$ There are an infinite number of highly composite numbers, and the first ten are $1, 2, 4, 6, 12, 24, 36, 48, 60, 120.$ The corresponding numbers of divisors are $1, 2, 3, 4, 6, 8, 9, 10, 12, 16.$ Ramanujan listed 102 highly composite numbers up to $6746328388800,$ but in 1926, Tirukkannapuram Vijayaraghavan (1902–1955, India) found that he missed the number $293318625600,$ which has 5040 factors. If $Q(x)$ represents the number of highly composite numbers $\leq x,$ then Ramanujan showed that

$\lim_{x\to\infty} Q(x)/\ln x = \infty$. Ramanujan also showed that if $N = 2^{(a_2)}3^{(a_3)}\cdots q^{(a_q)}$ is the prime factorization of a highly composite number, then the primes $2, 3, \cdots, q$ form a string of consecutive primes; the exponents are nonincreasing, so $a_2 \geq a_3 \geq \cdots \geq a_q$; the final exponent a_q is always 1, except for the two cases $N = 4 = 2^2$ and $N = 36 = 2^2 \cdot 3^2$, where it is 2. For example, $293318625600 = 2^6 \times 3^4 \times 5^2 \times 7^2 \times 11^1 \times 13^1 \times 17^1 \times 19^1$, whereas for the non-highly composite number $29331862560 = 2^5 \times 3^4 \times 5 \times 7^2 \times 11^1 \times 13^1 \times 17^1 \times 19^1$. Hardy aptly described a highly composite number by the phrase "as unlike a prime as a number can be." Ramanujan received his BA from the Cambridge University in March 1916 on the basis of his long paper on highly composite numbers. In 1944, Leonidas Alaoglu (1914–1981, Canada-USA) and Erdös showed that there exists a constant $a > 1$ such that $Q(x) \geq (\ln x)^a$, where in 1988, Jean-Louis Nicolas (France) proved that there exists a constant $b > 1$ such that $Q(x) \leq (\ln x)^b$. In recent years, Achim Flammenkamp (Germany) has posted a list of the first 779674 highly composite numbers. In his list, 15,000th highly composite number has the product of 230 primes.

- In any set of nine positive integers which differ from each other only in the unit's place, e.g., $1, 2, \cdots, 9$ or $11, 12, \cdots, 19$ or $21, 22, \cdots, 29$, there can be at most four primes. In fact, in the set $\{1, 2, \cdots, 9\}$, there are exactly four primes $2, 3, 5$, and 7. In any other such set, four of the numbers will be even and one will be divisible by 5, and so we are left with only four numbers (ending in $1, 3, 7$, or 9) which can be prime. We list nine quads which are such primes. These are the only cases of "four primes out of nine numbers" up to 5000.

2	3	5	7
11	13	17	19
101	103	107	109
191	193	197	199
821	823	827	829
1481	1483	1487	1489
1871	1873	1877	1879
2081	2083	2087	2089
3251	3253	3257	3259
3461	3463	3467	3469

- In 1963, Ulam drew with 1 a square spiral as shown in the following figure. He was amused to see that the primes tended to fall on diagonal lines. We have encircled the odd primes to show the pattern.

Encouraged by the pattern displayed by the above square spiral, Ulam, Mark Brimhall Wells (1929–2018, USA), and Myron Stein (1919–2020, USA) studied such spirals starting with whole numbers other than $1.$ The following figure gives one such square spiral starting with $41.$ In this also, we notice that primes tend to fall on diagonals. In particular, all the 20 entries on the principal diagonal are primes (421 through 383).

In 1963, the MANIAC II mainframe computer at Los Alamos was used to store the first 90 million primes. It was also used to draw a square spiral diagram for all primes below 10 million and in that many primes were located on its diagonal lines.

Prime numbers are considered the "building blocks" of the natural numbers because every natural number, excluding the number 1, is either a prime number or a product of prime numbers. This fact is known as the *fundamental theorem of arithmetic*; it is at the core of number theory, the study of integers. The essence of this result first appeared in Euclid's *Elements* (VII:32 and IX:14), but its proof was completed by Gauss in his *Disquisitiones Arithmeticae*. For this, we need the following preliminary results:

Lemma 4.1 (Euclid's Lemma) *Let $a, b,$ and c be integers, where $a \neq 0.$ If $a|bc$ and $\gcd(a, b) = 1,$ then $a|c.$*

Proof Since $a|bc,$ there is some integer *q* such that $bc = aq.$ Since *a* and *b* are relatively prime, from Corollary 3.3, there exist integers *x* and *y* such

that $1 = ax + by$. Thus,
$c = c \cdot 1 = c(ax + by) = a(cx) + (bc)y = a(cx) + (aq)y = a(cx + qy)$.
Since $cx + qy$ is an integer, $a|c$. ■

Corollary 4.1 *If* $ca \equiv cb (\text{mod } n)$, *then* $a \equiv b (\text{mod } n/d)$, *where* $d = \gcd(c, n)$.

Proof There exists some *k* such that $ca - cb = kn$. Since $d = \gcd(c, n)$ there exist *s* and *t* such that $\gcd(s, t) = 1$, which implies that $c = ds, n = dt$. Thus, it follows that $dsa - dsb = kdt$, and hence $s(a - b) = kt$, which means $t|s(a - b)$. Thus, in view of Lemma 4.1, $t|(a - b)$, which is the same as $a \equiv b (\text{mod } n/d)$. ■

For an illustration of Corollary 4.1, consider the congruence $33 \equiv 15 (\text{mod } 9)$, which is the same as $3 \cdot 11 \equiv 3 \cdot 5 (\text{mod } 9)$. Since $\gcd(3, 9) = 3$, it follows that $11 \equiv 5 (\text{mod } 3)$.

Corollary 4.2 *Let b and c be integers and p a prime. If* $p|bc$, *then either* $p|b$ *or* $p|c$.

Proof If $p|b$ the corollary holds, so we assume that $p \not| b$. Since the only positive integer divisors of *p* are 1 and p, it follows that $\gcd(p, b) = 1$. Thus, by Lemma 4.1, $p|c$. ■

Corollary 4.3 *Let* $a_1, a_2, \cdots, a_n$, *where* $n \geq 2$, *be integers and let p be a prime. If* $p|a_1 a_2 \cdots a_n$, *then* $p|a_i$ *for some* $i,\ 1 \leq i \leq n$.

Proof We proceed by induction. For $n = 2$, this is simply a restatement of Corollary 4.2. Assume that if a prime *p* divides the product of *k* integers $(k \geq 2)$, then *p* divides at least one of the integers. Now let $a_1, a_2, \cdots, a_{k+1}$ be $k + 1$ integers, where $p|a_1 a_2 \cdots a_{k+1}$. We show that $p|a_i$ for some $i,\ 1 \leq i \leq k + 1$. Let $b = a_1 a_2 \cdots a_k$. So $p|ba_{k+1}$. By Corollary 4.2, either $p|b$ or $p|a_{k+1}$. If $p|a_{k+1}$, then the proof is complete. Otherwise, $p|b$,, that is $p|a_1 a_2 \cdots a_k$. However, by the induction hypothesis, $p|a_i$ for some $i,\ 1 \leq i \leq k$. In any case, $p|a_i$ for some $i,\ 1 \leq i \leq k + 1$. ■

From Corollary 4.3, it is clear that if all $a_1, a_2, \cdots, a_n$ are prime, then $p = a_i$ for some $i,\ 1 \le i \le n.$

Lemma 4.2 *Let $a, b,$ and c be integers, where a and b are relatively prime. If $a|c$ and $b|c,$ then $ab|c.$*

Proof Since $a|c$ and $b|c$, there exist integers *q* and *r* such that $c = aq$ and $c = br.$ Furthermore, since *a* and *b* are relatively prime, there exist integers *x* and *y* such that $1 = ax + by.$ Multiplying by *c* and substituting, we obtain

$$c = c \cdot 1 = c(ax + by) = c(ax) + c(by) = (br)(ax) + (aq)(by) = ab(xr + qy).$$

Since $(xr + qy)$ is an integer, $ab|c.$ ■

Theorem 4.1 (Fundamental Theorem of Arithmetic) *Every integer $n \ge 2$ is either prime or can be expressed as a product of primes, that is,*

$$n = p_1 p_2 \cdots p_m \tag{4.1}$$

where $p_1, p_2, \cdots, p_m$ are primes (some or all of these primes may be repeated). Furthermore, this factorization is unique except possibly for the order in which the factors occur.

Proof If a number $n \ge 2$ is composite by definition, it can be written as $n = p_1 n_1$ where $1 < p_1 < n$ and $1 < n_1 < n.$ We can choose p_1 to be the least divisor of *n* and hence must be prime. Next we find p_2 the least divisor of n_1 and continue the process, which cannot continue indefinitely, and finally leads to product of primes of $n,$ i.e., (4.1) holds. This process is called the *prime factorization*. To prove that such a factorization is unique, we assume to the contrary that there is an integer $n \ge 2$ that can be expressed as a product of primes in two different ways, say

$$n = p_1 p_2 \cdots p_s = q_1 q_2 \cdots q_t$$

where in each factorization, the primes are arranged in nondecreasing order, i.e., $p_1 \le p_2 \le \cdots \le p_s$ and $q_1 \le q_2 \le \cdots \le q_t.$ Since the factorizations are different, there must be a smallest positive integer *r* such that $p_r \ne q_r.$ After canceling, we have

$$p_r p_{r+1} \cdots p_s = q_r q_{r+1} \cdots q_t. \tag{4.2}$$

Now consider the integer $p_r.$ Either $s = r$ and the left side of (4.2) is exactly p_r or $s > r$ and $p_{r+1}p_{r+2}\cdots p_s$ is an integer that is the product of $s-r$ primes. In either case, $p_r | q_r q_{r+1} \cdots q_t$. Therefore, by Corollary 4.3, $p_r | q_j$ for some *j* with $r \le j \le t$. Because q_j is prime $p_r = q_j$. Since $q_r \le q_j$ it follows that $q_r \le p_r$. By considering the integer q_r (instead of p_r) we can show that $p_r \le q_r$. Therefore, $p_r = q_r$. But this contradicts the fact that $p_r \ne q_r$. Hence, every integer $n \ge 2$ has a unique factorization.■

If an integer $n \ge 2$ is expressed as a product $p_1 p_2 \cdots p_m$ of primes, then the primes $p_1, p_2, \cdots, p_m$ need not be distinct. Consequently, we can group equal prime factors and express *n* in the form

$$n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k} = \prod_{i=1}^{k} p_i^{a_i} \tag{4.3}$$

where $p_1, p_2, \cdots, p_k \ (k \le m)$ are primes such that $p_1 < p_2 < \cdots < p_k$, and each exponent a_i is a positive integer. We call this the *canonical factorization* of n. From the Fundamental Theorem of Arithmetic, every integer $n \ge 2$ has a unique canonical factorization.

- Since $12 = 3 \times 4 = 1 \times 3 \times 4$ from the uniqueness of the prime factorization, it immediately follows that 1 is not a prime number.
- From (4.3), it follows that the total number of divisors of *n* is $\prod_{i=1}^{k}(a_i+1)$; the total number of divisors of *n* is odd if *n* is a perfect square, and even if *n* is not a perfect square; and the sum of all the divisors of *n* is

$$\sum_{d|n} d = (1 + p_1 + \cdots + p_1^{a_1})(1 + p_2 + \cdots + p_2^{a_2}) \cdots (1 + p_k + \cdots + p_k^{a_k})$$
$$= \prod_{i=1}^{k} \frac{(p_i^{a_i+1} - 1)}{(p_i - 1)}.$$

  In fact, for $n = 360 = 2^3 \cdot 3^2 \cdot 5$, the divisors are $\{1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15,$ $18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180, 360\}$. They are even (24) in number, and their sum is 1170. Now from the formulas, we have the total number of divisors of $360 = (3+1)(2+1)(1+1) = 24$, which is even because 360 is not a perfect square, and sum of the divisors of

$360 = [(2^4 - 1)/(2-1)][(3^3 - 1)/(3-1)][(5^2 - 1)/(5-1)] = 1170.$ In particular, if *p* is prime, then $\sigma(p^m) = (p^{m+1} - 1)/(p-1).$

- For a given number *n*, it is interesting to investigate how many distinct prime factors it possesses. Clearly, primes have only a single distinct prime divisor; numbers 9 and 10 are not prime, and have, respectively, one distinct prime factor 3 and two distinct prime factors, 2 and $5;$ numbers 30 and 60 have three distinct prime factors, $2, 3, 5;$ whereas the number 100 has only two distinct prime factors, 2 and $5.$ Around 1939, Kac realized that just as the probability that a tossed coin will come up heads is unaffected by the outcome of previous tosses, the probability that a number is divisible by one prime is independent of whether it is divisible by any other. Since the number of heads and tails expected after a large number of coin tosses obeys a normal distribution, it seemed to Kac that the number of distinct prime factors should obey a similar law. His observation in 1940 led to a famous Erdös-Kac theorem, also known as the fundamental theorem of probabilistic number theory (after 10 years, it induced to a new branch in number theory), which states that if $\omega(n)$ is the number of distinct prime factors of $n,$ then, approximately the probability distribution of
$$\frac{\omega(n) - \ln\ln n}{\sqrt{\ln\ln n}}$$
is the standard normal distribution. This result is an extension of the Hardy-Ramanujan theorem of 1917, which states that the normal order of $\omega(n)$ is $\ln\ln n$ with a typical error of size $\sqrt{\ln\ln n}.$

Euclid in his *Elements* (IX:20) used, likely to be first time in written, the method of contradiction to demonstrate that there are infinitely many prime integers. This not only makes the study of prime numbers interesting but also retains many questions, which are still unsolved.

Theorem 4.2 *There are infinitely many prime numbers.*

Proof Assume to the contrary that the number of primes is finite. Let $P = \{p_1, p_2, \cdots, p_m\}$ be the set of all primes in increasing order. Consider the integer $n_m = p_1 p_2 \cdots p_m + 1.$ Because $n_m \geq 2,$ we can use Theorem 4.1 to conclude that either n_m is prime, which contradicts *P* contains all primes, or n_m has a prime factor which belongs to $P,$ i.e., there is a prime $p_i (1 \leq i \leq n_m)$ such that $p_i | n_m.$ Hence, $n_m = p_i k$ for some integer k. Let $\ell = p_1 p_2 \cdots p_{i-1} p_{i+1} \cdots p_m$. Then

$$1 \;=\; n_m - p_1 p_2 \cdots p_m \;=\; p_i k - p_i \ell = p_i(k-\ell).$$

Since $(k-\ell)$ is an integer, $p_i|1$, which is impossible.■

- The numbers $n_m,\ m \geq 1$ defined above are called *Euclid numbers*. The first ten Euclid numbers are 3, 7, 31, 211, 2311, 30031, 510511, 9699691, 223092871, 6469693231. It is an open question whether all terms of this sequence are square free. Theorem 4.2 demonstrates the existence of some prime larger than p_m, but we do not necessarily arrive at the very next prime after p_m. For example, $n_6 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 + 1 = 30,031 = 59 \times 509$ yields 59 as a prime beyond 13. Similarly, $n_7 = 510511 = 19 \times 97 \times 277$. Generally, there are many primes between p_m and n_m.
- In 1737, the connection between the zeta function (*s* real) and prime numbers was discovered by Euler, who with argument $s > 1$ proved the identity

$$\zeta(s) \;=\; \sum_{n=1}^{\infty} \frac{1}{n^s} \;=\; \prod_{p \text{ prime}} \left(\frac{1}{1-p^{-s}} \right). \tag{4.4}$$

  This identity results from expanding the factor involving *p* as a geometric series

$$\frac{1}{1-1/p^s} \;=\; 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \cdots .$$

  On multiplying these series for all primes p, we get a sum of terms of the form

$$\frac{1}{(p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r})^s}$$

  where $p_1, p_2, \cdots, p_r$ are distinct primes and $k_1, k_2, \cdots, k_r$ are positive integers. By the canonical factorization, the products $p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ so obtained yield precisely all the positive integers (only once), allowing us to conclude that the sum in question is simply $\sum_{n=1}^{\infty} 1/n^s$. (The convergence of this series is guaranteed for $s > 1$.) Because the series for $\zeta(1)$ diverges, Euler's formula as $s \to 1$ implies the existence of an infinitude of prime numbers; for if there were only finitely many primes, then the product on the right–hand side of (4.4) for $s = 1$ would be a finite product and hence would have a finite value.

- In 1556, Tartaglia claimed that $(2^{n+1} - 1), (2^{n+2} - 1), n \geq 2$ are alternatively prime and composite. However, for $n = 7,$ we get $255, 511$ which are composite.
- Euler used (4.4) to prove heuristically a striking result showing that the sum of the reciprocals of all the primes diverges. His heuristic arguments are as follows:

$$\begin{aligned}
\ln\left(\sum_{n=1}^{\infty}\frac{1}{n}\right) &= \ln\left(\prod_{p}\frac{1}{1-1/p}\right) = -\sum_{p}\ln\left(1-\frac{1}{p}\right)\\
&= \sum_{p}\left(\frac{1}{p}+\frac{1}{2p^2}+\frac{1}{3p^2}+\cdots\right)\\
&< \sum_{p}\frac{1}{p}+\sum_{p}\frac{1}{2p^2}\left(1+\frac{1}{p}+\frac{1}{p^2}+\frac{1}{p^3}+\cdots\right)\\
&= \sum_{p}\frac{1}{p}+\sum_{p}\frac{1}{2p^2}\frac{p}{p-1}\\
&< \sum_{p}\frac{1}{p}+\frac{1}{2}\sum_{n=2}^{\infty}\frac{1}{n(n-1)} = \sum_{p}\frac{1}{p}+\frac{1}{2}
\end{aligned}$$

and hence

$$\sum_{p}\frac{1}{p} > \ln\left(\sum_{n=1}^{\infty}\frac{1}{n}\right)-\frac{1}{2} = \ln(\infty)-\frac{1}{2} = \infty. \tag{4.5}$$

In 1874, Mertens rigorously proved this result. In fact, by slightly modifying the arguments, one can show that

$$\sum_{p\leq n}\frac{1}{p} > \ln(\ln n)-\frac{1}{2}$$

from which (4.5) is immediate. As a consequence of Euler's result, Theorem 4.2 is immediate. The number

$$M = \lim_{n\to\infty}\left(\sum_{p\le n}\frac{1}{p} - \ln(\ln n)\right) \simeq 0.2614972128476427837554$$

is called *Meissel-Mertens Constant* after Daniel Friedrich Ernst Meissel (1826–1895, Germany) and Mertens.

- A remarkable generalization of Theorem 4.2 was established by Dirichlet (known as *Dirichlet Theorem*), who succeeded in showing in 1837 that every arithmetic sequence, $a + nd,\ n = 1, 2, \cdots$, in which *a* and *d* are relatively prime, i.e., $\gcd(a, d) = 1,$ contains an infinitude of primes (a generalization of the result of Euclid). He proved that the sum of the reciprocals of all such primes is divergent. The condition $\gcd(a, d) = 1$ is necessary, e.g., the sequence $\{3 + 6n\}$ contains only one prime, whereas $\{4 + 6n\}$ contains none. In 1949, Selberg gave a simple proof (without using complex analysis) of Dirichlet's theorem. As an application of this result, it follows that there are infinity of primes of the form $4n + 1, 4n + 3, 6n + 1,\ n = 0, 1, 2, \cdots$. In fact, in Sect. 4.18, we shall prove that there are infinite number of primes of the form $4n + 1$ and $4n + 3.$ It is easier to show that all terms of any sequence $a + nd,\ n = 1, 2, \cdots$ cannot be only primes; in fact, it must also contain infinitely many composite numbers. For this, let $a + nd = p$ be a prime, and consider the integers $n_k = a + kp,\ k = 1, 2, \cdots$. Then, the n_k th term of our sequence is $a + n_kd = a + (n + kp)d = (a + nd) + kpd = p + kpd = p(1 + kd),$ i.e., $p|(a + n_kd).$ In particular, although the last digit of each term of the sequence $3 + 10n,\ n = 1, 2, \cdots$ is 3, it contains infinite number of prime as well as composite numbers. In conclusion, Dirichlet's theorem gives some assurance to the existence of infinite primes of particular form, e.g., primes with last digit 3. In addition to Dirichlet's result, in 1944, Yuri Vladimirovich Linnik (1915–1972, Russia) asserted that when $1 \le a \le d - 1,$ then for the least prime in the arithmetic sequence denoted as $p(a, d)$, there exist positive computable constants *c* and *L* such that $p(a, d) < cd^L.$ In the literature, this result is known as Linnik's theorem. Later investigations could only decrease the value of the constant $L.$

- In 1845, François Bertrand asserted that between any number and its double there exists at least one prime. He based this assertion on an empirical study of a table of primes for all $n \leq 3000000.$ One surprising consequence of this proposition is that there are at least three primes having exactly *n* digits, where *n* is any positive integer. For over fifty years, this proposition was known as the *Bertrand's postulate*. It was finally proved in 1852 by Chebyshev who has also shown that there are primes even between much narrower limits, but his proof was very complicated. The postulate is now known as the Bertrand-Chebyshev Theorem. In 1899, V.I. Stanevich (Russia) proved that between *n* and 2*n*, there is at least one prime of the form $4k+1$ if $n > 15/2,$ and at least one prime of the form $4k-1$ if $n > 9/2.$ In 1911, Sylvester proved that for all sufficiently large *n*, there exists at least one prime between *n* and $1.092n.$ Again in 1911, A. Bonolis (Italy) advanced this problem considerably by giving an approximate formula for the number of primes between *n* and $3n/2.$ According to this formula, there are not fewer than a million primes between $100{,}000{,}000$ and $150{,}000{,}000.$ Later, Ramanujan in 1919 sharpened this result, Erdös in 1932 showed [same as Stanevich] the existence of two primes between any number greater than 7 and its double, one of the form $4n+1$ and another of the form $4n+3,$ i.e., $4n-1$), Denis Hanson (Canada) in 1973 proved that there exists a prime between 3*n* and $4n,$ Mohamed El Bachraoui (Morocco) in 2006 showed that there exists a prime between 2*n* and $3n,$ and Andy Loo (Hong Kong) in 2011 established that as *n* tends to infinity, the number of primes between 3*n* and 4*n* also goes to infinity. As a consequence of all these results, Theorem 4.2 is immediate.

  The following two problems were posed by Landau in 1912: Does for every $n \geq 1,$, there exists a prime *p* such that $n^2 < p < (n+1)^2$ (originally due to Legendre)? Are there infinitely many primes of the form n^2+1 (initiated by Euler in 1752)? The prime values of n^2+1 less than 10000 are 2, 5, 17, 37, 101, 197, 257, 401, 577, 677, 1297, 1601, 2917, 3137, 4357, 5477, 7057, 8101, 8837. It is also not known whether for all $n \geq 4,$ there is a prime *p* such that $n < p < n + \sqrt{n};$ there are infinite primes of the form $n^2-2,\ n \geq 2;$ of the form $n^2+n+1;$ of the form $n^2+(n+1)^2+(n+2)^2;$ of the form $\binom{2n}{n}$ not divisible by 3 or 5 (due to Graham); and of the form (such as 101) for which $p-k!$ is composite whenever $1 \leq k! < p.$

- Kaprekar Conjecture: Consider the prime $31.$ Add its digits to the number itself. We get $31+3+1=35$ which is a composite. Thus, we get a composite number from a prime in just one step-adding all the individual digits on the prime to itself. Some more primes of this type are $17, 41, 109, 2003$, and $3001.$ Now consider the prime $127.$ Add to it all its digits to get $127+1+2+7=137$ which is also a prime. Do the same thing to it to get $137+1+3+7=148$ which is a composite number. Here we get a composite number, starting from a prime, in two steps. Some other examples of primes of this type are $307, 587, 1009, 1061$, and $1087.$ So our procedure is to take a prime, add all its individual digits to get a new number, repeat the same procedure on it if it is a prime, and continue doing this till one gets a composite number. Some examples of primes for which we need three steps to arrive at a composite number are $11, 101, 149, 167, 367, 479$, and $1409.$ Some examples of primes for which we need four steps to reach a composite number are 277 and $1559.$

  Kaprekar made the conjecture that there exit no prime for which we need more than four steps to get a composite number by the above-described procedure. Kaprekar's conjecture was disproved by V.H. Joshi (India) in 2001. He produced many primes which require five or six steps to reach a composite number. From Joshi's collection, we give here four examples—two requiring five steps and two requiring six steps. They are (i) prime number 37783 which becomes successively $37783+28=37811,\ 37811+20=37831,\ 37831+22=37853,\ 37853+26=37879,\ 37879+34=37913.$ Till the fourth step, we get primes; at the fifth step, we get a composite number 37913 whose factors are 31 and $1223.$ (ii) Prime number 85601 at the end of the fifth step becomes composite $85726=2\times 42863.$ All the previous four steps produced primes: $85601 \to 85621 \to 85643 \to 85669 \to 85703.$ (iii) Prime number 516493 yields primes $516521, 516541, 516563, 516589, 516623$ in the first five consecutive steps, and the sixth step produces the composite $516646=2\times 13\times 31\times 641.$ (iv) When the required procedure of digits addition is applied to the prime $1885943,$ we successively get $1885981, 1886021, 1886047, 1886081, 1886113$ which are primes. Then the sixth step gives the composite $1886141=12491\times 151.$
- First 20 palindromic primes (sometimes called a palprime) are 2, 3, 5, 7, 11, 101, 131, 151, 181, 191, 313, 353, 373, 383, 727, 757, 787, 797, 919, 929. In the November 1980 issue of *Crux Mathematicorum* appears a

table of all 93 five-digit and all 668 seven-digit palindromic primes. The editor of the same Journal Léo Sauvé later (1927–2016, Canada) claimed that there are 5172 nine-digit palindromic primes, out of which 345676543 is unique because of the five consecutive digits. As we have seen every palindrome number with an even number of digits is divisible by 11, so 11 is the only palindromic prime with an even number of digits. It is conjectured that there are an infinite number of palindromic primes in base 10. The largest known as of October 2021 is $10^{1888529} - 10^{944264} - 1,$ which has $1{,}888{,}529$ digits and was found on October 18, 2021, by Ryan Propper and Serge Batalov (Russia).

37	36	35	34	33	32	**31**
38	**17**	16	15	14	**13**	30
39	18	**5**	4	**3**	12	**29**
40	**19**	6	1	2	**11**	28
41	20	**7**	8	9	10	27
42	21	22	**23**	24	25	26
43	44	45	46	**47**	48	49

421	420	**419**	418	417	416	415	414	413	412	411	410	**409**	408	407	406	405	404	403	402
422	**347**	346	345	344	343	342	341	340	339	338	**337**	336	335	334	333	332	**331**	330	**401**
423	348	**281**	280	279	278	**277**	276	275	274	273	272	**271**	270	**269**	268	267	266	329	400
424	**349**	282	**223**	222	221	220	219	218	217	216	215	214	213	212	**211**	210	265	328	399
425	350	**283**	224	**173**	172	171	170	169	168	**167**	166	165	164	**163**	162	209	264	327	398
426	351	284	225	174	**131**	130	129	128	**127**	126	125	124	123	122	161	208	**263**	326	**397**
427	352	285	226	175	132	**97**	96	95	94	93	92	91	90	121	160	207	262	325	396
428	**353**	286	**227**	176	133	98	**71**	70	69	68	**67**	66	**89**	120	159	206	261	324	395
429	354	287	228	177	134	99	72	**53**	52	51	50	65	88	119	158	205	260	323	394
430	355	288	**229**	178	135	100	**73**	54	**43**	42	49	64	87	118	**157**	204	259	322	393
431	356	289	230	**179**	136	**101**	74	55	44	**41**	48	63	86	117	156	203	258	321	392
432	357	290	231	180	**137**	102	75	56	45	46	**47**	62	85	116	155	202	**257**	320	391
433	358	291	232	**181**	138	**103**	76	57	58	**59**	60	**61**	84	115	154	201	256	319	390
434	**359**	292	**233**	182	**139**	104	77	78	**79**	80	81	82	**83**	114	153	200	255	318	**389**
435	360	**293**	234	183	140	105	106	**107**	108	109	**110**	111	112	**113**	152	**199**	254	**317**	388
436	361	294	235	184	141	142	143	144	145	146	147	148	149	150	**151**	198	253	316	387
437	362	295	236	185	186	187	188	189	190	**191**	192	**193**	194	195	196	**197**	252	315	386
438	363	296	237	238	**239**	240	**241**	242	243	244	245	246	247	248	249	250	**251**	314	385
439	364	297	298	299	300	301	302	303	304	305	306	**307**	308	309	310	**311**	312	**313**	384
440	365	366	**367**	368	369	370	371	372	**373**	374	375	376	377	378	**379**	380	381	382	**383**

4.3 Prime Factorization of Composite Numbers

According to Gauss "The problem of distinguishing prime numbers from composite numbers and of resolving the latter into prime factors is known to be one of the most important and useful in arithmetic. The dignity of the science itself seems to require that every possible means be explored for the solution of a problem so elegant and so celebrated." A similar characteristic remark is due to Dickson [164] "To tell whether a given number of 15 or 20 digits is prime or not, all time would not suffice for the test, whatever use is made of what is already known." In fact, while it is relatively easy to determine whether a small positive integer is prime or composite and, if it is composite, to express it as a product of primes, so far no practicable procedure is known for testing large numbers for primality, and the effort spent on testing certain special numbers has been enormous. We mention some tests for divisibility by certain integers, especially for all primes up to $50.$

Divisibility by $2, 4,$ and other powers of 2. An integer *n* is divisible by 2 iff if *n* is even (or the last digit of *n* is even). Further, *n* is divisible by 4 iff the two-digit number consisting of the last two digits of *n* is divisible by $4,$ the integer *n* is divisible by 8 iff the three-digit number consisting of the last three digits of *n* is divisible by $8,$ and so on. Therefore, the number 14236 is divisible by 4 since 36 is dividable by $4,$ but it is not divisible by 8 since 236 is not divisible by $8.$

Divisibility by 3 and 9. An integer *n* is divisible by 3 iff the sum of its digits is divisible by $3.$ Further, an integer is divisible by 9 iff the sum of its digits is divisible by $9.$ This procedure stops with $9,$ however, that is, it does not extend to $27.$ For example, the sum of the digits of the integer 68877 is $36,$ which is divisible by 9, and hence 68877 is divisible by $9;$ however, 36 is not divisible by 27 but 68877 is divisible by 27.

Divisibility by 5. An integer is divisible by 5 iff its last digit is 5 or $0.$ Thus, there is only one prime number whose last digit is $5,$ namely itself.

Divisibility by 7. Double the last digit of the number *n* and subtract it from the remaining truncated number. If the result is divisible by $7,$ then *n* is divisible by $7.$ Continue this process if necessary. For the example, for the

number 203189, successively, we have
$20318 - 18 = 20300,\ 2030 - 0 = 2030,\ 203 - 0 = 203,\ 20 - 6 = 14,$ and hence 203189 is divisible by $7,$ whereas for $29027,$ we find
$2902 - 14 = 2888,\ 288 - 16 = 272,\ 27 - 4 = 23,$ and hence 29027 is not divisible by $7.$ (In fact, 29027 is prime.)

Divisibility by 11. Take the alternating sum of the digits of the number $n,$ from left to right. If that is divisible by $11,$ so is the number $n.$ For example, for the number $319297,$ we have $3 - 1 + 9 - 2 + 9 - 7 = 11,$ and hence the number 319297 is divisible by $11,$ whereas for $29027,$ we find $2 - 9 + 0 - 2 + 7 = -2,$ and hence 29027 is not divisible by $11.$

Alternatively, from the number *n*, subtract the last digit from the remaining truncated number. If the result is divisible by $11,$ then *n* is divisible by $11.$ Continue this process if necessary. Again for the number 319297, successively, we have
$31929 - 7 = 31922,\ 3192 - 2 = 3190,\ 319 - 0 = 319,\ 31 - 9 = 22$, and hence the number 319297 is divisible by $11,$ whereas for $29027,$ we find
$2902 - 7 = 2895,\ 289 - 5 = 284,\ 28 - 4 = 24,$ and hence 29027 is not divisible by $11.$

Divisibility by 13. In the number *n*, add four times the last digit to the remaining truncated number. If the result is divisible by $13,$ then *n* is divisible by $13.$ Continue this process if necessary. For example, for the number $377351,$ successively, we have
$37735 + 4 = 37739,\ 3773 + 36 = 3809,\ 380 + 36 = 416,\ 41 + 24 = 65,$ and hence the number 377351 is divisible by $13,$ whereas for $29027,$ we find $2902 + 28 = 2930,\ 293 + 0 = 293,\ 29 + 12 = 41,$ and hence 29027 is not divisible by $13.$

Divisibility by 17. From the number *n*, subtract five times the last digit to the remaining truncated number. If the result is divisible by $17,$ then *n* is divisible by $17.$ Continue this process if necessary. For example, for the number $493459,$ successively, we have
$49345 - 45 = 49300,\ 4930 - 0 = 4930,\ 493 - 0 = 493,\ 49 - 15 = 34,$ and hence the number 493459 is divisible by $17,$ whereas for $29027,$ we find $2902 - 35 = 2867,\ 286 - 35 = 251,\ 25 - 1 = 24,$ and hence 29027 is not divisible by $17.$

Divisibility by 19. In the number *n*, add two times the last digit to the remaining truncated number. If the result is divisible by $19,$ then *n* is divisible by $19.$ Continue this process if necessary. For example, for the number $551513,$ successively, we have
$55151+6=55157,\ 5515+14=5529,\ 552+18=570,\ 57+0=57,$
and hence the number **551513** is divisible by $19,$ whereas for $29027,$ we find $2902+14=2916,\ 291+12=303,\ 30+6=36,$ and hence **29027** is not divisible by $19.$

Divisibility by 23. In the number *n*, add seven times the last digit to the remaining truncated number. If the result is divisible by $23,$ then *n* is divisible by $23.$ Continue this process if necessary. For example, for the number $667621,$ successively, we have
$66762+7=66769,\ 6676+63=6739,\ 673+63=736,\ 73+42=115,\ 11+35=46,$
and hence the number **667621** is divisible by $23,$ whereas for $29027,$ we find $2902+49=2951,\ 295+7=302,\ 30+14=44,$ and hence **29027** is not divisible by $23.$

Divisibility by 29. In the number *n*, add three times the last digit to the remaining truncated number. If the result is divisible by $29,$ then *n* is divisible by $29.$ Continue this process if necessary. For example, for the number $841783,$ successively, we have
$84178+9=84187,\ 8418+21=8439,\ 843+27=870,\ 87+0=87,$
and hence the number **841783** is divisible by $29,$ whereas for $29027,$ we find $2902+21=2923,\ 292+9=301,\ 30+3=33,$ and hence **29027** is not divisible by $29.$

Divisibility by 31. From the number *n* subtract three times the last digit to the remaining truncated number. If the result is divisible by $31,$ then *n* is divisible by $31.$ Continue this process if necessary. For example, for the number $899837,$ successively, we have
$89983-21=89962,\ 8996-6=8990,\ 899-0=899,\ 89-27=62,$
and hence the number **899837** is divisible by $31,$ whereas for $29027,$ we find $2902-21=2881,\ 288-3=285,\ 28-15=13,$ and hence **29027** is not divisible by $31.$

Divisibility by 37. From the number *n*, subtract 11 times the last digit to the remaining truncated number. If the result is divisible by $37,$ then *n* is

divisible by $37.$ Continue this process if necessary. For example, for the number $1073999,$ successively, we have
$107399-99=107300,\ 10730-0=10730,\ 1073-0=1073,\ 107-33=74,$ and hence the number **1073999** is divisible by $37,$ whereas for $29027,$ we find $2902-77=2825,\ 282-55=227,\ 22-77=-55,$ and hence **29027** is not divisible by $37.$

Divisibility by 41. From the number *n*, subtract four times the last digit to the remaining truncated number. If the result is divisible by $41,$ then *n* is divisible by $41.$ Continue this process if necessary. For example, for the number $1190107,$ successively, we have
$119010-28=118982,\ 11898-8=11890,\ 1189-0=1189,\ 118-36=82,$ and hence the number **1190107** is divisible by $41,$ whereas for $29027,$ we find $2902-28=2874,\ 287-16=271,\ 27-4=23,$ and hence **29027** is not divisible by $41.$

Divisibility by 43. From the number *n*, subtract 30 (or add 13) times the last digit to the remaining truncated number. If the result is divisible by $43,$ then *n* is divisible by $43.$ Continue this process if necessary. For example, for the number $1248161,$ successively, we have
$124816-30=124786,\ 12478-180=12298,\ 1229-240=989,\ 98+117=215,\ 21+65=86,$ and hence the number **1248161** is divisible by $43,$ whereas for $29027,$ we find $2902-210=2692,\ 269-60=209,$ and hence **29027** is not divisible by $43.$

Divisibility by 47. From the number *n*, subtract 14 times the last digit to the remaining truncated number. If the result is divisible by $47,$ then *n* is divisible by $47.$ Continue this process if necessary. For example, for the number $1364269,$ successively, we have
$136426-126=136300,\ 13630-0=13630,\ 1363-0=1363,\ 136-42=94,$ and hence the number **1364269** is divisible by $47,$ whereas for $29027,$ we find $2902-98=2804,\ 280-56=224,$ and hence **29027** is not divisible by $47.$

- Some of the above divisibility results have been proved by Hindus and Arabs in the tenth century, al-Karkhi, Fibonacci, Chuquet, Pacioli, Juan Martínez Silíceo (1486–1557, Spain), Tartaglia, Christopher Clavius (1537–1612, Germany), and Pierre Forcadel (1500–1572, France). In

1654, Pascal and Lagrange, about hundred years later, established very general theorems about the divisibility, but these results are only of mathematical elegance rather than of practical value. The above tests can easily be applied to check if a given number up to 2500 is prime. The above tests show that the numbers $569, 2441, 2447$ are prime, whereas the following numbers are composite

$$\begin{aligned} 360 &= 8 \times 5 \times 9 = 2^3 \cdot 3^2 \cdot 5 \\ 34056 &= 8 \times 9 \times 11 \times 43 = 2^3 \cdot 3^2 \cdot 11 \cdot 43 \\ 22148280 &= 8 \times 9 \times 5 \times 7 \times 11 \times 17 \times 47 = 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 17 \cdot 47. \end{aligned}$$

- We shall show that the integer *n* divides every entry of the *n*th row of Pascal's triangle, except the initial and final 1's, iff *n* is prime. Indeed, if *n* is prime and $0 < k < n,$ then in (2.3) the denominator $k!(n-k)!$ does not contain *n* as a prime factor, thus in the reduction of $\binom{n}{k}$ the factor *n* remains in the numerator. Conversely, we assume that *n* is composite and *p* is a prime factor of $n.$ Then it suffices to show that $n \not| \binom{n}{p}.$ Indeed, if $n|\binom{n}{p},$ then there exists an integer *m* such that $\binom{n}{p} = nm,$ whereas from (2.3), $m = [(n-1)(n-2)\cdots(n-p+1)]/p!.$ Now since *p* divides $n,$ the next smaller multiple of *p* is $n-p.$ But then, the factor *p* in the denominator of *m* cannot divide any of the integers $n-1, n-2, \cdots, n-p+1$ in the numerator. This contradicts *m* is an integer.
- In 1643, Mersenne sent the number $2{,}027{,}651{,}281$ to Fermat to find its factors; he responded immediately that it is the product of two primes $44{,}021$ and $46{,}061$ (some authors have claimed that Mersenne sent the number $100, 895, 598, 169$ and Fermat factorized it to the product of two primes $898, 423$ and $112, 303).$ For this, he used his newly developed method of factorizing large numbers, which for a given number *n* can be summarized as follows: Assume *n* is odd, because powers of 2 can be easily recognized and removed; assume that $n = pq,$ where *p* and *q* are odd and $p \geq q \geq 1,$ so that $n = x^2 - y^2,$ or $x^2 - n = y^2,$ where $x = (p+q)/2$ and $y = (p-q)/2$ are nonnegative integers; find the smallest integer *k* such that $k^2 \geq n,$ i.e., $k = \lceil \sqrt{n} \rceil$ (ceiling function); compute the sequence $(k+\ell)^2 - n, \ \ell = 0, 1, 2, \cdots$ until an integer $m \geq \sqrt{n}$ is found for which $(k+m)^2 - n$ is a perfect square. Clearly,

$x^2 = (k+m)^2$ and $y^2 = (k+m)^2 - n$. This process cannot be infinite, because eventually it will satisfy the equation

$$\left(\frac{n+1}{2}\right)^2 - n = \left(\frac{n-1}{2}\right)^2.$$

In the process of computation, the following facts help: All perfect squares always end in $1, 4, 5, 6, 9,$ or an even number of zeros; for a number that ends in $1, 4, 9$, its tens digit will always be even $2, 4, 6, 8, 0$; if it ends with 6, its tens digit will be odd $1, 3, 5, 7, 9;$ if it ends with 5, its tens digit will be $2.$ Since $45029^2 < 2{,}027{,}651{,}281 < 45030^2 = k^2$, we find

$$\begin{aligned}
45030^2 - 2{,}027{,}651{,}281 &= 49619 \\
45031^2 - 2{,}027{,}651{,}281 &= 139680 \\
45032^2 - 2{,}027{,}651{,}281 &= 229743 \\
45033^2 - 2{,}027{,}651{,}281 &= 319808 \\
45034^2 - 2{,}027{,}651{,}281 &= 409875 \\
45035^2 - 2{,}027{,}651{,}281 &= 499944 \\
45036^2 - 2{,}027{,}651{,}281 &= 590015 \\
45037^2 - 2{,}027{,}651{,}281 &= 680088 \\
45038^2 - 2{,}027{,}651{,}281 &= 770163 \\
45039^2 - 2{,}027{,}651{,}281 &= 860240 \\
45040^2 - 2{,}027{,}651{,}281 &= 950319 \\
45041^2 - 2{,}027{,}651{,}281 &= 1040400 = 1020^2
\end{aligned}$$

and hence $m = 11, x = 45041, y = 1020$ which gives $p = 46,061$ and $q = 44,021.$ It is obvious that if the factors of *n* are close, then *m* can be found only in a few steps.

- Recently, to check primality and find prime factorization of a number, several algorithms have been written and tested on high-speed computers. The website https://www.numberempire.com/numberfactorizer.php provides the results in a fraction of a second for any number up to 70 digits. For example, it can be easily checked that both of the following numbers of 70 digits are prime:

$$1,826,532,766,084,\ 201,676,844,941,657,870,604,197,738,441,723,$$
$$730,267,373,351,555,779,418,229$$

$$2,079,273,417,152,\ 470,561,359,973,687,476,836,967,914,909,117,\\ 042,518,961,718,623,081,462,807$$

Further, before the computer era, for the following four numbers, prime factorization was a big challenge:

$$101,137,139,149,199,227,257 = 3 \times 29,027 \times 1,161,414,535,308,497$$
$$173,191,223,229,233,239,251 = 3^3 \times 13 \times 27,271 \times 18,093,296,483,$$
$$531109,131,157,167,193,197,211,241 = 11\times31\times37\times89\times1,019\times95,\\ 373,663,814,603$$
$$112,329,374,143,475,359,677,173,798,397,103,113,151,163,179,181\\ = 19 \times 41 \times 53 \times 2,046,885,593,227 \times 1,329,188,094,362,967,\\ 720,668,210,443,169.$$

- The monster group (also known as the Fischer-Griess monster, or the friendly giant) is the largest sporadic simple group, having the order

$$808,017,424,794,\ 512,875,886,459,904,961,710,757,005,754,\\ 368,000,000,000 \simeq 8 \times 10^{53}$$

  and it has the prime factorization

$$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$$

  This monster was predicted by Bernd Fischer (1936–2020, Germany) in about 1973 and Robert Louis Griess (born 1945, USA) in 1976.

- Banks encrypt their electronic messages for transferring funds so that unauthorized person getting access to the communication lines cannot decipher the messages or introduce their own transfer messages. Insurance companies, stock brokers, and various government agencies also need to secure communication channels. In *Scientific American, August 1977, pp 120–124*, Ronald Linn Rivest (born 1947, USA), Adi Shamir (born 1952, Israel), and Leonard Adleman (born 1945, USA) described the method (without explaining), perhaps it depends on multiplying two huge prime numbers (up to 50 digits). There is no danger in revealing the product of two primes publicly, and indeed there are reasons to do this. An unauthorized receiver of an encrypted message cannot decipher it without knowing the two prime factors. Factoring the known product was then virtually impossible.
- In 2002, Manindra Agrawal (born 1966, India), Neeraj Kayal (born 1979, India), and Nitin Saxena (born 1981, India) developed the AKS

(Agarwal-Kayal-Saxena) primality test algorithm—a deterministic primality–proving method— which is general, polynomial-time, deterministic, and unconditional all at the same time. Earlier algorithms developed over the centuries achieved at most three of these properties but not all four simultaneously. The algorithm is based upon the following theorem: an integer $n(\geq 2)$ is prime if and only if the polynomial congruence $(x-\alpha)^n \equiv (x^n - \alpha) \pmod{n}$ holds for all integers α coprime to *n* (or even just for some such integer $\alpha,$ in particular for $\alpha = 1$), where *x* is a free variable.

4.4 Mersenne Primes

Mersenne in the preface to his *Cogitata Physica-Mathematica* of 1644 stated that the numbers $M_n = 2^n - 1$ (which stood for the primordial God and several divinities; when *n* is zero the expression denotes zero; there is nothing; when *n* is 1, the expression denotes unity, the Infinite God; when *n* is $2,$ the expression denotes Trinity; when *n* is $3,$ the expression denotes $7,$ the saptha Rishis, and so on) were prime for $n = 2, 3, 5, 7, 13, 17, 19, 31, 67, 127$, and 257 and also were composite for all other positive integers $n < 257.$ However, while there are mistakes in this statement, it is provocative. In fact, $n = 67{,}257$ do not give primes, and he missed $n = 61{,}89{,}107.$ As an honor to Mersenne, $M_n,\ n \geq 1$ are called Mersenne numbers, and among these numbers those which are prime are called Mersenne primes. Thus, the solution of the Tower of Brahma/Tower of Hanoi problem is the Mersenne number $M_{64} = 2^{64} - 1.$

In 1640, Fermat communicated to Mersenne in a letter that if *n* is composite, then M_n is composite, i.e, if M_n is prime, the *n* must be prime. For this, if $n = ab,$ then

$$2^n - 1 \ = \ (2^a)^b - 1 \ = \ [2^a - 1][(2^a)^{b-1} + (2^a)^{b-2} + \cdots + (2^a) + 1]$$

of which $2^a - 1$ is obviously a factor. For example, if $n = 6 = 2 \times 3,$ we have $2^6 - 1 = (2^2)^3 - 1 = [2^2 - 1][(2^2)^2 + 2^2 + 1].$ In 1494, Pacioli claimed that $M_{27} = 134217727$ is prime; however, $M_{27} = 7 \times 73 \times 262657.$ Fermat's result does not imply that if *n* is prime then so is $M_n.$ For this, in 1536, Dutch mathematician Hudalrichus Regius (1598–1679) made the first breakthrough by showing that $2^{11} - 1 = 2047 = 23 \times 89,$ i.e., M_{11} is composite (it also shows Mersenne primes is a proper subset of all primes). Thus, it becomes a matter of curiosity to determine prime *n* for which M_n is a prime. As a result of this, whenever a new prime is found, the probability is very high that the new prime number is of the form $M_n.$ The verification of the primality of $M_2 = 3, M_3 = 7, M_5 = 31, M_7 = 127$ have been known since time immemorial, whereas $M_{13} = 8191$ was discovered anonymously before 1461. It was obvious to other mathematicians that Mersenne could not have tested for primality all the numbers which he had announced as prime. The primality of M_{17} and M_{19} was found by Pietro Antonio Cataldi (1552–1626, Italy) in 1588. In 1732, Euler tested M_{31} and confirmed it to be prime by examining all prime numbers up to $46{,}339$ as possible divisors. The Mersenne prime M_{31} remained the largest known prime for the next century. In 1738, Euler noticed that Mersenne number $2^{83} - 1$ is divisible by $167.$ For this, we notice that

$$\begin{aligned} &2^8 = 256 \equiv 89 (\text{mod } 167) \Longrightarrow 2^{16} \equiv 89^2 = 7921 \equiv 72 (\text{mod } 167) \\ &\Longrightarrow 2^{32} \equiv 72^2 = 5184 \equiv 7 (\text{mod } 167) \Longrightarrow 2^{64} \equiv 49 (\text{mod } 167), \end{aligned}$$

and so $2^{67} = 2^3 \cdot 2^{64} \equiv 8 \cdot 49 = 392 \equiv 58 (\text{mod } 167).$ Combining these results, we obtain

$$2^{83} - 1 = 2^{16} \cdot 2^{67} - 1 \equiv 72 \cdot 58 - 1 = 4167 - 1 = 4166 \equiv 0 (\text{mod } 167),$$

which shows that $167 | 2^{83} - 1.$

In 1876, Lucas devised a special method to test the primality of Mersenne numbers. He defined the so-called Lucas sequence $\{S_m\}$ generated by the difference equation $S_m = S_{m-1}^2 - 2, \ S_1 = 4, \ m > 1$ and claimed that if the $(n-1)$th term of this sequence is divisible by $2^n - 1$ without a remainder, then M_n is prime. Further, if a term of the Lucas sequence gets larger than the M_n being tested, the term is divided by M_n and the remainder, if any, is used to continue the sequence. For example, to test the primality of $M_7 = 2^7 - 1 = 127,$ we compute $S_1 = 4, S_2 = 14, S_3 = 194.$ Since 194 is greater than $127,$ we divide it by 127 and get a remainder of $67.$ Then the fourth term is $67^2 - 2 = 4487,$ which on dividing by 127 gives a remainder of $42;$ the fifth term is $42^2 - 2 = 1762,$ which on dividing by 127 gives a remainder of $111;$ the sixth and the last term is $111^2 - 2 = 12321$, which on dividing by 127 gives no remainder. Therefore, M_7 is prime. Using this method, Lucas proved that M_{127} is prime; however, in 1891, in his *Théorie des Nombres*, he changed his mind and listed the number M_{127} as "undecided" (it is clear that even with his sequence no one would tackle the larger Mersenne numbers without the assistance of an electronic computer). However, in 1913, it was confirmed that M_{127} indeed is a prime, and it remained till 1951 the largest known prime of any type. In 1883, M_{61} was determined to be prime by Ivan Mikheevich Pervushin (1827–1900, Russia).

In 1903, during the meeting of the American Mathematical Society, Frank Nelson Cole (1861–1926, USA) perplexed the audience by his wordless presentation of $M_{67} = 147, 573, 952, 589, 676, 412, 927 = 193, 707, 721 \times 761, 838, 257, 287$, i.e., M_{67} is composite. His great discovery for which he spent three years of Sundays received standing ovation and unrestricted applause. The number M_{71} was factorized by the most powerful computer, of the time, in the world, the Cray, as

$$\begin{aligned} M_{71} &= 3^2 \times 241, 573, 142, 393, 627, 673, 576, 957, 439, 049 \times \\ &\quad 45, 994, 811, 347, 886, 846, 310, 221, 728, 895, 223, 034, 301, 839. \end{aligned}$$

The primality for M_{89} and M_{107} was confirmed by Ralph Ernest Powers (1875–1952, USA) in 1911 and 1914, respectively. In 1934, Powers verified that M_{241} is composite. In 1922, Maurice Borisovich Kraitchik (1882–1957, Belgium), and in 1931, Lehmer showed that M_{257} is also not prime. In 1948, Horace Scudder Uhler (1872–1956, USA) proved that both M_{193} and M_{227} are composite. Thus, up to the limit 257 set be Mersenne there are only 12 Mersenne numbers M_n which are prime for $n = 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127$ and rest are composite.

In 1952, Raphael Mitchel Robinson (1911–1995, USA) used computing machine SWAC to test the primality of M_n by Lucas's method and found two more larger prime numbers, namely, M_{521} and M_{607}. Later in the same year, he found three more larger primes, M_{1279}, M_{2203}, and M_{2281}. Now fast algorithms for finding Mersenne primes M_n are available, and as of March 2022, 51 Mersenne primes for the following *n* are known: 2, 3, 5, 7, 13, 17, 19, 31, 61, 89 (last to be discovered by hand calculations), 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253, 4423, 9689, 9941, 11213, 19937, 21701, 23209, 44497, 86243, 110503, 132049, 216091, 756839, 859433, 1257787, 1398269, 2976221, 3021377, 6972593, 13466917, 20996011, 24036583, 25964951, 30402457, 32582657, 37156667, 42643801, 43112609, 57885161, 74207281, 77232917, 82589933. Mersenne prime $M_{82589933}$ discovered in 2018 by Patrick Laroche (born 1984, USA) has 24862048 digits (it would fill up thousands of pages), and it is the largest known prime number of any kind till 2023. Paulo Ribenboim (born 1928, Brazil-Canada) in his book [432] has listed prime number records till 1995. It is an open question as to whether there exist infinitely many Mersenne primes. It is also not known if every Mersenne number is square free, and if there are infinitely many composite Mersenne numbers.

Does an exponent of a Mersenne prime is also a Mersenne prime? It is true for the first four cases: For the exponent $3,$ which is $2^2-1;\ 7,$ which is $2^3-1;\ 31,$ which is $2^5-1;$ and $127,$ which is $2^7-1.$ However, for the Mersenne prime $2^{13}-1=8191$, it turns out that $2^{8191}-1$ is not a prime, which is of 2466 digits. Clearly, M_{13} is the smallest Mersenne prime whose exponent is not a Mersenne prime. In conclusion, this conjuncture is false.

- Sharavyn Myangat (around 1692–1763, China) also known as Ming Antu/Minggatu around 1730 established and used the infinite sequence $1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, \cdots$ in his book *Ge Yuan Mi Lu Jie Fa* (The Quick Method for Obtaining the Precise Ratio of Division of a Circle), which was completed by one of his students in 1774, but published 60 years later. Euler rediscovered this sequence while counting the number of triangulations of convex polygons. In 1759, Johann Andreas von Segner (1704–1777, Hungary-Slovakia-Germany) found that the number C_n of triangulations of a convex polygon satisfies the recursive formula

$$C_n \ = \ C_0C_{n-1} + C_1C_{n-2} + \cdots + C_{n-1}C_0, \qquad (4.6)$$

  where $C_0 = 1,$ which generates Minggatu's sequence. In 1838, Catalan discovered the same sequence in connection to parenthesized expressions during his exploration of the Towers of Brahma/Hanoi problem. In 1958, John Francis Riordan (1903–1988, USA) in his work named $C_n,\ n \geq 0$ as *Catalan numbers*, which continues in the literature. The Catalan number C_n can be expressed in terms of binomial coefficients (2.3) as

$$C_n \ = \ \frac{1}{n+1}\binom{2n}{n} \ = \ \frac{(2n)!}{(n+1)!n!} \ = \ \prod_{k=2}^{n}\frac{n+k}{k}, \quad n \geq 0, \qquad (4.7)$$

for which several proofs are known. Catalan numbers occur in astonishingly many combinatorial problems and computer science. In particular, they appear in binary trees, multiplication ordering, lattice path problem, geodesy, cryptography, and even in medicine. Further, to find Catalan numbers besides (4.7), several other recurrence relations are known, and their extensive properties have been recorded. Here we shall prove two properties which are due to Koshy and Salmassi [313].

For $n > 0$, the Catalan number C_n is odd iff *n* is a Mersenne number: from (4.6), we have

$$C_n = \begin{cases} 2(C_0C_{n-1} + C_1C_{n-2} + \cdots + C_{(n/2)-1}C_{n/2}) & \text{if } n \text{ is even} \\ 2(C_0C_{n-1} + C_1C_{n-2} + \cdots + C_{(n-3)/2}C_{(n+1)/2}) + C_{(n-1)/2}^2 & \text{if } n \text{ is odd.} \end{cases}$$

Thus, C_n is odd iff both *n* and $C_{(n-1)/2}$ are odd. The same argument implies that C_n is odd iff $(n-1)/2$ and $C_{(n-3)/4}$ are both odd or $(n-1)/2 = 0$. Continuing this method of infinite descent (in written the earliest uses of the method of infinite descent appear in Euclid's Elements, Book VII, Proposition 31. In this method, one assumes the existence of a solution, which is related to one or more integers, and shows the existence of another solution which is related to smaller one or more integers, and this process infinitely continues. This leads to a contradiction. This method was later developed/popularized by Fermat. For details, see Bussey [112]); it follows that C_n is odd iff $C_{[n-(2^m-1)]/2^m}$ is odd, where $m \geq 1$. But the least value of *k* for which C_k is odd is $k = 0$. Thus, the sequence of these statements terminates iff when $[n - (2^m - 1)]/2^m = 0$, i.e., $n = 2^m - 1$, a Mersenne number.

The only prime Catalan numbers are C_2 and C_3: from (4.7), it follows that $(n+2)C_{n+1} = (4n+2)C_n$. Assume that C_n is prime for some n. Now from (4.6), for $n > 3$, we find $(n+2)/C_n < 1$, and hence $C_n > n+2$. Thus, $C_n | C_{n+1}$, so $C_{n+1} = kC_n$ for some positive integer k. Then, $4n + 2 = k(n+2)$, which implies that $1 \leq k \leq 3$ and thus $n \leq 4$. In conclusion, C_2 and C_3 are the only Catalan numbers which are prime.

4.5 Perfect Numbers

An integer $n \geq 2$ is said to be *perfect* (the nomenclature is due to Pythagoras) if it is equal to the sum of its proper divisors (excluding itself and including 1). For example, 6 is a perfect number, because its divisors are $1, 2, 3, 6$ and $6 = 1 + 2 + 3.$ Similarly, 28 is a perfect number, for, its divisors are $1, 2, 4, 7, 14, 28$ and $1 + 2 + 4 + 7 + 14 = 28.$ The smallest perfect numbers 6 and 28 were known to the Hindus as well as to the Hebrews. But 18 is not a perfect number for its positive divisors are $1, 2, 3, 6, 9, 18$ and $1 + 2 + 3 + 6 + 9 = 21 \neq 18.$ Although perfect numbers are regarded as arithmetical curiosities, their study has helped to develop the theory of numbers. Besides 6 and $28,$ Pythagoreans knew two more prefect numbers, 496 and $8, 128.$ The basic similarity of these numbers is apparent when they are represented algebraically. They are of the form

$2^{n-1}(2^n - 1) : 6 = 2^1(2^2 - 1);\ 28 = 2^2(2^3 - 1);\ 496 = 2^4(2^5 - 1); and\ 8, 128 = 2^6(2^7 - 1).$

We shall denote the sum of all divisors of a natural number *n* by $\sigma(n),$ so that a number is perfect if $\sigma(n) - n = n,$ i.e., if $\sigma(n) = 2n.$ Thus, the sum of the reciprocals of all divisors of a perfect number is always $2.$ Indeed, it follows from the fact that if $d|n,$ then $n = dd'$ for some $d';$ hence, the sum

$$\sum_{d|n} \frac{1}{d} = \sum_{d'|n} \frac{d'}{n} = \frac{1}{n} \sum_{d'|n} d' = \frac{2n}{n} = 2.$$

As an example, since the positive divisors of 6 are $1, 2, 3,$ and $6,$ and we have

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 2.$$

The problem of finding perfect numbers is simply that of solving the equation $\sigma(n) = 2n.$ To find solutions of this equation, we need the following lemma.

Lemma 4.3 *If m and n are relatively prime, then* $\sigma(mn) = \sigma(m)\sigma(n).$

Proof Using Theorem 4.1, if $m = \prod_{j=1}^{\ell} q_j^{b_j}$ and $n = \prod_{i=1}^{k} p_i^{a_i}$, then no *q* is *p* and no *p* is q. Thus, it follows that

$$mn = \prod_{j=1}^{\ell} q_j^{b_j} \prod_{i=1}^{k} p_i^{a_i}$$

and as we have seen earlier, it follows that

$$\sigma(mn) = \prod_{j=1}^{\ell} \frac{(q_j^{b_j+1} - 1)}{(q_i - 1)} \prod_{i=1}^{k} \frac{(p_i^{a_i+1} - 1)}{(p_i - 1)} = \sigma(m)\sigma(n).$$

■

Euclid in *Elements* (IX:36) proved the following general result:

Theorem 4.3 *If* $2^n - 1$ *is prime, then* $2^{n-1}(2^n - 1)$ *is a perfect number.*

Proof Let $s = 2^{n-1}(2^n - 1)$. Since $2^n - 1$ is prime, $\sigma(2^n - 1) = 2^n$. Further, since $2^{n-1},\ 2^n - 1$ are relatively prime, from Lemma 4.3, we have

$$\sigma(2^{n-1}(2^n - 1)) \ = \ \sigma(2^{n-1})\sigma(2^n - 1),$$

or

$$\sigma(s) \ = \ (2^n - 1)2^n \ = \ 2 \cdot 2^{n-1}(2^n - 1) = 2s,$$

and hence $\sigma(s) = 2s$, so that *s* is a perfect number.■

Alternatively, if $p = 2^n - 1$ is prime, then from Theorem 4.1, the divisors of $s = 2^{n-1}p$ are

$$1, 2, 2^2, \cdots, 2^{n-1}, p, 2p, 2^2p, \cdots, 2^{n-1}p.$$

The sum of these divisors is thus

$$\begin{aligned}\sigma(s) \ &= \ (1 + 2 + 2^2 + \cdots + 2^{n-1})(1 + p) \\ &= \ (2^n - 1)(1 + p) \ = \ 2(2^{n-1}(2^n - 1)) \ = \ 2s.\end{aligned}$$

Philo of Alexandria (born 25 BC, Egypt) in his book *On the Creation* mentions perfect numbers, claiming that the world was created in 6 days and the moon orbits in 28 days because 6 and 28 are perfect. Nicomachus in his famous text [394] classified integers based on the concept of perfect numbers. He divided integers into three classes: perfect numbers which have the property that the sum of their divisors is equal to the number, *abundant numbers* which have the property that the sum of their divisors is greater than the number (Boethius calls such numbers as *superfluous*, around 1934, Erdös and others showed that the density of abundant numbers is greater than zero), and *deficient numbers* (Boethius calls such numbers as *diminished*) which have the property that the sum of their divisors is less than the number. He also analyzed moral implications of the three types of numbers. According to him, perfect numbers are remarkable and rare, "even as fair and excellent things are... while ugly and evil ones are widespread." In the twelfth century, the study of perfect numbers was recommended in a program for the "healing of souls." If we define $s(n) = \sigma(n) - 2n,$ then *n* is perfect if $s(n) = 0,$ abundant if $s(n) > 0,$ and deficit if $s(n) < 0.$ Clearly, $12, 24,$ and 945 (all divisors are $1, 3, 5, 7, 9, 15, 21, 27, 35, 45, 63, 105, 135, 189, 315$ and their sum is 975) are abundant, whereas $8, 14,$ and 944 (all divisors are $1, 2, 4, 8, 16, 59, 118, 236, 472$ and their sum is 916) are deficit. If *p* is a prime, then p^n is deficient, indeed the sum of the proper divisors of p^n is $(p^n - 1)/(p - 1).$ The smallest odd abundant number 945 is due to Claude Gaspar Bachet Sieur de Méziriae (1581–1638, France) and it corrects Nemorarius statement that all abundant numbers are even. Now since all multiples of 945 are also abundant, and every alternate multiple is odd, it follows that there are an infinite number of odd abundant numbers. Origen of Alexandria (around 185–254, Egypt) and Didymus the Blind (around 313–398, Egypt) added the observation that there are only four perfect numbers that are less than 10,000. St. Augustine defines perfect numbers in his book *City of God* repeating the claim that God created the world in 6 days because 6 is the smallest perfect number. Early commentators on the Old Testament argued that the perfection of the Universe is represented by 28, the number of days the moon takes to circle the Earth.

Al-Haytham in his unpublished work, *Treatise on Analysis and Synthesis*, showed that perfect numbers satisfying certain conditions had to be of the form $2^{k-1}(2^k - 1)$, where $2^k - 1$ is prime. ibn Qurra wrote the *Treatise on Amicable Numbers* in which he examined that the numbers of the form $2^n p,$ where *p* is prime, can be perfect. Arab mathematician Ismail ibn Ibrahim ibn Fallus (1194–1239) mentioned the next three (5th, 6th, and 7th) perfect numbers
$2^{12}(2^{13} - 1) = 33{,}550{,}336;\ 2^{16}(2^{17} - 1) = 8{,}589{,}869{,}056;\ 2^{18}(2^{19} - 1) = 137{,}438{,}691{,}328$

and listed next three more which are now known to be incorrect. The fifth perfect numbers were later written down in a manuscript dated 1461 by an unknown European mathematician. The same year it is also written by Johann Regiomontanus (1436–1476, Germany), also known as Johannes Müller. Charles de Bouvelles (1471–1553, France) published a book on perfect numbers in 1509. In it he claimed that Euclid's formula gives a perfect number for all odd integers $n,$ which is obviously wrong. He also gave a erroneous proof showing that every perfect number is even. In 1588, Cataldi identified the sixth and the seventh perfect numbers and proved that every perfect number obtained from Euclid's rule ends with a 6 or an $8.$ Nicomachus and later Iamblichus based on known first four perfect numbers during their time concluded that perfect numbers follow certain patterns: They alternately end in a 6 or an $8,$ (not true since fifth $33{,}550{,}336$ as well as sixth $8{,}589{,}869{,}056$ perfect numbers ends with a six); however, it is true that every even perfect number must end in 6 or 28, and if it ends in 6, the digit proceeding it must be odd; there is one perfect number in each interval from 1 to $10;$ 10 to $100;$ 100 to $1{,}000$; and $1{,}000$ to $10{,}000$ (not true because fifth perfect number has eight digits rather than five); Euclid's characterization gives all perfect numbers (not yet known), and there are infinitely many perfect numbers (not yet known).

One of the most intensively studied questions which remains unresolved is whether all perfect numbers are even; however, it is known that if an odd perfect number exists, it must satisfy the following necessary conditions:

(a). It must leave a remainder of 1 when divided by 12 or a remainder of 9 when divided by $36.$

(b). It must have at least six different prime divisors.

(c). It must have the form $n = p^{4m+1} q_1^{2a_1} \cdots q_n^{2a_n},$ where *p* is a prime of only the form $4k+1$ while the *q*'s may be any odd primes.

(d). Also *n* cannot be perfect if all the *a*'s are equal to $2.$

(e). If all the exponents of *q*'s are increased by $1,$ the resulting exponent cannot have as a common divisor $9, 21, 33,$ or $51.$

(f). If the exponent $4m+1$ of *p* is $5,$ then none of the *a*'s may equal 1 or $2.$

(g). If *n* is divisible by 3, it must have at least 9 different prime divisors, and if not divisible by 21, it must have at least 11 such divisors. If not divisible by $15,$ it must have at least 14 different prime divisors and if not divisible by 105, it must have at least 27 such divisors. This requires *n* to be greater than $10^{44}.$

(h). If *n* has exactly *r* different prime divisors, then the smallest of them will be smaller than $r+1.$ Thus, should *n* (if it exists) have 28 different prime divisors the smallest would not exceed $23.$

(i). In 2012, no odd perfect number exists up to 10^{1500} has been checked by Ochem and Rao [398],

Thus, if an odd perfect number exists, all the above conditions must be satisfied, so that it will be so large that it cannot be found by a guesswork. One day, perhaps someone will discover the first odd perfect number or will emerge with a proof for its nonexistence. Till then, we need to believe on the statement of Sylvester, he made in 1888 "Greek's succeeded in discovering a class of perfect numbers which in all probability are the only numbers which are perfect."

In 1757, Euler proved the following stronger result (posthumously published in 1862), which shows that every even perfect number must be of Euclid's form.

Theorem 4.4 *The numbers* $2^{n-1}(2^n-1)$ *with* 2^n-1 *prime are the only even perfect numbers.*

Proof Let *n* be any even perfect number. Then, $n=2^e m,$ where *m* is odd and $e\geq 1.$ Since $\sigma(m)>m,$ we can let $\sigma(m)=m+s$ with $s>0.$ Now, it follows that

$$\begin{aligned} n \text{ is perfect } &\Rightarrow \sigma(n) = 2n \Rightarrow \sigma(2^e m) = 2\cdot 2^e m \Rightarrow \sigma(2^e)\sigma(m) = 2^{e+1}m \\ &\Rightarrow (2^{e+1}-1)(m+s) = 2^{e+1}m \\ &\Rightarrow 2^{e+1}m+2^{e+1}s-m-s = 2^{e+1}m \\ &\Rightarrow m = (2^{e+1}-1)s. \end{aligned}$$

Thus *s* is a divisor of *m* and $s<m.$ But $\sigma(m)=m+s,$ thus *s* is the sum

of all divisors of *m* that are less than m, i.e., *s* is the sum of a group of numbers that includes s. This is possible only if the group consists of one number alone, namely s. Therefore, the set of divisors of *m* smaller than *m* contains only one element and that element must be s. Since 1 is a divisor of m, it follows that *s* must be 1. Hence, $m = 2^{e+1} - 1$ is prime, and $n = 2^e(2^{e+1} - 1)$. ■

- An important feature of Theorem 4.4 is that every Mersenne prime generates a perfect number, and hence if Mersenne primes are infinite so are even perfect numbers. Thus, till March 2022, only 51 even perfect numbers are known, and the largest known 51st even perfect number is $2^{82,589,932}(2^{82,589,933} - 1)$ which has $49,724,095$ digits.
- Using Theorem 4.4, we shall show that an even perfect number *N* ends in the digit 6 or 8; that is, $N \equiv 6 \text{ or } 8 (\text{mod } 10)$. For this, first we note that any positive integer N, expressed in decimal notation, takes the form $N = \sum_{k=0}^{m} a_{m-k} \cdot 10^{m-k}$, where each a_i is an integer satisfying $0 \leq a_i \leq 9$, and the coefficients $a_0, a_1, \cdots, a_m$ are the digits of the number N. Thus, $N \equiv a_0 (\text{mod } 10)$. Now in view of Theorem 4.4, *N* must be of the form $N = 2^{n-1}(2^n - 1)$, where the factor $2^n - 1$ is a prime, and hence *n* must also be a prime. If $n = 2$, then $N = 6$, and the conclusion holds. If $n > 2$, we need to consider two cases, namely, $n = 4k + 1$ and $n = 4k + 3$. If $n = 4k + 1$, then
$$N = 2^{4k}(2^{4k+1} - 1) = 2^{8k+1} - 2^{4k} = 2 \cdot 16^{2k} - 16^k.$$
However, since $16^s \equiv 6 (\text{mod } 10)$ for any positive integer s, it follows that $N \equiv 2 \cdot 6 - 6 = 6 (\text{mod } 10)$. Now if $n = 4k + 3$, then
$$N = 4^{4k+2}(2^{4k+3} - 1) = 2^{8k+5} - 2^{4k+2} = 2 \cdot 16^{2k+1} - 4 \cdot 16^k,$$
which immediately gives $N \equiv 2 \cdot 6 - 4 \cdot 6 = -12 \equiv 8 (\text{mod } 10)$.
- No power of a prime can be a perfect number, because since $\sum_{\ell=0}^{k-1} p^\ell = (p^k - 1)/(p - 1) < p^k - 1 < p^k$ it follows that $\sigma(p^k) = \sum_{\ell=0}^{k-1} p^\ell + p^k < 2p^k$. The product of two odd primes is never a perfect number, because $\sigma(pq) = 1 + p + q + pq < 2pq$.
- In 1944, the concept of *superabundant numbers* was created. A natural number *n* is superabundant iff $\sigma[n]/n > \sigma[k]/k$ for all $k < n$. It is known that there are infinitely many superabundant numbers. Other

numbers related to perfect, deficient, and abundant numbers that have been introduced in recent times are *almost perfect numbers, practical numbers, quasiperfect numbers, semiperfect numbers*, and *weird numbers*. We mentioned these concepts to illustrate how ancient number work has inspired related modern investigations.

- All known perfect numbers except 6 have digital roots of $1,$ i.e., the ultimate sum of their digits equals $1.$ For example, for the perfect numbers $2^{18}(2^{19}-1)=137,438,691,328 \ \rightarrow \ 55 \ \rightarrow \ 10 \ \rightarrow \ 1.$
- Even perfect numbers are the sums of the successive powers of $2,$ for example, $6=2^1+2^2, \ 28=2^2+2^3+2^4, \ 496=2^4+2^5+2^6+2^7+2^8.$ In general, $2^{k-1}(2^k-1)=2^{k-1}+2^k+2^{k+1}+\cdots+2^{2k-2}.$
- Every even perfect number except 6 is also the sum of odd cubes, for example, $28=1^3+3^3, \ 496=1^3+3^3+5^3+7^3, \ 8128=1^3+3^3+\cdots+15^3.$ In general, $2^{2k}(2^{2k+1}-1)=1^3+3^3+5^3+\cdots+(2^{k+1}-1)^3$ for all $k\geq 0.$
- For any even perfect number $n=2^{k-1}(2^k-1)$, it follows that $2^k|\sigma(n^2)+1.$ Indeed, since $n^2=2^{2k-2}p^2,$ where $p=2^k-1$ is prime, $\sigma(n^2)=(1+2+2^2+\cdots+2^{2k-2})(1+p+p^2)=(2^{2k-1}-1)(2^{2k}-2^{k+1}+2^k+1).$ Thus, $\sigma(n^2)+1=2^kN$ for some $N.$
- For more details of perfect numbers, see Dickson [165] and Shoemaker [476].

4.6 Fermat Numbers

Fermat in 1640 also wrote to Mersenne that the numbers of the form $F_n=2^{2^n}+1, \ n\geq 0$ without exception are prime. Indeed, he tested $F_0=2^{2^0}+1=3, F_1=2^{2^1}+1=5, F_2=2^{2^2}+1=17, F_3=2^{2^3}+1=257, F_4=2^{2^4}+1=65{,}537,$ but testing of $F_5=2^{2^5}+1=4{,}294{,}967{,}297$ although was beyond him, in 1659 he wrote that he had found a proof of the primality of all $F_n.$ However, in 1732 Euler showed that $F_5=641\times 6700417,$ and hence he revealed that Fermat had been wrong. In fact, Euler's result in terms of congruences appears as $2^{32}+1\equiv 0(\mathrm{mod}\ 641).$ For this, since $5\cdot 2^7=640\equiv -1(\mathrm{mod}\ 641),$ in view of Theorem 3.6(6), it follows that $5^4\cdot 2^{28}\equiv(-1)^4(\mathrm{mod}\ 641) \ \equiv \ 1(\mathrm{mod}\ 641).$ Next, since

$5^4 = 625 \equiv -16 (\text{mod } 641) = -(2^4)(\text{mod } 641)$, which is the same as $2^4 \equiv -(5^4)(\text{mod } 641)$, from Theorem 3.6, we find

$$2^{32} + 1 = 2^4 \cdot 2^{28} + 1 \equiv -(5^4)2^{28} + 1 \equiv (-1+1) = 0(\text{mod } 641).$$

Thus, $641 | 2^{2^5} + 1$. Euler further showed that if any Fermat number has a factor, it must be a prime of the form $2^{n+1}k + 1$. Thus, a factor of F_5, if such existed, would have to be a number of the form $2^{5+1}k + 1$, or $64k + 1$. The first few primes of this type are $193, 257, 449, 577,$ and 641. Since Fermat made his conjecture, as of 2022, no other Fermat primes F_n with $n > 4$ have been found. Zerah Colburn (1804–1839), the American lightning-calculating boy, when asked whether the fifth number of Fermat's was prime or not, replied after a short mental calculation that it was not, as it had the divisor 641. He was unable to explain the process by which he reached his correct conclusion. As of 2014, it is known that F_n is composite for $5 \le n \le 32$. The largest Fermat number known to be composite is $F_{18233954}$ and its prime factor $7 \times 2^{18233956} + 1$ was discovered in October 2020. Factoring Fermat numbers is extremely difficult as a result of their large size. As of 2022, only F_0 to F_{11} have been completely factored, and the number of respective factors are $1, 1, 1, 1, 1, 2, 2, 2, 2, 3, 4, 5$. There are no known prime factors for $n = 20$ and $n = 24$. While heuristics suggest that for all $n > 4$ all Fermat numbers are composite, a formal proof is awaited. Gauss did not answer the question whether F_4 is the last prime. If not, and there are more, is their number finite or infinite? Related with Fermat numbers, in 1844, Eisenstein conjectures that the numbers $2^2 + 1, 2^{2^2} + 1, 2^{2^{2^2}} + 1, \cdots$ are all prime. While first three terms of this sequence have been tested to be prime, it is not known whether any of the remaining are prime.

- Fermat numbers satisfy the difference equation $F_{n+1} = F_0 F_1 \cdots F_n + 2$. Indeed, since $2^{2^0} - 1 = 1$, we have

$$
\begin{aligned}
F_0F_1\cdots F_n &= (2^{2^0}-1)(2^{2^0}+1)\prod_{k=1}^{n}(2^{2^k}+1)\\
&= (2^{2^1}-1)(2^{2^1}+1)\prod_{k=2}^{n}(2^{2^k}+1)\\
&\quad \cdots\\
&= (2^{2^n}-1)(2^{2^n}+1) \;=\; (2^{2^{n+1}}-1)\\
&= (2^{2^{n+1}}+1)-2 \;=\; F_{n+1}-2.
\end{aligned}
$$

- Any two distinct Fermat numbers F_m and F_n with $m > n$ are relatively prime: Suppose that $d > 0$ is a common divisor of F_m and $F_n,$ then *d* divides $2 = F_m - F_0F_1\cdots F_{m-1}.$ Thus, $d = 1$ or $d = 2;$ however, since F_m and F_n both are odd, we must have $d = 1.$ Therefore, for $m > n$, the Fermat numbers F_m and F_n are relatively prime.
- Polya in his book of 1945 *How to Solve It* gave a neat proof of Euclid's theorem that the number of primes is infinite. In fact, since there are infinitely many distinct Fermat numbers, each of which is divisible by an odd prime, and from the above observation, two Fermat numbers are relatively prime, these odd primes must all be distinct and infinite.
- For every integers $n \geq 0$ from $2^{2^n} - 1 = 2^{2^n} + 1 - 2 = F_n - 2 = F_0F_1\cdots F_{m-1}$, it follows that $2^{2^n} - 1$ is divisible by at least *n* different primes.

4.7 Fermat's Little Theorem

Fermat in the same letter of 1640 to Mersenne also wrote two propositions: 1. If *p* is an odd prime, then 2*p* divides $2^p - 2,$ or *p* divides $2^{p-1} - 1,$ and with the same hypothesis, 2. The only possible divisors of $2^p - 1$ are of the form $2pk + 1.$ Then in a letter written a few months later to de Bessy, he stated a more general result of which these two propositions are easy corollaries (whose proof had to wait till 1683 by Leibniz and then 1736 by Euler), which was fundamental to the progress of number theory, is now stated as:

Theorem 4.5 (Fermat's Little Theorem) *If p is prime and a any positive integer, then p divides* $a^p - a.$ *(This result is often written in the form*

$a^p \equiv a(\text{mod } p)$, *or, adding the condition that a and p are relatively prime, in the form* $a^{p-1} \equiv 1(\text{mod } p)$).

Proof There are many proofs of this theorem. One of them uses mathematical induction on *a* to prove that if *p* is prime, then $a^p \equiv a(\text{mod } p)$. For this, we use the binomial theorem (2.4)

$$(x+1)^p = x^p + \binom{p}{1}x^{p-1} + \binom{p}{2}x^{p-2} + \cdots + \binom{p}{p-1}x + 1.$$

Since

$$\binom{p}{k} = \frac{p!}{k!\,(p-k)!} = \frac{p(p-1)\cdots(p-k+1)}{k(k-1)\cdots 1}$$

is an integer. If $0 < k < p$, then *p* is not a factor of $k(k-1)\cdots 1$. Since *p* is a factor of

$$\binom{p}{k}k(k-1)\cdots 1 = p(p-1)\cdots(p-k+1)$$

it follows that *p* is a factor of $\binom{p}{k}$ when $0 < k < p$. Hence

$$(x+1)^p = x^p + 1 + mp$$

for some integer m. If $x = 1$, we obtain $2^p = 2 + m_1 p$, and this means $2^p \equiv 2(\text{mod } p)$. If $x = 2$, we get $3^p = 2^p + 1 + m_2 p = 2 + m_1 p + 1 + m_2 p = 3 + (m_1 + m_2)p$, and this means $3^p \equiv 3(\text{mod } p)$. Continuing in the same way, we see that, for all $a,\ a^p = a + rp$ for some integer r, and hence $a^p \equiv a(\text{mod } p)$. Now if *a* and *p* are relatively prime, then $a(a^{p-1} - 1) = rp$ implies that $a^{p-1} - 1$ is divisible by p, so $a^{p-1} \equiv 1(\text{mod } p)$. ■

As an example, from Theorem 4.5 it follows that $7^{11-1} \equiv 1(\text{mod } 11)$. Thus, from Theorem 3.6(6), it follows that $7^{222} = (7^{10})^{22} \cdot 7^2 \equiv (1)^{22} \cdot 49(\text{mod } 11) \equiv 5(\text{mod } 11)$. As an another example for the case $p = 5$, we have $a^5 - a = a(a^4 - 1) = a(a^2 + 1)(a^2 - 1)$ and on considering the cases $a = 5k, 5k+1, 5k+2, 5k+3, 5k+4$, it is easily seen that one of the three factors must be a multiple of five, no matter what *a* may be.

- If *a* is a negative integer, there is no difficulty. For $a = -b$, where $b > 0$, so that

$$a^p - a \;=\; (-b)^p - (-b) \;=\; -(b^p - b).$$

Because *b* is positive, we know already that *p* divides $b^p - b$.

- Fermat's first proposition follows by letting $a = 2$ in the above theorem. As an example, $2^{5-1} - 1 = 15 = 5 \times 3$. To show his second proposition, we assume that *q* is a prime divisor of $2^p - 1$. Theorem 4.5, then implies that *p* divides $q - 1$, i.e., $q - 1 = hp$ for some integer h. Since $q - 1$ is even, 2 must divide *hp* and therefore must divide h. It follows that $h = 2k$ and hence $q = 2kp + 1$. As an example, $2047 = 2^{11} - 1 = 23 \times 89$, and $23 = 2 \times 11 + 1,\ 89 = 2 \times 11 \times 4 + 1$.
- The converse of Theorem 4.5 does not hold. For this, we note that $341 = 11 \times 31$ is composite, $\gcd(2, 341) = 1$, and $2^{10} = 1024 \equiv 1 (\mathrm{mod}\ 341),\ 2^{341-1} = (2^{10})^{34} \equiv 1^{34} \equiv 1 (\mathrm{mod}\ 341)$. However, the contrapositive form of Theorem 4.5 is important: If $a^p \not\equiv a (\mathrm{mod}\ p)$, then *p* is not prime, i.e., composite. For example, we have $7^3 = 343 \equiv 2 (\mathrm{mod}\ 341)$. Thus, $7^{341} = (7^3)^{113} \cdot 7^2 \equiv 2^{113} \cdot 7^2 \equiv 2^{110} \cdot 2^3 \cdot 7^2 \equiv 8 \cdot 49 \equiv 392 \equiv 51 (\mathrm{mod}\ 341) \not\equiv 7 (\mathrm{mod}\ 341)$. The number 341 is indeed composite.
- Almost 2500 years back, Chinese either guessed or proved the special case of Theorem 4.5. However, their conjecture: *n* is prime iff $n|(2^n - 2)$, which is true for all integers $n \le 340$ (as late as 1680, Leibniz also made this very statement) fails for the composite $n = 341$. Indeed, $2^{341} - 2 = 2((2^{10})^{34} - 1) \equiv 2(1 (\mathrm{mod}\ 341) - 1) \equiv 0 (\mathrm{mod}\ 341)$, i.e., $341|(2^{341} - 2)$. This observation of 1819 is due to Pierre Frédéric Sarrus (1798–1861, France). A composite integer *n* is called *pseudoprime* if $n|(2^n - 2)$. It has been shown that there are infinite number of pseudoprimes. The first ten pseudoprimes are 341, 561, 645, 1105, 1387, 1729, 1905, 2047, 2465, 2701.
- The following assertion follows from Theorem 4.5: If $a \ge 2$ and *p* and *q* are primes such that $q|(a^p - 1)$, then $q|(a - 1)$ or $p|(q - 1)$. Theorem 4.5 has been extended and generalized in several ways, by no means with insignificant proofs. In 1760, Euler proved that if $\gcd(a, n) = 1$, then *n* divides $a^{\phi(n)} - 1$, i.e., $a^{\phi(n)} \equiv 1 (\mathrm{mod}\ n)$; here $\phi(n)$ is the Euler's totient function (the number of positive integers less than *n* and

relatively prime to *n*). For example, such positive integers less than 16 are $1, 3, 5, 7, 9, 11, 13, 15,$ it follows that $\phi(16) = 8.$ Now let $a = 15,$ so that $\gcd(15, 16) = 1,$ then Euler's result asserts that $15^8 - 1$ is divisible by $16.$ In fact, we have

$$15^8 - 1 \;=\; 2562890625 - 1 \;=\; 2562890624 \;=\; 160180664 \times 16.$$

Further, since $\gcd(23, 16) = 1,$ we find

$$23^8 - 1 \;=\; 78310985281 - 1 \;=\; 78310985280 \;=\; 4894436580 \times 16.$$

4.8 Futile Formulas to Generate Primes

A dream of number theorists is to find a function $f(n)$ that yields only prime numbers for $n \in \mathcal{Z},$ and the sequence of primes so obtained is infinite. In 1772, Euler pointed out that the formula $f(n) = n^2 - n + 41$ gives primes when $n = 1, 2, \cdots, 40.$ These primes, respectively, are 41, 43, 47, 53, 61, 71, 83, 97, 113, 131, 151, 173, 197, 223, 251, 281, 313, 347, 383, 421, 461, 503, 547, 593, 641, 691, 743, 797, 853, 911, 971,1033, 1097, 1163, 1231, 1301, 1373, 1447, 1523, 1601. Clearly, $f(41) = 41^2$ is composite. (Euler's *lucky numbers* are positive integers *k* such that for all integers *n* with $1 \leq n < k,$ the polynomial $n^2 - n + k$ produces a prime number. There are only seven Euler's lucky numbers exist, namely, $1, 2, 3, 5, 11, 17$ and $41.$ Note that Euler's lucky numbers have to be prime.) It is interesting to note that if we take the formula $g(n) = n^2 + n + 41$, it also gives same primes for $n = 0, 1, 2, \cdots, 39$ and yields the same primes for $n = -1, -2, \cdots, -40$ since $g(0) = g(-1), g(1) = g(-2), \cdots, g(39) = g(-40).$ Thus, $n^2 + n + 41$ yields primes for 80 consecutive integers $x = -40, -39, \cdots, 38, 39.$ Alternatively, the function $p(n) = n^2 - 79n + 1601$ gives the same primes generated by the above-mentioned functions $f(n)$ and $g(n),$ for $n = 0, 1, 2, \cdots, 79.$ The computer MANIAC II has found that for the primes under 10 million, Euler's formula $g(n)$ generated primes 47.5% of the time. For values on *n* below $2398,$ this formula has 50% probability of getting a prime. For values between 1 and $100,$ this formula yields 86 primes and only 14 composites (these being for the values $40, 41, 44, 49, 56, 65, 76, 81, 82, 84, 87, 89, 91,$ and 96 of *n*). Other quadratic equations, like that of Euler's are $h(n) = 103n^2 - 3945n + 34381$ (known as Ruby's polynomial) found in 1988, give 43 distinct primes for $0 \leq n \leq 42$, the absolute value of $j(n) = 36n^2 - 810n + 2753$ (known as Fung and Ruby's polynomial), given in 2009, gives 45 distinct primes for $0 \leq n \leq 44$, Ulam and his colleagues in 1963 gave $q(n) = 4n^2 + 170n + 1847.$ Its success rate is 46.6%, and it generates 760 primes below 10 million that are not generated by Euler's formula. One more such formula is $\phi(n) = 4n^2 + 4n + 59,$ whose success rate is $43.7\%.$ It generates 1500 primes not generated by Euler's $g(n)$ or Ulam's formulas $q(n).$

In 1743, Goldbach observed that a polynomial $P_n(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ with integer coefficients $a_0, a_1, \ldots, a_n$ cannot represent primes only, that is, the integers $f(0), f(1), f(2), \cdots$ are not all prime. For contrary, let $P_n(x)$ assume only primes. Let $a_0 \neq 0,$ and for a fixed $x = k,\ P_n(k) = p$ be a prime number, also ℓ be an arbitrary integer, then from (2.4), we have

$$\begin{aligned} P_n(k + \ell p) &= a_0(k + \ell p)^n + a_1(k + \ell p)^{n-1} + \cdots + a_{n-1}(k + \ell p) + a_n \\ &= (a_0k^n + a_1k^{n-1} + \cdots + a_{n-1}k + a_n) + pQ_n(\ell) \\ &= p + pQ_n(\ell) \ = \ p(1 + Q_n(\ell)), \end{aligned}$$

where $Q_n(\ell)$ is a polynomial of degree *n* with integer coefficients, and hence $1 + Q_n(\ell)$ is an integer. Thus, it follows that $p|P_n(k + \ell p),$ but since $P_n(x)$ assumes only primes, it is necessary that $P_n(k + \ell p) = p$ for any arbitrary integer $t.$ However, since $P_n(x)$ is a polynomial of degree *n*, it can assume the same value at most *n* times. In recent years, several formulas to generate infinite (perhaps all) primes have been suggested; however, most of them are only mathematically elegant, e.g., in 1947, William Harold Mills (1921–1964, USA) proved the existence of the smallest positive real number *A* such that $\lfloor A^{3^n} \rfloor$ ($\lfloor x \rfloor$ is the floor function) is a prime number for all natural numbers $n.$ However, it does not generate all prime numbers. The first five Mills primes are 2, 11, 1361, 2521008887, 16022236204009818131831320183. In 2017, László Tóth (Hungary) proved that the floor function in Mills's formula could be replaced with the ceiling function, so that there exists a constant *B* such that $\lceil B^{r^n} \rceil$, where $r > 2.106 \cdots$.

- Mersenne as well as Fermat formulas for generating prime numbers were based on visualizing the patterns. As another example of similar nature, consider the sequence whose initial numbers are 31, 331, 3331, 33331, 333331, 3333331, 33333331. All these are primes, so one can propose that the sequence generated by $a_n = (10^{n+1} - 7)/3,\ n \geq 1$ gives prime numbers. However, for $n = 8,$ we get $333333331 = 17 \times 19607843,$ a composite number. For the sequence $\{a_n\}$, it follows that $a_{15k+1} \equiv 0 (\mathrm{mod}\ 31),\ k \geq 0$ i.e., every 15th term of $\{a_n\}$ is divisible by 31. In general, it has been proved that all sequences such as $ab, abb, abbb, abbbb, \cdots$ or $ab, aab, aaab, aaaab, \cdots$ where *a* and *b* are integers are periodically divisible by the number $ab.$
- The following exact formula of finding $\pi(x)$ (the number of primes less than or equal to *x*) is due to Legendre: For a given positive integer *n* let $S_n = \{2, 3, 5, \cdots, q\}$ be the set of all primes *p* such that $p \leq \sqrt{n}.$ Suppose that S_n has *k* elements, i.e., $k = \pi(\sqrt{n}).$ Then

$$\pi(n) \ = \ n + k - 1 - \sum \left\lfloor \frac{n}{p_1} \right\rfloor + \sum \left\lfloor \frac{n}{p_1 p_2} \right\rfloor - \sum \left\lfloor \frac{n}{p_1 p_2 p_3} \right\rfloor + \cdots;$$

here, each sum is taken over all products of one or more primes from $S_n.$ As an example, we consider $n = 10,$ so that $S_n = \{2, 3, 5, 7\}.$ Then the formula gives

$$\begin{aligned}\pi(100) \ &= \ 100 + 4 - 1 - \left(\left\lfloor \frac{100}{2} \right\rfloor + \left\lfloor \frac{100}{3} \right\rfloor + \left\lfloor \frac{100}{5} \right\rfloor + \left\lfloor \frac{100}{7} \right\rfloor \right) \\ &\quad + \left(\left\lfloor \frac{100}{6} \right\rfloor + \left\lfloor \frac{100}{10} \right\rfloor + \left\lfloor \frac{100}{14} \right\rfloor + \left\lfloor \frac{100}{15} \right\rfloor + \left\lfloor \frac{100}{21} \right\rfloor + \left\lfloor \frac{100}{35} \right\rfloor \right) \\ &\quad - \left(\left\lfloor \frac{100}{30} \right\rfloor + \left\lfloor \frac{100}{42} \right\rfloor + \left\lfloor \frac{100}{70} \right\rfloor \right) \\ &= \ 103 - (50 + 33 + 20 + 14) + (16 + 10 + 7 + 6 + 4 + 2) \\ &\quad - (3 + 2 + 1) \ = \ 25.\end{aligned}$$

- The following known formulas for finding p_n are also infeasible to compute in practice: Willans's formula of 1964, [536]

$$p_n = 1 + \sum_{i=1}^{2^n} \left[\left(\frac{n}{\sum_{j=1}^{n} \left[\left(\cos \frac{(j-1)!+1}{j} \pi \right)^2 \right]} \right)^{1/n} \right];$$

Gandhi's formula of 1971, [207]

$$p_n = \left\lfloor 1 - \log_2 \left(-\frac{1}{2} + \sum_{d|P_{n-1}} \frac{\mu(d)}{2^d - 1} \right) \right\rfloor,$$

where $P_n = p_1 p_2 \cdots p_n$ and $\mu(d)$ is the Möbius function; and Matthew Frank's (USA) recurrence relation (around 2002)

$$a(n) = a(n-1) + \gcd(n, a(n-1)), \quad a(1) = 7$$

for which in 2008, Eric Rowland (USA) proved that the sequence generated by this recurrence relation contains only ones and (not all) prime numbers.

- In a memoir of 1747 Euler said, "Till now the mathematicians tried in vain to discover some order in the sequences of the prime numbers and we have every reason to believe that there is some mystery which the human mind shall never penetrate$\cdots$." According to Erdös, it will be another million years, at last, before we understand the primes.

4.9 Wilson's Theorem

Ever since Euclid proved that the number of primes is infinite, mathematicians have been seeking for a test which would determine whether or not a given number is prime. But no test applicable to all numbers has been found. In 1770, Waring published a book under the title *Meditationes Algebraica*. One passage from this work reads: "If *n* is a prime, then the quantity $((n-1)!+1)/n$ is a whole number $\cdots$. This elegant theoretical property of prime numbers is the discovery of John Wilson." It is very likely that this result Waring dedicated to Wilson to pay of a "political" debt. Waring further wrote "Theorems of this kind will be very hard to prove, because of the absence of a notation to express prime numbers." Commenting on this passage, Gauss remarked "in questions of this kind it was not nomenclature, but conception that mattered." Around 1000 AD, the same result was stated by al-Haytham and was known to Leibniz who never published it. Despite Waring's pessimistic forecast in 1771, Lagrange proved that *n* is prime iff *n* divides $(n-1)!+1.$ His proof is based on finding all integral solutions of the Diophantine equation $x^2 - ay^2 = 1,$ and the solution of numerous problems posed by Fermat on how certain primes could be represented in particular ways (typical of these was the result that every prime *n* of the form $n = 8k+1$ could be written as $n = a^2 + 2b^2$ for suitable integers *a* and *b*). Waring also supplied a proof in the third edition of his *Meditationes Algebraica* which was published in 1782. We note that this result, now known as Wilson's criterion, can be stated as *n* is a prime, iff, $(n-1)!+1 \equiv 0(\mathrm{mod}\ n)$ or $(n-1)! \equiv -1(\mathrm{mod}\ n).$ As examples, for $n=6$, this gives $120+1=121$ which is not divisible by 6, so 6 is not prime; for $n=7,\ 720+1=721,$ divisible by 7, so 7 is prime. However, $n!$ the notation which was first used in Indian mathematics becomes gigantic even for small numbers, e.g., $(100)! \simeq 10^{158}$ (for upper bounds, see Mahmoud et. al. [356]), a direct verification as to whether $(n-1)!+1$ has *n* for a divisor is as difficult/impossible as testing directly whether *n* is prime, when *n* is a large number. Thus, Wilson/Waring/Lagrange theorem is only of theoretical interest.

The following proof of Wilson's Theorem is immediate: The result is obvious for *n* is 2 or 3, so let us assume that $n > 3.$ If *n* is composite, then its positive divisors are among the integers $1, 2, 3, 4, \cdots, n-1$ so $\gcd((n-1)!, n) > 1,$ but then $(n-1)! \equiv -1(\mathrm{mod}\ n)$ is impossible. Alternately, assume that $n > 4,$ and *n* is composite but not the square of a prime; then it is the product of two distinct integers, both less than $n-1,$ and therefore divisors of $(n-1)!.$ Next assume that $n = q^2,$ where *q* is an odd prime: then *q* is less than $n-1,$ and so is $2q.$ Hence, both *q* and 2*q* enter into $(n-1)!;$ it follows that the latter is divisible by $2q^2,$ and, consequently, by *n*). Next, if *n* is prime, then each of the integers $1, 2, 3, 4, \cdots, n-1$ is relatively prime to $n.$ Thus, for each of these integers *a*, there is another *b* such that $ab \equiv 1(\mathrm{mod}\ n).$ It is clear that this *b* is unique modulo $n,$ and that since *n* is prime, $a \equiv b(\mathrm{mod}\ n)$ iff *a* is 1 or $n-1.$ Now if we omit 1 and $n-1,$ then the others can be grouped into pairs whose product satisfies $2 \cdot 3 \cdot 4 \cdots (n-2) \equiv 1(\mathrm{mod}\ n),$ which is the same as $(n-2)! \equiv 1(\text{mod } n)$. Multiplying this equality by $n-1$, we obtain the required relation $(n-1)! \equiv -1(\text{mod } n)$.

An integer is called *Wilson's prime* iff it satisfies Wilson's criterion. So far, the only known Wilson primes are 5, 13, and 563; also there are no others less than 500,000,000. It is conjectured that the number of Wilson primes is infinite and that the number of such primes between *x* and *y* should be about $\ln(\ln y / \ln x)$. Wilson's theorem guarantees that $(n-2)! - 1$ and $(n-1)! + 1$ are composite for every prime number $n > 5$. Thus, there exists an infinitude of composite numbers of the form $n! + 1$. However, it is not known if $n! + 1$ is prime for infinitely many values of n. The largest known such prime is $422429! + 1$, which is 2193027 digits long and discovered in 2022. The largest known factorial prime of the form $n! - 1$ is $208003! - 1$, which is 1015843 digits long and discovered in 2016. In conclusion, Wilson's theorem is more beautiful than useful.

Wilson's theorem is equivalent to the relation $(n-1)! = Wn - 1$ where *W* is an integer. If $n > 3$, we can take *W* greater than 1. We set $W = G + 1$, to obtain $(n-1)! = Gn + (n-1)$. Thus, Wilson's theorem can be stated as: If *n* is a prime, the remainder of division of $(n-1)!$ by *n* is $(n-1)$. Further, since $(n-1)! = (n-1)(n-2)!$ it is equivalent to saying that the remainder of division of $(n-2)!$ by the prime, *n* is 1.

4.10 Goldbach's Conjecture

Goldbach in 1742 wrote a letter to Euler speculating that every *integer* $n > 5$ *is the sum of three primes*. Euler replied that this is equivalent to *every even* $n > 2$ *is the sum of two, not necessarily distinct, primes*, which became known as Goldbach's conjecture. To show the equivalence, first we shall show that Euler's statement implies Goldbach's: for an odd integer $n > 5$ since $n - 3$ is even, there exist primes p_1 and p_2 such that $n - 3 = p_1 + p_2$ and hence $n = p_1 + p_2 + 3$; now to show that Goldbach's statement implies Euler's: for an even integer *n* since $n + 2$ is even, there exist primes $p_1, p_2,$ and p_3 such that $n + 2 = p_1 + p_2 + p_3$, but then from the same parity, it follows that one of the p_i must be 2, say $p_3 = 2$, which means $n = p_1 + p_2$. Thus, for example, $4 = 2 + 2,\ 28 = 23 + 5,\ 96 = 89 + 7$. In 1896, Sylvester rephrased Goldbach's conjecture as: Every even integer 2*n* greater than 4 is the sum of two primes, one larger than $n/2$ and the other smaller than $3n/2.$ Erdös liked to point out that Goldbach's conjecture had actually been anticipated about a hundred years earlier by Descartes. "I feel that the name Goldbach's conjecture should remain," Erdös reasoned, demonstrating his strong sense of justice. "First of all, Goldbach popularized it by writing to Euler. And also, Goldbach is so poor and Descartes is so rich, it would be like taking candy from a baby." Since every even number can be written as the sum of two odd numbers in several different ways, and every prime greater than 2 is odd, apparently the conjecture seems reasonable enough, and almost obvious (according to Hardy, any idiot could have made this assumption). However, it remains one of the best-known unsolved problems in number theory.

In 1930, Schnirelmann proved that every positive integer > 2 can be written as the sum of not more than *k* primes, where *k* is an effectively computable constant. However, this was an existence theorem, which gives no indication about the actual magnitude of $k.$ Every odd integer > 5 can be expressed as the sum of three primes is known as *weak Goldbach conjecture*, for example, $31 = 5 + 7 + 19.$ It is called weak because if Goldbach's conjecture is proven, then this would also be true. The weak Goldbach conjecture shows that prime numbers play a fundamental role in the *multiplicative* representation of an odd number by means of primes. In 1922, Hardy and Littlewood used generalized Riemann hypothesis to show that weak Goldbach conjecture holds for sufficiently large odd integers, and consequently every sufficiently large even integer could be written as the sum of four primes. In 1937, Vinogradov used the method of trigonometric sums to prove the same result. In 1956, Konstantin Vasil'evich Borozdin (1912–1987, Russia) showed that $3^{3^{15}} = 3^{14,348,907}$ ($6,846,170$ digits) is an upper bound for sufficiently large. In 1995, Olivier Ramaré (France) proved that every even number is a sum of at most six primes. In 1996, Chen and Wang [125] showed that this bound can be replaced by the small bound $e^{e^{9.715}}.$ In 2013, Harald Andrés Helfgott (born 1977, Peru) proposed the proof (which is broadly accepted, but not published) of the weak Goldbach conjecture. In 2014, Terence Tao has proved that every odd number greater than 1 is the sum of at most five primes.

In 1948, Rényi proved that every large enough even number was a prime plus a product of at most *r* primes. He was only able to prove that *r* was a very large, but nevertheless, finite number. A significant reduction was made in 1965 when A.A. Buhstab (Russia) and Vinogradov independently proved every even number was a prime plus the product of at most three primes. Finally, in 1966, Jingrun Chen proved that every sufficiently large even number can be written as the sum of either two primes, or a prime and a *almost prime/semiprime* (i.e., the product of two primes). However, in China during sixties, there was a "cultural revolution," and so this kind of mathematics was frowned for being far removed from any conceivable application to industry or agriculture. Because Jingrun stubbornly stuck to his esoteric research at the risk of neglecting his teaching, he was discriminated against during the reign of the so-called gang of four and may have lost his academic position. After the overthrow of the gang of four, he was rehabilitated and even declared a "hero of the revolution."

For small values of $n,$ Goldbach's conjecture (and hence the weak Goldbach conjecture) can be verified directly. In fact, in 1938, Nils Johan Pipping (1890–1982, Finland) vigorously verified the conjecture up to $n = 10^5.$ With the advent of supercomputers, by 1993, Goldbach's conjecture was verified up to 400 million. In 1998, Jörg Richstein (Germany) extended this up to $4 \times 10^{14}.$ He also showed that $389, 965, 026, 819, 938 = 5569 + 389, 965, 026, 814, 369.$ In 2013, Tomás Oliveira e Silva (Portugal) verified Goldbach's conjecture up to 4×10^{18} (and double-checked up to 4×10^{17}). An interesting observation during the verification he made is that $3, 325, 581, 707, 333, 960, 528$ is the smallest number that cannot be written as a sum of two primes where one is smaller than $9781,$ in fact, $3, 325, 581, 707, 333, 960, 528 = 9781 + 3, 325, 581, 707, 333, 950, 747.$

Similar problems to Goldbach's conjecture exist in which primes are replaced by other particular sets of numbers, such as Euler conjectured that any even number greater than 6 that is of the form $4n + 2$ is the sum of two primes of the form $4n + 1$ (taking 1 as a prime of this latter type, where necessary). For instance, because 30 is an even number of the form $4n + 2,$ we can write $30 = 13 + 17.$ In 1775, Lagrange asserted that every odd integer greater than 5 can be written as a sum $p_1 + 2p_2,$ where both p_1 and p_2 are prime. For example, $51 = 47 + 2 \cdot 2,\ 53 = 47 + 2 \cdot 3,\ 55 = 41 + 2 \cdot 7,\ 57 = 53 + 2 \cdot 2,\ 59 = 53 + 2 \cdot 3.$ Hardy and Littlewood in 1922 conjectured that every large integer is a sum of a prime and two squares, an assertion that was subsequently verified. For any positive number *n*, there exists an even integer *a*, which is representable as the sum of two odd primes in *n* different ways. For example, the integers $66, 96,$ and 108 can be written as the sum of two primes in six, seven, and eight ways, respectively. In fact,

$$66 = 5 + 61 = 7 + 59 = 13 + 53 = 19 + 47 = 23 + 43 = 29 + 37,$$

$$96 = 7 + 89 = 13 + 83 = 17 + 79 = 23 + 73 = 29 + 67 = 37 + 59 = 43 + 53$$

and

$$108 = 5 + 103 = 7 + 101 = 11 + 97 = 19 + 89 = 29 + 79 = 37 + 71 = 41 + 67 = 47 + 61.$$

Consider the primes arranged in their natural order $2, 3, 5, 7, \cdots$. It is conjectured that beginning with $3,$ every other prime can be composed of the addition and subtraction of all smaller primes (and 1), each taken once. For example,

$$3 = 1 + 2,\ 7 = 1 - 2 + 3 + 5,\ 13 = 1 + 2 - 3 - 5 + 7 + 11 = -1 + 2 + 3 + 5 - 7 + 11,\ 29 = 23 + 19 + 17 - 13 - 11 - 7 - 5 + 3 + 2 + 1,\ 37 = 31 - 29 + 23 - 19 + 17 - 13 + 11 + 7 + 5 + 3 + 2 - 1.$$

- de Polignac Conjecture: In 1848, Alphonse de Polignac (1826–1863, France) claimed that “Every odd number can be expressed as the sum of a power of 2 and a prime.” He claimed that he checked his assertion for all odd numbers up to 3 million. Since no positive power of 2 can have an odd number in its prime factorization, had it been true, Polignac’s assertion would have been a marvelous result. But alas, it is patently wrong. Polignac said that he verified his claim for many numbers, but actually missed a simple counterexample, namely $127.$ Let us express 127 as a sum of a power of 2 and the remaining number in all possible ways:

$$\begin{aligned} 127 &= 1 + 126 &&= 2^0 + (9 \times 14) \\ &= 2 + 125 &&= 2 + (5 \times 25) \\ &= 4 + 123 &&= 2^2 + (3 \times 41) \\ &= 8 + 119 &&= 2^3 + (7 \times 17) \\ &= 16 + 111 &&= 2^4 + (3 \times 37) \\ &= 32 + 95 &&= 2^5 + (5 \times 19) \\ &= 64 + 63 &&= 2^6 + (7 \times 9). \end{aligned}$$

Since $2^7 = 128 > 127,$ our verification is over; we see that the second component of 127 is not a prime in each case. So Polignac's conjecture is disproved.

4.11 Twin Primes Conjecture

The largest pair of consecutive integers, known as *Siamese twins* which are both prime is $(2, 3)$ because any other two consecutive numbers will have an even number, which is not prime. Primes of the form *p* and $p + 2$ are called twin primes; the term was coined by Paul Gustav Samuel Stäckel (1862–1919, Germany). From among the primes between 1 and 100 (which are 25 in number), we have eight pairs of twin primes. They are $(3, 5), (5, 7), (11, 13), (17, 19), (29, 31), (41, 43), (59, 61),$ and $(71, 73).$ Among the primes between 101 and 200 (which are 21 in number), the twin primes are $(101, 103), (107, 109), (137, 139), (149, 151), (179, 181), (191, 193),$ and $(197, 199).$ Since every prime greater than 3 is of the form $6n + 1$ or $6n - 1 (= 6k + 5)$. The sum of twin primes is $(6n - 1) + (6n + 1) = 12n,$ and hence the sum of twin primes greater than 3 is divisible by 12. If *n* and $n + 2$ are prime, then $n + 1$ is divisible by 6. In fact, among three consecutive numbers, one of them must be divisible by 3, and since *n* and $n + 2$ are prime, $n + 1$ must also be even, so it has to be divisible by 6. In 1919, Viggo Brun (1885–1978, Norway) proved a remarkable theorem showing that the sum of the reciprocals of the twin primes converges, i.e., *Brun's constant*

$$B_2 \;=\; \left(\frac{1}{3}+\frac{1}{5}\right)+\left(\frac{1}{5}+\frac{1}{7}\right)+\left(\frac{1}{11}+\frac{1}{13}\right)+\left(\frac{1}{17}+\frac{1}{19}\right)+\cdots$$

is approximately $1.90216058 \pm 7 \cdot 81 \times 10^{-10}$. While Brun's result does not prove the well-known twin primes conjecture *there are infinitely many twin primes*, it confirms that the twin primes become sparser and sparser as we move out into the larger integers. In fact, computation shows

Range	Twin primes
$10^9 - \ 10^9 + 150,000$	461
$10^{10} - 10^{10} + 150,000$	374
$10^{11} - 10^{11} + 150,000$	309
$10^{12} - 10^{12} + 150,000$	259
$10^{13} - 10^{13} + 150,000$	221
$10^{14} - 10^{14} + 150,000$	191
$10^{15} - 10^{15} + 150,000$	166

Twin conjecture has resisted all attempts at proof for hundreds of years, and most mathematicians do not feel that a proof will occur any time soon. The largest known twin prime as of August 2022 discovered in September 2016 is $2996863034895 \times 2^{1290000} \pm 1,$ with $388,342$ decimal digits. In 1966, Jingrun Chen proved a weaker version of the twin primes conjecture, namely, there exist infinitely many pairs of numbers which differ by two in which the first number of the pair is a prime and the second is either a prime or a semiprime. The number $10 = 2 \times 5$ is almost prime, whereas $30 = 2 \times 3 \times 5$ is not an almost prime. Using computers, billions of twin primes have been found, e.g., below 10^{18}, there are $808,675,888,577,436$ twin primes pairs. Primes of the form *p* and $p+4$ are called *cousin primes*, for example, between 1 and 100, there are eight pairs of cousin primes $(3,7),(7,11),(13,17),(19,23),(37,41),(43,47),(67,71),(79,83).$ Like Brun's constant, in 1996, Marek Wolf (born 1956, Poland) for cousin primes, leaving the initial term (3,7), showed that

$$B_4 = \left(\frac{1}{7}+\frac{1}{11}\right)+\left(\frac{1}{13}+\frac{1}{17}\right)+\left(\frac{1}{19}+\frac{1}{23}\right)+\cdots.$$

converges approximately to $1.1970449.$ In 2022, Batalov found the largest-known pair of cousin primes having 51,934 digits with $p = 29055814795 \times (2^{172486} - 2^{86243}) + 2^{86245} - 3.$ Primes of the form *p* and $p+6$ are called *sexy primes*, for example, between 1 and 100, there are 15 pairs of sexy primes $(5,11),(7,13),(11,17),(13,19),(17,23),(23,29),(31,37),(37,43),(41,47),$ $(47,53),(53,59),(61,67),(67,73),(73,79),(83,89),$ whereas between 101 and 200, there are ten pairs of sexy primes $(101,107),(103,109),(107,113),\ (131,137),$

$(151, 157), (157, 163), (167, 173), (173, 179), (191, 197), (193, 199)$. Between 1 and 200 besides these sexy pairs there is another pair $(97, 103)$. In 2022, Batalov also found the largest-known pair of sexy primes having 51,934 digits with $p = 11922002779 \times (2^{172486} - 2^{86243}) + 2^{86245} - 5$.

In 1849, a more general conjecture was made by Polignac that for every natural number k, there are infinitely many primes *p* such that $p + 2k$ is a prime. Clearly, for $k = 1, 2,$ and $3,$ respectively, it reduces to twin, cousin, and sexy primes conjecture. Primes of the form $p, p + 2,$ and $p + 4$ are called triplet primes. In this case, there is only one prime triplet, namely $(3, 5, 7)$. In fact, $p = 2$ is obviously impossible, and for $p > 3$, the only possibilities are $p = 3q + 1$ or $p = 3q + 2,\ q > 1$ but then $p + 2 = 3q + 1 + 2 = 3(q + 1)$ or $p + 4 = 3q + 2 + 4 = 3(q + 2)$. Primes of the form *p* and $p + 2,$ or $p + 4$ and $p + 6$ are also called prime triplets. For such type of triplets examples are $(5, 7, 11), (7, 11, 13), (11, 13, 17), (13, 17, 19), (17, 19, 23), (37, 41, 43), (41, 43, 47), (67, 71, 73)$. A prime can be a member of up to three prime triplets—for example, 103 is a member of $(97, 101, 103), (101, 103, 107)$, and $(103, 107, 109)$. This also leads to prime quintuplet $(97, 101, 103, 107, 109)$. A prime quadruplet $(p, p + 2, p + 6, p + 8)$ can be obtained from two overlapping prime triplets, $(p, p + 2, p + 6)$ and $(p + 2, p + 6, p + 8)$, for example, $(5, 7, 11), (7, 11, 13)$ give $(5, 7, 11, 13)$, and other prime quadruplet are of the form $(30n + 11, 30n + 13, 30n + 17, 30n + 19)$. Between 1 and 10000, there are only 12 prime quadruplet $(5, 7, 11, 13), (11, 13, 17, 19), (101, 103, 107, 109), (191, 193, 197, 199), (821, 823, 827, 829), (1481, 1483, 1487, 1489), (1871, 1873, 1877, 1879), (2081, 2083, 2087, 2089), (3251, 3253, 3257, 3259), (3461, 3463, 3467, 3469), (5651, 5653, 5657, 5659), (9431, 9433, 9437, 9439)$. However, the smallest prime quadruplet of a special type is $(2, 3, 5, 7)$. Bombieri is known for the distribution of prime numbers in arithmetic progressions. His work was based on a new development of the large sieve introduced by Linnik in 1941 and signaled a turning point in analytic number theory. In 2004 (published in 2008 [225]), Ben Joseph Green (born 1977, England) and Terence Tao proved that there are arbitrarily long arithmetic progressions of prime numbers. As an example, the following six sequences consist of primes in arithmetic progression

(1)	(2)	(3)	(4)	(5)	(6)
7	107	7	47	71	199
37	137	157	257	2381	409
67	167	307	467	4691	619
97	197	457	677	7001	829
127	227	607	887	9311	1039
157	257	757	1097	11621	1249
		907	1307	11931	1459
					1669
					1879
					2089

In September 2019, Rob Gahan (Ireland) found the first known case of 27 primes in an arithmetic progression as part of PrimeGrid's AP27 Search subproject

$$224,584,605,939,537,911 + 18,135,696,597,948,930 \times k, \quad 0 \le k \le 26.$$

Here $224,584,605,939,537,911$ is prime, whereas $18,135,696,597,948,930$ is composite with factorization $2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19 \times 23 \times 81292139$. There is a considerable interest in proving if there are infinitely many sets of three consecutive primes in arithmetic progression. As an example, 22 known primes *n* such that $n, n+60, n+120$ are consecutive primes are 4911251, 5309539, 9113263, 11355797, 11397103, 13940057, 14306203, 14313527, 14585089, 17172521, 21126109, 24419281, 24581803, 24861631, 24922291, 25308799, 26241751, 26722523, 27408193, 28740919, 29675137, 30045811 (see https://oeis.org/A089234).

- In 1949, Paul Arnold Clement (born 1917, USA) proved that *n* and $n+2$ are twin primes iff $4[(n-1)!+1]+n \equiv 0(\text{mod } n(n+2))$. This result is obviously connected to Wilson's theorem. Obviously, $n+2$ is not prime when $n=2$, so we exclude $n=2$, i.e., we assume that *n* and $n+2$ are odd primes. We know *n* is prime iff $(n-1)!+1 \equiv 0(\text{mod } n)$, and hence iff $4[(n-1)!+1]+n \equiv 0(\text{mod } n)$. Now $n+2$ is prime iff

$(n+1)!+1 \equiv 0(\bmod\ (n+2))$. Next, since $n+1 \equiv -1(\bmod\ (n+2))$ and $n \equiv -2(\bmod\ (n+2))$, it follows that $(n+1)! \equiv (-1)(-2)(n-1)! \equiv 2(n-1)!(\bmod\ (n+2))$, and therefore $4[(n-1)!+1]+n \equiv 2(n+1)!+2+n+2 \equiv 0(\bmod\ (n+2))$. Now since *n* and $n+2$ are relatively prime, it follows that *n* and $n+2$ are twin primes iff $4[(n-1)!+1]+n \equiv 0(\bmod\ n(n+2))$.

- In 1971 [468], Sergusov showed that *n* and $n+2$ are twin primes if and only if $\phi(m)\sigma(m)=(m-3)(m+1)$, where $m=n(n+2)$; here $\phi(m)$ is Euler's totient function and $\sigma(m)$ is the sum of positive divisors of m, as defined earlier.
- In 1986, Dorin Andrica (born 1956, Romania) conjectured that the gaps between prime numbers satisfy the inequality $r_n=\sqrt{p_{n+1}}-\sqrt{p_n}<1$. If $g_n=p_{n+1}-p_n$ denotes the *n*th prime gap, then Andrica's conjecture can also be rewritten as $g_n<2\sqrt{p_n}+1$. This inequality obviously holds for twin, cousin, and sexy primes. This conjecture has been verified up to $p_n=4\times 10^{18}$. It follows that if $r_n<1$, then there must be a prime between n^2 and $(n+1)^2$. Andrica also claimed that $\lim_{n\to\infty}(\sqrt{p_{n+1}}-\sqrt{p_n})=0$.
- On April 17, 2013, Yitang Zhang (born 1955, China-USA) proved *weak conjecture of twin primes*: There exists an even integer $M \geq 2$ with the property that there are infinitely many primes of the form $(p, p+M)$. In fact, there exists an *M* with $M \leq 7\times 10^7$. In July 2013, this huge upper bound was replaced by Terence Tao to merely 4680. In November 2013, this upper bound was further sharpened by James Alexander Maynard (born 1987, England) to 600. As of April 2014, the Polymath project 8 (using the methods of Maynard and Terence Tao) lowered the upper bound to 246.

4.12 Prime Number Theorem

When we trace a sophisticated theorem to its origin, we often find its formulation to have been prompted by certain "circumstantial evidence" which render such a result plausible. Prime numbers seem to appear rather haphazardly. In fact, between any two numbers *a* and b, which numbers are prime? How many of them are there? How far apart do they appear? Let us consider the following observations: between 1 and 100

(100 numbers), there are 25 primes; there are 5 primes between 101 and $113,$ but none between 114 and $126;$ between 1001 and 1100, there are 16 primes; between 8401 and 8500, there are only 8 primes, and these 8 are crowded in the interval 8418 to $8460;$ between 10001 and 10100, there are 11 primes; there are 13 primes between 89501 and $89600;$ between 9,999,900 and 10,000,000, there are 9 primes; and between 10,000,000 and 10,000,100, there are only 2 primes. Thus, the distribution of primes seems to be highly irregular. Landau commented that the study of the distribution of primes should be considered as one of the most important chapters of mathematical sciences.

In 1797/98, Legendre conjectured that $\pi(x)$ can be approximated by the function $x/(A\ln x+B),$ where *A* and *B* are unspecified constants. His conjecture was based on the tables prepared earlier by Felkel and Baron Jurij Bartolomej Vega (1754–1802, Germany). In 1808, Legendre provided specific values of unspecified constants in his function as $A=1$ and $B=-1.08366.$ According to his own recollection in 1849, when Gauss was a boy of 15/16, he also examined a table of prime numbers (divided the natural numbers into intervals of 1000) smaller than $102,000$ complied by Lambert, looking for patterns and counted the number primes. Around 1800, Gauss conjectured that asymptotically (as *x* increases to infinity) the ratio $x/\ln x$ equals $\pi(x),$ equivalently, $\lim_{x\to\infty}\pi(x)/(x/\ln x)=1,$ or the *density* of the primes among the first *x* integers, is approximated by $\pi(x)/x=1/\ln x.$ (The concept of density is used to compare the size of sets of numbers, even when these sets are infinitely large. The even numbers, for example, have a density of one half since exactly half the numbers are even; multiples of 5 have a density of one fifth. Although the number of primes is infinite the density of primes is zero! That is, the primes are spread so thinly among the natural numbers that the probability that a randomly chosen number is prime is vanishingly small; most numbers are not prime.) In 1823, Abel characterized the prime number theorem (referring to Legendre) as perhaps the most remarkable theorem in all mathematics. In 1838, Dirichlet came up with the approximating function involving logarithmic integral Li$(x)=\int_2^x dt/\ln t,$ which he communicated to Gauss. Later, Gauss showed that both Legendre's and Dirichlet's formulas imply the same conjectured asymptotic equivalence of $\pi(x)$ and $x/\ln(x);$ however, Dirichlet's approximation provides considerably closer numerical approximations.

In 1852, Chebyshev was the first to make any progress toward a proof of Gauss's conjecture. His theorem was an extension of his proof of Bertrand's assertion. Chebyshev actually proved that for sufficiently large *x*

$$0.92129 \; < \; \frac{\pi(x)}{(x/\ln x)} \; < \; 1.10555$$

and

$$0.89 \; \mathrm{Li}(x) \; < \; \pi(x) \; < \; 1.11 \; \mathrm{Li}(x);$$

however, he was unable to prove Gauss's conjecture. Chebyshev's bounds were improved by several mathematicians including Sylvester in 1892 as $0.95695 < \pi(x)/(x/\ln x) < 1.04423.$ Riemann in his 1959 memoir extended Euler's formula (4.4) from real to complex numbers analytically and showed that prime numbers are intimately connected with the zeros of the extended Riemann zeta function (3.26). His work is considered as an important contribution to the distribution of prime numbers; in fact, it made the field analytic number theory sufficiently promising. Gauss conjecture was not proved until 1896, when Hadamard and Poussin proved it independently. Since then, the conjecture became famous as the *Prime Number Theorem*, and it remains one of the supreme achievements of mathematics. The proof was not elementary and made use of Hadamard's theory of integral functions applied to the Riemann zeta function (3.26) and a simple trigonometric identity. It is to be remarked that prime number theorem is equivalent to the statement that the *n*th prime number p_n satisfies $p_n \sim n \ln n.$

After Hadamard and Poussin, several proofs of prime number theorem were offered, among these the most noticeable are due to Norbert Wiener (1894–1964, USA), who deduced the result almost as a corollary from his work on Tauberian theorems of 1927–1932, and its reformation by Landau in 1932. Then without involving functions of a complex variable, an "elementary"—though not easy—independent proofs of Selberg and Erdös (boyhood dream) in 1949 (which also sparked bitter confrontation between them; in Hungary, the proof is known as the Erdös–Selberg proof, whereas in Princeton Selberg–Erdös proof), and nonelementary proof of Donald Joseph Newman (1930–2007, USA) proof in 1980. Later, the technique of their proofs was implemented by several number theorists to deal with conjectures previously considered too profound.

Let us observe the following table in which $\pi(x)$ denotes the number of prime numbers between 1 and x.

x	$\pi(x)$	$x/\ln x$	$\pi(x)/x$	$f(x)=\pi(x)/(x/\ln x)$
10	4	4	0.4	
10^2	25	22	0.25	
10^3	168	145	0.168	1.159
10^4	1229	1086	0.1229	1.132
10^5	9592	8686	0.09592	1.104
10^6	78498	72382	0.078498	1.084
10^7	664579	620420	0.0664579	1.071
10^8	5761455	5428681	0.05761455	1.061
10^9	50847534	48254942	0.050847534	1.054
10^{10}	455052511	434294482	0.0455052511	1.048
10^{11}	4118054813	3948131654	0.0411805481	1.043
10^{12}	37607912018	36191206825	0.0376079120	1.039

We note that the fourth column gives the density of prime numbers. It indicates that $\pi(x)/x \to 0$ as $x \to \infty$, which can be proved analytically. This shows that almost all of the positive integers are composite. Now multiply these numbers by $1, 2, 3, \cdots$ respectively, i.e., $\log_{10} x$; we get a list of numbers converging to a number *c* between 0.4 and 0.5. Thus,

$$\frac{\pi(x)}{x} \times \log_{10} x \ \sim \ c, \qquad \text{or} \qquad \pi(x) \ \sim \ c \times \frac{x}{\log_{10} x}.$$

In fact, $c = \log_{10} e$, where $e = 2.71828\cdots$ is nothing else but the base of the natural logarithm. Thus, $\pi(x) \sim \frac{x}{\ln x}$. Further, the fifth column indicates that the convergence of $f(x)$ to its limiting value of 1 is slow. For example, $f(10^{20}) = 1.023$ and $f(10^{25}) = 1.018$. This means that relative error of 2.3% results when $\pi(10^{20})$ is estimated by $10^{20}/\ln 10^{20}$, and 1.8% when $\pi(10^{25})$ is estimated by $10^{25}/\ln 10^{25}$.

- Ramanujan, in his first letter to Hardy on January 16, 1913, had written: “I have found a function which exactly represents the number of prime numbers less than *x*.” Actually, he was mistaken, he had not found the correct function. Ramanujan’s formula was for an infinite series. He gave three versions of it in his second letter to Hardy on February 8, 1913. For values up to 1000, Ramanujan’s formula virtually gave exact agreement. It was known that there were $602,489$ primes below nine million. Ramanujan's formula gave the figure off by just 53. This was better than the performance of the prime number theorem. Ramanujan relied too much on the low values of *x* for which he had tried his formula. The error for higher values of *x* was much larger than he thought. Hardy still found Ramanujan's approach very illuminating. He wrote, "Ramanujan's theory was what the theory might be if the zeta function had no complex zeros."

- In 1923, Hardy and Littlewood conjectured (known as the second Hardy-Littlewood conjecture) that $\pi(x+y) \leq \pi(x) + \pi(y)$ for all integers *x* and *y* with $2 \leq y \leq x.$ This inequality is the same as $\pi(x+y) - \pi(y) \leq \pi(x),$ and hence no interval $y+1 \leq k \leq x+y$ of length *x* can contain as many primes as there are in the interval $0 < k \leq x.$ For example, for $x = 8, y = 2$, we have $\pi(10) - \pi(2) = 3, \pi(8) = 4$ and for $x = 90, y = 10$ we have $\pi(100) - \pi(10) = 21,\ \pi(90) = 24.$ First significant work on this conjecture is of Sanford Leonard Segal (1937–2010, USA) in 1962. In 1975, V.S. Udrescu (Romania) showed that for every $\epsilon > 0,\ \pi(x+y) \leq (1+\epsilon)(\pi(x) + \pi(y))$ holds whenever $x, y \geq 17$ and $x + y \geq 1 + e^{4(1+1/\epsilon)}$. Thus, the conjecture holds for x, y sufficiently large, but no effective bounds on the region of validity could be determined. In 1998, Daniel Gordon (USA) and Gene Rodemich (USA) showed that the conjecture is valid for $2 \leq \min(x,y) \leq 1731.$ In 2002, Pierre Dusart (France) proved that the conjecture holds for $2 \leq y \leq x \leq (7/5) y \ln y \ln\ln y.$ Computationally, this conjecture has been verified for $x + y \leq 100,000;$ however, general consensus is there may be some rare exceptions on which conjecture may not be true because the function $\pi(x)$ is irregular.

The special case $\pi(2x) \leq 2\pi(x)$ is known as Landau's inequality, which he studied in 1901; he also showed that $\lim_{x\to\infty}(\pi(2x) - 2\pi(x)) = -\infty.$ Landau's inequality was proved for all integers $x \geq 2$ in 1975 by John Barkley Rosser (1907–1989, USA) and Lowell Schoenfeld (1920–2002, USA). In 1971, Chr Karanikolov (Serbia) showed that if $a \geq e^{1/4}$ and $x \geq 364,$ then $\pi(ax) < a\pi(x).$ Some of these inequalities have been improved and several new added in 2000 by Laurentiu Panaitopol (1940–2008, Romania). A related exercise is to show $\pi(xy) \geq \pi(x)\pi(y)$ for all

$x, y \geq 8.$ Since $\pi(35) = 11, \pi(7)\pi(5) = 4 \times 3 = 12$ inequality does not hold for $x = 7, y = 5.$ Similarly, it does not hold for $x = y = 7.$ For sufficiently large *x* and $y,$ Sylvester bounds show that inequality certainly holds provided

$$xy \ln x \ln y \ > \ \frac{1.04423^2}{0.95695} xy \ln(xy),$$

which on letting $a = \ln x, b = \ln b$ is the same as $ab > 1.1395(a+b).$ This inequality holds for all $a, b \geq 3,$ and hence both *x* and *y* are greater than $21.$ In conclusion, the inequality $\pi(xy) \geq \pi(x)\pi(y)$ holds for all $x, y \geq 21$ and as long as Sylvester bounds hold. In 2018, Dao Thanh Oai (Vietnam) checked the validity of the inequality for all $8 \leq x, y \leq 10^9.$

- In 1901, Niels Fabian Helge von Koch (1870–1924, Sweden) showed that the Riemann hypothesis is equivalent to

$$|\mathrm{Li}(x) - \pi(x)| \ \leq c\sqrt{x} \ln x$$

  for some constant $c.$ In 1976, Schoenfeld assumed Riemann hypothesis to show that

$$|\pi(x) - li(x)| \ \leq \ \frac{\sqrt{x}\ \ln x}{8\pi}$$

  for all $x \geq 2657.$ Here $li(x) = \int_0^x dt/\ln t$ for $0 < x < 1,$ and principal value (PV) $\int_0^x dt/\ln t$ for $x > 1.$ Clearly, $li(x) = \mathrm{Li}(x) + li(2) \sim \mathrm{Li}(x) + 1.0451637801174.$
- In 1891, Lars Edvard Phragmén (1863–1937, Sweden) expressed the idea by saying "there is no limit beyond which the difference $\pi(x) - (\mathrm{Li}(x) - \ln 2)$ does not change sign." In 1914, Littlewood gave an existence proof to show that the function $\pi(x) - \mathrm{Li}(x)$ changes sign infinitely often as *x* increases to infinity. In 1933, Stanley Skewes (1899–1988, South Africa) provided an upper bound, known as *First Skewes Number* $Sk_1 = 10^{10^{10^{34}}}$ which is the smallest integer *x* for which $\pi(x) > \mathrm{Li}(x).$ In 1955, he used the upper bound as *Second Skewes Number* $Sk_2 = 10^{10^{10^{1000}}}.$ In 1966, Russell Sherman Lehman (born 1930, USA) improved the upper bound to $1.65 \times 10^{1165}.$ In 1987, Te Riele improved upper bound to $7 \times 10^{370}.$ In 2000, a better upper bound

1.39822×10^{316} was discovered by Carter Bays (USA) and Richard Howard Hudson (USA). They also showed that at least 10^{153} consecutive integers exist somewhere near this value where $\pi(x) > \mathrm{Li}(x)$. After Bays and Hudson's work some mild improvements on the upper bound have been obtained. In the following table, we note that $\mathrm{Li}(x) - \pi(x) > 0$ and increasing for all calculated values of x.

x	$\mathrm{Li}(x) - \pi(x)$	x	$\mathrm{Li}(x) - \pi(x)$
10	2	10^{13}	108971
10^2	5	10^{14}	314890
10^3	10	10^{15}	1052619
10^4	17	10^{16}	3214632
10^5	38	10^{17}	7956589
10^6	130	10^{18}	21949555
10^7	339	10^{19}	99877775
10^8	754	10^{20}	222744644
10^9	1701	10^{21}	597394254
10^{10}	3104	10^{22}	1932355208
10^{11}	11588	10^{23}	7250186216
10^{12}	38263	10^{24}	17146907278
		10^{25}	55160980939

- In 1923, Hardy and Littlewood also conjectured that $\pi_2(x)$ increases much like the function

$$L_2(x) \;=\; C_2 \int_2^x \frac{dt}{(\ln t)^2},$$

where $C_2 = 1.3203236316937\cdots$ is known as *twin-prime constant*, and $\pi_2(x)$ is the number of twin primes less than x. In the following table, we note that $\mathrm{L}_2(x) - \pi_2(x)$ changes sign even for low values of *x*

x	$\pi_2(x)$	$\mathrm{L}_2(x)$	$\mathrm{L}_2(x) - \pi_2(x)$
10^3	35	46	11
10^4	205	214	9
10^5	1224	1249	25
10^6	8169	8248	79
10^7	58980	58754	− 226
10^8	440312	440368	56
10^9	3424506	3425308	802
10^{10}	27412679	27411417	− 1262
10^{11}	224376048	224368865	− 7183

In 2011, Marek Wolf have shown that for $x < 2^{48} = 2.81 \cdots \times 10^{14}$, there are 477118 sign changes of this difference.

4.13 Amicable Numbers

If each of the numbers m, n is equal to the sum of the proper divisors of the other, it is called an "amicable number pair," and m, n are "amicable" or "friendly" numbers. Thus, (m, n) is an amicable number pair if $\sigma(n) = n + m = \sigma(m)$. The smallest amicable numbers are 220 and $284.$ In fact, the proper divisors of 284 are $1, 2, 4, 71, 142,$ and their sum is $220,$ and the proper divisors of 220 are $1, 2, 4, 5, 10, 11, 20, 22, 44, 55, 110,$ and their sum is $284.$ According to Iamblichus, when a person asked Pythagoras what a friend was, and he replied, "One who is another I, such as 220 and 284." The Pythagoreans regarded this intimate union between amicable numbers as the very essence of friendship and the innermost soul of harmony. There is sufficient evidence that the amicable numbers were known to the Hindus before the days of Pythagoras. Also, certain passages of the Bible seem to indicate that the Hebrews attached a good omen to such numbers. In Muslim mathematical writings, the amicable numbers occur repeatedly. They play a role in magic and astrology, in the casting of horoscopes, in sorcery, in talismans, and concocting love potions. There is an unauthenticated medieval story of a prince whose name was from the standpoint of gematria equivalent to $284.$ He sought a bride whose name would represent $220,$ believing that this would be Heaven's guarantee of a happy marriage.

Amicable numbers were one of the hobbies of Abu Zaid Abdel Rahman ibn Khaldun (1332–1406, Tunisia-Egypt). He wrote that persons who have concerned themselves with talismans affirm that the amicable numbers 220 and 284 have an influence to establish a union or close friendship between two individuals. About 850, ibn Qurra derived and proved a general formula which leads to certain types of amicable pairs: If the three numbers $p = 3 \times 2^{n-1} - 1,\ q = 3 \times 2^n - 1,$ and $r = 9 \times 2^{2n-1} - 1$ are all prime and $n \geq 2,$ then $2^n pq$ and $2^n r$ are amicable numbers. This formula was rediscovered by Descartes and Fermat and in Europe ascribed to them. For $n = 2, 4,$ and 7, this formula gives

n	p	q	r	$2^n pq$	$2^n r$
2	5	11	71	220	284
4	23	47	1151	$17,296 = 2^4 \times 23 \times 47$	$18,416 = 2^4 \times 1151$
7	191	383	$73,727$	$9,363,584 = 2^7 \times 191 \times 383$	$9,437,056 = 2^7 \times 73727$

The amicable pair $(17296, 18416)$ was discovered by al-Marrakushi ibn Al-Banna (1256–1321, Morocco) and rediscovered by Fermat in 1636. Similarly, the amicable pair $(9363584, 9437056)$ was discovered by Muhammad Baqir Yazdi (died in 1637, Iran) and rediscovered by Descartes in 1638. Other Arab mathematicians who contributed to amicable numbers are Abu al-Qasim Maslama ibn Ahmad al-Majriti (950–1007, Spain), Abu Mansur ibn Tahir al-Baghdadi (980–1037, Iraq), and Kamal al-Din Abu'l Hasan Muhammad al-Farisi (1260–1320, Iran). In 1985, Bartel Leendert van der Waerden (1903–1996, the Netherlands) provided a modern mathematical proof of Thabit's formula. It is clear that Thabit's result severely restricts the possible values of $n.$ For example, for $n = 3,$ we get $r = 287,$ which is not a prime number. Thus, it is not known if Thabit's formula generates infinitely many amicable pairs, but it is known that there are some amicable pairs it does not generate, such as the second smallest pair $(1184, 1210),$ which was discovered in 1866 by the 16-year old student B. Nicolo I. Paganini (born in 1850, Italy). This pair had eluded his more illustrious predecessors.

Before 1747, only above three pairs of amicable were known, but then within three years Euler found 58 new pairs, so the number reached to 61. Based on the patterns of amicable numbers, he developed a formula that would produce amicable pairs; however, it did not generate every amicable pair. Euler also extended Thabit's formula to: For integers $n > m > 0$ if the numbers

$p_1 = (2^{n-m}+1)\times 2^m - 1,\ q_1 = (2^{n-m}+1)\times 2^n - 1,\ r_1 = (2^{n-m}+1)^2\times 2^{m+n} - 1$ are prime numbers, then $2^n p_1 q_1$ and $2^n r_1$ are a pair of amicable numbers. Thabit's formula corresponds to the case $m = n - 1$. Euler's formula provides two additional amicable pairs for $(m, n) = (1, 8), (29, 40)$ with no others being known. For $m = 1, n = 8$ the pair $(2172649216, 2181168896)$ was computed by Legendre in 1830. Extensive generalizations of Thabit and Euler formulas have been given by Walter Borho (born 1945, Germany) in a series of papers since 1972.

In recent years, more than $1,227,319,870$ amicable pairs have been computed with the help of supercomputers. So far, even theoretically, it is not known if the number of amicable pairs is finite or infinite. Of course, much harder problem is to find a general formula to generate all amicable pairs. It is also not known if there exists an amicable number whose the pairs have opposite parity, also if there exist pairs of relatively prime amicable numbers. In 1955, Erdös showed that the density of amicable numbers, relative to the positive integers, was 0. In fact, there are 1947667 amicable pairs less than 10^{19}. We list here first 20 smallest and the largest known (due to Paul Jobling in 2005 with each member having 24073 decimal digits) amicable pairs

(220,284), (1184,1210), (2620,2924), (5020,5564), (6232,6368), (10744,10856), (12285,14595), (17296,18416), (63020,76084), (66928,66992), (67095,71145), (69615,87633), (79750,88730), (100485,124155), (122265,139815), (122368, 123152), (141664,153176), (142310,168730), (171856,176336), (176272,180848).

For the known largest amicable pair, we define

$$a = 2\times 5\times 11, \quad S = 37\times 173\times 409\times 461\times 2136109\times 2578171801921099\times 68340174428454377539$$

$$p = 925616938247297545037380170207625962997960453645121$$

$$q = 21095843021805411767901860198505910768098870743702508192267359$$
9999

$$q_1 = (p+q)p^{235} - 1, \quad q_2 = (p-S)p^{235} - 1,$$

then $p, q, q_1,$ and q_2 are all primes, and $(aSp^{235}q_1, aqp^{235}q_2)$ is the required amicable pair.

- In 1981, Carl Bernard Pomerance (born 1944, USA) showed that the sum of the reciprocals of all amicable numbers, P, is a constant. In 2010, Jonathan Bayless (USA) showed that $0.0119841556 < P < 6.56\times 10^8$.

In a recent publication of 2019, Hanh My Nguyen (USA) and Pomerance have shown that this upper bound can be reduced to just 215.

- There are amicable pairs in which the sum of the digits of both the members is equal. In fact, out of first 5000 amicable pairs, there are 427 such pairs. For example, for the amicable pair $(69615, 87633)$, we have $6+9+6+1+5=27=8+7+6+3+3.$ The same property holds for the pairs $(100485, 124155)$ and $(1358595, 1486845)$.
- In 1986, Te Riele found 37 pairs of amicable pairs having the same-pair sum. The first such pair is $(609928, 686072)$ and $(643336, 652664)$, which has the pair-sum $1296000.$ In 1993, David Moews (USA) and Paul Moews (USA) found the following four triples of amicable pairs with the same pair-sum

$$\begin{gathered}(29912035725, 34883817075), (31695652275, 33100200525),\\ (32129958525, 32665894275),\ \text{sum } 64795852800\\ (54666647145, 57100392855), (54853467435, 56913572565),\\ (55171066784, 56595973216),\ \text{sum } 111767040000\\ (52025880375, 65160720585), (53734975875, 63451625085),\\ (55477298835, 61709302125),\ \text{sum } 117186600960\\ (57826671370, 60486723830), (58557943665, 59755451535),\\ (58906037421, 59407357779),\ \text{sum } 118313395200\end{gathered}$$

In 1995, Te Riele found a quadruple. In November 1997, a quintuple and sextuple were detected. The sextuple is

(1953433861918, 2216492794082), (1968039941816, 2201886714184), (1981957651366, 2187969004634), (1993501042130, 2176425613870), (2046897812505, 2123028843495), (2068113162038, 2101813493962),

all having pair-sum $4169926656000.$ It is interesting, the sextuple is smaller than any known quadruple or quintuple and is perhaps smaller than any quintuple.

- Harshad amicable pair is an amicable pair in which both the numbers are Harshad numbers. In fact, out of first 5000 amicable pairs, there are 192 Harshad amicable pairs, e.g., see the articles of Mayadhar Swain (born 1956, India) entitled facinating amicable numbers of 2013 and Murty et. al. [381]. For example, in the amicable pair $(2620, 2924)$, the number 2620 is divisible by $2+6+2+0=10$ and 2924 is divisible by $2+9+2+4=17.$ Similarly,

$(10634085, 14084763), (23389695, 25132545),$ and $(34256222, 35997346)$ are Harshad amicable pairs.

- If we iterate the process of summing, the squares of the digits of a number and if the process terminates in 1, then the original number is called a *happy number*. For example, 7 is a happy number because $7 \to 7^2 = 49 \to 4^2 + 9^2 = 97 \to 9^2 + 7^2 = 130 \to 1^2 + 3^2 + 0^2 = 10 \to 1^2 + 0^2 = 1.$ However, 11 is not a happy number, because $11 \to 2.$ The first 30 happy numbers are 1, 7, 10, 13, 19, 23, 28, 31, 32, 44, 49, 68, 70, 79, 82, 86, 91, 94, 97, 100, 103, 109, 129, 130, 133, 139, 167, 176, 188, 190. An amicable pair is called happy if both members of the pair are happy numbers. There are 111 happy amicable pairs in first 5000 amicable pairs (see http://www.shyamsundergupta.com/aphappy.htm). For example, $(10572550, 10854650)$ is a happy amicable pair because $10572550 \to 129 \to 86 \to 100 \to 1$ and $10854650 \to 167 \to 86 \to 100 \to 1.$ Other examples are $(32685250, 34538270), (35361326, 40117714), (35390008,$ $39259592), (186878110, 196323170).$
- In 1968, Gardner noted that most even amicable pairs known at his time have sums divisible by $9.$ For example, for the pair $(2172649216, 2181168896)$ the sum 4353818112 and the sum of its digits is 36 which is divisible by 9. The smallest known even amicable pair $(666030256, 696630544)$ whose sum 1362660800 (with the sum of its digits 32) is nondivisible by 9 was discovered by Paul Poulet (1887–1946, Belgium), published in 1948 posthumously.

The earliest known odd amicable numbers all were divisible by 3, e.g., $(12285, 14595)$ and $(69615, 87633).$ This led Paul Bratley (Canada) and John McKay (England-Canada) in 1968 to conjecture that there are no amicable pairs coprime to 6 (i.e., divisible by either 2 or 3). However, Stefan Battiato (Germany) and Borho in 1988 found 15 counterexamples of amicable pairs not divisible by 6, each member of their smallest pair has 36 digits. The smallest known example of this kind is the odd amicable pair $(42262694537514864075544955198125, 42405817271188606697466971841875),$ each member of which has 32 digits with sums 148 and 158 (non-divisible by 3), respectively.

- From the tables of amicable pairs, it can be seen that the smallest pairs in which both the members have the same last digits are (79750, 88730), (1558818261, 1596205611), (106930732, 1142071892), (664747083, 673747893), (196724, 202444), (12285, 14595), (17296, 18416), (290142314847, 292821792417), (469028, 486178), and (68606181189, 70516785339).

- Let $n_1 < n_2 < \cdots < n_k$ be *k* positive integers. An *amicable k tuple* denoted as $(n_1, n_2, \cdots, n_k)$ and defined as $\sigma(n_1) = \sigma(n_2) = \cdots = \sigma(n_k) = n_1 + n_2 + \cdots + n_k$. Clearly, an amicable 2 tuple is an amicable pair. For example, $(1980, 2016, 2556)$ is an amicable 3 tuple, because $\sigma(1980) = \sigma(2016) = \sigma(2556) = 6552 = 1980 + 2016 + 2556$; $(3270960, 3361680, 3461040, 3834000)$ is an amicable 4 tuple; and $(227491164588441600, 228507506351308800, 229862628701798400, 230878970464665600, 243752632794316800)$ is an amicable 5 tuple found by Yasutoshi Kohmoto (Japan) in 2008.
- A cyclic sequence of three or more numbers such that the sum of the proper divisors of each is equal to the next in the sequence is known as a *sociable chain* of numbers. They are generalizations of the concepts of amicable numbers and perfect numbers. In 1970, Henri Cohen (born 1947, France) obtained nine social chains of order 4, the smallest one of these is $1264460 \to 1547860 \to 1727636 \to 1305184 \to 1264460$. In fact, the sum of the proper divisors of 1264460 is
$$1 + 2 + 4 + 5 + 10 + 17 + 20 + 34 + 68 + 85 + 170 + 340 + 3719 + 7438 + 14876 + 18595 + 37190 + 63223 + 74380 + 126446 + 252892 + 316115 + 632230 = 1547860,$$
the sum of the proper divisors of 1547860 is
$$1 + 2 + 4 + 5 + 10 + 20 + 193 + 386 + 401 + 772 + 802 + 965 + 1604 + 1930 + 2005 + 3860 + 4010 + 8020 + 77393 + 154786 + 309572 + 386965 + 773930 = 1727636,$$
the sum of the proper divisors of 1727636 is
$1 + 2 + 4 + 521 + 829 + 1042 + 1658 + 2084 + 3316 + 431909 + 863818 = 1305184,$
and the sum of the proper divisors of 1305184 is
$1 + 2 + 4 + 8 + 16 + 32 + 40787 + 81574 + 163148 + 326296 + 652592 = 1264460.$
Till 1918, only following two sociable chains involving numbers below 10^6 were known of order 5 and 28 due to Poulet:
$12496 \to 14288 \to 15472 \to 14536 \to 14264 \to 12496,$ and
$$14316 \to 19116 \to 31704 \to 47616 \to 83328 \to 177792 \to 295488 \to 629072 \to 589786 \to 294896 \to 358336 \to 418904 \to 366556 \to 274924 \to 275444 \to 243760 \to 376736 \to 381028 \to 285778 \to 152990 \to 122410 \to 97946 \to 48976 \to 45946 \to 22976 \to 22744 \to 19916 \to 17716 \to 14316.$$
A sociable chain of order 3 is called a *crowd*; no crowds have yet been found.

4.14 Fibonacci Numbers/Sequence

The Fibonacci numbers are defined as

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, \cdots,$$

where each integer after the first two is the sum of the two integers immediately preceding it. It is to be noted that these numbers first appeared under the name matrameru (mountain of cadence) in the treatise *Chandahshastra* (The Art of Prosody) written by Pingala. Then, Virahanka (sixth century, India) showed how the Fibonacci sequence arose in the analysis of metres with long and short syllables, and Jain philosopher Hemachandra (around 1088–1172, India) composed a well-known text on these. Fibonacci numbers were also known to Egyptians and Greeks. The name Fibonacci numbers to this sequence was given by Lucas in 1877. Fibonacci's name is attached with this sequence due to his popular problem in *Liber Abaci*: A certain man had one pair of rabbits together in a certain enclosed place and one wishes to know how many are created from the pair in one year when it is the nature of them in a single month to bear another pair, and in the second month those born to bear also. If we let $\mathcal{F}_n$ denote the *n*th Fibonacci number, then it is clear that $\mathcal{F}_n$ satisfies the initial value problem (in posthumously published work of 1634 by Girard)

$$\begin{aligned} \mathcal{F}_{n+2} &= \mathcal{F}_{n+1} + \mathcal{F}_n, \quad n = 1, 2, 3, \cdots \\ \mathcal{F}_1 &= 1, \quad \mathcal{F}_2 = 1. \end{aligned}$$

For this, the characreristic equation of the difference equation is $\lambda^2 - \lambda - 1 = 0,$ which gives $\lambda = (1 \pm \sqrt{5})/2.$ Hence, the general solution of the difference equation is

$$C_1 \left(\frac{1+\sqrt{5}}{2} \right)^n + C_2 \left(\frac{1-\sqrt{5}}{2} \right)^n.$$

By using the initial conditions, we find $C_1 = -C_2 = 1/\sqrt{5},$ so

$$\mathcal{F}_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n, \quad n \geq 1 \tag{4.8}$$

Although $\sqrt{5}$ is predominant in this formula, all these numbers must be integers. Formula (4.8) was derived by Jacques Philippe Marie Binet (1786–1856, France) in 1843, although the result was known to Daniel

Bernoulli, Euler, and De Moivre more than a century earlier. Fibonacci numbers can be extended to zero and negative integers as follows: from $\mathcal{F}_{n+2} = \mathcal{F}_{n+1} + \mathcal{F}_n$ we have $\mathcal{F}_n = \mathcal{F}_{n+2} - \mathcal{F}_{n+1}$ and hence $\mathcal{F}_0 = \mathcal{F}_2 - \mathcal{F}_1 = 1 - 1 = 0,\ \mathcal{F}_{-1} = \mathcal{F}_1 - \mathcal{F}_0 = 1,\ \mathcal{F}_{-2} = \mathcal{F}_0 - \mathcal{F}_{-1} = -1,\ \mathcal{F}_{-3} = \mathcal{F}_{-1} - \mathcal{F}_{-2} = 1 - (-1) = 2,\ \cdots, \mathcal{F}_{-n} = (-1)^{n+1}\mathcal{F}_n.$ Indeed, by induction, we have

$$\begin{aligned}\mathcal{F}_{-(n+1)} = \mathcal{F}_{-(n-1)} - \mathcal{F}_{-n} = (-1)^n \mathcal{F}_{n-1} - (-1)^{n+1}\mathcal{F}_n &= (-1)^{n+2}(\mathcal{F}_{k-1} + \mathcal{F}_n)\\ &= (-1)^{n+2}\mathcal{F}_{n+1}.\end{aligned}$$

Kepler was the first who noticed that Fibonacci numbers occur in nature in many surprising ways. He claimed that if we count the number of petals of different flowers, we often come across Fibonacci numbers to such an extent which can no longer be regarded as pure coincidence. Examples include flowers: Iris and lili have 3 petals; columbine, primrose, buttercup, dog rose or Alpine rose, larkspur, and aquilegia have 5 petals each; delphinium has 8 petals, corn marigold has 13 petal, aster, black-eyed Susan and coffee weed have 21 petals each; daisies have 13, 21, or 34 petals; plantain and chrysanthemum have 34 petals each; and New York aster have 55 or 89 petals. The number of spiral arms: on sunflower are three of 21, 34, and 55 arms; on pine cone are two of 5 and 8 arms; on pineapple are three of 5, 8, and 13 arms. Fibonacci numbers are also found in the arrangement of leaves, twigs, and stems. These numbers can also be found in a varied number of fields such as music, human body, in coding (computer algorithms, interconnecting parallel, and distributed systems), high energy physical science, quantum mechanics, logarithmic spiral, cryptography, etc. Male honeybees hatch from eggs that have not been fertilized, so a male bee has only one parent, a female. On the other hand, female honeybees hatch from fertilized eggs, so a female has two parents, one male and one female. The number of ancestors in consecutive generations of bees follows the Fibonacci sequence. Fibonacci numbers are of compelling interest to mathematicians to have a Journal devoted to the study of their properties. For leisure reading, see the book of Vorobyov [525].

- The Euclidean algorithm (Theorem 3.3) gives an impression that to find the greatest common divisor of two positive integers *a* and *b* requires finitely many divisions $N.$ In addition, Lamé showed that $N \leq 1 + \log_\varphi b,$ where $\varphi = (\sqrt{5}+1)/2,$ (Golden Section, see Sect. 4.16). However, we shall show that if the integers *a* and *b* are suitably chosen, then *N* can be arbitrarily large, i.e., Lamé's upper bound loses its practical importance. For this, we let $a = \mathcal{F}_{N+2}$ and $b = \mathcal{F}_{N+1}$. Then the Euclidean algorithm for obtaining $\gcd(\mathcal{F}_{N+2}, \mathcal{F}_{N+1})$ leads to the following system of equations:

$$\begin{aligned}
\mathcal{F}_{N+2} &= 1 \cdot \mathcal{F}_{N+1} + \mathcal{F}_N \\
\mathcal{F}_{N+1} &= 1 \cdot \mathcal{F}_N + \mathcal{F}_{N-1} \\
&\vdots \\
\mathcal{F}_4 &= 1 \cdot \mathcal{F}_3 + \mathcal{F}_2 \\
\mathcal{F}_3 &= 2 \cdot \mathcal{F}_2 + 0.
\end{aligned}$$

Thus, the number of necessary divisions is $N.$ From Theorem 3.4, it also follows that $\gcd(\mathcal{F}_{N+2}, \mathcal{F}_{N+1}) = \mathcal{F}_2 = 1,$ and hence the successive terms of the Fibonacci sequence are relatively prime.

- The following identities follow by induction, or the explicit relation (4.8)

$$\begin{aligned}
\mathcal{F}_{m+n} &= \mathcal{F}_{m-1}\mathcal{F}_n + \mathcal{F}_m\mathcal{F}_{n+1} \quad \text{for all} \quad m \geq 2 \\
\mathcal{F}_1 + \mathcal{F}_2 + \cdots + \mathcal{F}_n &= \mathcal{F}_{n+2} - 1 \quad \text{(Lucas)} \\
\mathcal{F}_2 + \mathcal{F}_4 + \cdots + \mathcal{F}_{2n} &= \mathcal{F}_{2n+1} - 1 \\
\mathcal{F}_1 + \mathcal{F}_3 + \cdots + \mathcal{F}_{2n-1} &= \mathcal{F}_{2n} \\
\mathcal{F}_1 + 2\mathcal{F}_2 + \cdots + n\mathcal{F}_n &= (n+1)\mathcal{F}_{n+2} - \mathcal{F}_{n+4} + 2 \\
\mathcal{F}_{n-1}\mathcal{F}_{n+1} - \mathcal{F}_n^2 &= (-1)^n \\
\mathcal{F}_{n+2}^2 - \mathcal{F}_n^2 &= \mathcal{F}_{2n+2} \\
\mathcal{F}_n^2 + \mathcal{F}_{n-1}^2 &= \mathcal{F}_{2n-1} \\
\mathcal{F}_n^2 - \mathcal{F}_{n-m}\mathcal{F}_{n+m} &= (-1)^{n-m}\mathcal{F}_m^2 \\
\mathcal{F}_1^2 + \mathcal{F}_2^2 + \cdots + \mathcal{F}_n^2 &= \mathcal{F}_n\mathcal{F}_{n+1}.
\end{aligned}$$

We also have sum of odd and even sequences of products of consecutive Fibonacci numbers

$$\sum_{j=1}^{2n-1} \mathcal{F}_j\mathcal{F}_{j+1} = \mathcal{F}_{2n}^2, \quad \sum_{j=1}^{2n} \mathcal{F}_j\mathcal{F}_{j+1} = \mathcal{F}_{2n+1}^2 - 1.$$

- The generating functions for all, even, and odd Fibonacci numbers are

$$\frac{x}{1-x-x^2} = \sum_{n=0}^{\infty} \mathcal{F}_n x^n, \quad \frac{x}{1-3x+x^2} = \sum_{n=0}^{\infty} \mathcal{F}_{2n} x^n,$$
$$\frac{1-x}{1-x+x^2} = \sum_{n=0}^{\infty} \mathcal{F}_{2n+1} x^n.$$

- For all $n \geq 1,$ by induction, the following relation between Fibonacci numbers and the binomial coefficients (2.3) follows:
$$\mathcal{F}_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k}$$
$$= \binom{n-1}{0} + \binom{n-2}{1} + \cdots + \binom{n-j}{j-1} + \binom{n-j-1}{j}, j = \left\lfloor \frac{n-1}{2} \right\rfloor.$$
This also shows that Fibonacci sequence is related to Pascal's triangle in that the sum of the shallow diagonals of Pascal's triangle are equal to the corresponding Fibonacci sequence term.

- The only Fibonacci number which are prime (Fibonacci primes) up to $\mathcal{F}_{1000}$ are $\mathcal{F}_k$ where $k = 3, 4, 5, 7, 11,$ $13, 17, 23, 29, 43, 47, 83, 131, 137, 359, 431, 433, 449, 509, 569, 571.$ As of January 2022, the largest known Fibonacci prime is $\mathcal{F}_{148091},$ which has 30949 digits. In 2018, Henri Lifchitz (born 1948, France) speculated that $\mathcal{F}_{3340367}$ is Fibonacci prime, with $698, 096$ digits, which has not been confirmed. It is conjectured that there are infinitely many Fibonacci primes.
- Let *a* and *b* be any two numbers. Consider first ten terms of the Fibonacci-like sequence, i.e., $a+b, a+2b, 2a+3b, 3a+5b, 5a+8b, 8a+13b, 13a+21b, 21a+34b, 34a+55b, 55a+89b,$ the sum of these ten terms gives $143a + 231b = 11(13a + 21b),$ i.e., the sum of first ten terms is the same as 11 times the seventh term.
- In 1972, Edouard Zeckendorf (1901–1983, Belgium) published a result which he rediscovered in 1939, whose origin appears to be in medieval India. In the literature, this result is now known as Zeckendorf's Theorem: Every positive integer *N* can be written uniquely as a sum of distinct, pairwise nonconsecutive Fibonacci numbers. The proof is by induction. Thus, Fibonacci numbers are building blocks for the natural numbers through addition. For example, $108 = 89 + 21 + 8 = \mathcal{F}_{11} + \mathcal{F}_8 + \mathcal{F}_6.$ Notice that $\mathcal{F}_{11} < 108 < \mathcal{F}_{12}.$ In

fact, from the above identities, it can be shown that if $\mathcal{F}_k < N < \mathcal{F}_{k+1}$, then in the representation of *N* as the sum of Fibonacci numbers, the number $\mathcal{F}_k$ has to be one of them.

- The following statements follow by induction: $\mathcal{F}_{3n}$ is an even number, $\mathcal{F}_{4n}$ is a multiple of $3,$ $\mathcal{F}_{5n}$ is a multiple of $5,$ $\mathcal{F}_{6n}$ is a multiple of $8,$ $\mathcal{F}_{7n}$ is a multiple of $13,$ in general for all $k, n \in \mathcal{N},$ $\mathcal{F}_{kn}$ is divisible by $\mathcal{F}_n.$ This follows immediately by the relation $\mathcal{F}_{m+n} = \mathcal{F}_{m-1}\mathcal{F}_n + \mathcal{F}_m\mathcal{F}_{n+1}$ and mathematical induction by letting $m = kn.$ Thus, for any prime $p,$ there are infinitely many Fibonacci numbers that are divisible by *p* and that lie at equal distances from one another in the Fibonacci sequence.
- We shall show that $\mathcal{F}_m|\mathcal{F}_n$ iff $m|n,\ n \geq m \geq 3.$ For this we shall use the relation $\mathcal{F}_{m+n} = \mathcal{F}_{m-1}\mathcal{F}_n + \mathcal{F}_m\mathcal{F}_{n+1}$ and the mathematical induction. If $m|n,$ then there exists an integer *j* such that $n = jm.$ Clearly, $\mathcal{F}_m|\mathcal{F}_{jm}$ for $j = 1.$ We assume that $\mathcal{F}_m|\mathcal{F}_{km}$ then the relation with $n = km$ gives $\mathcal{F}_{m+km} = \mathcal{F}_{(k+1)m} = \mathcal{F}_{m-1}\mathcal{F}_{km} + \mathcal{F}_m\mathcal{F}_{km+1},$ and hence $\mathcal{F}_m|\mathcal{F}_{(k+1)m}.$ Conversely, let $\mathcal{F}_m|\mathcal{F}_n,$ but $m \not| n.$ Then, we may write $n = jm + r,$ where $0 < r < m.$ Thus from the relation, we have $\mathcal{F}_n = \mathcal{F}_{jm+r} = \mathcal{F}_{jm-1}\mathcal{F}_r + \mathcal{F}_{jm}\mathcal{F}_{r+1}.$ Now since $\mathcal{F}_m|\mathcal{F}_{jm+r}$ and $\mathcal{F}_m|\mathcal{F}_{jm}$ it follows that $\mathcal{F}_m|\mathcal{F}_{jm-1}\mathcal{F}_r.$ Next since Fibonacci numbers $\mathcal{F}_{jm-1}$ and $\mathcal{F}_{jm}$ are relatively prime and $\mathcal{F}_m|\mathcal{F}_{jm},$ it follows that $\mathcal{F}_m|\mathcal{F}_r,$ which means $\mathcal{F}_m \leq \mathcal{F}_r.$ But this contradicts the fact that Fibonacci sequence is increasing and $m \geq 3,\ 0 < r < m.$
- In 1876, Lucas reported that the greatest common divisor of two Fibonacci numbers is also a Fibonacci number, i.e., $\gcd(\mathcal{F}_n, \mathcal{F}_m) = \mathcal{F}_d,$ where $d = \gcd(n, m).$ To prove this, we shall use the relation $\mathcal{F}_{m+n} = \mathcal{F}_{m-1}\mathcal{F}_n + \mathcal{F}_m\mathcal{F}_{n+1}$ and all steps of Theorem 3.3 with $a = n, b = m$ and assume that $n > m.$ We have

$$\begin{aligned}
\gcd(\mathcal{F}_n, \mathcal{F}_m) &= \gcd(\mathcal{F}_{q_1m+r_1}, \mathcal{F}_m) \\
&= \gcd(\mathcal{F}_{q_1m-1}\mathcal{F}_{r_1} + \mathcal{F}_{q_1m}\mathcal{F}_{r_1+1}, \mathcal{F}_m) \\
&= \gcd(\mathcal{F}_{q_1m-1}\mathcal{F}_{r_1}, \mathcal{F}_m).
\end{aligned}$$

Here we have used the fact that $\mathcal{F}_m|\mathcal{F}_{q_1m}\mathcal{F}_{r_1+1}$, and hence any common divisor of $\mathcal{F}_m$ and $\mathcal{F}_{q_1m-1}\mathcal{F}_{r_1}+\mathcal{F}_{q_1m}\mathcal{F}_{r_1+1}$ is a common divisor of $\mathcal{F}_m$ and $\mathcal{F}_{q_1m-1}\mathcal{F}_{r_1}$. Now since $\mathcal{F}_m|\mathcal{F}_{q_1m}$ implies that $\mathcal{F}_m$ and $\mathcal{F}_{q_1m-1}$ are relatively prime, it follows that

$$\gcd(\mathcal{F}_n,\mathcal{F}_m) \;=\; \gcd(\mathcal{F}_{r_1},\mathcal{F}_m).$$

Repeating this argument leads to

$$\gcd(\mathcal{F}_n,\mathcal{F}_m) \;=\; \gcd(\mathcal{F}_{r_1},\mathcal{F}_m) \;=\; \gcd(\mathcal{F}_{r_2},\mathcal{F}_{r_1}) \;=\; \cdots \;=\; \gcd(\mathcal{F}_{r_N},\mathcal{F}_{r_{N-1}}).$$

Finally, since $r_N|r_{N-1}$, it follows that $\gcd(\mathcal{F}_n,\mathcal{F}_m)=\mathcal{F}_{r_N}=\mathcal{F}_d$. As an example, we have $\gcd(\mathcal{F}_7,\mathcal{F}_{14})=\gcd(13,377)=13=\mathcal{F}_7$, and $\gcd(7,14)=7$.

- As a corollary of a result of Yuri Vladimirovich Matiyasevich (born 1947, Russia), it follows that if *x* and *y* are positive integers and $z=y(2-(x^2+xy-y^2)^2)$ then $z>0$ iff *z* is a Fibonacci number. The one side proof of this is easy. For this, we let $x=\mathcal{F}_n$ and $y=\mathcal{F}_{n+1}$, and recall the above relations, to get

$$\begin{aligned} z &= \mathcal{F}_{n+1}(2-(\mathcal{F}_n^2-\mathcal{F}_{n+1}[\mathcal{F}_{n+1}-\mathcal{F}_n])^2) \\ &= \mathcal{F}_{n+1}(2-(\mathcal{F}_n^2-\mathcal{F}_{n+1}\mathcal{F}_{n-1})^2) \\ &= \mathcal{F}_{n+1}(2-((-1)^{n+1})^2) \;=\; \mathcal{F}_{n+1}. \end{aligned}$$

4.15 Lucas Numbers

Lucas numbers denoted as $L_n,\ n\geq 1$ form a Fibonacci-like sequence

$$1,3,4,7,11,18,29,47,76,123,199,322,521,843,\cdots.$$

In fact, L_n satisfies the initial value problem

$$\begin{aligned} L_{n+2} &= L_{n+1}+L_n, \quad n=1,2,3,\cdots \\ L_1 = 1&, \quad L_2=3. \end{aligned}$$

The closed form solution of this problem is

$$L_n \;=\; \left(\frac{1+\sqrt{5}}{2}\right)^n+\left(\frac{1-\sqrt{5}}{2}\right)^n. \tag{4.9}$$

The relation $L_{n-2} = L_n - L_{n-1}$ can be used to obtian $L_0 = 2,$ so that L_0, L_1, L_2, L_3 form the basis of tetraktys, and for all negative indices, it follows that

$$L_{-n} = (-1)^n L_n.$$

- The following identities follow by induction

$$L_n^2 - L_{n-1}L_{n+1} = 5(-1)^n, \quad \sum_{k=0}^{n} L_k = L_{n+2} - 1, \quad \sum_{k=0}^{n} L_k^2 = L_n L_{n+1} + 2,$$

$$\mathcal{F}_{n-m} + \mathcal{F}_{n+m} = \begin{cases} \mathcal{F}_n L_m & (m \text{ even}) \\ \mathcal{F}_m L_n & (m \text{ odd}), \end{cases} \quad L_{n-m} + L_{n+m} = \begin{cases} L_n L_m & (m \text{ even}) \\ 5\mathcal{F}_n \mathcal{F}_m & (m \text{ odd}), \end{cases}$$

$$\mathcal{F}_{n+m} - \mathcal{F}_{n-m} = \begin{cases} \mathcal{F}_n L_m & (m \text{ odd}) \\ \mathcal{F}_m L_n & (m \text{ even}), \end{cases} \quad L_{n+m} - L_{n-m} = \begin{cases} L_n L_m & (m \text{ odd}) \\ 5\mathcal{F}_n \mathcal{F}_m & (m \text{ even}). \end{cases}$$

  Thus, in particular, we have relations between Fibonacci and Lucas numbers $L_n = \mathcal{F}_{n-1} + \mathcal{F}_{n+1}$ and $5\mathcal{F}_n = L_{n-1} + L_{n+1}.$

- The generating function for all Lucas numbers is

$$\frac{2-x}{1-x-x^2} = \sum_{n=0}^{\infty} L_n x^n.$$

- The only Lucas number which are prime (Lucas primes) up to L_{1000} are L_k where $k = 2, 4, 5, 7, 8, 11, 13, 16,$ $17, 19, 31, 37, 41, 47, 53, 61, 71, 79, 113, 313, 353, 503, 613, 617, 863.$ As of September 2015, the largest known Lucas prime is $L_{148091},$ which has 30950 digits. In 2017, Henri Lifchitz and Renaud Lifchitz (born 1982, France) speculated that $L_{2316773}$ is Lucas prime, with 484177 digits, which has not been confirmed. It is conjectured that there are infinitely many Lucas primes.

4.16 Golden Section/Ratio

In Euclid's *Elements* (II and VI), we find a definition of a particular type of partition of a line segment in two uneven parts. According to Euclid, a line segment *AB* can be divided by an interior point $C.$ If this is the case, then the line segment *AB* is equicontinuously divided if the quotient AC/CB is equal to the quotient AB/AC, i.e.,

$$\varphi = \frac{AC}{CB} = \frac{AC+CB}{AC} \implies \varphi = 1 + \frac{1}{\varphi} \implies \varphi^2 - \varphi - 1 = 0,$$

which gives $\varphi = (1+\sqrt{5})/2$, and $\varphi = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}$. The symbol φ named after the Greek sculptor Phidias (around 480–430 BC), who incorporated the ratio into many of his sculptures. The earliest known treatise dealing with this number is *Divina Proportione* (divine proportion) by Pacioli of 1509. In his work, he concluded that the number was a message from God and a source of secret knowledge about the inner beauty of things. In 1815, Martin Ohm (1792–1872, Germany) named it as *golden ratio* or *golden section*. A classical ruler-and-compass construction for the golden section of a segment *AB* is as follows (see Fig. 4.1): Draw a perpendicular *BD* with $BD = AB/2$ and draw the hypotenuse AD. Draw an arc with center at *D* and radius BD. This arc intersects the hypotenuse at the point E. Draw an arc with center *A* and radius AE. This arc intersects the line *AB* at the required point C. In fact, from the Pythagorean theorem, $AD = (\sqrt{5}/2)AB = AE + ED = AC + AB/2$, and hence
$AB/AC = 2/(\sqrt{5}-1) = (\sqrt{5}+1)/2 = \varphi \simeq 1.61803398874989.$

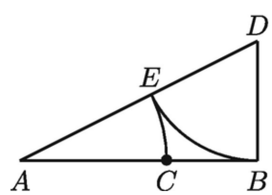

Fig. 4.1 Golden section/ratio

It was Kepler who discovered that the golden ratio (1.618034) is connected to Fibonacci numbers (the first known calculation of the inverse golden ratio, i.e., $(\sqrt{5}-1)/2$ as a decimal of about 0.6180340 was written in 1597 by Michael Maestlin, 1550–1631, Germany, in a letter to Kepler). In the 1750s, Robert Simson (1687–1768, Scotland) noted that the ratio of each term in the Fibonacci sequence to the previous term approaches, with greater accuracy the higher terms a ratio of approximately $1 : 1.6180339887$. If fact, from (4.8), we have

$$\frac{\mathcal{F}_{n+1}}{\mathcal{F}_n} = \frac{\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^n}{1 - \left(\frac{1-\sqrt{5}}{1+\sqrt{5}}\right)^n} \to \left(\frac{1+\sqrt{5}}{2}\right) = \varphi$$

as $n \to \infty$. Similarly, from (4.9) it follows that $L_{n+1}/L_n \to \varphi$ as $n \to \infty$.

- For the equation $f(u) = u^2 - u - 1 = 0,\ \varphi$ is a root. For this equation, Newton's method is

$$u_{n+1} = u_n - \frac{u_n^2 - u_n - 1}{2u_n - 1} = \frac{u_n^2 + 1}{2u_n - 1}.$$

Thus, with $u_1 = 1$ we have

$$u_1 = 1 = \frac{\mathcal{F}_2}{\mathcal{F}_1}, \quad u_2 = \frac{1+1}{2-1} = \frac{2}{1} = \frac{\mathcal{F}_3}{\mathcal{F}_2}, \quad u_3 = \frac{4+1}{4-1} = \frac{5}{3} = \frac{\mathcal{F}_5}{\mathcal{F}_4},$$

$$u_4 = \frac{25/9 + 1}{10/3 - 1} = \frac{34}{21} = \frac{\mathcal{F}_9}{\mathcal{F}_8}, \quad u_5 = \frac{(34/21)^2 + 1}{2(34/21) - 1} = \frac{1597}{987} = \frac{\mathcal{F}_{17}}{\mathcal{F}_{16}}.$$

We claim that $u_n = \mathcal{F}_{2^{n-1}+1}/\mathcal{F}_{2^{n-1}},\ n \geq 1$. For this, we need the relations $\mathcal{F}_{2n+1} = \mathcal{F}_{n+1}^2 + \mathcal{F}_n^2,\ \mathcal{F}_{2n} = (2\mathcal{F}_{n-1} + \mathcal{F}_n)\mathcal{F}_n$. Now the inductive step is

$$u_{n+1} = \frac{u_n^2 + 1}{2u_n - 1} = \frac{\mathcal{F}_{2^{n-1}+1}^2/\mathcal{F}_{2^{n-1}}^2 + 1}{2\mathcal{F}_{2^{n-1}+1}/\mathcal{F}_{2^{n-1}} - 1} = \frac{\mathcal{F}_{2^{n-1}+1}^2 + \mathcal{F}_{2^{n-1}}^2}{(2\mathcal{F}_{2^{n-1}+1} - \mathcal{F}_{2^{n-1}})\mathcal{F}_{2^{n-1}}}$$

$$= \frac{\mathcal{F}_{2^{n-1}+1}^2 + \mathcal{F}_{2^{n-1}}^2}{(2\mathcal{F}_{2^{n-1}-1} + \mathcal{F}_{2^{n-1}})\mathcal{F}_{2^{n-1}}} = \frac{\mathcal{F}_{2\cdot 2^{n-1}+1}}{\mathcal{F}_{2\cdot 2^{n-1}}}.$$

- By mathematical induction, we shall show that for all $n \geq 1,\ \varphi^{n-2} \leq \mathcal{F}_n \leq \varphi^{n-1}$. Since $\varphi^2 - \varphi - 1 = 0$, we have $\varphi^2 = \varphi + 1$. Let $p(n) : \varphi^{n-2} \leq \mathcal{F}_n$. Now $p(1)$ is $\varphi^{-1} \leq \mathcal{F}_1 = 1$, which is true, and $p(2)$ is $\varphi^0 \leq \mathcal{F}_2 = 1$, which is also true. Next we assume that $p(k)$ is true for all $k,\ 1 \leq k \leq n$. We will show that $p(n+1)$ is true, i.e., $\varphi^{n-1} \leq \mathcal{F}_{n+1}$. By induction hypothesis, both $p(n)$ and $p(n-1)$ are true. Hence, $\varphi^{n-2} \leq \mathcal{F}_n$ and $\varphi^{n-3} \leq \mathcal{F}_{n-1}$. Thus, it follows that

$$\mathcal{F}_{n+1} = \mathcal{F}_n + \mathcal{F}_{n-1} \geq \varphi^{n-2} + \varphi^{n-3} = \varphi^{n-3}(\varphi + 1) = \varphi^{n-3}(\varphi^2) = \varphi^{n-1}.$$

Therefore, $p(n+1)$ is true. The demonstration of the other part is similar. In addition, from (4.8), it follows that $\mathcal{F}_n$ is nearest to $\varphi^n/\sqrt{5}$. For example, the tenth Fibonacci number is the integer nearest to $55.003636,$ in other words, 55. Similarly, from (4.9), we find that Lucas number L_n is nearest to $\varphi^n.$ By induction, it also follows that $\mathcal{F}_n = [\varphi^n - (1-\varphi)^n]/\sqrt{5}.$

A golden rectangle is a rectangle whose sides are in the ratio $(\sqrt{5}+1)/2.$ Ancient Greeks (and psychologists) considered φ to be the most aesthetically pleasing to the eye ratio, practically worshiped it, and used it for the measurement of the facade of the Parthenon (in Athens) and other Greek temples. It is believed to govern the dimensions of everything from the Great Pyramid at Gizeh. φ keeps on popping up throughout the nature (shells, spiral galaxies, hurricanes, faces, fingers, animal bodies, animal fight patterns, and DNA molecules), art, architecture (e.g., united nations building), biology, books, eggs, music, mythology, painting, philosophy, poetry, religion, science, and mathematics (geometry, and for the secant method to solve nonlinear equations the rate of convergence is the golden ratio). For further importance of the golden ratio, see Livio [349], and Sen and Agarwal [460, 464].

- The equation

$$2\sin\left(\frac{108^\circ}{2}\right) = \varphi$$

results in the golden ratio. This relation can be restated as the chord of 108 degrees is $\varphi.$ We also note that $0.5/\sin(0.1\pi) = \varphi,\ \pi = 5\cos^{-1}(\varphi/2),\ \pi \simeq 4/\sqrt{\varphi}.$

- Using *decimal parity*, we can break numbers down into single digits. For example, the decimal parity equivalent of the number 3456 is $3+4+5+6=18$ and $1+8=9.$ So the decimal parity equivalent of 3456 is $9.$ In many ancient cultures, India and Egypt decimal parity was used as a way to understand the truth of numbers. If we apply decimal parity to the Fibonacci sequence, we find that there is a repeating series of 24 digits: $(0), 1, 1, 2, 3, 5, 8, 4, 3, 7, 1, 8, 9, 8, 8, 7, 6, 4, 1, 5, 6, 2, 8, 1.$ The addition of these 24 numbers, i.e.,

$0+1+1+2+3+5+8+4+3+7+1+8+9+8+8+7+6+4+1+5+6+2+8+1$ is the number 108. For this number besides the properties, we have discussed in Sect. 2.9, it is astonishing that the 108 constant growth rate the nautilus uses to build its spiral shell involves the same pattern which repeats every 24 numbers of the Fibonacci sequence.

4.17 Quadratic Congruence and Reciprocity

The congruence of the type

$$Ay^2 + By + C \equiv 0 (\bmod\ p), \tag{4.10}$$

where *p* is an odd prime and $\gcd(A, p) = 1$ is called a *quadratic congruence*. Since $4A^2y^2 + 4ABy + B^2 - (B^2 - 4AC) = (2Ay + B)^2 - (B^2 - 4AC)$, the quadratic congruence (4.10) on multiplying by 4*A* can be written as

$$x^2 \equiv a (\bmod\ p), \tag{4.11}$$

where $x = 2Ay + B$ and $a = B^2 - 4AC$. Since for $x^2 \equiv 0(\bmod\ p)$ the only solution is $x \equiv 0(\bmod\ p)$, in what follows we shall consider (4.11) with $0 < a < p$. We begin with the following result. It is clear that the congruence (4.10) either has no solutions or two solutions. In fact, If *x* is a solution, so is $-x$.

Theorem 4.6 *Let p be an odd prime. Then,* $x^2 \equiv a^2(\bmod\ p)$ *iff* $x \equiv \pm a(\bmod\ p)$.

Proof If $x \equiv \pm a(\bmod\ p)$, then Theorem 3.6(6) implies that $x^2 \equiv a^2(\bmod\ p)$. Conversely, if $x^2 \equiv a^2(\bmod\ p)$, then $p|(x^2 - a^2)$, and hence $p|(x-a)(x+a)$. But, then by Corollary 4.2, $p|(x-a)$ or $p|(x+a)$, which means $x - a \equiv 0(\bmod\ p)$ or $x + a \equiv 0(\bmod\ p)$, and therefore $x \equiv \pm a(\bmod\ p)$. ■

As an example for $x^2 \equiv 2^2(\bmod\ 3)$ solutions are $x = 1, 2, 4, 5, 7, 8, 10, \cdots$. Note that if *q* is composite and $x^2 \equiv a^2(\bmod\ q)$, then it does not follow that $x \equiv \pm a(\bmod\ q)$. For example, $10^2 \equiv 7^2(\bmod\ 51)$, but $10 \not\equiv \pm 7(\bmod\ 51)$.

Now we shall answer the following question: Is the congruence $x^2 \equiv -1 (\text{mod } p)$ solvable, where *p* is an odd prime. From $(x^2+1)/n$ it follows that $2/p, 5/p, 10/p, 17/p, 26/p, 37/p, 50/p, 65/p, 82/p, 101/p, \cdots$ and hence among $p = 3, 5, 7, 11, 13, \cdots$ congruence has solutions only for 5 and $13,$ which are of the form $p = 4k+1,$ i.e., $p \equiv 1 (\text{mod } 4).$ In the following result, we shall prove the general result.

Theorem 4.7 *Let p be an odd prime. Then,* $x^2 \equiv -1 (\text{mod } p)$ *iff* $p \equiv 1 (\text{mod } 4).$

Proof Since $x^2 \equiv -1 (\text{mod } p)$ implies that $p \not| x,$ in view of Theorem 4.5, we have

$$x^{p-1} \equiv (x^2)^{\frac{p-1}{2}} \equiv (-1)^{\frac{p-1}{2}} (\text{mod } p) \equiv 1 (\text{mod } p),$$

which is true provided $p = 4k+1$ or $p \equiv 1 (\text{mod } 4).$ In fact, if $p = 4k+3,$ then it gives the impossibility $-1 \equiv 1 (\text{mod } p).$ Conversely, if $p \equiv 1 (\text{mod } 4),$ then since $p - k \equiv -k (\text{mod } p),$ for all $1 \le k \le (p-1)/2$ and

$$\begin{aligned} (p-1)! &= (1)(2)\cdots\left(\frac{p-1}{2}\right)\left(\frac{p+1}{2}\right)\cdots(p-2)(p-1) \\ &\equiv (1)(-1)(2)(-2)\cdots\left(\frac{p-1}{2}\right)\left(-\frac{p-1}{2}\right) \quad (\text{mod } p) \\ &\equiv (-1)^{\frac{p-1}{2}}\left(\left(\frac{p-1}{2}\right)!\right)^2 \quad (\text{mod } p) \end{aligned}$$

Wilson's theorem confirms that $\left(\left(\frac{p-1}{2}\right)!\right)^2 \equiv -1 (\text{mod } p)$ provided $p = 4k+1.$ This means $p \equiv 1 (\text{mod } 4)$ implies the existence of $x^2 \equiv -1 (\text{mod } p).$ ■

Next for the congruence (4.11), we state a general result.

Theorem 4.8 (Euler's Criterion) *Let p be an odd prime and* $\gcd(a, p) = 1.$ *Then, the congruence* (4.11) *has a solution or does not have a* *solution according as* $a^{\frac{p-1}{2}} \equiv 1 (\mathrm{mod}\ p)$ *or* $a^{\frac{p-1}{2}} \equiv -1 (\mathrm{mod}\ p)$.

As an example consider the congruence $x^2 \equiv 7 (\mathrm{mod}\ 31)$. Since $a = 7$ and $p = 31$, it follows that $(a^{\frac{p-1}{2}} - 1)/31 = (7^{15} - 1)/31 = 153147145482$, and hence from Theorem 4.8, there is a solution of the given congruence. In fact, by inspection $x = 10 (\mathrm{mod}\ 31)$ and $21 (\mathrm{mod}\ 31)$ are the solutions of this congruence. Similarly, for the congruence $x^2 \equiv 14 (\mathrm{mod}\ 31)$, Theorem 4.8 confirms the existence of a solution, and $x = 13 (\mathrm{mod}\ 31)$ and $18 (\mathrm{mod}\ 31)$ are the solutions. However, since for the congruence $x^2 \equiv 21 (\mathrm{mod}\ 31)$, we have $(a^{\frac{p-1}{2}} + 1)/31 = (21^{15} + 1)/31 = 219749414783715104\,2$, and hence in view of Theorem 4.8, it has no solution.

If the congruence (4.11) has a solution (no solution), we say *a* is a *quadratic residue* (*quadratic nonresidue*) $(\mathrm{mod}\ p)$. Thus, $a = 7$ is a quadratic residue $(\mathrm{mod}\ 31)$, whereas $a = 21$ is a quadratic nonresidue $(\mathrm{mod}\ 31)$. For *p* an odd prime and $\gcd(a, p) = 1$, a continent notation known as *Legendre symbol* is defined as

$$\left(\frac{a}{p}\right) = (a/p) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue} \quad (\mathrm{mod}\ p) \\ -1 & \text{if } a \text{ is a quadratic nonresidue} \quad (\mathrm{mod}\ p) \end{cases}$$

Thus, $(7/31) = 1$, whereas $(21/31) = -1$. The following result provides interesting properties of the Legendre symbol.

Theorem 4.9 *Let p be an odd prime and the integers a and b are such that* $p \nmid ab$. *Then, the following hold:*

(i) $(a^2/p) = 1$.

(ii) *If* $a \equiv b (\mathrm{mod}\ p)$, *then* $(a/p) = (b/p)$.

(iii) $(ab/p) = (a/p)(b/p)$.

(iv) $(a/p) \equiv a^{\frac{p-1}{2}} (\mathrm{mod}\ p)$.

(v) $(1/p) = 1$ *and* $(-1/p) = (-1)^{\frac{p-1}{2}}$.

The following lemma Gauss proved in 1808 and again in 1818.

Lemma 4.4 (Gauss Lemma) *Let p be an odd prime and* $\gcd(a, p) = 1$. *Consider the least positive residues* $(\bmod\ p)$ *of the integers* $a, 2a, 3a, \cdots, (\frac{p-1}{2})a$. *If n denotes the least number of positive residues that exceed* $p/2$, *then* $(a/p) = (-1)^n$.

As an example, we consider $a = 7$ and $p = 31$. Since $(31 - 1)/2 = 15$, we need first 15 multiples of 7, namely $7, 14, 21, 28, 35, 42, 49, 56, 63, 70, 77, 84, 91, 98, 105$. For these numbers, the least positive residues $(\bmod\ 31)$ are $7, 14, 21, 28, 4, 11, 18, 25, 1, 8, 15, 22, 29, 5, 12$. From this list, 6 exceeds $31/2$. Thus, Lemma 4.4 concludes that $(7/31) = (-1)^6 = 1$.

Corollary 4.4 *If p is an odd prime, then*

$$\left(\frac{2}{p}\right) = \left\{ \begin{array}{rl} 1 & \text{if } \ p \equiv 1 \ \text{ or } \ -1 \equiv 7 (\bmod\ 8) \\ -1 & \text{if } \ p \equiv 3 \ \text{ or } \ -3 \equiv 5 (\bmod\ 8) \end{array} \right\} = (-1)^{\frac{p^2-1}{8}}.$$

Proof To apply Lemma 4.4, we need the first $(p - 1)/2$ multiples of 2, namely $2, 4, 6, \cdots, p - 1$. Since all of these are less than p, they are already least residues $(\bmod\ p)$. Among these, we need to find the least number of positive residues that exceed $p/2$. If $p = 8k + 1$, i.e., $p/2 = 4k + (1/2)$, then such numbers are $4k + 2, 4k + 4, \cdots, 8k$, whereas if $p = 8k + 7$, i.e., $p/2 = 4k + (7/2)$, then such numbers are $4k + 4, 4k + 6, \cdots, 8k + 6$. Since in the first case these residues are $2k$ whereas in the second case $2k + 2$ in number, it follows that $(2/p) = (-1)^{2k} = (-1)^{2k+2} = 1$. Similarly, if $p = 8k + 3$, then we have $4k + 2, 4k + 4, \cdots, 8k, 8k + 2$, whereas when $p = 8k + 5$, then $4k + 4, 4k + 6, \cdots, 8k + 4$. Since in these both cases these residues are $2k + 1$ in number, we find $(2/p) = (-1)^{2k+1} = -1$. To complete the proof, it suffices to note that $(p^2 - 1)/8$ for $p = 8k + 1$ is $2(4k^2 + k)$ (even), for $p = 8k + 3$ is $2(4k^2 + 3k) + 1$ (odd), for $p = 8k + 5$ is $2(4k^2 + 5k + 1) + 1$ (odd), and for $p = 8k + 7$ is $2(4k^2 + 7k + 3)$ (even).■

Now we shall state a theorem known as *The Law of Quadratic Reciprocity*. It was discovered by Euler, but he did not prove it. Then at the age of 18, completely unaware of the work of Euler, Gauss rediscovered the law on his own and proved it next year. Gauss called the law as the gem of arithmetic, and he remained fascinated by it throughout his life. For this result till 2019, 246 proofs were known, out of which 8 belongs to Gauss, one in 1963 by Murray Gerstenhaber (born 1927, USA) one-page proof in the *American Mathematical Monthly*, jocularly giving it the title "The 152nd proof of the law of quadratic reciprocity" and in the same Journal in 1990 half-page paper entitled "Another proof of the quadratic reciprocity theorem?" by Richard Gordon Swan (born 1933, USA).

Theorem 4.10 (The Law of Quadratic Reciprocity) *If p and q are distinct odd primes, then*

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\left(\frac{p-1}{2}\right)\left(\frac{q-1}{2}\right)}. \tag{4.12}$$

Following are alternative forms of the law of quadratic reciprocity:

- If *p* and *q* are distinct odd primes, then

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1 \text{ or } q \equiv 1 (\text{mod } 4) \\ -1 & \text{if } p \equiv q \equiv 3 (\text{mod } 4). \end{cases} \tag{4.13}$$

- If *p* and *q* are distinct odd primes, then

$$\left(\frac{p}{q}\right) = \begin{cases} (q/p) & \text{if } p \equiv 1 \text{ or } q \equiv 1 (\text{mod } 4) \\ -(q/p) & \text{if } p \equiv q \equiv 3 (\text{mod } 4). \end{cases} \tag{4.14}$$

Corollary 4.5 *If $p \neq 3$ is an odd prime, then*

$$\left(\frac{3}{p}\right) = \begin{cases} 1 & \text{if } p \equiv \pm 1 (\text{mod } 12) \\ -1 & \text{if } p \equiv \pm 5 (\text{mod } 12). \end{cases}$$

Proof If $p = 12s + t$, where $t = \pm 1, \pm 5$, then $p \equiv 1 (\text{mod } 4)$ for $t = 1$ or 5, and $p \equiv 3 (\text{mod } 4)$ for $t = -1$ or -5. Hence, for $t = 1$ or $t = 5$ from (4.14) and Theorem 4.8, we have

$$\left(\frac{3}{p}\right) = \left(\frac{p}{3}\right) = \left(\frac{t}{3}\right) \equiv \begin{cases} 1 & \text{if } t = 1 \\ -1 & \text{if } t = 5 \end{cases}$$

and for $t = -1$ or $t = -5$, it follows that

$$\left(\frac{3}{p}\right) = -\left(\frac{p}{3}\right) = -\left(\frac{t}{3}\right) \equiv \begin{cases} 1 & \text{if } t = -1 \\ -1 & \text{if } t = -5. \end{cases}$$

These observations imply the required result. ■

From the above results, the following can be computed easily:

a	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	29	21	22	23	24	25	26	27	28	29	30
$(a/31)$	1	1	−1	1	1	−1	1	1	1	1	−1	−1	−1	1	−1	1	−1	1	1	1	−1	−1	−1	−1	1	−1	−1	1	−1	−1

In the above table from Theorem 4.9, several entries for larger values of *a* can be obtained from the smaller values of a. For example,

$$\left(\frac{15}{31}\right) = \left(\frac{3}{31}\right)\left(\frac{5}{31}\right) = -1 \cdot 1 = -1 \quad \text{and} \quad \left(\frac{28}{31}\right) = \left(\frac{2}{31}\right)\left(\frac{2}{31}\right)\left(\frac{7}{31}\right)$$
$$= 1 \cdot 1 \cdot 1 = 1.$$

4.18 Characterization of Primes of the Form $4n + 1$ and $4n + 3$

Clearly, all odd primes have to be of the form $4n + 1$ or $4n - 1 (= 4(n - 1) + 3)$, i.e., $4m + 3$. In the following two results, we shall show that there are infinite number of primes in each of these categories.

Theorem 4.11 *There are infinite number of primes of the form* $4n - 1$.

Proof First we note by induction that if $\prod_{i=1}^{m} a_i = 4k - 1$, then for some factor $a_j = 4\ell - 1$. Now as in Theorem 4.2, we assume that $p_1, p_2, \cdots, p_m$ are the only primes of the form $4n - 1$. We let $N = 4\prod_{i=1}^{m} p_i - 1$, which is of the form $4n - 1$, and hence in view of Theorem 4.1, *N* must have a prime factor *q* of the form $4n - 1$. But then there exists an index *j* such that $q = p_j$. Thus $q | N$ as well as $q | (N + 1)$, which is possible only when

$q|1$. But this is impossible. Hence, there are infinite number of primes of the form $4n-1$. Dirichlet also noted that the reciprocal of all primes of the form $4n-1$ diverges, i.e., the series

$$\frac{1}{3}+\frac{1}{7}+\frac{1}{11}+\frac{1}{19}+\cdots$$

diverges.■

Theorem 4.12 *There are infinite number of primes of the form* $4n+1$.

Proof Assume that $n>1$, and define $N=(n!)^2+1$. Let *p* be the smallest prime divisor of N. Since *N* is odd, *p* has to be an odd prime (either $4m+1$ or $4m+3$). We also note that *p* has to be larger than n, otherwise, $p|1$. If we can show that *p* is of the form $4m+1$, then we can repeat the procedure replacing *n* with *p* and continue to produce an infinite sequence of primes of the form $4m+1$. Now since $p|N$, it follows that $(n!)^2\equiv -1(\mathrm{mod}\ p)$, and hence $(n!)^{p-1}\equiv(-1)^{\frac{p-1}{2}}(\mathrm{mod}\ p)$. But in view of Theorem 4.5, we have $(-1)^{\frac{p-1}{2}}\equiv 1(\mathrm{mod}\ p)$, and therefore if *p* was of the form $4m+3$, we get $-1\equiv 1(\mathrm{mod}\ p)$, which leads to the contradiction $p|2$. As for the case $4n-1$ the reciprocal of all primes of the form $4n+1$ diverges, i.e., the series

$$\frac{1}{5}+\frac{1}{13}+\frac{1}{17}+\frac{1}{29}+\cdots$$

diverges.■

- Primes of the form $4n-1$ and $4n+1$ less than 200, respectively, are

  $$3,7,11,19,23,31,43,47,59,67,71,79,83,103,107,127,131,139,151,163,\\ 167,179,191,199$$
  $$5,13,17,29,37,41,53,61,73,89,97,101,109,113,137,149,157,173,181,\\ 193,197.$$

  The function $\pi_{d,a}(x)$ counts the number of primes of the form $p=a+nd\le x$. Thus, $\pi_{4,-1}(100)=13,\ \pi_{4,1}(100)=11,\ \pi_{4,-1}(200)=24$, and $\pi_{4,1}(200)=21$. In 1853, Chebyshev remarked that $\pi_{4,1}(x)\le\pi_{4,-1}(x)$ for all integers x, which is indeed true for small values of x. However, in 1914, Littlewood theoretically proved that the inequality fails infinitely often. It was only in 1957 when with the help of computer it was shown that $x = 26861$ is the smallest prime for which $1473 = \pi_{4,1}(26861) > \pi_{4,-1}(26861) = 1472.$ The next such number is $x = 616841.$

Fermat in his letter of 1640 to Mersenne points out the following two theorems showing that the primes in these two categories $4n+1$ and $4n+3$ behave in mutually exclusive ways.

Theorem 4.13 *Any number of the form (and hence any odd prime of the form)* $4n+3$ *cannot be expressed as a sum* a^2+b^2 *of two perfect squares.*

Proof Let *a* and *b* be positive integers. Consider the sum a^2+b^2. Clearly, *a* and *b* cannot be both odd or both even because then a^2+b^2 would be even and $4n+3$ is odd. So we can assume that $a = 2s+1$ and $b = 2t$ for some integers *s* and t, but then $a^2+b^2 = 4(s^2+s+t^2)+1$ which in not of the form $4n+3$. ■

Theorem 4.14 (Fermat's Two Square Theorem) *If n is a prime number, then it can be expressed as a unique (except the order) sum of two squares iff either:* $n = 2,$ *or* $n = 4k+1,$ *i.e.,* $n \equiv 1 (\mathrm{mod}\ 4).$

Proof Clearly, $2 = 1^2 + 1^2$, and if $n = a^2 + b^2$, then from Theorem 4.13, *n* has to be of the form $n = 4k + 1$. Conversely, we shall show that every prime congruent to $n \equiv 1(\mathrm{mod}\ 4)$ can be written as a sum of two squares. For this, first we shall show that there always exist nonzero *a* and *b* such that $a^2 + b^2 = kn$ for some $k \geq 1$. Since $n \equiv 1(\mathrm{mod}\ 4)$, Theorem 4.7 ensures the existence of some *a* such that $a^2 \equiv -1(\mathrm{mod}\ n)$, i.e., $n|(a^2 + 1)$, which is the same as $a^2 + 1 = kn$ for some $k \geq 1$. This confirms that $a^2 + b^2 = kn$ (we can take $b = 1$) always has a solution for some $k \geq 1$. In fact, as in Theorem 4.7, we can take $\left(\left(\frac{n-1}{2}\right)!\right)^2 \equiv -1(\mathrm{mod}\ n)$; however, usually a number much smaller than $\left(\frac{p-1}{2}\right)!$ works. Now we define *u* and *v* such that

$$u \equiv a(\mathrm{mod}\ k), \quad v \equiv b(\mathrm{mod}\ k), \qquad -\frac{k}{2} < u, v \leq \frac{k}{2}. \tag{4.15}$$

Then, it follows that

$$u^2 + v^2 \equiv a^2 + b^2(\mathrm{mod}\ k),$$

which in view of $a^2 + b^2 = kn$ implies that $u^2 + v^2 \equiv 0(\mathrm{mod}\ k)$, or $u^2 + v^2 = k_1 k$ for some k_1. Thus, we have

$$(u^2 + v^2)(a^2 + b^2) \;=\; (k_1 k)(kn) = k_1 k^2 n.$$

Now from the Diophantus (also known as Brahmagupta-Fibonacci) identity (which shows that if two numbers are representable as the sum of two squares, then their product is also representable as the sum of two squares)

$$(a^2 + b^2)(c^2 + d^2) \;=\; (ac + bd)^2 + (ad - bc)^2 \;=\; (ad + bc)^2 + (ac - bd)^2, \tag{4.16}$$

which follow directly by squaring the the right sides, or using the fact

$$|a \mp bi|^2 |c + di|^2 \;=\; |(ac \pm bd) + i(ad \mp bc)|^2$$

it follows that

$$(ua + vb)^2 + (ub - va)^2 \;=\; k_1 k^2 n. \tag{4.17}$$

Next, from (4.15), we have

$$ua + vb \equiv u^2 + v^2 \equiv 0(\mathrm{mod}\ k) \quad \text{and} \quad ub - va \equiv uv - vu \equiv 0(\mathrm{mod}\ k).$$

Thus, it means (4.17) can be divided by k^2, to obtain

$$\left(\frac{ua+vb}{k}\right)^2 + \left(\frac{ub-va}{k}\right)^2 \;=\; k_1 n.$$

Finally, to complete the proof, we use the method of infinite descent, i.e., show that $k_1 < k$ and continue. For this, from (4.15), we have $u^2+v^2 \le (k/2)^2 + (k/2)^2 = k^2/2$, however, since $u^2+v^2 = k_1 k$, it follows that $k_1 k \le k^2/2$, i.e., $k_1 \le k/2$ and hence $k_1 < k$. It is clear that $k_1 \ge 0$, otherwise $u = v = 0$.

To prove the uniqueness, we assume that prime $n = 4k+1 = a^2+b^2 = c^2+d^2$, where $a > b > 0$ and $c > d > 0$. Since

$$\begin{aligned}(ad-bc)(ad+bc) \\ = a^2d^2 - b^2c^2 \;=\; (n-b^2)d^2 - b^2(n-d^2) \;=\; n(d^2-b^2) \equiv 0(\mathrm{mod}\ n)\end{aligned}$$

it follows that $(ad-bc)(ad+bc) \equiv 0(\mathrm{mod}\ n)$. Thus, from Corollary 4.2, it follows that $n|(ad-bc)$ or $n|(ad+bc)$. First, we consider the case $n|(ad+bc)$. Clearly, each of a^2, b^2, c^2, d^2 must be less than n. Hence, $0 < ad+bc < 2n$, which in turn implies that $ad+bc = n$. Now in view of (4.16), it follows that $n^2 = n^2 + (ac-bd)^2$. But this is possible only when $ac-bd = 0$. However, since $a > b$ and $c > d$, we have $ac > bd$. This contradiction shows that $n \not| (ad+bc)$. Now we consider the case $n|(ad-bc)$. From the earlier argument, we have $-n < ad-bc < n$ which is possible only when $ad = bc$. Now $\gcd(a,b) = 1$ because in contrary a^2+b^2 will have a common divisor greater than 1 and less than n, which in view of *n* being prime is impossible. So Euclid's Lemma 4.1 ensures that $a|c$, or $c = \ell a$ and so $ad = bc$ gives $d = \ell b$. But this gives $n = c^2+d^2 = \ell^2(a^2+b^2) = \ell^2 n$ and hence $\ell = 1$. This means $a = c$ and $b = d$. ■

The following results complements Theorem 4.14.

Theorem 4.15 *Let the positive integer n can be written as* $n = m^2 k$, *where k is square free. Then n can be written as the sum of two squares iff k does not contain prime factors of the form* $4\ell+3$. *(If* $k=1$, *then* 0^2 *is permitted, and hence then* $m^2 = m^2 + 0^2$.*)*

Theorem 4.16 (Fermat-Lagrange) *If n is an odd prime, then* $n = a^2 + 2b^2$ *iff* $n \equiv 1 \text{ or } n \equiv 3 (\text{mod } 8)$. *Further,* $n = a^2 + 3b^2$ *iff* $n \equiv 1 (\text{mod } 3)$.

Theorem 4.17 (Euler-Lagrange) *If n is an odd prime, then* $n = a^2 + 5b^2$ *iff* $n \equiv 1 \text{ or } n \equiv 9 (\text{mod } 20)$. *Further,* $2n = a^2 + 5b^2$ *iff* $n \equiv 3 \text{ or } n \equiv 7 (\text{mod } 20)$.

- Diophantus knew Theorem 4.13, whereas Theorem 4.14 was known to Girard in 1625, several years before Fermat. The first proof of Theorem 4.14 was given by Euler in 1749, which was later simplified by several prominent mathematicians. In 1990, Don Bernard Zagier (born 1951, Germany-USA) presented a nonconstructive one-sentence proof. The converse of Theorem 4.13 is not true. In fact, it does not imply that if a number cannot be expressed as a sum of two perfect squares, it will be an odd number of the form $4n + 3$. It could be that such a number may be even, or odd of the form $4n + 1$. For example, $12 (= 11 + 1 = 10 + 2 = 9 + 3 = 8 + 4 = 7 + 5 = 6 + 6)$ cannot be expressed as a sum of two perfect squares, yet it is even. Similarly, $9 (= 8 + 1 = 7 + 2 = 6 + 3 = 5 + 4),\ 33 (= 25 + 8 = 16 + 17 = 9 + 24 = 4 + 29 = 1 + 32), 57 (= 49 + 8 = 36 + 21 = 25 + 32 = 16 + 41 = 9 + 48 = 4 + 53 = 1 + 56)$ cannot be expressed as a sum of two perfect squares, but these are odd of the form $4n + 1$. Theorem 4.14 also does not imply that if an integer can be expressed as the sum of two perfect squares, uniquely, then it will be a prime of the form $4n + 1$. In fact, it may be that such a number is of the form $4n + 1$, but not a prime or an even and non-prime. For example, $25 (= 24 + 1 = 23 + 2 = 22 + 3 = 21 + 4 = 20 + 5 = 19 + 6 = 18 + 7 = 17 + 8 = 16 + 9 = 15 + 10 = 14 + 11 = 13 + 12)$ can be expressed as the sum of two perfect squares in only one way, $4^2 + 3^2$, and it is of the form $4n + 1 (= 4 \times 6 + 1)$, but it is not a prime. Similarly, $10 (= 9 + 1 = 8 + 2 = 7 + 3 = 6 + 4 = 5 + 5)$ can be expressed as the sum of two perfect squares uniquely, $3^2 + 1^2$, but it is even and non-prime.

Fermat's Theorem 4.14 is cited in any discussion of mathematical beauty. Examples of 11 primes of the form $4n + 1$ below 100 and their unique representations as the sum of two perfect squares are $5 = 1^2 + 2^2, 13 = 2^2 + 3^2, 17 = 1^2 + 4^2, 29 = 2^2 + 5^2, 37 = 1^2 + 6^2, 41 = 4^2 + 5^2, 53 = 2^2 + 7^2, 61 = 5^2 + 6^2, 73 = 3^2 + 8^2, 89 = 5^2 + 8^2, 97 = 4^2 + 9^2.$ These primes are Pythagorean primes (because in view of Pythagorean theorem the square roots of these primes represent the length of the hypotenuse of right triangles). From Theorem 4.12, it is clear that Pythagorean primes are infinite. Fermat numbers $F_n = 2^{2^n} + 1,\ n \geq 1$ are of the form $4k+1$, and each can be written as a sum of two squares, in fact, we have $F_n = (F_{n-1} - 1)^2 + 1^2$. In view of Theorem 4.14, this representation is unique for prime Fermat numbers. However, for composite Fermat numbers, first we recall Theorem 4.7 to ensure that for an odd prime $p|F_n$, i.e., $(2^{2^{n-1}})^2 \equiv -1 (\mathrm{mod}\ p)$ implies that $p \equiv 1 (\mathrm{mod}\ 4)$. Thus, each factor of F_n is of the form $4k+1$ and hence as a consequence of Theorem 4.14 can be written as a sum of two squares uniquely. Now (4.16) provides at least two different representations of each composite F_n (depending on the number of factorizations), for example,

$$\begin{aligned} F_5 &= 641 \times 6700417 = (25^2 + 4^2)(2556^2 + 409^2) \\ &= (65536)^2 + 1^1 = (F_4 - 1)^2 + 1 \quad \text{and} \quad (20449)^2 + (62264)^2. \end{aligned}$$

It is well known that there are only 31 numbers that cannot be expressed as the sum of distinct squares:
$2, 3, 6, 7, 8, 11, 12, 15, 18, 19, 22, 23, 24, 27, 28, 31, 32, 33, 43, 44, 47, 48, 60, 67, 72, 76, 92, 96, 108, 112, 128.$
It is the contra-positive form of Theorem 4.14 which is important. It says if a number of the form $4n+1$ cannot be expressed uniquely as the sum of two perfect squares, then that number is not a prime. For example, the number 65 is of the form $4n + 1 (= 4 \times 16 + 1)$. It can be written as the sum of two perfect squares in two ways, namely, $65 = 8^2 + 1^1 = 7^2 + 4^2$. Therefore, by the contra-positive of Theorem 4.14, 65 cannot be a prime. In fact, $65 = 13 \times 5$. Similarly,
$37 \times 13 = 481 = 4 \times 120 + 1 = 20^2 + 9^2 = 15^2 + 16^2$, and
$29 \times 37 = 1073 = 4 \times 268 + 1 = 17^2 + 28^2 = 7^2 + 32^2$. It is interesting to note that for the numbers of the form 4*n*, sum of two squares may not exist, or can exist uniquely or non-uniquely. For example,
$12 (= 11 + 1 = 10 + 2 = 9 + 3 = 8 + 4 = 7 + 5 = 6 + 6)$ cannot be written as the the sum of two squares,
20(=19+1=18+2=17+3=16+4=15+5=14+6=13+7=12+8=11+9 =10+10) can be expressed as the sum of two perfect squares uniquely, $4^2 + 2^2$, and $200 = 10^2 + 10^2 = 2^2 + 14^2$, also $680 = 2^2 + 26^2 = 14^2 + 22^2$.

Similarly, for the numbers of the form $4n+2$ sum of two squares may not exist, or can exist uniquely or non-uniquely. For example,
$14 (= 13 + 1 = 12 + 2 = 11 + 3 = 10 + 4 = 9 + 5 = 8 + 6 = 7 + 7)$

cannot be written as the the sum of two squares, we have seen 10 can be written as a sum of two squares uniquely, the number $170 = 42(4) + 2$ has exactly two representations of the sum of two squares, namely, $1^2 + 13^2$ and $7^2 + 11^2$, also $530 = 132(4) + 2$ has two representations $23^2 + 1^2$ and $19^2 + 13^2$. As examples of Theorem 4.15, we have
$8 = 2^2 \cdot 2 = 2^2 + 2^2, 45 = 3^2 \cdot 5 = 3^2 + 6^2, 833 = 7^2 \cdot 17 = 7^2 + 28^2.$
However, $132 = 2^2 \cdot 3 \cdot 11$, $343 = 7^2 \cdot 7$ and $5040 = 12^2 \cdot 5 \cdot 7$ cannot be represented as the sum of two squares. Some examples illustrating Theorems 4.16 and 4.17 are
$73 = 9(8) + 1 = 1^2 + 2 \cdot 6^2,\ 67 = 8(8) + 3 = 7^2 + 2 \cdot 3^2,\ 61 = 20(3) + 1 = 7^2 + 3 \cdot 2^2,\ 61 = 3(20) + 1 = 4^2 + 5 \cdot 3^2,\ 89 = 4(20) + 9 = 3^2 + 5 \cdot 4^2,\ 63 = 20(3) + 3$
and hence $2 \times 63 = 126 = 1^2 + 5 \cdot 5^2 = 9^2 + 5 \cdot 3^2,\ 107 = 5(20) + 7$ and hence $214 = 13^2 + 5 \cdot 3^2$. From these examples, it is clear that in Theorems 4.76 and 4.77 uniqueness of the representations is not guaranteed.

- If an integer *m* can be written as $m = a^2 + b^2 = c^2 + d^2$, then the identity

$$m \;=\; \frac{((a-c)^2 + (b-d)^2)((a+c)^2 + (b-d)^2)}{4(b-d)^2}$$

  ensures that *m* is composite and provides two factors. From this identity, uniqueness in Theorem 4.14 is immediate. As an example, we know that $530 = 23^2 + 1^2 = 19^2 + 13^2$. Thus, for $a = 23, b = 1, c = 19, d = 13$ the identity gives $530 = 10 \times 53 = 5 \times 106.$

- Fermat also showed that every odd prime *n* can be written as the difference of two squares in one and only one way. In fact,

$$n \;=\; \left(\frac{n+1}{2}\right)^2 - \left(\frac{n-1}{2}\right)^2. \tag{4.18}$$

  For this, if $n = x^2 - y^2 = (x+y)(x-y)$, then $x + y = n$ and $x - y = 1$ (since n, being prime, has no factors except 1 and n.) This gives $x = (n+1)/2$ and $y = (n-1)/2$, which are successive integers. For example, $481 = (241)^2 - (240)^2$. This simple observation also holds for the square of every odd prime n, e.g., $7^2 = 49 = 25^2 - 24^2$. In fact, the relation (4.18) holds for all odd integers n, but then uniqueness is lost, e.g., $27 = 14^2 - 13^2 = 6^2 - 3^2$. The contra-positive form of (4.18) is important. It says if an odd number cannot be expressed uniquely as the difference of two perfect squares, then that number is not a prime. For example, $125 = 63^2 - 62^2 = 15^2 - 10^2$. If *n* is of the form $4k$, then the relation

$$n = \left(\frac{n+4}{4}\right)^2 - \left(\frac{n-4}{4}\right)^2 \qquad (4.19)$$

provides the difference of two squares, but not necessarily unique, e.g., $64 = 17^2 - 15^2 = 10^2 - 6^2$. It also follows that if *n* is of the form $4k+2$, then it cannot be written as the difference of two squares. Indeed, since for all integers $a,\ a^2 \equiv 0 \text{ or } 1 (\text{mod } 4)$, it suffices to note that $a^2 - b^2 \equiv 0, 1 \text{ or } 3 (\text{mod } 4) \not\equiv 2 (\text{mod } 4)$. Thus, the number 530 cannot be written as the difference of two squares. It is interesting to note that

$$2^{2n+1} = (2^{2n-k} + 2^{k-1})^2 - (2^{2n-k} - 2^{k-1})^2, \quad k = 1, 2, \cdots, n$$

and hence for any $n > 0$, there exists a positive integer that can be expressed in *n* distinct ways of the difference of two squares.

- Fermat also proved another result which states: A prime of the form $4n+1$ is only once the hypotenuse of a right triangle; its square is twice; its cube, three times; and so on. As an example of this result, in the case of five we have

$$\begin{aligned}
5^2 &= 3^2 + 4^2 \\
25^2 &= 15^2 + 20^2 \quad \text{and} \quad 7^2 + 24^2 \\
125^2 &= 75^2 + 100^2 \quad \text{and} \quad 35^2 + 120^2 \quad \text{and} \quad 44^2 + 117^2 \\
625^2 &= 375^2 + 500^2 \quad \text{and} \quad 175^2 + 600^2 \\
&\quad \text{and} \quad 220^2 + 585^2 \quad \text{and} \quad 336^2 + 527^2.
\end{aligned}$$

4.19 Legendre's Three-Square Theorem

Theorem 4.14 affirms that not every number can be written as the sum of two squares even if 0^2 is permitted. However, with the addition of 0^2, every number which can be represented as the sum of two squares can also be repressed as the sum of three squares. But the converse is not necessarily true, e.g., numbers $12, 14, 33, 57,$ cannot be written as the sum of two squares, but
$12 = 2^2 + 2^2 + 2^2,\ 14 = 3^2 + 2^2 + 1^2,\ 33 = 5^2 + 2^2 + 2^2,\ 57 = 7^2 + 2^2 + 2^2,$
whereas the number 73 and 137 can be written as the sum of two as well

as three squares $73 = 3^2 + 8^2 = 1^2 + 6^2 + 6^2,\ 137 = 4^2 + 11^2 = 3^2 + 8^2 + 8^2$. The following result provides necessary and sufficient condition when a number can be written as the sum of three squares.

Theorem 4.18 (Legendre's Three-Square Theorem) *An integer n can be represented as the sum of three squares of integers, i.e.,* $n = a^2 + b^2 + c^2$ *iff n is not of the form* $n = 4^h(8k+7)$ *for nonnegative integers h and* k.

Proof We are in the position to show only that if *n* is of the form $n = 4^h(8k+7)$ then it cannot be written as the sum of three squares (in fact, no simple proof of its converse is known). For this, first we consider the case $h = 1$. We assume that there exists a nonnegative integer *k* such that $a^2 + b^2 + c^2 = 8k + 7$, i.e., $a^2 + b^2 + c^2 \equiv 7(\mathrm{mod}\ 8)$. Since for any integer s, we have $s^2 \equiv 0, 1,$ or $4(\mathrm{mod}\ 8)$, it follows that $a^2 + b^2 + c^2 \not\equiv 7(\mathrm{mod}\ 8)$ (we can come as close as $1 + 1 + 4 \equiv 6(\mathrm{mod}\ 8)$ and $0 + 4 + 4 \equiv 8(\mathrm{mod}\ 8)$). Now suppose that $4^h(8k+7) = x^2 + y^2 + z^2$ for some $h \geq 1$, we assume such an *h* is the smallest. But then the only possibility is each of the integers a, b, c must be even, i.e., $a = 2u, b = 2v, c = 2w$. Thus, we have $4^h(8k+7) = a^2 + b^2 + c^2 = 4(u^2 + v^2 + w^2)$, and hence $4^{h-1}(8k+7) = (u^2 + v^2 + w^2)$, which shows $4^{h-1}(8k+7)$ is also a sum of three squares of integers. This contradiction completes the proof.■

- The necessary part of Theorem 4.18 which we have demonstrated above first appeared as a conjecture made by Diophantus, which was verified by Descartes in 1618. The complete theorem was stated by Fermat, and first proved by Legendre in 1797/98. Since then, several proofs of this result have been offered (but none of them is simple), more so, in 1801, Gauss obtained an extensive generalization of this theorem, which contains this result as a corollary.
- Theorem 4.14 confirms the representation of numbers in two squares uniquely, but in Theorem 4.18, the representation of numbers in three squares is not necessarily unique. For example, the number 126 can be represented as the sum of three squares in three different ways: $126 = 11^2 + 2^2 + 1^2 = 10^2 + 5^2 + 1^2 = 9^2 + 6^2 + 3^2$. The number 129 which is the sum of first ten prime numbers is the smallest number that

can be expressed as a sum of three squares in four different ways:
$129 = 11^2 + 2^2 + 2^2 = 10^2 + 5^2 + 2^2 = 8^2 + 8^2 + 1^2 = 8^2 + 7^2 + 4^2.$

4.20 Lagrange's Four-Square Theorem

From Theorem 4.18, it is clear that every number cannot be represented as the sum of three squares. For example, none of the numbers $7, 15, 23, 28, 31, 39, 47, 55, 60, 63, 71$ can be written as the sum of three squares. In fact, for each of these numbers we need minimum four squares, e.g.,
$7 = 2^2 + 1^2 + 1^2 + 1^2,\ 15 = 3^2 + 2^2 + 1^2 + 1^2,\ 23 = 3^2 + 3^2 + 2^2 + 1^2.$
The following result supplements Theorem 4.18.

Theorem 4.19 (Lagrange's Four-Square Theorem) *Every positive integer N can be written as the sum of four integer squares.*

Proof We divide the proof in the following six parts:

Part 1. Euler in 1748 gave the following Brahmagupta-Diophantus-Fibonacci (4.16) type four-square identity, whose proof is by direct expansion of each term

$$\begin{aligned}(a_1^2 + a_2^2 + a_3^2 + a_4^2)(b_1^2 + b_2^2 + b_3^2 + b_4^2) &= (a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4)^2 \\ &+(a_1b_2 - a_2b_1 + a_3b_4 - a_4b_3)^2 + (a_1b_3 - a_2b_4 - a_3b_1 + a_4b_2)^2 \\ &+(a_1b_4 + a_2b_3 - a_3b_2 - a_4b_1)^2.\end{aligned} \tag{4.20}$$

It shows that if two numbers are representable as the sum of four squares, then their product is representable as the sum of four squares.

Part 2. If $2n$ is a sum of two squares, then so is $n.$ Indeed, if $2n = a^2 + b^2,$ then either both a and b are even or odd. Thus, it follows that

$$n \;=\; \left(\frac{a+b}{2}\right)^2 + \left(\frac{a-b}{2}\right)^2.$$

Clearly, both fractions on the right side are integers.

Part 3. If p is an odd prime, then $1 + a^2 + b^2 \equiv 0 (\mathrm{mod}\ p)$ has a solution with $0 \le a \le (p-1)/2$ and $0 \le b \le (p-1)/2.$ For this, we consider the sets

$$S_1 = \left\{0^2, 1^2, 2^2, \cdots, \left(\frac{p-1}{2}\right)^2\right\}$$

and

$$S_2 = \left\{-1-0^2, -1-1^2, -1-2^2, \cdots, -1-\left(\frac{p-1}{2}\right)^2\right\}.$$

The numbers in S_1 (and similarly in S_2) are distinct $(\text{mod } p)$. Indeed, if $c, d \in S_1$, then $c^2 \equiv d^2 (\text{mod } p)$ implies that $c \equiv \pm d (\text{mod } p)$, but $p|(c+d)$ is impossible because $0 < c+d < p/2 + p/2 = p$ (unless $c = d = 0$), and hence $c = d$. Now S_1 and S_2 contain together $(p-1)/2 + 1 + (p-1)/2 + 1 = p+1$ numbers, and there are only *p* least resides $(\text{mod } p)$, thus in view of Dirichlet's pigeonhole principle, we must have one of the numbers in S_1 congruent to one of the numbers in S_2, i.e., $a^2 \equiv -1 - b^2 (\text{mod } p)$ for some *a* and *b* with $0 \le a \le (p-1)/2$ and $0 \le b \le (p-1)/2$.

Part 4. For every odd prime p, there is an odd integer $n < p$ such that $np = a^2 + b^2 + c^2 + d^2$ has a solution. In fact, as a consequence of Part 3, there exist *a* and *b* such that $mp = a^2 + b^2 + 1^2 + 0^2$ for some integer m. Since $0 \le a < p/2$ and $0 \le b < p/2$, we have

$$mp = a^2 + b^2 + 1^2 + 0^2 < p^2/4 + p^2/4 + 1 < p^2$$

and hence $m < p$. To show *m* is odd, we assume that $mp = a^2 + b^2 + c^2 + d^2$. If *m* is even, then for a, b, c, d, there are only three possibilities: all are even, two even, and none is even. Thus, we can organize a, b, c, d into two groups each containing these numbers of the same parity. But then Part 2, allows us to take $n = m/2$. If $m/2$ is even, we can repeat the process to represent $(m/4)p$ as a sum of four squares. Since $m \neq 0$, finally we will find an odd integer *n* such that *np* can be written as a sum of four squares.

Part 5. Any prime *p* can be written as the sum of four squares. The statement obviously holds for $p = 2 = 1^2 + 1^2 + 0^2 + 0^2$, so we can assume that *p* is an odd prime. Then, in view of Part 4, there exists an odd *smallest* integer $n < p$ such that $np = a^2 + b^2 + c^2 + d^2$. Thus, in this representation, we need to show that $n = 1$. If $n > 1$ (at least 3), we can

choose integers w, x, y, z such that
$|w| < n/2,\ |x| < n/2,\ |y| < n/2, |z| < n/2$ and
$w \equiv a(\mathrm{mod}\ n),\ x \equiv b(\mathrm{mod}\ n),\ y \equiv c(\mathrm{mod}\ n),\ z \equiv d(\mathrm{mod}\ n).$ (In fact, as an example, *w* can be chosen as the remainder *r* or $r - n$ when *a* is divided by *n* according as $r < n/2$ or $r > n/2.$) But now we have $w^2 + x^2 + y^2 + z^2 < 4(n^2/4) = n^2$, and hence $w^2 + x^2 + y^2 + z^2 \equiv a^2 + b^2 + c^2 + d^2 \equiv 0(\mathrm{mod}\ n)$, which implies the existence of some nonnegative integer n_1 such that $w^2 + x^2 + y^2 + z^2 = n_1 n.$ However, since $0 \leq n_1 n = w^2 + x^2 + y^2 + z^2 < n^2$, it follows that $n_1 < n.$ It is not possible to have $n_1 = 0$, because then $w = x = y = z = 0$, and so as a consequence, we will have $a \equiv b \equiv c \equiv d \equiv 0(\mathrm{mod}\ n)$, which implies $np = a^2 + b^2 + c^2 + d^2 = n^2 q$ (for some integer *q*). But this means $n|p$, which is impossible, because $1 < n < p.$ Thus, we have $0 < n_1 < n.$ Now in view of Part 1, we can write

$$\begin{aligned} n^2 n_1 p &= (np)(n_1 n) = (a^2 + b^2 + c^2 + d^2)(w^2 + x^2 + y^2 + z^2) \\ &= r^2 + s^2 + t^2 + u^2, \end{aligned}$$

where

$$\begin{aligned} r &= aw + bx + cy + dz \equiv a^2 + b^2 + c^2 + d^2 \equiv 0(\mathrm{mod}\ n) \\ s &= ax - bw + cz - dy \equiv 0(\mathrm{mod}\ n) \\ t &= ay - bz - cw + dx \equiv 0(\mathrm{mod}\ n) \\ u &= az + by - cx - dw \equiv 0(\mathrm{mod}\ n). \end{aligned}$$

Hence, it follows that

$$n_1 p = \left(\frac{r}{n}\right)^2 + \left(\frac{s}{n}\right)^2 + \left(\frac{t}{n}\right)^2 + \left(\frac{u}{n}\right)^2.$$

This means $n_1 p$ can also be represented as a sum of four squares $r/n, s/n, t/n,$ and $u/n,$ which are integers. But since $0 < n_1 < n$ it contradicts our assumption that *n* is the smallest such integer. In conclusion, *p* can be written as the sum of four squares.

Part 6. If $N = 1$, then clearly $1 = 1^2 + 0^2 + 0^2 + 0^2$ is the sum of four squares. If $N > 1,$ we use (4.1), to express *N* as a product of primes, i.e., $N = p_1 p_2 \cdots p_m$ (some or all of these primes may be repeated). From Part

5, each of these primes can be written as a sum of four squares. We use these sums for p_1 and p_2 in Part 1, to write p_1p_2 as the sum of four squares. For this sum of p_1p_2 and the sum of p_3 we again use Part 1 to get the sum of four squares. On continuing this process $m-1$ times, we will get the required four square sum for N. ■

- If $N = a^2+b^2+c^2+d^2$, where a, b, c, d are nonnegative integers, then the following immediate inequalities provide bounds on these numbers

$$0 \le \min\{a,b,c,d\} \le \sqrt{N}/2 \le \max\{a,b,c,d\} \le \sqrt{N}.$$

- In 1621, Bachet stated four-square theorem as a conjecture and checked its correctness up to 325. In 1636, Fermat asserted to have a proof. Part 3, which plays a crucial role in the proof, is also due to Euler; however, for more than 40 years (off and on), he unsuccessfully struggled to prove the theorem. Final steps were completed by Lagrange in 1770, which were simplified by Euler in 1773. In a Ramanujan conference talk, Ralph William Gosper Jr. (born 1943, USA) conjectured that every sum of four distinct odd squares is the sum of four distinct even squares. This conjecture was proved by Michael D. Hirschhorn (Australia) using the identity

$$\begin{aligned}(4a+1)^2+(4b+1)^2+(4c+1)^2+(4d+1)^2 \\ = 4[(a+b+c+d+1)^2+(a-b-c+d)^2 \\ +(a-b+c-d)^2+(a+b-c-d)^2],\end{aligned}$$

  where $a, b, c,$ and *d* are positive or negative integers. The following examples suggest that in Theorem 4.19, the representation of numbers in four nonzero squares is not necessarily unique

$$39 = 5^2+3^2+2^2+1^2 = 6^2+1^2+1^2+1^2,\ 55 = 5^2+5^2+2^2+1^2 = 6^2+3^2+3^2+1^2 = 7^2+2^2+1^2+1^2,\ 63 = 5^2+5^2+3^2+2^2 = 6^2+3^2+3^2+3^2 = 6^2+5^2+1^2+1^2 = 7^2+3^2+2^2+1^2,\ 111 = 6^2+5^2+5^2+5^2 = 7^2+6^2+5^2+1^2 = 7^2+7^2+3^2+2^2 = 9^2+5^2+2^2+1^2 = 10^2+3^2+1^2+1^2.$$

  All positive numbers that are not the sum of five nonzero squares are $1, 2, 3, 4, 6, 7, 9, 10, 12, 15, 18, 33,$ i.e., every integer $N > 33$ can be expressed as the sum of five nonzero squares, e.g., $34 = 4^2+3^2+2^2+2^2+1^2,\ 39 = 4^2+3^2+3^2+2^2+1^2,\ 40 = 5^2+3^2+2^2+1^2+1^2.$ The following numbers cannot be represented using fewer than five distinct squares: $55, 88, 103, 132, 172, 176, 192, 240, 268, 288, 304, 368, 384, 432, 448, 496,$ $512,$ and $752,$ together with all numbers obtained by multiplying these numbers by a power of 4. This gives all known such numbers less than

10^5. For example,
$55 = 5^2 + 4^2 + 3^2 + 2^2 + 1^2,\ 88 = 7^2 + 5^2 + 3^2 + 2^2 + 1^2,\ 132 = 9^2 + 5^2 + 4^2 + 3^2 + 1^2.$
All numbers $N > 188$ can be expressed as the sum of at most five distinct squares, except $124 = 7^2 + 6^2 + 5^2 + 3^2 + 2^2 + 1^2$ and $188 = 10^2 + 7^2 + 5^2 + 3^2 + 2^2 + 1^1$. The number 188 can also be represented using seven distinct squares $188 = 8^2 + 7^2 + 6^2 + 5^2 + 3^2 + 2^2 + 1^2$.

- Eisenstein in 1847 determined arithmetical representation of an integer as a sum of six or eight squares. This was followed in 1847 and 1850 by an arithmetical determination of the number of representations of an integer without square factors as a sum of five or seven squares. In 1844, he established a formula for the number of solutions of $x^2 + y^2 \leq n$ in integers $x, y,$ where *n* is given.

4.21 Carmichael Numbers

The positive integer $n(> 2)$ is called a Carmichael number (the term was coined in 1950 by Nicolaas George Wijnand Henri Beeger, 1884–1965, the Netherlands) if *n* is a composite number such that $b^n \equiv b(\bmod\ n)$, equivalently, $b^{n-1} \equiv 1(\bmod\ n)$, for all integers *b* such that $\gcd(b, n) = 1$. Carmichael numbers are also called *Fermat pseudoprimes* or *absolute Fermat pseudoprimes*. In what follows we shall provide a characterization of Carmichael numbers. For this, we need some preparatory results.

Lemma 4.5 *If* $n = \prod_{i=1}^{r} p_i$, *where* p_i*'s are distinct odd primes such that* $(p_i - 1)|(n - 1)$ *for each* i, *then n is a Carmichael number.*

Proof If *b* is an integer such that $\gcd(b, n) = 1$, then $\gcd(b, p_i) = 1$ for all i, which in view of Theorem 4.5 implies that $b^{p_i - 1} \equiv 1(\bmod\ p_i)$ for all i. Since $(p_i - 1)|(n - 1)$ for each *i* there exists some integer k_i such that $n - 1 = k_i(p_i - 1)$, which in turn implies that $b^{n-1} = b^{k_i(p_i-1)} = (b^{p_i-1})^{k_i} \equiv 1^{k_i} \equiv 1(\bmod\ p_i)$. Thus, from Theorem 3.10, it follows that $b^{n-1} \equiv 1(\bmod\ \prod_{i=1}^{r} p_i) \equiv 1(\bmod\ n)$, and hence *n* is a Carmichael number.■

Lemma 4.6 *Every Carmichael number n is square-free.*

Proof Assume that a Carmichael number *n* can be written as $n = p^k m$, where *p* is prime, $k \geq 1$, and $\gcd(p, m) = 1$. Clearly, if $k \geq 2$ then *n* is divisible by p^2. Now in view of Theorem 3.10, the system of congruences $x \equiv 1 + p(\text{mod } p^k),\ x \equiv 1(\text{mod } m)$ has a solution *a* such that $\gcd(a, n) = 1$. Again since *n* is Carmichael, we have $a^{n-1} \equiv 1(\text{mod } n)$, which in particular is $a^{n-1} \equiv 1(\text{mod } p^k)$, and hence $a^n \equiv a(\text{mod } p^2)$. Thus, it follows that $(1+p)^n \equiv 1 + p(\text{mod } p^2)$. Next using binomial theorem to expand $(1+p)^n$ and noticing that the first two terms of the expansion are 1 and $np(= p^{k+1}m)$ and the rest of the terms are divisible by p^2, to get $(1+p)^n \equiv 1(\text{mod } p^2)$. But this leads to a contradiction $1 \equiv 1 + p(\text{mod } p^2)$. ■

Lemma 4.7 *Every Carmichael number n is odd.*

Proof Since $\gcd(n-1, n) = 1$, we have $(n-1)^{n-1} \equiv 1(\text{mod } n)$, which in view of binomial theorem is the same as $(-1)^{n-1} \equiv 1(\text{mod } n)$. Now since $(-1)^{n-1} = \pm 1$, and $n > 2$ we have $-1 \not\equiv 1(\text{mod } n)$. Thus, $n-1$ must be even, and hence *n* must be odd. ■

To prove our next result, we need the following definition which is originally due to Euler: An integer *h* is a *primitive root modulo n* if for every integer *a* coprime to n, there is some integer *k* for which $h^k \equiv a(\text{mod } n)$. For example, 3 is a primitive root modulo 7 because
$3^1 \equiv 3(\text{mod } 7), 3^2 \equiv 2(\text{mod } 7), 3^3 \equiv 6(\text{mod } 7), 3^4 \equiv 4(\text{mod } 7), 3^5 \equiv 5(\text{mod } 7), 3^6 \equiv 1(\text{mod } 7), 3^7 \equiv 3(\text{mod } 7),$
and hence 3, 2, 6, 4, 5, 1, form a rearrangement of all required nonzero remainders modulo 7. However, 3 is not a primitive root modulo 11 because
$3^1 \equiv 3(\text{mod } 11), 3^2 \equiv 9(\text{mod } 11), 3^3 \equiv 5(\text{mod } 11), 3^4 \equiv 4(\text{mod } 11), 3^5 \equiv 1(\text{mod } 11),$
and then the sequence starts repeating. In fact, in view of modulo *n* produces a finite number of values, the sequence $h^k \equiv a(\text{mod } n)$ always repeats after some value of k. If *h* is a primitive root modulo *n* and *n* is prime, then the period of repetition is $n-1$.

Lemma 4.8 *Let p be an odd prime factor of n and* $h \geq 1$. *Then there exists an integer r such that* $\gcd(r, n) = 1$ *and r is a primitive root modulo* p^h.

Proof Assume that *g* is a primitive root modulo p^h such that $0 < g < p^h$. If $\gcd(g, n) = 1$, then we can choose $r = g$. Otherwise, we consider the arithmetic sequence $s_j = g + p^h j,\ j = 1, 2, \cdots$. Since $\gcd(g, p) = 1$, in view of Dirichlet's theorem in this sequence, there are infinitely many *j* such that the corresponding s_j is prime. We choose the least *j* such that s_j is prime and this exceeds all the prime factors of n. Then, $\gcd(s_j, n) = 1$ and this s_j is a primitive root modulo p^h, and so we can let $r = s_j$. ■

Lemma 4.9 *Let n be the product of two or more distinct primes* p_i. *If n is a Carmichael number, then* $(p_i - 1)|(n - 1)$ *for each index* i.

Proof In view of Lemma 4.8, for each index *i*, there exists an integer r_i such that $\gcd(r_i, n) = 1$ and r_i is a primitive root modulo p_i. Now since *n* is a Carmichael number $r_i^{n-1} \equiv 1 (\text{mod } n)$, and hence $r_i^{n-1} \equiv 1 (\text{mod } p_i)$. Now note that the order (smallest exponent) of r_i modulo p_i is $(p_i - 1)$ thus $(p_i - 1)|(n - 1)$. ■

Lemma 4.10 *Every Carmichael number n has at least three different prime factors.*

Proof Assume that $n = pq$, where *p* and *q* are primes. From Lemma 4.6, $p \neq q$. We suppose that $p > q$. From Lemma 4.9, it follows that $(p - 1)|(n - 1)$. However, since

$$\frac{n-1}{p-1} = \frac{pq-1}{p-1} = q + \frac{q-1}{p-1}$$

it follows that $(p - 1)|(q - 1)$. But it is possible only when $p - 1 < q - 1$, i.e., $p < q$. This contradiction shows that *n* must have at least three distinct prime factors. ■

Lemma 4.11 *Each prime factor of every Carmichael number n is less than* $\sqrt{n}$.

Proof Let *p* be a prime factor of n. Then, we have

$$\frac{n-1}{p-1} = \frac{p(n/p)-1}{p-1} = \frac{(p-1)(n/p)+(n/p)-1}{p-1} = \frac{n}{p} + \frac{(n/p)-1}{p-1}.$$

This means $(p-1)|(n/p-1)$, and hence $n/p \geq p$. But this inequality must be strict, because otherwise $n = p^2$, which contradicts Lemma 4.6. Hence, $p < \sqrt{n}$. ■

From the above Lemmas, the following result is immediate:

Theorem 4.20 (Korselt Criterion) *A composite number n is Carmichael number iff (i) n is square-free and (ii) for every prime $p|n$, also* $(p-1)|(n-1)$.

- Alwin Reinhold Korselt (1864–1947, Germany) proved Theorem 4.20 in 1899, but gave no example. This result was rediscovered by Carmichael in his works of 1910 and 1912. He also found the first five Carmichael numbers $561, 1105, 1729, 2465, 2821$. However, in addition to these numbers, next two Carmichael numbers $6601\ and\ 8911$ already appeared in the work of Václav Šimerka (1819–1887, Bohemia). In 2008, Richard G.E. Pinch (born 1954, England) verified that there are 20,138,200 Carmichael numbers between 1 and 10^{21} (approximately one in 50 billion numbers), in fact, as numbers become larger, Carmichael numbers become very rare. In 1956, Erdös heuristically argued there should be infinitely many Carmichael numbers. If $C(x)$ denotes the number of Carmichael numbers not exceeding x, then for sufficiently large *x* he also showed that

$$C(x) \ < \ x \exp\left(\frac{-k_2 \ln x \ln \ln \ln x}{\ln \ln x}\right)$$

  for some constant k_2, which Pinch found to be 1. In 1994, William Robert Alford (1937–2003, USA), Andrew James Granville (born 1962, England), and Pomerance proved the existence of infinitely many Carmichael numbers. They also proved that for sufficiently large *x* the upper bound $C(x) > x^{2/7}$ holds. This bound was improved to $C(x) > x^{0.33336704}$ by Glyn Harman (born 1956, England) in 2008.
- In 1935, Jack Chernick (1911–1971, USA) proved a theorem which can be used to construct a subset *JC* of Carmichael numbers. The number $(6k+1)(12k+1)(18k+1)$ is a Carmichael number if its three factors are all prime. For example, $561 = 3 \cdot 11 \cdot 17$ and $1105 = 5 \cdot 13 \cdot 17$ are not in JC, whereas $1729 = 7 \cdot 13 \cdot 19$ is in JC. Carmichael numbers

with more than three prime factors are also known, e.g., least from 4 to 10 factors are: $41041 = 7 \cdot 11 \cdot 13 \cdot 41$, $825265 = 5 \cdot 7 \cdot 17 \cdot 19 \cdot 73$, $321197185 = 5 \cdot 19 \cdot 23 \cdot 29 \cdot 37 \cdot 137$, $5394826801 = 7 \cdot 13 \cdot 17 \cdot 23 \cdot 31 \cdot 67 \cdot 73$, $232250619601 = 7 \cdot 11 \cdot 13 \cdot 17 \cdot 31 \cdot 37 \cdot 73 \cdot 163$, $9746347772161 = 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 31 \cdot 37 \cdot 41 \cdot 641$, $1436697831295441 = 11 \cdot 13 \cdot 19 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 71 \cdot 127.$ In 1996, Günter Löh (Germany) and Wolgang Niebuhr (Germany) found huge Carmichael numbers including one with $1,101,518$ factors and over 16 million digits. It is not known whether there exist infinitely many Carmichael numbers with a fixed number of prime factors.
- The second Carmichael number 1105 can be expressed as the sum of two squares in more ways than any smaller number. In fact, $1105 = 32^2 + 9^2 = 31^2 + 12^2 = 24^2 + 23^2.$ It is also the magic number of a 13×13 magic square. The third Carmichael number 1729 is the Hardy-Ramanujan taxicab number.

4.22 Ruth-Aaron Pairs

For several years, the number 714 ranked as the record for home runs of George Herman “Babe” Ruth (1895–1948, USA). His record was broken only on April 8, 1974, in Atlanta, Georgia, when Henry Louis Aaron (1934–2021, USA) hit his 715th home run. This event attracted so much of excitement that the numbers 714 and 715 were on millions lips. In 1974 itself Carol Nelson (USA), David Penney (USA), and Pomerance discovered several interesting properties of the 714 and $715.$ Consider first seven primes: $2, 3, 5, 7, 11, 13, 17.$ The first, second, fourth, and seventh primes are $2, 3, 7,$ and $17,$ and the remaining primes (the third, fifth, and sixth) are $5, 11,$ and $13.$ If we add the integers $1, 2, 4,$ and $7,$ we get the same result as when we add $3, 5,$ and $6,$ namely, $1 + 2 + 4 + 7 = 3 + 5 + 6 = 14.$ It is interesting to note that summing the primes that correspond to these two sets of integers also give the same result: $2 + 3 + 7 + 17 = 5 + 11 + 13 \ = \ 29.$ While the sums of these primes are equal, in view of Theorem 4.1 their products, namely $2 \cdot 3 \cdot 7 \cdot 17$ and $5 \cdot 11 \cdot 13,$ cannot be equal. However, these products are surprisingly close since $2 \cdot 3 \cdot 7 \cdot 17 = 714$ and $5 \cdot 11 \cdot 13 = 715.$ The sum of these numbers $714 + 715 = 1429$ is the year Columbus discovered

America, and interestingly rearrangements of the digits of this year gives $9241,\ 1249,\ 9421,$ and 4219 which are all primes.

Let P_k denote the product of the first *k* primes. For example, $2 = P_1,\ 2 \cdot 3 = P_2,\ 2 \cdot 3 \cdot 5 = 5 \cdot 6 = P_3,\ 2 \cdot 3 \cdot 5 \cdot 7 = 14 \cdot 15 = P_4$, and $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 = 714 \cdot 715 = P_7$. Thus, there are pairs of consecutive numbers whose product is P_k for some k. Nelson, Penney , and Pomerance computationally showed that $P_1, P_2, P_3, P_4,$ and P_7 are the only P_k which can be expressed as the product of two consecutive numbers in the range $1 \leq k \leq 3029.$ From this, they concluded that if there is any other pair of consecutive integers whose product is a $P_k,$ then these integers exceed 10^{6021}. They conjectured that the largest pair of consecutive integers whose product is also the product of the first *k* primes for some *k* is 714 and $715.$ Let $S(n)$ denote the sum of the prime divisors of *n* with multiplicity. Two consecutive integers $n, n+1$ are called *Ruth-Aaron pairs* of integers if $S(n) = S(n+1),$ and *n* is called the *Ruth-Aaron number*. Since $S(714) = S(715) = 29$ integers 714 and 715 is a Ruth-Aaron pair. First ten Ruth-Aaron pairs are (5,6), (8,9), (15,16), (77,78), (125,126), (714,715), (948,949), (1330,1331), (1520,1521), (1862,1863). If only distinct prime factors are counted, then the first ten Ruth-Aaron pairs are (5,6), (24,25), (49,50), (77,78), (104,105), (153,154), (369,370), (492,493), (714,715), (1682,1683). In 1978, Erdös and Pomerance showed that such pairs of integers may appear to be rare, i.e., have density $0.$ They showed that the number of Ruth-Aaron numbers up to *x* is $O(x \ln\ln x \ln\ln\ln x / \ln x)$ which they believed can be improved to $O(x/\ln x).$ In 2002, Pomerance improved even this bound to $O(x(\ln\ln x)^4/(\ln x)^2),$ which established the fact that the sum of the reciprocals of the Ruth-Aaron numbers is bounded. In fact,

$$\sum_{n \text{ is a Ruth-Aaron number}} \frac{1}{n} \sim 0.4207.$$

However, it remains to prove that there are infinitely many Ruth-Aaron pairs.

- A few Ruth-Aaron triplets are also known: when counting distinct prime factors

$$\begin{aligned}89460294 &= 2\cdot 3\cdot 7\cdot 11\cdot 23\cdot 8419\\ 89460295 &= 5\cdot 4201\cdot 4259\\ 89460296 &= 2\cdot 2\cdot 2\cdot 31\cdot 43\cdot 8389\end{aligned}$$

and
$2+3+7+11+23+8419=5+4201+4259=2+31+43+8389=8465,$

$$\begin{aligned}151165960539 &= 3\cdot 11\cdot 11\cdot 83\cdot 2081\cdot 2411\\ 151165960540 &= 2\cdot 2\cdot 5\cdot 7\cdot 293\cdot 1193\cdot 3089\\ 151165960541 &= 23\cdot 29\cdot 157\cdot 359\cdot 4021\end{aligned}$$

and
$3+11+83+2081+2411=2+5+7+293+1193+3089=23+29+157+359+4021=4589.$

When counting distinct prime repeated factors

$$\begin{aligned}417162 &= 2\cdot 3\cdot 251\cdot 277\\ 417163 &= 17\cdot 53\cdot 463\\ 417164 &= 2\cdot 2\cdot 11\cdot 19\cdot 499\end{aligned}$$

and
$2+3+251+277=17+53+463=2+2+11+19+499=533,$

$$\begin{aligned}6913943284 &= 2\cdot 2\cdot 37\cdot 89\cdot 101\cdot 5197\\ 6913943285 &= 5\cdot 283\cdot 1259\cdot 3881\\ 6913943286 &= 2\cdot 3\cdot 167\cdot 2549\cdot 2707\end{aligned}$$

and
$2+2+37+89+101+5197=5+283+1259+3881=2+3+167+2549+2707=5428.$

- We have seen that there is a partition of the set of the first seven primes into two subsets such that the sums of the elements in these two subsets are equal. However, this partition is not unique, e.g., we also have $2+3+11+13=5+7+17=29,$ but the sum remains the same. We have the following general result: When $n=2k+1$ is odd, prime list $P[n]$ can be partitioned into two nonoverlapping sublists, in which each sublist has equal sum total $[P[n]]/2.$ For example, the sum of first 25 primes is 1060, and the sum of one of the possible patricians is
$2+7+11+19+23+29+31+37+41+43+47+53+59+61+67=3+5+13+17+71+73+79+83+89+97=530.$
When $n=2k$ is even, prime list $P[n]$ can be partitioned into two nonoverlapping sublists, one sublist's sum is (total$[P[n]]-1)/2,$ the other's is (Total$[P[n]]+1)/2$). For example, the sum of first 24 primes is 963, and the sum of one of the possible patricians is

$3+5+13+17+19+23+29+31+37+41+43+47+53+59+61=481$ and $2+7+11+67+71+73+79+83+89=482$.

4.23 Special Prime Numbers

Several subsets of all prime numbers have been studied with a great interest. We discuss a few of them.

- Bell Primes: A partition of a set is a grouping of its elements into non-empty subsets, in such a way that every element is included in exactly one subset. For example, the set $\{1,2,3\}$ has the five partitions $\{\{1\},\{2\},\{3\}\},\ \{\{1,2\},\{3\}\},\ \{\{1,3\},\{2\}\},\ \{\{1\},\{2,3\}\},\ \{\{1,2,3\}\}$. The Bell numbers (after Eric Bell, 1938) denoted as B_n counts the number of different ways to partition a set that has exactly *n* elements. The first systematic study of Bell numbers was made by Ramanujan about 25–30 years prior to Bell's work, also the roots of these numbers go back further to medieval Japan. Starting with $B_0 = 1$ first fourteen Bell numbers are $1,1,2,5,15,52,203,877,4140,21147,115975,678570,4213597,27644437$. The Bell numbers satisfy the recurrence relation

$$B_{n+1} \ = \ \sum_{k=0}^{n} \binom{n}{k} B_k, \quad n \geq 1.$$

  Among B_n's the numbers which are prime are called Bell primes. Out of the first 14 Bell numbers, only four numbers $2,5,877,27644437$ are prime. The next two Bell primes are

$$\begin{aligned} B_{42} &= 35742549198872617291353508656626642567 \\ B_{55} &= 359334085968622831041960188598043661065388726959079837. \end{aligned}$$

  The largest known Bell prime $B_{2841} \sim 9.30740105 \times 10^{6538}$ was discovered in 2002 and confirmed in 2004 by Ignacio Larrosa Cañestro (Spain). In 1978, Gardner raised the question of whether infinitely many Bell numbers are also prime numbers.

- Sophie Germain Primes: A prime *n* is said to be a Sophie Germain prime if both *n* and $2n+1$ (known as safe prime) are prime. The first ten Sophie Germain primes are $2,3,5,11,23,29,41,53,83,89$. The largest known Sophie Germain prime pair $(n, 2n+1)$ is

$n = 2618163402417 \times 2^{1290000} - 1$ which was track down by James Scott Brown (USA) in 2016. In 1998, [265] Paul Hoffman (born 1956, USA) cast doubt if there are infinite numbers of Sophie Germain primes. In 2022, Marko Jankovic (born 1968, Serbia) has claimed to have proved Hoffman's proposition, see https://hal.science/hal-02169242v12. A generalization of Sophie Germain primes was introduced by Allan Joseph Champneys Cunningham (1842–1928, India-England): A Cunningham chain of the *first kind* of length *n* is a sequence of prime numbers $(p_1, \cdots, p_n)$ recursively defined by $p_i = 2^{i-1}p_1 + (2^{i-1} - 1),\ 1 \le i \le n.$ Similarly, a Cunningham chain of the *second kind* of length *n* is defined by $p_i = 2^{i-1}p_1 - (2^{i-1} - 1),\ 1 \le i \le n.$ A Cunningham chain is called complete if it cannot be further extended. The smallest prime beginning a complete Cunningham chain of the first kind of lengths $1 \le n \le 10$ are $13, 3, 41, 509, 2, 89, 1122659, 19099919, 85864769, 26089808579.$ The smallest prime beginning a complete Cunningham chain of the second kind of lengths $1 \le n \le 10$ are $11, 7, 2, 2131, 1531, 33301, 16651, 15514861, 857095381, 205528443121.$ The longest known Cunningham chains of first kind is of length 18 starting with 658189097608811942204322721, and of second kind is of length 19 starting with 79910197721667870187016101, both were obtained by Raanan Chermoni (Israel) and Jaroslaw Wroblewski (Poland) in 2014. It is widely believed that for every *n*, there are infinitely many Cunningham chains of length n.

- Balanced Primes: Primes with equal-sized prime gaps above and below them, so that they are equal to the arithmetic mean of the nearest primes above and below, i.e., $p_n = (p_{n-1} + p_{n+1})/2.$ For example, $p_{108} = (p_{107} + p_{109})/2 = (587 + 599)/2 = 593.$ The first ten balanced primes are $5, 53, 157, 173, 211, 257, 263, 373, 563, 593.$ As of March 2023, the largest known 15004 digits balanced prime $p_n = 2494779036241 \times 2^{49800} + 7,\ p_{n-1} = p_n - 6,\ p_{n+1} = p_n + 6$ is due to Batalov. It is conjectured that there are infinitely many balanced primes. A prime is called *weaker* (*stronger*) if the arithmetic mean of its two neighboring primes is smaller (greater). For example, $p_{33} = 137 > (p_{32} + p_{34})/2 = (131 + 139)/2 = 135,$ so p_{33} is a weaker prime, and $p_4 = 7 < (p_3 + p_5)/2 = (5 + 11)/2 = 8,$ so p_4 is a stronger prime.

- Eisenstein Real Primes: These are primes of the form $3n-1,\ n\geq 1$ or $3n+2,\ n\geq 0.$ The first ten Eisenstein primes are $2,5,11,17,23,29,41,47,53,59.$ As of March 2023, the largest known real Eisenstein prime is the ninth largest known prime $10223\times 2^{31172165}+1$ of 9383761 digits discovered by Péter Szabolcs (Hungary). All larger known primes are Mersenne primes. It is clear that there are infinite Eisenstein real primes.
- Primorial Primes: These primes are of the form $p_n^{\#}\pm 1,$ where $p_n^{\#}$ is the primorial (defined by Harvey Dubner, 1928–2019, USA) of $p_n,$ i.e., the product of the first *n* primes. Primes of the form $p_n^{\#}+1$ have been named as *Euclid primes*. The first ten *n* for which $p_n^{\#}-1$ is prime are $2,3,5,6,13,24,66,68,167,287$ and the first ten *n* for which $p_n^{\#}+1$ is prime are $0,1,2,3,4,5,11,75,171,172$ (here $p_0^{\#}=1$ because of the empty product). The first ten primorial primes are $2,3,5,7,29,31,211,2309,2311,30029.$ The largest known primorial prime of the form $p_n^{\#}-1$ is $3267113^{\#}-1\ (n=234725)$ with 1418398 digits, found in 2021 by the PrimeGrid project. As of March 2023, the largest known primorial prime of the form $p_n^{\#}+1$ is $392113^{\#}+1\ (n=33,237)$ with 169966 digits by Daniel Heuer (USA). It is not known whether there is an infinite number of primorial primes; however, such primes do seem to become scarce among them as *n* gets large. *Generalized Fermat primes* defined by Ribenboim in 1996 are of the form $a^{2^n}+1,\ (a>2).$ The 5th largest known such prime is $1372930^{2^{17}}+1$ with 804474 digits, discovered by Heuer.
- A *Fortunate number*, named after Reo Franklin Fortune (1903–1979, New Zealand) is the smallest integer $m>1$ such that, for a given positive integer $n,\ p_n^{\#}+m$ is a prime number. The first ten fortunate numbers for the first primorial are 3, 5, 7, 13, 23, 17, 19, 23, 37, 61. It has been conjectures that the only fortunate numbers that are also prime are 3, 5, 7, 13, 17, 19, 23, 37, 47, 59, 61, 67, 71, 79, 89, 101, 103, 107, 109, 127, 151, 157, 163, 167, 191, 197, 199, 223, 229, 233, 239, 271, 277, 283, 293, 307, 311, 313, 331, 353, 373, 379, 383, 397 (see Guy [241]).
- Among the first 30 happy numbers *happy primes* are 7, 13, 19, 23, 31, 79, 97, 103, 109, 139, 167. The palindromic prime $10150006+7426247\times 1075000+1$ is a happy prime with 150007

digits, discovered by Jobling in 2005. The largest known Mersenne prime that is also a happy prime is $2^{42643801}-1$, which has 12837064 digits, discovered by Chris Caldwell (USA).

- A *good prime* is a prime number whose square is greater than the product of any two primes at the same number of positions before and after it in the sequence of primes, i.e., $p_n^2 > p_{n-i}p_{n+i}$ for all $1 \leq i \leq n-1.$ The first ten good primes are $5, 11, 17, 29, 37, 41, 53, 59, 67, 71.$ In 2011, Pomerance proved that there are infinite number of good primes.
- Higgs introduced *Higgs primes* as follows: $p_{n+1}^H =$ smallest prime $> p_n$ such that $p_{n+1}^H - 1$ divides the product $(p_1 \cdots p_n)^2.$ The first ten Higgs primes are $2, 3, 5, 7, 11, 13, 19, 23, 29, 31.$ For example, 19 is a Higgs prime because $(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17)^2 = 260620460100$ is divisible by $18.$ However, 17 is not a Higgs prime because $(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13)^2 = 901800900$ which is not divisible by 16. It is not known if there are infinitely many Higgs primes, the same uncertainty holds if in the definition power 2 is replaced by any other integer greater than 2. However, if the power is just 1, then there are only four 2, 3, 7, and 43 such primes.
- *Ramanujan Primes*: In Sect. 4.2 we have remarked that in 1919 Ramanujan sharpened Bertrand's postulate. He showed that $\pi(x) - \pi(x/2) \geq 1, 2, 3, 4, 5, \cdots$ for all $x \geq 2, 11, 17, 29, 41, \cdots$. The converse of his result leads to the definition of Ramanujan primes: The *n*th Ramanujan prime is the least integer R_n for which $\pi(x) - \pi(x/2) \geq n$ for all $x \geq R_n.$ Thus, Ramanujan primes are the least integers R_n for which there are at least *n* primes between $x/2$ and *x* for all $x \geq R_n.$ The first ten Ramanujan primes are $2, 11, 17, 29, 41, 47, 59, 67, 71, 97.$ The smallest such number R_n must be prime, since the function $\pi(x) - \pi(x/2)$ can increase only at a prime. The case $\pi(x) - \pi(x/2) \geq 1$ for all $x \geq 2$ is Bertrand's postulate. In 2009 [491], Jonathan Sondow (1943–2020, USA) showed that for all $n \geq 1,\ 2n \ln 2n < R_n < 4n \ln 4n,$ and as *n* tends to infinity, R_n is asymptotic to $p_{2n},$ i.e., as $n \to \infty,\ \ R_n \to p_{2n}.$ For $n > 1,$ he also proved the inequality $p_{2n} < R_n$ and conjectures that $R_n < p_{3n},$ which was proved in 2010 by Shanta Laishram (India). In 2011, Sondow, John

Nicholson (USA), and Tony Noe (USA) improved this upper bound to $R_n \leq (41/47)p_{3n}$ which is optimal since the equality holds for $n = 5$.

4.24 What Is the Necessity to Find Next Larger Prime Number?

Throughout the history of prime numbers, strenuous efforts have been put down to find incredible size of giant prime numbers. Here are two main reasons:

1. For the Glory: The following extraordinary endeavors of the curious and courageous individuals will always be remembered and admired: Charles Augustus Lindbergh (1902–1974, USA) made the first solo flight across the Atlantic in 1927, had to solve many problems, such as the route over the ocean and the financing of the trip. Most importantly, the aircraft design had to be modified to carry enough fuel for the 3600–mile trip. To keep the load light, Lindbergh flew without a copilot, parachute, or radio and carried only five sandwiches and a quart of water on his 33-hour, 32-minute trip. On May 29, 1953, Edmund Percival Hillary (1919–2008, New Zealand) took the first step onto the summit of Mount Everest (the highest point on Earth, 29,028 ft) and his companion Tenzing Norgay (1914–1986, Nepal-India) also referred to as Sherpa Tenzing followed him. They spent about 15 minutes at the summit. On July 21 , 1969, Armstrong became the first person to walk on the Moon. Behind their glory, the rationale is the priority of achieving something sensational which required appropriate resources, time, monumental efforts, and luck.

   In mathematics, Euclid is recognized incessantly for his *Elements* because his work set the guidelines how mathematical proofs must be reported. Specially his monumental Theorem 4.2 ensures the existence of infinite number of prime numbers but gives no indication how to find them. Eratosthenes is consistently remembered for developing the first coherent process to separate the primes from the composite numbers up to 100. As we have noticed his method, Kulik used to find all primes less than $100,000,000$. Starting from Mersenne, Fermat, and Euler several prominent mathematicians tried to find formulas or the recurrence relations to generate, if not all, infinite number of primes; however, so far only futile theoretical results are known. Since the invention of computer technology, mathematicians and computer scientists have diverted their mind to find giant prime numbers. Several records have been made and broken, each time the person who discovers the next larger prime number gets glory (at least for few years) and finds a place in the literature. For example, in 1963, Donald Bruce Gillies (1928–1975, Canada-USA) showed that $M_{11213} = 2^{11213} - 1$ is prime, which was considered a great discovery, and used as a postage stamp in USA until M_{19937} was proved to be prime. In 2018, Patrick Laroche broke all previous records by showing that the number $M_{82589933}$ is prime, which will remain the largest till another giant prime is known. While for Lindbergh, Hillary, and Armstrong glory is sustaining, for the discoverers of giant primes it is temporary, more a curiosity, and spirit of man. What is important, uncovering monster prime numbers demand new and faster algorithm, and advanced technology, so that multiplication of large integers is possible, which keeps mathematicians and computer scientist busy. The Electronic Frontier Foundation has offered US$100,000 to anyone who can find the first 100–million–digit prime number!

2. Real Applications: Hardy is reported to have made the toast: “Here’s to pure mathematics! May it never have any use.” According to Dickson “Thank God that number theory is unsullied by applications.” It is true that in the past number theorists worked in this field because of its intrinsic interest and its distinctive beauty—and they did not care one way, or the other, whether their elegant theorems would, or would not have “useful” applications. However, in 1974, Donald Ervin Knuth (born 1938, USA) “father of the analysis of algorithms” vigorously said “$\cdots$ virtually every theorem in elementary number theory arises in a natural, motivated way in connection with the problem of making computers do high-speed numerical calculations.” In fact, since the early 1980s, topics in number theory such as congruences and recurrence relations are considered part of elementary discrete mathematics courses for computer scientists. In 1972, Stanislaw Krystyn Zaremba (1903–1990, Poland) collected in his edited book Applications of Number Theory to Numerical Analysis, Academic Press, New York, i.e., applications of number theory to the continuous problems. Prime numbers of 10 to 15 digits are useful in multiple-modulus residue arithmetic, floating-point modular arithmetic, finite field arithmetic, the Chinese remainder theorem for error-free

computation. For further methods and applications of error-free computation, see Gregory and Krishnamurthy [227]. In recent years, prime numbers have also been used in acoustics, boundary value problems, Brownian motion, modern coding, and cryptography (particularly, Sophie Germain primes) communications, computer security, cryptocurrency, data analysis, error-correcting codes, image compression, random number generation, digital signal processing, test of the hardware, and the list continues.

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5. Pythagorean Theorem

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5.1 Introduction

In (two-dimensional) Euclidean geometry *Pythagorean theorem*, also known as *Pythagoras's theorem*, states that: If a and b are the lengths of the two legs of a right triangle, and c is the length of the hypotenuse (Greek word with meaning: The side opposite to the right angle), then the sum of the areas of the two squares on the legs equals the area of the square on the hypotenuse (see Fig. 5.1), i.e.,

$$a^2 + b^2 = c^2 \tag{5.1}$$

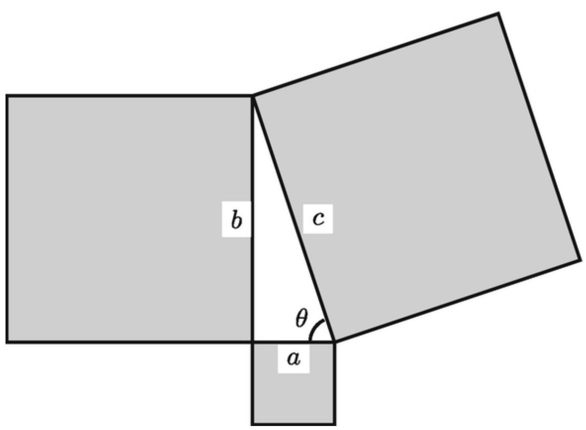

Fig. 5.1 Pythagorean theorem

Pythagorean theorem is an inherent union between geometry and arithmetic, which benefits use as we learn different ways to improve our society and infrastructure (see Agarwal [17]). In fact, it serves as the cornerstone of the Euclidean distance formula: If (x_1, y_1) and (x_2, y_2) are the Cartesian coordinates (due to Descartes) of two points *p* and *q* in a plane, then the Euclidean distance between these points is the length of the line segment given by $d(p, q) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. In (5.1), if we let $a = (x - y)/2, b = \sqrt{xy}, c = (x + y)/2,$ where $x > y > 0$, then from $c > b$ it immediately follows that AM $= (x + y)/2 > \sqrt{xy} =$ GM. In the literature, the fundamental/elegant relation (equation) (5.1) is often called *Pythagorean relation* (equation). From this relation, it is immediate that in any right triangle, the hypotenuse is greater than any of the other sides, but less than their sum. Further, if the length of any two sides is known, the length of the third side can be calculated. For a given complex number $z = x + iy$, the absolute value (modulus) is given by $r = |z| = \sqrt{x^2 + y^2}$, and hence, the numbers $x, y,$ and *r* satisfy the Pythagorean relation $r^2 = x^2 + y^2$. Here *r* is a nonnegative number, representing the distance from the origin to *z* in the complex plane, but *x* and *y* can be negative numbers. Pythagorean theorem is familiar (known by heart) to many people who studied it in high school.

According to Kepler, “Geometry has two great treasures: one is the theorem of Pythagoras, the other the division of a line into extreme and mean ratio (*golden section/ratio*). The first we may compare to a measure of gold; the second to a precious jewel,” whereas Lewis Carroll commented in 1895 “It is as dazzlingly beautiful now as it was in the day when Pythagoras first discovered it.” Further appreciating (5.1), Dantzig in 1955 wrote “No other proposition of geometry has exerted so much influence on so many branches of mathematics as has the simple quadratic formula known as the Pythagorean theorem. Indeed, much of the history of classical mathematics, and of modern mathematics, too, could be written around that proposition,” and Jacob Bronowski (1908–1974, Poland-England) commented “To this day, theorem of Pythagoras remains the most important single theorem in the whole of mathematics.” Michio Kaku (born 1947, USA) has reported “The Pythagorean theorem, of course, is the foundation of all architecture: every structure built on this planet is based on it.”

In recent years, Pythagorean theorem has been successfully applied in various branches of mathematics such as discrete, combinatorial, and computational geometry, e.g., in combinatorics to prove the famous Sylvester-Gallai-Erdös theorem (Tibor Gallai, 1912–1992, Hungary) Pythagorean theorem has been used: "Let *n* points be given in a plane, not all on a line. Join every pair of points by a line. At least *n* distinct lines are obtained in this way." Nicaragua issued a series of ten stamps commemorating mathematical formulas, including the Pythagorean theorem. In a survey in 2004, in the Journal Physical World, (5.1) ranked fourth place among the 20 most beautiful equations in science (Euler's equation (2.22) ranked first). Undoubtedly, if one has to select a mathematical theorem that enjoys "perpetual youth" which has a very long history as well as deep significance unto this day, the Pythagorean theorem is a robust nominee. It is of vital importance in problems ranging from carpentry and navigation to astronomy. However, Pythagorean theorem horribly mangled by the Scarecrow in *The Wizard of Oz*. One cannot think trigonometry without Pythagorean theorem, since trigonometric (circular, angle, or goniometric) functions are rather easily defined based on the sides of a right-angled triangle.

From the relation (5.1), the trigonometric identities so called *Pythagorean identities* $\cos^2\theta + \sin^2\theta = 1,\ 1 + \tan^2\theta = \sec^2\theta$ and $\cot^2\theta + 1 = \csc^2\theta$ are immediate. (The English word sine comes from a series of mistranslations of the Sanskrit jy$\overline{\text{a}}$-ardha (chord-half). Aryabhata frequently abbreviated this term to jy$\overline{\text{a}}$ or its synonym J$\overline{\text{i}}$v$\overline{\text{a}}$. When some of the Hindu works were later translated into Arabic, the word was simply transcribed phonetically into an otherwise meaningless Arabic word *jiba*. However, since Arabic is written without vowels, later writers interpreted the consonants *jb* as *jaib*, which means bosom or breast. In the twelfth century when an Arabic work of trigonometry was translated into Latin, the translator used the equivalent Latin word *sinus*, which means almost meant bosom, and by extension, fold (as a toga over a breast), or a bay or gulf. This Latin word has now become our English sine. The first abbreviation of sine to sin is due to Edmund Gunter (1581–1626, England) in 1624. Similarly, the Sanskrit word kotijy$\overline{\text{a}}$ in English has become cosine. The tangent, cotangent, secant, and cosecant functions made their appearance in Islamic works in the ninth century, perhaps earliest in the works of

Ahmad ibn Ábdallāh al-Marwazī Habas al-Hāsib (around 770–870) and Al-Battani, although the tangent function had already been used in China in the eighth century. An extensive discussion of these functions is available in the work of al-Biruni.) The pictorial representation, Fig. 5.1, of the Pythagorean theorem is known under many names, for example, bride's chair, Franciscan's cowl, the goose foot, the peacock's tail, the windmill, and the chase of the little married women.

According to one of the fables, "Pythagoras discovered *his* theorem while waiting in a palace hall to be received by Polycrates. Being bored, Pythagoras studied the stone square tiling of the floor and imagined the right triangles (half-squares) *hidden* in the tiling together with the squares erected over its sides. Having *seen* that the area of a square over the hypotenuse is equal to the sum of areas of squares over the legs, Pythagoras came to think that the same might also be true when the legs have unequal lengths." Throughout the history of mathematics, it has been claimed that Pythagoras (which made him immortal) gave first proof of Pythagorean theorem by deductive method. However, the earliest known mention of Pythagoras's name in connection with the theorem occurred five centuries after his death, in the writings of Marcus Tullius Cicero (106–43 BC, Italy) and Plutarch. It is very likely that one of the Pythagoreans proved the theorem, and as it was common in the ancient world, particularly in the Asian culture, out of respect for their leader, credited the proof to his famous teacher.

This result has been recorded as the Proposition 47 in Book I of Euclid's *Elements*. In Elements, the proposition reads: "In right-angled triangles the square on the side subtending the right angle is equal to the sum of the squares on the sides containing the right angle." Euclid provides two proofs of this proposition, first in Book I and second in Book VI. Its first proof uses knowledge about congruent triangles, and although it is not too demanding, many readers are puzzled by the strangeness of the acquired relations. For this proof, philosopher Arthur Schopenhauer (1788–1860, Germany) wrote "the same uncomfortable feeling that we experience after a juggling trick." But the well-known seventeenth-century English philosopher Thomas Hobbes (1588–1679), who never studied geometry, admired this proof. At the age of 40, Hobbes came across this theorem quite by chance on the page of an opened book while waiting at his friend's study. He wondered how it could that be possible. The proof, however, referred to a previous proposition whose proof in turn referred to more preceding propositions. After several hours of detailed investigation, he was finally convinced of the truth of Proposition 47. Hobbes not only finished Book I of Euclid's Elements, but started his life-long love for geometry. Denis Henrion (1580–1632, France) in 1615, commented: "Now it is said that this celebrated and very famous theorem was discovered by Pythagoras, who was so full of joy at his discovery that, as some say, he showed his gratitude to the Gods by sacrificing a Hecatomb of oxen. Others say he only sacrificed one ox..." But this is nonsense because being a follower of Buddha, Pythagoras must have been very scrupulous about shedding the blood of animals. In fact, Eudoxus writes "Pythagoras was distinguished by such purity and so avoided killing and killers that he not only abstained from animal foods, but even kept his distance from cooks and hunters."

5.2 Origin of Pythagorean Theorem

Pythagorean theorem was certainly known much before fourth century BC. Baudhayana contains one of the earliest references to this theorem (with a convincing valid proof): a rope that is stretched across the diagonal of a square produces an area double the size of the original square. This is a special case of the Pythagorean theorem for a 45° right triangle. Egyptian civilizations around 2500 BC used ropes to measure out distances to form right triangles that were in whole number ratios (Berlin Papyrus 6619, around 1990–1800 BC, Pyramids, and Cairo Mathematical Papyrus, unearthed in 1938 and first examined in 1962, dating from the early Ptolemeic dynasties founded in 305 BC). However, some prominent historians of mathematics: van der Waerden [526], Dirk Jan Struik (1894–2000, The Netherlands-USA) [500], and Heath [259] have suggested that Egyptians had no knowledge of Pythagorean theorem. There is sufficient evidence that Pythagorean theorem was known to Mesopotamian (tablet number 7289 in the Babylonian Collection of Yale University famous as "YBC 7289," and tablet number 322 in the Babylonian Collection of Columbia University popular as "Plimpton 322", written between 1790 and 1750 BC, during the time of Hammurabi, which was discovered by Edgar James Banks (1866–1945, USA) shortly after 1900, and sold to George Arthur Plimpton (1855–1936, USA) in 1922, for $10). Of more mathematical interest is a group of tablets uncovered by the French at Susa (Iraq) in 1936. These provide some of the oldest Babylonian examples of the use of the theorem of Pythagoras. One tablet computes the radius *r* of a circle that circumscribes an isosceles triangle of sides, $50, 50,$ and $60.$

The Apastamba gives a general statement of Pythagoras's theorem: The diagonal of a rectangle produces the sum of what the largest and the smallest sides produce separately. Apastamba was also familiar with the result that as a special case of this theorem, the diagonal of a square is the side of a square with twice the area of the original one. The Katyayana, written later, gives a more general version of the Pythagorean theorem: a rope that is stretched along the length of the diagonal of a rectangle produces an area that the vertical and horizontal sides make together. In other words, the square of the hypotenuse equals the sum of the squares of the sides. Chinese mathematician Tschou-Gun who lived in 1100 BC knew the characteristics of the right angle. In Chinese literature, the Pythagorean theorem is known as Gougu theorem (in Chinese gou means base, gu stands for shorter leg, and hypotenuse is called xian) and Shang Gao theorem (named after the Duke of Zhou's astronomer and mathematician). The theorem was also known to the Caldeans more than a thousand years before Pythagoras. Before and after Pythagoras, this theorem has been given numerous logically correct different proofs (almost 500)—possibly the most for any mathematical theorem (several false, and with little or no variations, proofs have also been published). These proofs are very diverse, including both geometric and algebraic proofs, some make use of vectors, while others are demonstrations based on physical devices. Some of these proofs are extremely complicated, while others are astonishingly simple. The English language edition of Euclid's *Elements* by Oliver Byrne (1810–1880, Ireland-England), described by one critic as "one of the oddest and most beautiful books of the whole (nineteenth) century," uses color and shapes to illustrate a proof of the Pythagorean theorem.

A lifelong project of Elisha Scott Loomis (1852–1940, USA) [350], a mathematics teacher, was to publish all available demonstrations of Pythagorean theorem in his book *Pythagorean Proposition* in 1927, which was written in 1907 and revised in 1940, the year of his death. The revised edition contains 371 proofs, a "Pythagorean Curiosity," "five Pythagorean magic squares," and an extensive bibliography. National Council of Teachers of Mathematics (Washington, D.C.) republished this book in 1968. According to him, "The Pythagorean theorem is regarded as the most fascinating Theorem of all of Euclid, so much so, that thinkers from all classes and nationalities, from the aged philosopher in his armchair to the young soldier in the trenches next to no-man's land have whiled away hours seeking a new proof of its truth." This book includes proofs of those of Leonardo da Vinci, a blind girl Miss E.A. Coolidge in 1888, a 16-year-old high school student Miss Ann Condit in 1938, and by the United States Representative Garfield, 5 years before he became the 20th President of the United States in 1881. In his book, Loomis remarked that in the Middle Ages (fifth to fifteenth century), it was required that a student taking Master's degree in mathematics offer a new and original proof of the Pythagorean theorem; this, he asserts that, has resulted in several new proofs. In the Foreword, the author rightly declares that the number of algebraic proofs is limitless as is also the number of geometric proofs, but that the proposition admits no trigonometric proof. However, in 2009, Jason Zimba (USA) gave a very clever trigonometric proof, which is followed by more trigonometric proofs by David Houston (USA) and Luc Gheysens (USA). Several websites (e.g., see [549]) deal with Pythagorean theorem and give fairly decent update of this theorem; however, https://www.cut-the-knot.org/pythagoras/ found by Alexander Bogomolny (1948–2018, Russia-Israel-USA) in 1996, is particularly interesting as it provides 118 different proofs. We also refer to the additional information provided in the monograph of Agarwal and Sen [14], and papers of Siu [481, 482] and Veljan [518].

5.3 Converse of Pythagorean Theorem

Euclid's *Elements* (Book I, Proposition 48) reads "If in a triangle the square on one of the sides equals the sum of the squares on the remaining two sides of the triangle, then the angle contained by the remaining two sides of the triangle is right." Thus, for any three positive numbers a, b and c such that $a^2 + b^2 = c^2$, there exists a triangle with sides a, b and c, and this triangle has a right angle between the sides of lengths a and b. The proof is by contradiction. Assume that the triangle has sides a, b, c such that $c^2 = a^2 + b^2$. We construct a right triangle with sides a and b and assume its hypotenuse to be d. But then by the Pythagorean theorem $a^2 + b^2 = d^2$, and this implies $a^2 + b^2 = c^2 = d^2$, and hence, $d = c$. Thus, for both the triangles, all the three sides are equal, and therefore, these triangles are congruent. Since (a, b, d) is a right triangle, the triangle (a, b, c) must also be a right triangle.

The aforementioned proof requires Pythagorean theorem; however, using several known results from geometry, the converse has also been proved by Stephen Casey [119] in 2008 (also see the work of Macro [355] in 1973) without employing the Pythagorean theorem. Here we give such an ingenious proof, which is due to Douglas Mitchell [374] in 2009.

We multiply each side of the triangle ABC by c and use $a^2 + b^2 = c^2$, to obtain a similar triangle GHI (see Fig. 5.2). Now by SAS (side angle side) postulate, $\triangle IHJ$ is congruent to $\triangle ABC$ scaled by the factor a, thus $\angle HJI = \angle BCA$. Similarly, by SAS postulate $\triangle GIJ$ is congruent to $\triangle ABC$ scaled up by the factor b, so $\angle IJG = \angle BCA$. This leads to $\angle HJI = \angle BCA = \angle IJG$ and since *HJG* is a side of $\triangle GHI$ it follows that $\angle HJI + \angle IJG = \pi$. But, then $\angle BCA = \pi/2$.

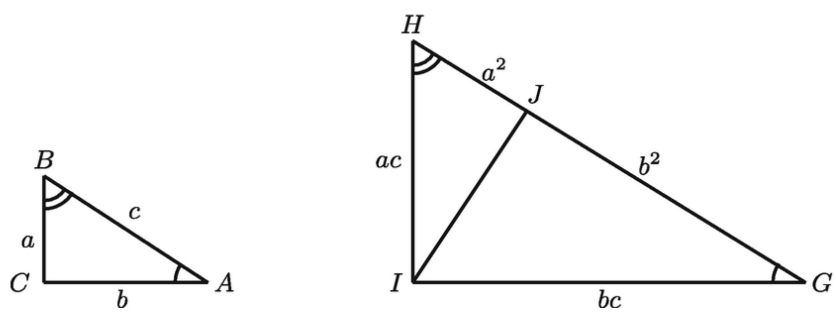

Fig. 5.2 Converse of Pythagorean theorem

As a consequence of the Pythagorean theorem's converse, we can determine whether a triangle is acute, right, or obtuse, as follows: Let c be chosen to be the longest of the three sides a, b, c and $a + b > c$. Then, the following Ernest Julius Wilczynski's (1876–1932, USA) statements of 1914 hold:

If $a^2 + b^2 > c^2$, then the triangle is acute.
If $a^2 + b^2 = c^2$, then the triangle is right.
If $a^2 + b^2 < c^2$, then the triangle is obtuse.

Dijkstra in [166] combined these statements in the following relation

$$\text{sgn}\ (\alpha + \beta - \gamma)\ =\ \text{sgn}\ (a^2 + b^2 - c^2),$$

where α is the angle opposite to side $a,\ \beta$ is the angle opposite to side $b,\ \gamma$ is the angle opposite to side c, and $\text{sgn}\ (x) = \begin{cases} 1 & \text{if}\ \ x > 0 \\ 0 & \text{if}\ \ x = 0 \\ -1 & \text{if}\ \ x < 0. \end{cases}$

5.4 Hippocrates's Generalizations of Pythagorean Theorem

A nontrivial generalization of Pythagorean theorem far off the areas of squares on the three sides (see Fig. 5.1) to similar figures (figures which are of the same shape, but not necessarily of the same size, for example, two *N*-sided polygons are similar if the ratios of their corresponding sides are all equal) was known to Hippocrates, see Figs. 5.3 and 5.4 (multiplying relation (5.1) by $\pi/2,$ Fig. 5.3 immediately follows). Recall that the area *A* of a regular polygon is $A = S^2N/4\tan{(180/N)},$ where *S* is the length of any side, *N* is the number of sides, and $\tan$ is the tangent function calculated in degrees. Further, the length of a side s_{2n} of a 2*n*-sided regular polygon circumscribing a circle of radius 1 in terms of the length of a side s_n of an *n*-sided circumscribing polygon is $s_{2n} = (2\sqrt{4 + s_n^2} - 4)/s_n$. Archimedes used Pythagorean theorem to obtain this formula and employed it to a series of inscribed and circumscribing polygons to compute an approximate value of $\pi,$ he showed that $3\frac{10}{71} < \pi < 3\frac{10}{70},$ see Sect. 8.11. Hippocrates's result is included in Euclid's Elements in Book VI as Proposition VI 31. It reads "In right-angled triangles the figure on the side subtending the right angle is equal to the similar and similarly described figures on the sides containing the right angle," (see Heath [257]). This extension presumes that the sides of the original triangle are the corresponding sides of the three similar figures (so the common ratios of sides between the similar figures are $a : b : c$). Euclid's proof applies only to convex polygons; however, in 2003, John Frank Putz (1921–2022, USA) and Timothy Sipka (USA) [422] have shown that the result also applies to concave polygons and even to similar figures that have curved boundaries. To show this result for a simple case, we recall that the area of a plane figure is proportional to the square of any linear dimension and in particular is proportional to the square of the length of any side. Now we erect similar figures with areas A, B and C on sides with corresponding lengths a, b and c.

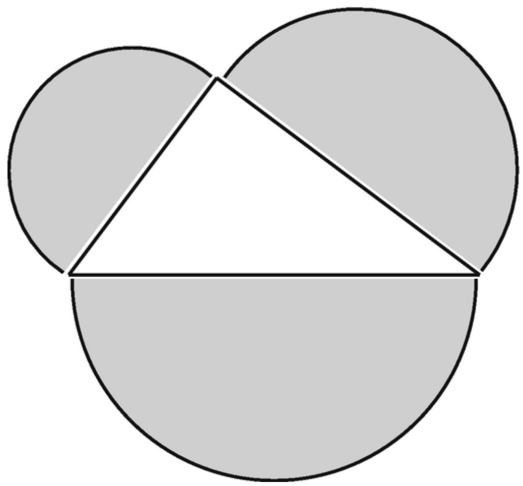

Fig. 5.3 Hippocrates's generalization 1 of Pythagorean theorem

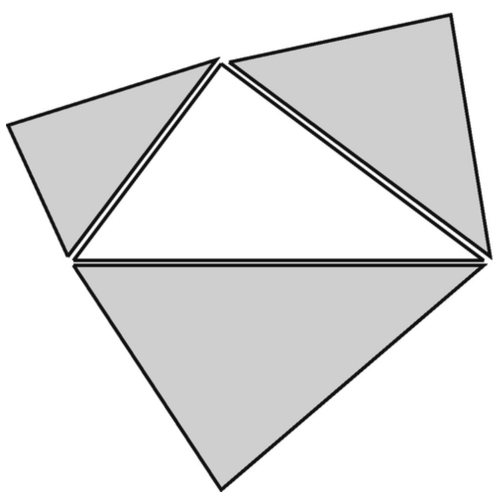

Fig. 5.4 Hippocrates generalization 2 of Pythagorean theorem

Then, it follows that $A/a^2 = B/b^2 = C/c^2$, which implies

$$A+B = \frac{a^2}{c^2}C+\frac{b^2}{c^2}C = \frac{a^2+b^2}{c^2}C = C.$$

Conversely, as for the converse of Pythagorean theorem, if the sides of a triangle are corresponding parts in three similar figures such that the area of one is the sum of the areas of the other two, then the triangle is a right triangle.

Reconstructing Fig. 5.3 as Fig. 5.5, we find Hippocrates's famous result: The sum of the areas of two lunes (a lune is basically a crescent-shaped figure that is attached by the arcs of two circles) is equal to the area of the triangle, i.e., Area of $I+$ Area of $II=$ Area of A. Encouraged with this result, Hippocrates unsuccessfully tried to square the circle.

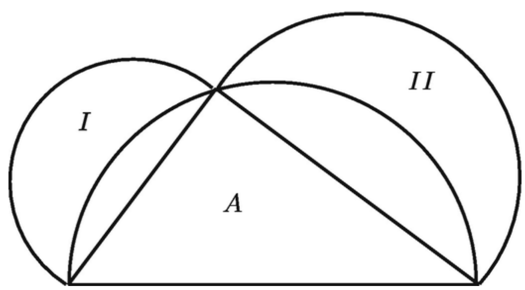

Fig. 5.5 Areas of two lunes

It is interesting to note that Hippocrates gave the first example of constructing a rectilinear area equal to an area bounded by one or more curves. It is known that only five particular lunes can be squared by Euclidean tools. Three of these were described by Hippocrates himself, and two more were discovered in 1771 by Euler, but according to Heath, all five squarable lunes were given in a dissertation by Martin Johan Wallenius (1731–1773, Finland) in 1766. In 1934, N.G. Tschebatorew and in 1947, A.W. Dorodnow completed the study of quadrable lunules, proving that these five are in fact the only ones in existence.

5.5 Historical Proofs of Pythagorean Theorem

We shall detail five different proofs (in modern terminology with necessary changes) of the Pythagorean theorem, which are fairly easy and have some historical importance.

Proof **1.** Euclid perhaps acknowledging the complications in the proof given in Book I (Proposition 47), he himself derived (according to Proclus) a simpler proof in Book VI (Proposition 31). A semi-algebraic version of Euclid's proof appeared in Legendre's textbook *Eléments de géométrie* in 1794 [339]. His book was translated into English in 1858 by Charles Davis (1798–1876, USA), which became very popular in America. This version of Euclid's proof is now very popular all over the world: In Fig. 5.6, $\angle ACB$ is the right angle. We draw the perpendicular *CD* from *C* on the hypotenuse $AB,$ so that $\angle ADC$ and $\angle BDC$ are right angles. We also note that $\angle DAC = \angle DCB$ ($\angle ACD = \angle DBC$). Thus, ΔADC and ΔCDB are similar to each other, and both are similar to $\Delta ACB.$ Hence, it follows that $AC/AB = AD/AC$ and $BC/AB = BD/BC,$ and therefore $AC^2 = AB \times AD$ and $BC^2 = AB \times BD.$ Finally, adding these relations, we get

$$AC^2 + BC^2 \;=\; AB(AD + BD) \;=\; AB \times AB \;=\; AB^2. \tag{5.2}$$

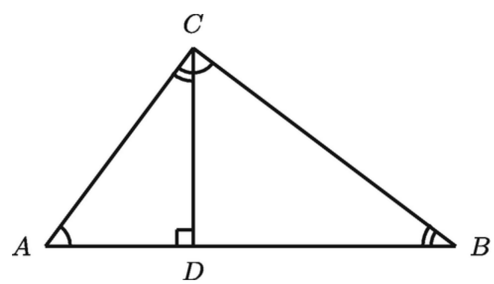

Fig. 5.6 Euclid's proof of Pythagorean theorem

Einstein, when he was 12-year-old, succeeded in "proving" Pythagorean theorem (without claiming its originality) by using the similarity of the triangles. However, unfortunately, he left no such record of his childhood proof. The general consensus among Einstein's biographers is that he probably rediscovered Euclid's proof or found one of its variants. However, Walter Seff Isaacson (born 1952, USA), Jeremy Bernstein (born 1929, USA), and Banesh Hoffman (1906–1986, England) showed some resistance to this conclusion. Ten years later, Einstein discovered four-dimensional form of Pythagorean theorem and used it in his special theory of relativity. After a few years, he expanded this theorem further and used it in his study of general relativity.

In Fig. 5.6, we also note that $A_{ABC} = AC \times BC/2 = AB \times CD/2$, and hence, $AB = AC \times BC/CD$. Using this relation in (5.2), we find

$$\frac{1}{AC^2} + \frac{1}{BC^2} = \frac{1}{CD^2}. \tag{5.3}$$

Eli Maor in his book [361] of 2007 calls the relation (5.3) as the *Little Pythagorean theorem*, but in the literature, it is better known as the *Reciprocal Pythagorean theorem*.

Proof **2.** In the ancient Chinese text *Zhou Bi Suan Jing* (The Arithmetical Classic of the Gnomon and the Circular Paths of Heaven) of Zhou dynasty (1046–256 BC), there is a passage that gives the following dissection proof of Pythagorean theorem: Rotate the given right-angled triangle (ABC) about the center of the square on the hypotenuse to form triangles FCY, GYX, and EXB as in Fig. 5.7. Then, it is easy to see that $A_{AFGE} = A_{CDHF} + A_{BEMD} + 2A_{ABDC}$ and $A_{AFGE} = A_{BXYC} + 4A_{ABC}$. From this, one sees that the area of the square on the hypotenuse (BC) is the sum of the areas of the squares on the other two sides (AB and CA) of the right-angled triangle (ABC).

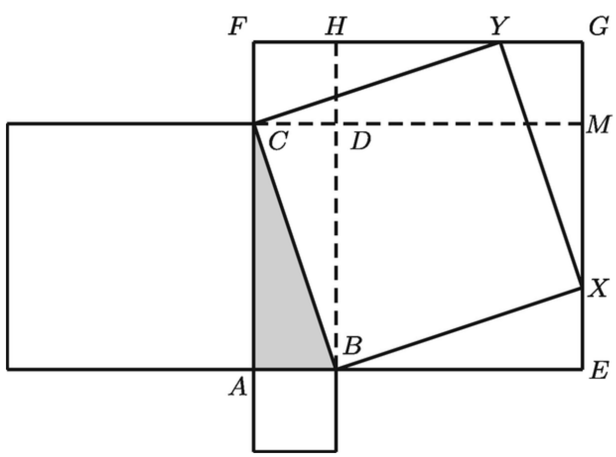

Fig. 5.7 Chinese proof of Pythagorean theorem

During the time of the Three Kingdoms (third century AD) in China, the Wu mathematician Zhao Shuang provided a similar proof in his annotation of Zhou Bi Suan Jing (see Fig. 5.8).

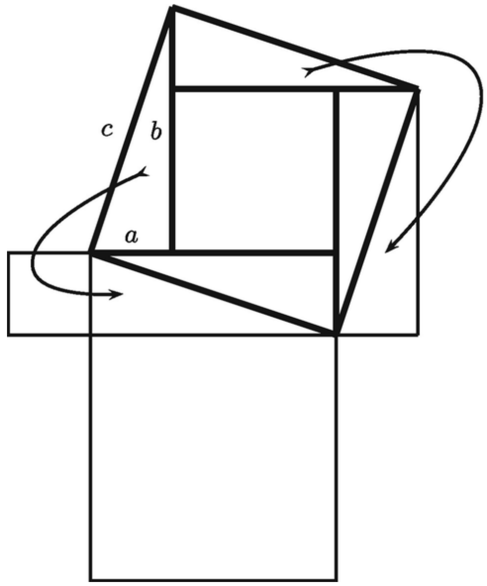

Fig. 5.8 Shuang's proof of Pythagorean theorem

Proclus conjectures that the following variation of the Chinese proof by *dissection* is due to Pythagoras (see Fig. 5.9):

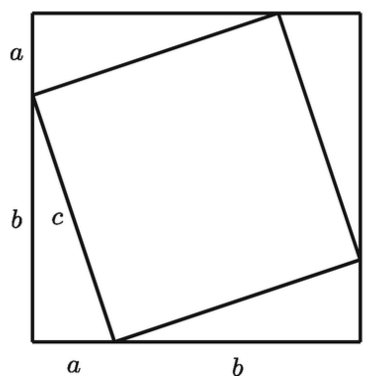

Fig. 5.9 Variation of Chinese proof of Pythagorean theorem

From the Fig. 5.9 it follows that

$$c^2 + 4\frac{ab}{2} = (a+b)^2 = a^2 + b^2 + 2ab,$$

and hence (5.1) holds.

Another similar idea was proposed by Bhaskara II (Fig. 5.10a). It is amusing to note that, besides the diagram, Bhaskara's proof consists only of a single exclamation: "Behold"! This is perhaps the first *visual proof* (a proof without words of an identity or mathematical statement, which can be demonstrated as self-evident by a diagram without any accompanying explanatory text), for more details of such proofs, see (Nelsen [390]). Coolidge's proof is similar to Bhaskara's proof.

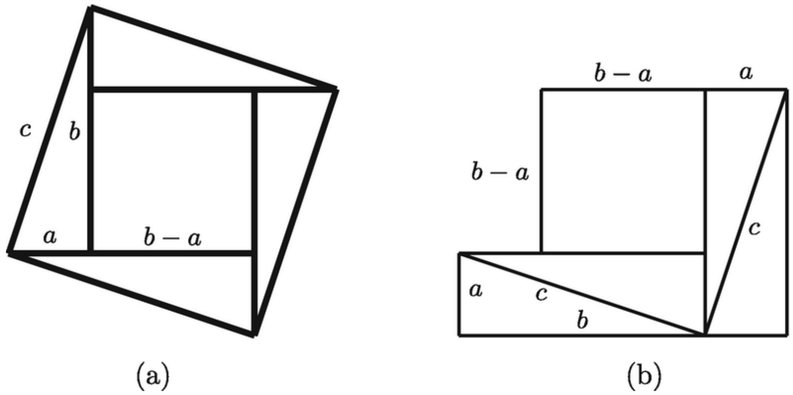

Fig. 5.10 Bhaskara's proof of Pythagorean theorem

Algebraically, from Fig. 5.10a, b it follows that

$$c^2 = 2ab + (b - a)^2 = a^2 + b^2.$$

Proof **3.** From Fig. 5.11 and Hippocrates's Generalization of Pythagorean theorem (5.1) is immediate. This proof was originally given by R.P. Lamy in 1685, which was rediscovered by Stanley Jashemski in 1934 at the age of 19, and then after 70 years by Eli Maor, who calls it as the *Folding Bag* in his book of 2007.

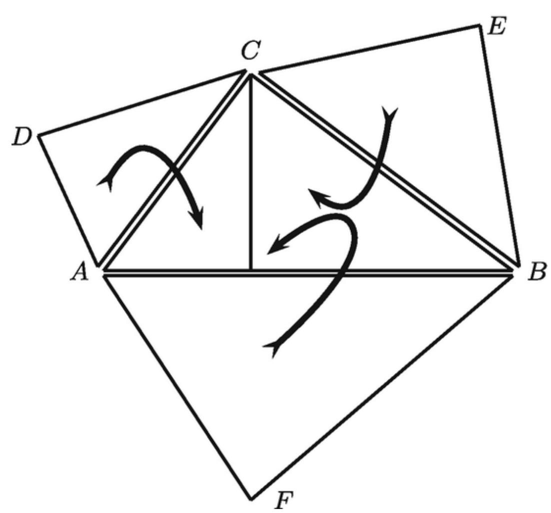

Fig. 5.11 Lamy's proof of Pythagorean theorem

Proof **4.** The following direct proof is due to President Garfield. It appeared in 1876 in the *New England Journal of Education*. Figure 5.12

shows three triangles forming half of a square with sides of length $a+b$. The angles $A,\ B$ and *D* satisfy the relations

$$A+B\ =\ 90^\circ \quad \text{and} \quad A+B+D\ =\ 180^\circ.$$

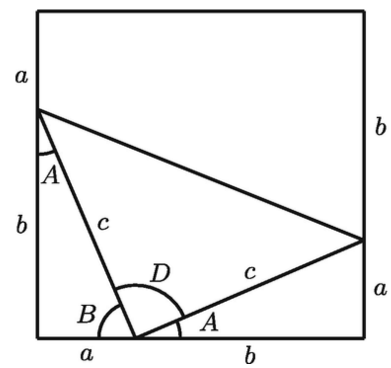

Fig. 5.12 Garfield's proof of Pythagorean theorem

Thus, $D=90^\circ$, and hence all the three triangles are right triangles. The area of the half square is

$$\frac{1}{2}(a+b)^2\ =\ \frac{1}{2}(a^2+2ab+b^2),$$

while the equivalent total area of the three triangles is

$$\frac{1}{2}ab+\frac{1}{2}c^2+\frac{1}{2}ab.$$

Equating these two expressions, we get

$$a^2+2ab+b^2\ =\ ab+c^2+ab \quad \text{or} \quad a^2+b^2\ =\ c^2.$$

Proof **5.** Varahamihira gave several trigonometric formulae that correspond to $\sin x = \cos(\pi/2 - x),\ (1 - \cos 2x)/2 = \sin^2 x,$ and $\sin^2 x + \cos^2 x = 1$, which is the same as the Pythagorean theorem (identity). The following proof is one of the easiest trigonometric proofs of the Pythagorean theorem, which was deduced by Landau from the Cosine addition formula

$$\cos(x + y) \ = \ \cos x \cos y - \sin x \sin y. \qquad (5.4)$$

This formula was known to Bhaskara II. Landau's proof of (5.4) is based on infinite series representations of $\sin x$ and $\cos x$. From Fig. 5.13, in which $\angle AFD = 90^\circ$, we have

$$\cos(x + y) \ = \ \frac{AE}{AD} = \frac{AG}{AD} - \frac{HF}{AD} = \frac{AG}{AF}\frac{AF}{AD} - \frac{HF}{FD}\frac{FD}{AD}$$

$$= \ \cos x \cos y - \sin x \sin y.$$

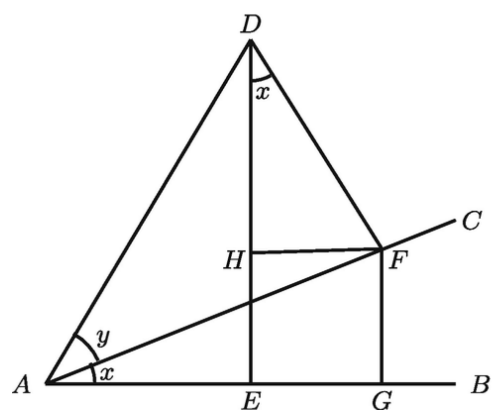

Fig. 5.13 Cosine addition formula

In (5.4) we let $y = -x$, to obtain

$$\cos 0 \ = \ \cos x \cos(-x) - \sin x \sin(-x),$$

which is the same as $1 = \cos^2 x + \sin^2 x$. Thus, Pythagorean Identity is hidden in (5.4).

Finally, we note that the relation (5.4) immediately follows from the formula $e^{i\theta} = \cos\theta + i \sin\theta$ of Euler. In fact, from this formula, we have

$$e^{i(x+y)} \ = \ \cos(x + y) + i \sin(x + y)$$

and

$$\begin{aligned} e^{i(x+y)} = e^{ix}e^{iy} &= (\cos x + i\sin x)(\cos y + i\sin y) \\ &= (\cos x\cos y - \sin x\sin y) + i(\sin x\cos y + \cos x\sin y). \end{aligned}$$

Now comparing the real parts of the aforementioned two equations (5.4) follows.

5.6 Ptolemy's Generalization of Pythagorean Theorem

Ptolemy proved that in any cyclic quadrilateral (vertices all lie on a single circle) *ABCD* (see Fig. 5.14)

$$AB \times CD + BC \times DA = AC \times BD. \tag{5.5}$$

This result appears in his great work *Syntaxis Mathematica* (more popularly known by its Arabian title as *Almagest*) of 150 AD.

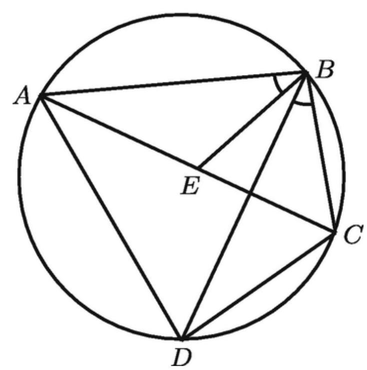

Fig. 5.14 Ptolemy's generalization of Pythagorean theorem

In Fig. 5.14, *E* on *AC* is such that $\angle ABE = \angle CBD$. On *BC* the angles $\angle BAC = \angle BDC$ and on AB, $\angle ADB = \angle ACB$. Thus, $\triangle ABE$ is similar to $\triangle DBC$, and $\triangle EBC$ is similar to $\triangle ABD$. Hence, it follows that

$$\frac{AE}{AB} = \frac{DC}{DB} \quad \text{and} \quad \frac{EC}{BC} = \frac{AD}{BD},$$

which are the same as

$$AE \times DB \;=\; AB \times DC \quad \text{and} \quad EC \times BD \;=\; BC \times AD.$$

An addition of these relations gives

$$(AE + EC) \times BD \;=\; AB \times DC + BC \times AD.$$

But, since $AE + EC = AC$ the above relation is the same as (5.5). Pythagorean theorem follows as a special case when *ABCD* is a rectangle.

5.7 Pappus's Generalization of Pythagorean Theorem

Pappus showed that for an arbitrary triangle with arbitrary parallelograms drawn to its two sides how to construct a parallelogram on the third side whose area is equal to the sum of the areas of the other two parallelograms. This extension of Pythagorean theorem has been of considerable interest, e.g., see Howard Whitley Eves (1911–2004, USA) in 1958, Eli Maor in 2007, and Claudi Alsina (born 1952, Spain) and Roger Bain Nelsen (born 1942, USA) in 2010.

Let *ABC* be any triangle, and let *ABDE* and *ACFG* be two parallelograms built on the sides *AB* and $AC,$ respectively (see Fig. 5.15). Extend DE and FG until they intersect at $H.$ Draw $BL = CM,$ each parallel and equal to *AH*. This produces the parallelogram $BLMC.$ Pappus's construction says that

$$A_{ABDE} + A_{ACFG} \;=\; A_{BLMC}. \tag{5.6}$$

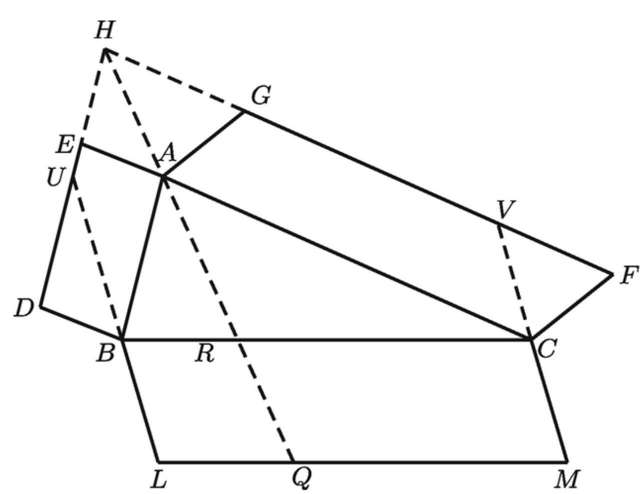

Fig. 5.15 Pappus's generalization of Pythagorean theorem

To show (5.6) it suffices to notice that the parallelograms *ABDE* and *ABUH* have the same base length and height and hence, have the same area. A similar argument holds for the parallelograms *ACFG* and $ACVH$, *ABUH* and $BLQR$, and *ACVH* and $RCMQ$. This gives

$$A_{ABDE} + A_{ACFG} = A_{ABUH} + A_{ACVH} = A_{BLQR} + A_{RCMQ} = A_{BLMC}.$$

5.8 ibn Qurra's Generalization of Pythagorean Theorem

Qurra in an arbitrary triangle *ABC* with $\angle BAC \geq 90°$ drew two straight lines *AP* and *AQ* so that $\angle APB = \angle AQC = \angle BAC$, (see Fig. 5.16). Thus, the triangles ABC, PBA and *QAC* are similar. Hence, it follows that

$$\frac{AB}{BC} = \frac{PB}{AB} \quad \text{and} \quad \frac{AC}{BC} = \frac{QC}{AC},$$

which are the same as

$$AB^2 = BC \times PB \quad \text{and} \quad AC^2 = BC \times QC.$$

An addition of these relations give

$$AB^2 + AC^2 = BC(PB + QC). \tag{5.7}$$

If $\angle BAC = 90°$, then the points *P* and *Q* are the same, and thus $PB + QC = BC$. Hence, in this case (5.7) reduces to Pythagorean theorem.

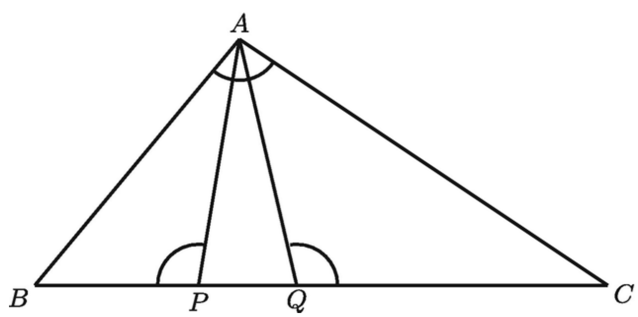

Fig. 5.16 ibn Qurra's generalization of Pythagorean theorem

5.9 The Law of Cosines

The law of cosines states that for any triangle ABC, with sides a, b, c (see Fig. 5.17)

$$c^2 \;=\; a^2 + b^2 - 2ab\cos C. \tag{5.8}$$

The nontrigonometric form of the Law of Cosines is available in Euclid's Book II (Propositions 12 and 13): "In any triangle, the sum of squares of two sides is equal to the square of the third side increased by twice the product of the first side with orthogonal projection of the second to the first side." If $C = \pi/2$, then the cosines law (5.8) reduces to Pythagorean relation (5.1).

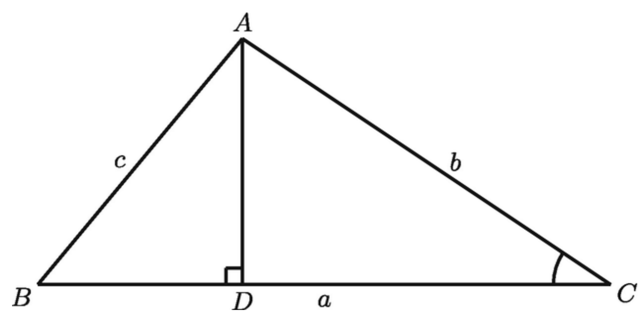

Fig. 5.17 The law of cosines

In the triangle ACD, we have

$$DC \;=\; b\cos C \quad \text{and} \quad AD \;=\; b\sin C,$$

and hence

$$BD \;=\; a - DC \;=\; a - b\cos C.$$

Now in the triangle ABD, Pythagorean theorem and the above relations give

$$\begin{aligned} c^2 \;=\; BD^2 + AD^2 \;&=\; (a - b\cos C)^2 + (b\sin C)^2 \\ &=\; a^2 + b^2(\cos^2 C + \sin^2 C) - 2ab\cos C \\ &=\; a^2 + b^2 - 2ab\cos C. \end{aligned}$$

If in (5.8), $\angle C > 90^\circ$, then $\cos C < 0$ which implies $c^2 > a^2 + b^2$, and if $\angle C < 90^\circ$, then $\cos C > 0$, which gives $c^2 < a^2 + b^2$. Hence, the converse of Pythagorean theorem also follows from the Law of Cosines.

al-Kashi provided the first explicit statement of the Law of Cosines in a form suitable for triangulation. Viéte popularized this law in the Western world. Finally, at the beginning of the nineteenth century, this law was written in its current symbolic form.

5.10 Pythagorean Theorem in Vector Spaces

We need the following definitions, see Agarwal and Flaut [15].

A *vector space V* over a field *F* denoted as (V, F) is a nonempty set of elements called *vectors* together with two binary operations, addition of vectors and multiplication of vectors by scalars so that the following axioms hold:

B1. *Closure property of addition:* If $u, v \in V$, then $u + v \in V$.

B2. *Commutative property of addition:* If $u, v \in V$, then $u + v = v + u$.

B3. *Associativity property of addition:* If $u, v, w \in V$, then $(u + v) + w = u + (v + w)$.

B4. *Additive identity:* There exists a zero vector, denoted by 0, in *V* such that for all $u \in V$, $u + 0 = 0 + u = u$.

B5. *Additive inverse:* For each $u \in V$, there exists a vector *v* in *V* such that $u + v = v + u = 0$. Such a vector *v* is usually written as $-u$.

B6. *Closure property of multiplication:* If $u \in V$ and $a \in F$, then the product $a \cdot u = au \in V$.

B7. If $u, v \in V$ and $a \in F$, then $a(u + v) = au + av$.

B8. If $u \in V$ and $a, b \in F$, then $(a + b)u = au + bu$.

B9. If $u \in V$ and $a, b \in F$, then $ab(u) = a(bu)$.

B10. *Multiplication of a vector by a unit scalar:* If $u \in V$ and $1 \in F$, then $1u = u$.

The spaces $(V, \mathcal{R})$ and $(V, \mathcal{C})$ will be called *real* and *complex vector spaces*, respectively.

The *n*-tuple space Let *F* be a given field. We consider the set *V* of all ordered *n*-tuples $u = (a_1, \cdots, a_n)$ of scalars (known as *components*) $a_i \in F.$ If $v = (b_1, \cdots, b_n)$ is in $V,$ the addition of *u* and *v* is defined by $u + v = (a_1 + b_1, \cdots, a_n + b_n),$ and the product of a scalar $c \in F$ and vector $u \in V$ is defined by $cu = (ca_1, \cdots, ca_n).$ It is to be remembered that $u = v,$ if and only if, their corresponding components are equal, i.e., $a_i = b_i, \ \ i = 1, \cdots, n.$ With this definition of addition and scalar multiplication it is easy to verify all the axioms B1–B10, and hence, this (V, F) is a vector space. If $F = \mathcal{R},$ then *V* is denoted as $\mathcal{R}^n,$ which for $n = 2$ and 3 reduces, respectively, to the two- and three-dimensional usual vector spaces. Similarly, if $F = \mathcal{C}$, then *V* is written as $\mathcal{C}^n$.

The space of polynomials Let *F* be a given field. We consider the set $\mathcal{P}_n, \ n \geq 1$ of all polynomials of degree *at most* $n - 1,$ i.e.,

$$\mathcal{P}_n \ = \ \left\{ a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} = \sum_{i=0}^{n-1} a_i x^i : a_i \in F, \quad x \in \mathcal{R} \right\}.$$

If $u = \sum_{i=0}^{n-1} a_i x^i, \ \ v = \sum_{i=0}^{n-1} b_i x^i \in \mathcal{P}_n,$ then the addition of vectors *u* and *v* is defined by

$$u + v \ = \ \sum_{i=0}^{n-1} a_i x^i + \sum_{i=0}^{n-1} b_i x^i \ = \ \sum_{i=0}^{n-1} (a_i + b_i) x^i,$$

and the product of a scalar $c \in F$ and vector $u \in \mathcal{P}_n$ is defined by

$$cu \ = \ c \sum_{i=0}^{n-1} a_i x^i \ = \ \sum_{i=0}^{n-1} (ca_i) x^i.$$

This $(\mathcal{P}_n, F)$ is a vector space. We remark that the set of all polynomials of degree exactly $n-1$ is not a vector space. In fact, if we choose $b_{n-1} = -a_{n-1}$, then $u+v$ is a polynomial of degree $n-2$.

The space of functions Let *F* be a field of complex numbers, and $X \subseteq F$. We consider the set *V* of all functions from the set *X* to F. The sum of two functions $u, v \in V$ is defined by $(u+v)$, i.e., $(u+v)(x) = u(x) + v(x),\ x \in X$, and the product of a scalar $c \in F$ and function $u \in V$ is defined by cu, i.e., $(cu)(x) = cu(x)$. This (V, F) is a vector space. In particular, $(C[X], F)$, where $C[X]$ is the set of all continuous functions from *X* to F, with the same vector addition and scalar multiplication is a vector space.

An *inner product* on $(V, \mathcal{C})$ is a function that assigns to each pair of vectors $u, v \in V$ a complex number, denoted as (u, v), or simply by $u \cdot v$, which satisfies the following axioms:

C1. *Positive definite property:* $(u, u) > 0$ if $u \neq 0$, and $(u, u) = 0$ if and only if $u = 0$.

C2. *Conjugate symmetric property:* $(u, v) = \overline{(v, u)}$.

C3. *Linear property:* $(c_1 u + c_2 v, w) = c_1(u, w) + c_2(v, w)$ for all $u, v, w \in V$ and $c_1, c_2 \in \mathcal{C}$.

The vector space $(V, \mathcal{C})$ with an inner product is called a *complex inner product space*. From C2 we have $(u, u) = \overline{(u, u)}$ and hence (u, u) must be real. Further, from C2 and C3 it immediately follows that $(w, c_1 u + c_2 v) = \overline{c}_1(w, u) + \overline{c}_2(w, v)$. The definition of a *real inner product space* $(V, \mathcal{R})$ remains the same as above except now for each pair $u, v \in V,\ (u, v)$ is real, and hence in C2 complex conjugates is omitted. In $(V, \mathcal{C})$ the angle between the vectors u, v is defined by the relation

$$\cos\theta = \frac{Re(u, v)}{(u, u)^{1/2}(v, v)^{1/2}}, \tag{5.9}$$

where $Re(u, v)$ is the real part of (u, v). In (5.9), the right-hand side lies between -1 and 1 (Cauchy-Schwarz inequality).

Inner Product in $\mathcal{C}^n$ and $\mathcal{R}^n$ Let $u = (z_1, \cdots, z_n),\ v = (w_1, \cdots, w_n) \in \mathcal{C}^n$. The standard inner product in $\mathcal{C}^n$ is defined as

$$(u, v) \ = \ z_1\overline{w}_1 + \cdots + z_n\overline{w}_n \ = \ \sum_{i=1}^{n} z_i\overline{w}_i.$$

The vector space $\mathcal{C}^n$ with the above inner product is called a *unitary space*. Similarly, for $u = (a_1, \cdots, a_n),\ v = (b_1, \cdots, b_n) \in \mathcal{R}^n$, the inner product in $\mathcal{R}^n$ is defined as

$$(u, v) \ = \ a_1b_1 + \cdots + a_nb_n \ = \ \sum_{i=1}^{n} a_ib_i.$$

The inner product in $\mathcal{R}^n$ is also called *dot product* and sometimes denoted as $u \cdot v$. The vector space $\mathcal{R}^n$ with the aforementioned inner product is simply called an inner product, or dot product, or *Euclidean n-space*.

Inner Product in $C_{\mathcal{C}}[a, b]$ and $C_{\mathcal{R}}[a, b]$ For the functions $u(t) = f(t) + ig(t)$ and $v(t) = p(t) + iq(t),\ t \in [a, b]$ in the vector space of complex-valued continuous functions $C_{\mathcal{C}}[a, b]$ an inner product is defined as

$$(u, v) \ = \ \int_a^b (f(t) + ig(t))(p(t) - iq(t))dt.$$

Similarly, for the functions $u(t) = \phi(t)$ and $v(t) = \psi(t),\ t \in [a, b]$ in the vector space of real-valued continuous functions $C_{\mathcal{R}}[a, b]$ an inner product is defined as

$$(u, v) \ = \ \int_a^b \phi(t)\psi(t)dt.$$

A subset *S* of an inner product space (V, F) is said to be *orthogonal* if and only if for every pair of vectors $u, v \in S,\ u \neq v$ the inner product $(u, v) = 0.$ From (5.9) two vectors $u, v \in (V, \mathcal{R})$ are orthogonal if and only if $(u, v) = 0,$ i.e., $\theta = \pi/2.$ Thus, orthogonality naturally generalizes the geometric concept perpendicular in $\mathcal{R}^2$.

A *norm* (or *length*) on a vector space (V, F) is a function that assigns to each vector $u \in V$ a nonnegative real number, denoted as $\|u\|$, which satisfies the following axioms:

D1. *Positive definite property:* $\|u\| \geq 0,$ and $\|u\| = 0$ if and only if $u = 0,$

D2. *Homogeneity property:* $\|cu\| = |c|\|u\|$ for each scalar $c,$

D3. *Triangle inequality:* $\|u+v\| \leq \|u\| + \|v\|$ for all $u, v \in V.$

A vector space (V, F) with a norm $\|\cdot\|$ is called a *normed linear space* and is denoted as $(V, F, \|\cdot\|).$ In what follows we shall use only the Euclidean norm defined as $\|u\| = (u, u)^{1/2}.$ In the vector space $\mathcal{C}^n$ for two vectors $u = (z_1, \cdots, z_n),\ v = (w_1, \cdots, w_n)$ the Euclidean distance is denoted and defined as

$$\|u - v\| \ = \ (|z_1 - w_1|^2 + \cdots + |z_n - w_n|^2)^{1/2} \ = \ \left(\sum_{j=1}^n |z_j - w_j|^2\right)^{1/2}.$$

Similarly, in $C_{\mathcal{C}}[a, b]$ for two functions $u(t) = f(t) + ig(t)$ and $v(t) = p(t) + iq(t)$ the Euclidean distance is defined as

$$\|u - v\| \ = \ \left(\int_a^b (|f(t) - p(t)|^2 + |g(t) - q(t)|^2)dt\right)^{1/2}.$$

The subset $\hat{S}$ is called *orthonormal* if $\hat{S}$ is orthogonal and for every $\hat{u} \in \hat{S},\ \|\hat{u}\|^2 = (\hat{u}, \hat{u}) = 1.$

The subset $S_1 = \{u^1, u^2, u^3\} = \{(1, 2, 0, -1), (5, 2, 4, 9), (-2, 2, -3, 2)\}$ of $\mathcal{R}^4$ is

orthogonal. The subset
$S_2 = \{w^1, w^2, w^3, w^4\} = \{(1+i, 1, 1-i, i), (1+5i, 6+5i, -7-i, 1-6i), (-7+34i, -8-23i, -10+22i, 30+13i), (-2-4i, 6+i, 4+3i, 6-i)\}$
of $\mathcal{C}^4$ is orthogonal. The set $S_3 = \left\{\frac{1}{\sqrt{\pi}}, \sqrt{\frac{2}{\pi}} \cos nx,\ n = 1, 2, \cdots\right\}$ is orthonormal on $0 < x < \pi.$ The set $S_4 = \left\{\sqrt{\frac{2}{\pi}} \sin nx,\ n = 1, 2, \cdots\right\}$ is orthonormal on $0 < x < \pi.$ The set $S_5 = \left\{\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos nx, \frac{1}{\sqrt{\pi}} \sin nx,\ n = 1, 2, \cdots\right\}$ is orthonormal on $-\pi < x < \pi.$ The set $S_6 = \{P_n(x),\ n = 0, 1, 2, \cdots\}$, where $P_n(x)$ is the Legendre polynomial of degree *n* defined in (see Agarwal and O'Regan [11])

$$P_n(x) \ = \ \sqrt{\frac{2n+1}{2}} \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \frac{(2n-2k)!}{2^n \ k! \ (n-k)! \ (n-2k)!} x^{n-2k}$$

is orthonormal on $-1 < x < 1.$ In particular, we have

$$P_0(x) \ = \ \frac{1}{\sqrt{2}}, \quad P_1(x) \ = \ \sqrt{\frac{3}{2}} x, \quad P_2(x) = \sqrt{\frac{5}{2}} \cdot \frac{1}{2}(3x^2 - 1),$$

$$P_3(x) \ = \ \sqrt{\frac{7}{2}} \cdot \frac{1}{2}(5x^3 - 3x), \quad P_4(x) \ = \ \sqrt{\frac{9}{2}} \cdot \frac{1}{8}(35x^4 - 30x^2 + 3).$$

We are now in the position to state the following *Generalized Pythagorean Theorem*: Let $\{u^1, \cdots, u^r\}$ be an orthogonal subset of an inner product space $(V, F).$ Then, the following holds

$$\|u^1 + \cdots + u^m\|^2 \ = \ \|u^1\|^2 + \cdots + \|u^m\|^2. \tag{5.10}$$

Indeed, from the definition of inner product and orthogonality of $\{u^1, \cdots, u^m\}$, it follows that

$$\begin{aligned}\|u^1 + \cdots + u^m\|^2 &= ((u^1 + \cdots + u^m), (u^1 + \cdots + u^m)) \\ &= (u^1, u^1) + (u^1, u^2) + \cdots + (u^1, u^m) \\ &\quad + \cdots \\ &\quad + (u^m, u^1) + (u^m, u^2) + \cdots + (u^m, u^m) \\ &= (u^1, u^1) + \cdots + (u^m, u^m) \\ &= \|u^1\|^2 + \cdots + \|u^m\|^2.\end{aligned}$$

For $m = 3$ Eq. (5.10) immediately extends Pythagorean relation (5.1) to rectangular solids. Indeed, from Fig. 5.18 and Pythagorean theorem twice, it follows that

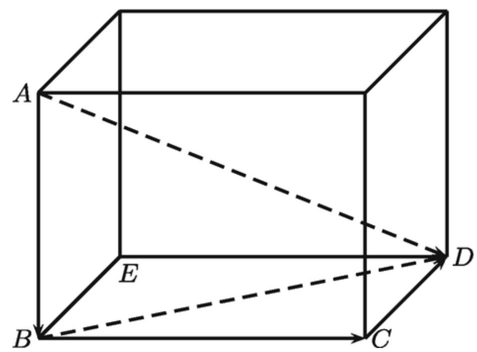

Fig. 5.18 Pythagorean theorem for rectangular solids

$$|\vec{BD}|^2 = |\vec{BC}|^2 + |\vec{CD}|^2$$

and

$$|\vec{AD}|^2 = |\vec{AB}|^2 + |\vec{BD}|^2.$$

Thus, it follows that

$$|\vec{AD}|^2 = |\vec{AB}|^2 + |\vec{BC}|^2 + |\vec{CD}|^2.$$

In particular for the vectors $\vec{BC} = (6, 0, 0)$, $\vec{BE} = \vec{CD} = (0, 2, 0)$ and $\vec{BA} = (0, 0, 3)$, we have $7^2 = 3^2 + 6^2 + 2^2$.

As further examples, for the vectors in the sets S_1 and S_2, respectively, we have

$$\|(1, 2, 0, -1) + (5, 2, 4, 9) + (-2, 2, -3, 2)\|^2 = \|(4, 6, 1, 10)\|^2 = 153$$
$$= \|(1, 2, 0, -1)\|^2 + \|(5, 2, 4, 9)\|^2 + \|(-2, 2, -3, 2)\|^2 = 6 + 126 + 21 = 153$$

and

$$\|w^1+w^2+w^3+w^4\|^2 \;=\; \|(-7+36i, 5-17i, -12+23i, 37+7i)\|_2 = 3750$$
$$= \|w^1\|^2+\|w^2\|^2+\|w^3\|^2+\|w^4\|^2 \;=\; 6+174+3451+119 = 3750.$$

Clearly in (5.10) if the set $\{u^1, \cdots, u^m\}$ is orthonormal, then it becomes

$$\|u^1+\cdots+u^m\|^2 \;=\; \|u^1\|^2+\cdots+\|u^m\|^2 \;=\; m. \tag{5.11}$$

Thus, for the vectors in the sets S_3 and $S_6,$ respectively, we have

$$\int_0^\pi \left(\frac{1}{\sqrt{\pi}}+\sum_{k=1}^{m-1}\sqrt{\frac{2}{\pi}}\cos kx\right)^2 dx$$
$$= \int_0^\pi \left(\frac{1}{\sqrt{\pi}}\right)^2 dx + \sum_{k=1}^{m-1}\int_0^\pi\left(\sqrt{\frac{2}{\pi}}\cos kx\right)^2 dx = m$$

and

$$\int_{-1}^1 \left(\sum_{k=0}^{m-1} P_k(x)\right)^2 dx \;=\; \sum_{k=0}^{m-1}\int_{-1}^1 P_k^2(x)dx \;=\; m.$$

We note that in (5.10), *m* can be infinite provided $\sum_{k=1}^\infty \|u^k\|^2$ converges (finite). For this, as an example we note that the set $\bar{S}_4 = \left\{\sqrt{\frac{2}{\pi}}\frac{\sin nx}{n},\ n=1,2,\cdots\right\}$ is orthogonal on $0<x<\pi,$ and we have

$$\begin{aligned}\int_0^\pi \left(\sum_{k=1}^\infty \sqrt{\frac{2}{\pi}}\frac{\sin kx}{k}\right)^2 dx \;&=\; \sum_{k=1}^\infty \frac{1}{k^2}\int_0^\pi\left(\sqrt{\frac{2}{\pi}}\sin kx\right)^2 dx\\ &=\; \sum_{k=1}^\infty \frac{1}{k^2} \;=\; \zeta(2) \;=\; \frac{\pi^2}{6}.\end{aligned} \tag{5.12}$$

Another generalization of Pythagorean theorem in inner product spaces is known as *Parallelogram Law*: For any pair of vectors u, v in an inner product space $(V, F),$

$$\|u+v\|_2^2 + \|u-v\|_2^2 \;=\; 2\|u\|_2^2 + 2\|v\|_2^2, \tag{5.13}$$

i.e., the sum of the squares of the lengths of the four sides of a parallelogram equals the sum of the squares of the lengths of the two diagonals.

In R^2 from Fig. 5.19, the relation (5.13) is the same as

$$|\vec{AC}|^2 + |\vec{DB}|^2 \;=\; 2|\vec{AB}|^2 + 2|\vec{BC}|^2,$$

which for a rectangle is the same as (5.1).

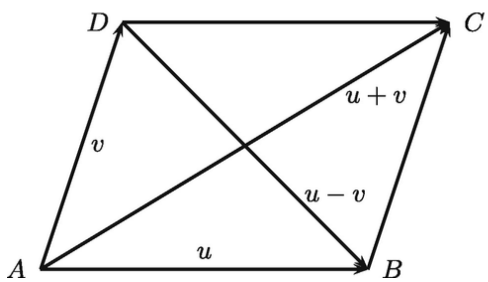

Fig. 5.19 Parallelogram law

De Gua's Theorem In the year 1783, Gua de Malves (1713–1785, France) showed that given three right triangles with leg lengths such that we can form a tetrahedron, the sum of the squares of the areas of the three right triangles is equal to the square of the area of the base. From Fig. 5.20, it means that

$$A_{ABC}^2 \;=\; A_{ABO}^2 + A_{ACO}^2 + A_{BCO}^2. \tag{5.14}$$

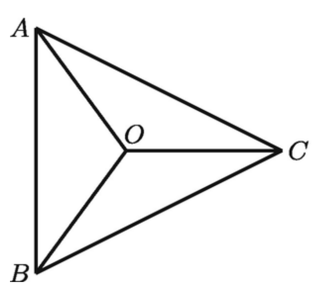

Fig. 5.20 De Gua's theorem

To show the relation (5.14) algebraically, in 2017, Hartzer [251] cleverly used the formula of Bhaskara I (before 123 BC, India) $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = (a+b+c)/2$ and A is the area of the triangle with sides $a, b,$ and $c.$ This formula now known in the literature as Heron's formula. Substituting *s* in the formula of *A* and squaring, we find

$$A^2 \;=\; \frac{2(a^2b^2+a^2c^2+b^2c^2)-(a^4+b^4+c^4)}{16}.$$

Now let $d=OA,\;\; e=OB,\;\; f=OC,$ and $a=AB,\;\; b=BC,\;\; c=CA.$ Then by Pythagorean theorem three times, it follows that

$$a^2 \;=\; d^2+e^2, \quad b^2 \;=\; e^2+f^2, \quad c^2=f^2+d^2$$

and now these relations give

$$A^2 \;=\; \frac{d^2e^2+f^2d^2+e^2f^2}{4} \;=\; \left(\frac{de}{2}\right)^2+\left(\frac{fd}{2}\right)^2+\left(\frac{ef}{2}\right)^2,$$

which is the same as (5.14).

Around the same year 1783, De Gua proved his theorem, a slightly more general version was published by Charles de Tinseau d'Amondans (1748–1822, France). This theorem was also known much earlier to Johann Faulhaber (1580–1635, Germany) and Descartes. We now state a result that was proved in 1974, which is a far-reaching generalization of De Gua's theorem.

Conant and Beyer's Theorem [135] Let *U* be a measurable subset of an *n*-dimensional affine subset of $\mathcal{R}^m,\;\; n\le m.$ For any subset $I\subseteq\{1,\cdots,m\}$ with exactly *n* elements, let U_I be the orthogonal projection of *U* onto the linear span of $e^{i_1},\cdots,e^{i_n},$ where $I=\{i_1,\cdots,i_n\}$ and $e^1,\cdots,e^m$ is the standard basis for $\mathcal{R}^m$ (e^i is one in the *i*th position and zeros everywhere else). Then,

$$\mathrm{vol}_n^2(U) \;=\; \sum_I \mathrm{vol}_n^2(U_I), \tag{5.15}$$

where $\mathrm{vol}_n(U)$ is the *n*-dimensional volume of *U* and the sum is over all subsets $I\subseteq\{1,\cdots,m\}$ with exactly *n* elements.

Next, we state a theorem of Atzema [43] which he proved in 2000 for $m\times n$ matrices, where $n\le m.$ In the year 2010 Frohman [203] used a different method to prove Atzema's result, which is more transparent.

Related results also established by Alvarez [28] in 2018 and Wong [537] in 2002.

Atzema's Theorem For an $m \times n,\ n \leq m$ matrix *A* the following relation holds

$$\det(A^t A) \ = \sum_{I \subset \{1,\cdots,m\}\ |I|=n} \det(A_I)^2, \tag{5.16}$$

where A^t is the transpose of $A,\ |I|$ represents the cardinality of $I,\ A_I$ denotes the $n \times n$ matrix made from the rows of *A* corresponding to the subset *I*, and the summation is taken on all possible combinations I.

As (5.15) the relation (5.16) geometrically can be interpreted as follows: the square of the content of the parallelepiped spanned by *A* is equal to the sum of the squares of the orthogonal projections of the parallelepiped into the *n*-dimensional coordinate hyperplanes.

To illustrate the relation (5.16), we consider the matrix

$$A \ = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \\ 3 & 4 & 9 \end{pmatrix}$$

for which the left-hand side is

$$|A^t A| \ = \begin{vmatrix} 12 & 18 & 36 \\ 18 & 30 & 56 \\ 36 & 56 & 110 \end{vmatrix} \ = \ 24$$

and the right side is

$$\begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{vmatrix}^2 + \begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 3 & 4 & 9 \end{vmatrix}^2 + \begin{vmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \\ 3 & 4 & 9 \end{vmatrix}^2 + \begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 4 & 9 \end{vmatrix}^2 = 0^2 + 2^2 + 4^2 + 2^2 = 24.$$

Relation Between Cross and Inner Products Recall that in $\mathcal{R}^3$, the cross product between two vectors *u* and *v* is a vector *w* denoted and defined as

$$w = u \times v = \|u\|\|v\|(n)\sin\theta; \tag{5.17}$$

here, $0^\circ \le \theta \le 180^\circ$ is the angle between *u* and *v* in the plane *P* containing them, and (n) is a unit vector perpendicular to the plane *P* in the direction given by the right-hand rule. It is clear that *w* is orthogonal to both *u* and *v*, and if *u* and *v* are parallel, then the angle θ is either 0° or 180°.

Now inner product between *u* and *v* from (5.9) can be written as

$$(u, v) = \|u\|\|v\|\cos\theta. \tag{5.18}$$

Squaring and adding both sides of (5.17) and (5.18), and using the Pythagorean Identity $\sin^2\theta + \cos^2\theta = 1,$ we have the following relation

$$\|u \times v\|^2 + (u, v)^2 = \|u\|^2\|v\|^2. \tag{5.19}$$

Inner and cross products were introduced in 1881 by Josiah Willard Gibbs (1839–1903, USA) and independently by Oliver Heaviside (1850–1925, England).

5.11 Pythagorean Theorem in Non-Euclidean Geometry

To draw parallel lines from a point *P* not on a line ℓ, there are several possibilities:

1. There is one and only one parallel line through *P*. This statement is equivalent to the Parallel Postulate and leads to the Euclidean geometry.

2. There is no parallel line through *P*. This possibility leads to a non-Euclidean geometry known as *spherical geometry* (which is crucial in navigation by sea). As an example, we can consider the geometry of the surface of the earth or the celestial sphere. A line on a sphere is the shortest distance between two points on the sphere. If a line is

extended, it forms a *great circle*. A great circle is the end of the lines path. A great circle revolves around the entire sphere with its radius as the radius of the sphere. There are infinitely many great circles on a sphere. The points that are exactly opposite of each other on the sphere, such as poles, are called *antipodal points*. Thus, two great circles will always cross paths at antipodal points. In spherical geometry, angles are defined between great circles, and a triangle is formed by three great circles intersecting. It is clear that in spherical geometry the sum of the interior angles of a triangle always lies between 180° and 540°. Further, the size of an angle increases according as the size of the triangle increases. In antiquity, in India, several astronomical rules for spherical triangles were discovered that are scattered all over ancient astronomical texts such as *Surya Siddhanta* and its commentaries. For example, on a sphere of radius *R*, the spherical law of cosine is given as

$$\cos\left(\frac{c}{R}\right) = \cos\left(\frac{a}{R}\right)\cos\left(\frac{b}{R}\right)+\sin\left(\frac{a}{R}\right)\sin\left(\frac{b}{R}\right)\cos C, \quad (5.20)$$

where A, B, C are the angles of a spherical triangle, of which the opposite sides are $a, b,$ and $c,$ respectively (see Fig. 5.21).

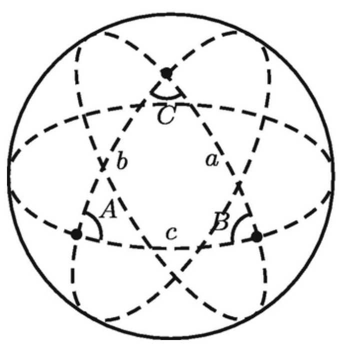

Fig. 5.21 Spherical law of cosine

If $\angle C$ is a right angle, then spherical law of cosine (5.20) reduces to

$$\cos\left(\frac{c}{R}\right) = \cos\left(\frac{a}{R}\right)\cos\left(\frac{b}{R}\right), \quad (5.21)$$

which is the same as

$$\sin^2\left(\frac{c}{2R}\right) = \sin^2\left(\frac{a}{2R}\right) + \sin^2\left(\frac{b}{2R}\right) - 2\sin^2\left(\frac{a}{2R}\right)\sin^2\left(\frac{b}{2R}\right).$$

Thus, in view of $\sin^2\theta = \theta^2 - O(\theta^4)$ for small θ; here, the symbol *O* (called "big-*O*," which tells how fast a function grows or declines) is due to Landau, it follows for large *R* that

$$\left(\frac{c}{2R}\right)^2 - O\left(\left(\frac{c}{2R}\right)^4\right) = \left(\frac{a}{2R}\right)^2 - O\left(\left(\frac{a}{2R}\right)^4\right) + \left(\frac{b}{2R}\right)^2 - O\left(\left(\frac{b}{2R}\right)^4\right)$$

$$-2\left[\left(\frac{a}{2R}\right)^2 - O\left(\left(\frac{a}{2R}\right)^4\right)\right]\left[\left(\frac{b}{2R}\right)^2 - O\left(\left(\frac{b}{2R}\right)^4\right)\right],$$

which is the same as

$$\left(\frac{c}{2R}\right)^2 = \left(\frac{a}{2R}\right)^2 + \left(\frac{b}{2R}\right)^2 + O\left(\left(\frac{1}{2R}\right)^4\right) \quad \text{as} \quad R \to \infty.$$

Hence, we find

$$c^2 = a^2 + b^2 + O\left(\frac{1}{R^2}\right) \quad \text{as} \quad R \to \infty. \tag{5.22}$$

Thus, in the limit, we get back the Pythagorean relation (5.1) as the radius *R* of the sphere tends to infinity.

3. In *hyperbolic geometry*, there are more than one parallel line through P. This possibility leads to the sum of angles in a triangle less than 180 degrees. A modern use of this geometry known as is in the theory of special relativity. Hyperbolic law of cosine was first known to Taurinus in 1826 and then Lobachevsky in 1830. Here for this law, we shall need the following representation, which Jane Gilman [213] has presented in 1995.

$$\cosh\left(\frac{c}{K}\right) = \cosh\left(\frac{a}{K}\right)\cosh\left(\frac{b}{K}\right) - \sinh\left(\frac{a}{K}\right)\sinh\left(\frac{b}{K}\right)\cos C. \tag{5.23}$$

Here A, B, C are the angles of a hyperbolic triangle, of which the opposite sides are $a, b,$ and c, respectively, and $-1/K^2$ is the *Gaussian curvature*: an intrinsic measure of curvature, depending only on distances that are measured within or along the surface (see Fig. 5.22).

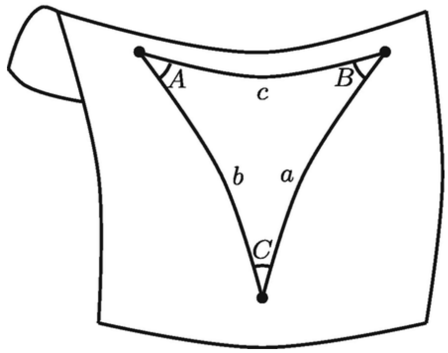

Fig. 5.22 Hyperbolic law of cosine

If $\angle C$ is a right angle, then hyperbolic law of cosine (5.23) reduces to

$$\cosh\left(\frac{c}{K}\right) = \cosh\left(\frac{a}{K}\right)\cosh\left(\frac{b}{K}\right), \tag{5.24}$$

which is the same as

$$\sinh^2\left(\frac{c}{2K}\right) = \sinh^2\left(\frac{a}{2K}\right) + \sinh^2\left(\frac{b}{2K}\right) + 2\sinh^2\left(\frac{a}{2K}\right)\sinh^2\left(\frac{b}{2K}\right).$$

Thus, in view of $\sinh^2\theta = \theta^2 + O(\theta^4)$ for small θ, it follows for large *K* that

$$c^2 = a^2 + b^2 + O\left(\frac{1}{K^2}\right) \quad \text{as} \quad K \to \infty. \tag{5.25}$$

Thus, in the limit, we get back the Pythagorean relation (5.1) as *K* tends to infinity.

4. In *elliptic geometry*, all lines perpendicular to one side of a given line intersect at a single point called the *absolute pole* of that line. The perpendiculars on the other side of the given line also intersect at a point. However, unlike in spherical geometry, the poles on either side are the same. In elliptic geometry also, the sum of the interior angles of a triangle is greater than $180°$. The Pythagorean theorem fails in elliptic geometry. For this, on a sphere of radius *R* consider a spherical triangle with three right angles $A = B = C = \pi/2$, and

sides $a, b, c,$ as in Fig. 5.23. Since the arc length of each side is $L = R\theta,$ where θ is the angle from the origin to each endpoint of the arc, it follows that $L_a = RA = R\pi/2, L_b = RB = R\pi/2, L_c = RC = R\pi/2,$ where L_a represents the length of side $a.$ If we assume that (5.1) holds, and as usual let a and b be the sides of the right triangle and c be the hypotenuse, then we must have $(R\pi/2)^2 + (R\pi/2)^2 = (R\pi/2)^2$, which leads to a contradiction.

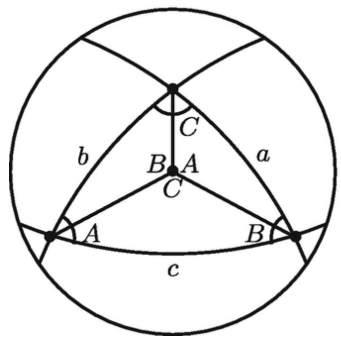

Fig. 5.23 Pythagorean theorem fails in elliptic geometry

5. *Riemannian geometry* is a very broad and abstract generalization of the *differential geometry* of surfaces. It enabled the formulation of Einstein's general theory of relativity. In particular, in *n*-dimensional space *V* on an infinitesimal level Pythagorean theorem takes the following quadratic form (Riemann introduced this in his doctoral address in 1854, also see Tai Chow [128])

$$ds^2 = \sum_{ij}^{n} g_{ij} dx_i dx_j; \tag{5.26}$$

here, *ds* is the line element (the differential of arc length) in *V*, g_{ij} is the matrix tensor, and $(dx_1, \cdots, dx_n)$ are the components of the vector separating the two pints. For the rectangular coordinates, we have $(dx_1, dx_2, dx_3) = (dx, dy, dz),$ and

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

so that (5.26) reduces to

$$ds^2 \;=\; dx^2 + dy^2 + dz^2.$$

Similarly, for the spherical coordinates we have $(dx_1, dx_2, dx_3) = (dr, d\theta, d\phi)$, and

$$g_{ij} \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

so that (5.26) becomes

$$ds^2 \;=\; dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2.$$

5.12 Applications of Pythagorean Theorem

We conclude this chapter with the following historical interesting problems and an example, which require Pythagorean theorem.

1. The following problems are from old Babylonian tablets: (i) A door. Height: half a Ninda (1 ninda = 12 cubits) and 2 cubits. Width: 2 cubits. What is its diagonal? [It requires the calculation of an approximate square root of 68.] (ii) Given that the circumference of a circle is 60 units and the length of a perpendicular from the center of a chord of the circle to the circumference is 2 units, find the length of the chord. [In solving this problem, take $\pi = 3$ to get 12.] (iii) Find the area of an isosceles trapezoid whose sides are 30 units long and whose bases are 14 and 50. (768) (iv) A patu (beam) of length $0;30$ stands against a wall. The upper end has slipped down a distance $0;6.$ How far did the lower end move?

2. The Berlin Papyrus contains two problems, the first stated as "the area of a square of 100 is equal to that of two smaller squares. The side of one is $1/2 + 1/4$ the side of the other." The interest in the question may suggest some knowledge of the Pythagorean theorem, though the papyrus only shows a straightforward solution to a single second-degree equation in one unknown. In modern terms, the simultaneous equations $x^2 + y^2 = 100$ and

$x = (3/4)y$ reduce to the single equation in $y : ((3/4)y)^2 + y^2 = 100$, giving the solution $y = 8$ and $x = 6$.

3. The Cairo Papyrus contains 40 problems of a mathematical nature, of which 9 deal exclusively with the Pythagorean theorem. One, for instance, translates as, "A ladder of 10 cubits has its foot 6 cubits from a wall; to what height will it reach"? [Height 8.] One other problem is concerned with a rectangle having area 60 square cubits and diagonal 13. One is required to find the lengths of their sides. Writing, say, the first of the problems in modern notation, we have the system of equations $x^2 + y^2 = 13^2,\ xy = 60.$ The scribe's method of solution amounts to adding and subtracting $2xy = 120$ from the equation $x^2 + y^2 = 169$, to get $(x + y)^2 = 289,\ (x - y)^2 = 49$; or equivalently, $x + y = 17,\ x - y = 7.$ From this it is found that $2y = 10$, or $y = 5$, and as a result $x = 17 - 5 = 12.$

4. A bamboo 36 cubits tall is broken (bent) by the wind so that the top touches the ground 12 cubits from the stem. Tell the height of the break. (Babylonia and China) [The height of the break is 16 cubits.]

5. In a pond, the flower of a water lily is 2 cubits (cubit was a linear measurement from one's elbow to the tip of the longest (middle) finger, usually 17–21 inches) above the water. When it is bent by a gentle breeze, it touches the water at a distance of 4 cubits. Tell the depth of the water. (China) [The depth of the water is 3 cubits.]

6. A chain suspended from an upright post has a length of 9 cubits lying on the ground. When stretched out to its full length so as to just touch the ground, the end is found to be 21 cubits from the post. What is the length of the chain? (China) [The length of the chain is 29 cubits.]

7. A snake's hole is at the foot of a pillar which is 24 cubits high with a peacock perched on its summit. Seeing the snake at a distance of 48 cubits gliding toward its hole, the peacock pounces on it. Say quickly (perhaps means mentally) now at how many cubits from

the snake's hole they meet, both proceeding an equal distance. (India) [They meet 18 cubits from the hole.]

8. Two magicians live on a cliff of height 40 cubits. There is a stream at a distance of 120 cubits from the foot of the cliff. One magician climbs down and walks to the stream. The other levitates directly up a short distance and then directly to the stream. If both magicians travel the same distance, tell how high the second one flies. (India) [The magician flies 24 cubits high.]

9. The height of a door (say, x) is 6 *chi* 8 *cun* (say, a) greater than its width (say, y) and that the opposite corners are 1 *zhang* (say, d) apart. Find the height and the width of the door. (China) [$x = \frac{1}{2}(a + \sqrt{2d^2 - a^2}), \; y = \frac{1}{2}(-a + \sqrt{2d^2 - a^2})$.]

10. Find the area of the following pointed field whose sides and one diagonal are labeled as in Fig. 5.24.

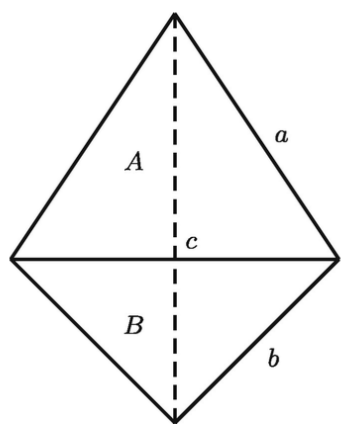

Fig. 5.24 Area of a pointed field

Using Pythagorean theorem the area of the lower triangle is given by $B = (c/2) \times \sqrt{b^2 - (c/2)^2}$ and that of the upper triangle by $A = (c/2)\sqrt{a^2 - (c/2)^2}$. Then the area x of the entire field is given by $x = A + B$. It follows that x satisfies the fourth degree polynomial equation $-x^4 + 2(A^2 + B^2)x^2 - (A^2 - B^2)^2 = 0$. If $a = 39, \; b = 25$, and $c = 30$, this equation becomes

$$-x^4 + 763{,}200x^2 - 40{,}642{,}560{,}000 \; = \; 0. \qquad (5.27)$$

Chiu-Shao in his book *Mathematical Treatise in Nine Sections* (1247) solved polynomial equations up to tenth degree, particularly, he found a root of (5.27) as $x = 840$ by using the method *fan fa* which is now known as Horner's method (William George Horner, 1786–1837, England, this method was also known to Viéte in 1600). The other three roots of (5.27) are $-840, 240, -240$, but for this geometric problem only the solution 840 is meaningful.

11. Chinese mathematician Chu Shih-Chieh (1249–1314) in his book *Precious Mirror of the Four Elements* (1303) considered the following problem: "Given that the length of the diameter of a circle inscribed in a right triangle multiplied by the product of the lengths of the two legs equals 24, and the length of the vertical leg added to the length of the hypotenuse equals 9, what is the length of the horizontal leg?" For this, let *a* stand for the vertical leg, *b* the horizontal leg, *c* the hypotenuse, and *d* the diameter of the circle (see Fig. 5.25). The problem can be translated into the two equations $dab = 24$ and $a + c = 9.$ Chu in addition assumed the two known equations $a^2 + b^2 = c^2$ and $d = b - (c - a)$, where the second gives the relationship between the diameter of the inscribed circle and the lengths of the sides of the triangle. From $b^2 = c^2 - a^2 = (c - a)(c + a)$ and $c + a = 9$, we conclude that $b^2 = 9(c - a)$. Next, we multiply the equation $(c + a) - (c - a) = 2a$ by 9 to get $9(c + a) - 9(c - a) = 18a.$ Thus, it follows that $81 - b^2 = 18a$ and

$$18ab \;=\; 81b - b^3. \tag{5.28}$$

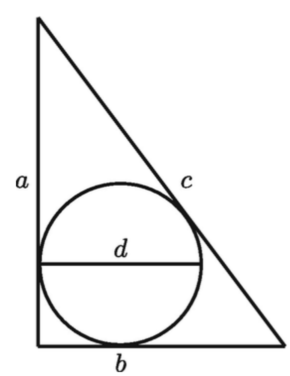

Fig. 5.25 Circle inscribed in a right triangle

Now we multiply $d = b - (c - a)$ by 9 to get $9d = 9b - 9(c - a)$, or

$$9d \;=\; 9b - b^2. \tag{5.29}$$

Multiplying together Eqs. (5.28) and (5.29) gives

$$162dab \;=\; 729b^2 - 81b^3 - 9b^4 + b^5.$$

Because $dab = 24$, Chu had to solve the fifth degree equation in *b*:

$$b^5 - 9b^4 - 81b^3 + 729b^2 - 3888 \;=\; 0.$$

However, he did not illustrate his method of solution, Chu merely wrote that $b = 3$. The other approximate values of *b* are $10.367,\ 6.6143,\ \ -8.8439$, and -2.1372.

12. **(Laws of reflection and refraction of light).** Fermat's principle in optics states that light travels from one point to another along a path that minimizes the travel time. An immediate consequence of this is that in a homogeneous medium light travel in a straight line, since a straight line gives the shortest distance between two points.

Consider a mirror lying horizontally as shown in Fig. 5.26a. Light travels from a source at point *A* to point *B* after reflecting from the mirror at P. We shall find the point of reflection *P*, which requires the light to travel the shortest possible total distance.

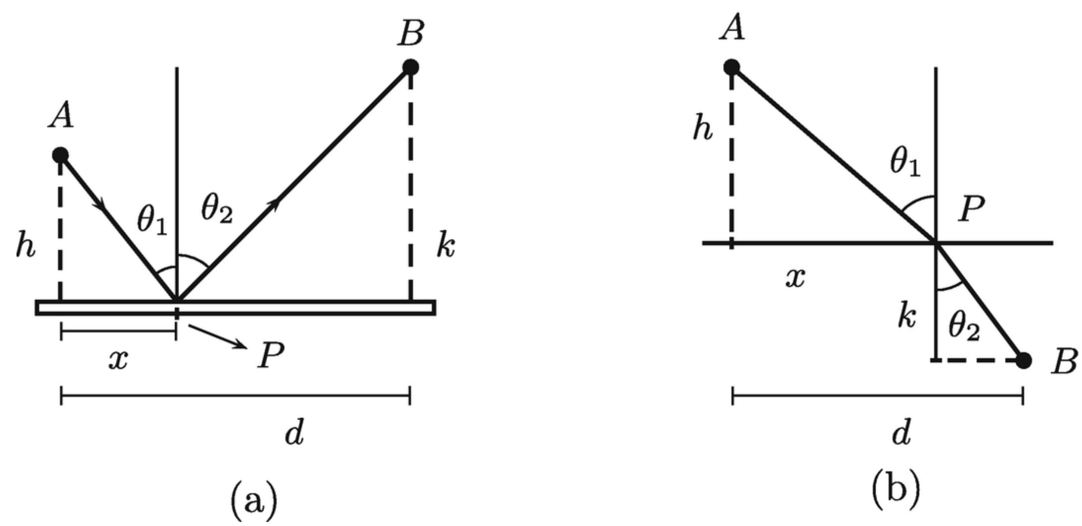

Fig. 5.26 Laws of reflection and refraction

From Fig. 5.26a it is clear that the total distance *D* for the light to travel from *A* to *B* is

$$D = \sqrt{h^2 + x^2} + \sqrt{k^2 + (d - x)^2}.$$

Thus, we have

$$\frac{dD}{dx} = \frac{x}{\sqrt{h^2 + x^2}} - \frac{d - x}{\sqrt{k^2 + (d - x)^2}},$$

$$\frac{d^2D}{dx^2} = \frac{h^2}{(h^2 + x^2)^{3/2}} + \frac{k^2}{(k^2 + (d - x)^2)^{3/2}} > 0.$$

Hence, *D* is minimum when $\sin\theta_1 = \sin\theta_2$, i.e., $\theta_1 = \theta_2$.

The ray of light *AP* that hits the mirror is called the incident ray, the ray *PB* is the reflected ray, θ_1 is the angle of incidence, and θ_2 the angle of reflection. Thus, we have shown that the angle of incidence equals the angle of reflection. This well-known law in physics is known as *reflection law*.

Now consider the problem of refraction of light from a source *A* in vacuum to point *B* in medium of refractive index μ. If light travels with velocity *v* is vacuum, then it travels with velocity v/μ in the second medium. From Fig. 5.26b it is clear that the total time *T* of light to travel from *A* to *B* is

$$T = \frac{\sqrt{h^2+x^2}}{v} + \frac{\sqrt{k^2+(d-x)^2}}{v/\mu}.$$

Thus, we have

$$v\frac{dT}{dx} = \frac{x}{\sqrt{h^2+x^2}} - \mu\,\frac{d-x}{\sqrt{k^2+(d-x)^2}},$$

$$v\frac{d^2T}{dx^2} = \frac{h^2}{(h^2+x^2)^{3/2}} + \mu\frac{k^2}{(k^2+(d-x)^2)^{3/2}}.$$

Hence, T is minimum when $\sin\theta_1 = \mu\sin\theta_2$. This is called *Snell's law*; however, it is now known that this law was first discovered by Ibn Sahl (940–1000, Persia) in 984.

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6. Pythagorean Triples

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6.1 Introduction

A set of three positive integers $a, b,$ and *c*, which satisfies Pythagorean relation (5.1) is called *Pythagorean triple* (or triad) and written as an ordered triple $(a, b, c).$ For convenience it is always assumed that $0 < a < b < c.$ A triangle whose sides (line segments whose lengths are denoted by integers) form a Pythagorean triple is called a *Pythagorean triangle*, it is clearly a right triangle. Pythagorean triangles tell us which pairs of points with whole-number coordinates on the horizontal and vertical direction are also whole-number distance apart. Thus, there is a one-to-one correspondence between Pythagorean triangles and Pythagorean triples. Therefore, we can use Pythagorean triangle and Pythagorean triple interchangeably. A Pythagorean triangle (a, b, c) is said to be *primitive* (sometimes reduced) if a, b, c have no common divisor other than $1.$ Thus, each primitive Pythagorean triple has a unique representation (a, b, c). It is obvious that every primitive Pythagorean triangle can lead to infinitely many non-primitive triangles, for, if (a, b, c) is a primitive Pythagorean triangle, then (ka, kb, kc) is a non-primitive Pythagorean triangle, where $k > 1$ is an integer. Conversely, every non-primitive Pythagorean triangle gives rise to a primitive Pythagorean triangle. Therefore, it suffices to study only primitive Pythagorean triangles. In this chapter, we have provided necessary and sufficient conditions with detailed proofs for the construction of Pythagorean triples. This is followed by a list of several properties and patterns, extensions, and some problems. In recent years, primitive Pythagorean triples have been used in cryptography as random sequences and for the generation of keys, see Kak and Prabhu [292].

6.2 Origin of Pythagorean Triples

Pythagorean triples $(3, 4, 5), (12, 16, 20), (15, 20, 25), (5, 12, 13),$ $(8, 15, 17),$ and $(12, 35, 37)$ first appeared in Apastamba. These triples were used for the precise construction of altars. Datta [152] claims that “early Hindus recognized that if

(a, b, c) is a Pythagorean triple, then (pa, pb, pc) and $(a + na, b + nb, c + nc)$ are also Pythagorean triples," where *p* is any rational number, integer, or fraction, and *n* is any rational number. In 1943, Plimpton 322 was classified as "commercial account." However, 2 years later, two prominent historians of mathematics Neugebauer and Abraham Sachs (1915–1983, USA) made a startling discovery that the content of Plimpton 322 is a list of Pythagorean triples. Table 6.1 reproduces the text in modern notation with base $60.$ There are four columns, of which the rightmost, headed by the words "its name" in the original text, merely gives the sequential number of the lines 1–15. The second column and third column (counting from the right to left) are headed "solving number of the diagonal" and "solving number of the width," respectively; that is, they give the length of the diagonal and of the short side of a rectangular, or equivalently, the length of the hypotenuse and the short leg of a right triangle. We will label these columns with letters *c* and a, respectively. The leftmost column is the most curious of all. Its heading again mentions the word "diagonal," but the exact meaning of the remaining text is not entirely clear. However, when one examines its entries an unexpected fact emerges: this column gives $(c/b)^2$, that is, the value of $\csc^2 \theta,$ where θ is the angle opposite side of *b* and $\csc$ is the cosecant function (see Fig. 5.1). As an example, in the third line we read $a = 1, 16, 41 = 1 \times 60^2 + 16 \times 60 + 41 = 4601,$ and $c = 1, 50, 49 = 1 \times 60^2 + 50 \times 60 + 49 = 6649,$ and hence, $b = \sqrt{6649^2 - 4601^2} = \sqrt{23040000} = 4800,$ giving us the triple $(4601, 4800, 6649).$

Table 6.1 Numbers in the brackets are wrong

$(c/b)^2$	***a***	***c***	
(1,)59,00,15	1,59	2,49	1
(1,)56,56,58,14,50,06,15	56,07	(3,12,01)1,20,25	2
(1,)55,07,41,15,33,45	1,16,41	1,50,49	3
(1,)53,10,29,32,52,16	3,31,49	5,09,01	4
(1,)48,54,01,40	1,05	1,37	5
(1,)47,06,41,40	5,19	8,01	6
(1,)43,11,56,28,26,40	38,11	59,01	7
(1,)41,33,59,03,45	13,19	20,49	8
(1,)38,33,36,36	(9,01)8,01	12,49	9
(1,)35,10,02,28,27,24,26,40	1,22,41	2,16,01	10
(1,)33,45	45	1,15	11
(1,)29,21,54,02,15	27,59	48,49	12
(1,)27,00,03,45	(7,12,01)2,41	4,49	13
(1,)25,48,51,35,06,40	29,31	53,49	14
(1,)23,13,46,40	56	(53)1,46	15

Unfortunately, the table contains some obvious errors. In line 2 we have $a = 56, 07 = 56 \times 60 + 7 = 3367$ and

$c = 3, 12, 01 = 3 \times 60^2 + 12 \times 60 + 1 = 11521$ and these do not form a Pythagorean triple (the third number *b* not being an integer). But if we replace $3, 12, 01$ by $1, 20, 25 = 1 \times 60^2 + 20 \times 60 + 25 = 4825$, we get an integer value of $b = \sqrt{4825^2 - 3367^2} = \sqrt{11943936} = 3456$, which leads to the Pythagorean triple $(3367, 3456, 4825)$. In line 9 we find $a = 9, 1 = 9 \times 60 + 1 = 541$ and $c = 12, 49 = 12 \times 60 + 49 = 769$, and these do not form a Pythagorean triple. But if we replace the $9, 1$ by $8, 1 = 481,$ we do indeed get an integer value of $b;\ b = \sqrt{769^2 - 481^2} = \sqrt{360000} = 600,$ resulting in the triple $(481, 600, 769)$. Again in line 13 we have $a = 7, 12, 1 = 7 \times 60^2 + 12 \times 60 + 1 = 25921$ and $c = 4, 49 = 4 \times 60 + 49 = 289$, and these do not form a Pythagorean triple; but we may notice that 25921 is the square of 161, and the numbers 161 and 289 do form the triple $(161, 240, 289)$. And in row 15 we find $c = 53$, whereas the correct entry should be twice that number, that is, $106 = 1, 46,$ producing the triple $(56, 90, 106)$. This, however, is not a primitive triple, since its members have the common factor $2;$ it can be reduced to the simple triple $(28, 45, 53)$. Similarly, row 11 is not a primitive triple $(45, 60, 75)$, since its members have the common factor $15;$ it can be reduced to $(3, 4, 5)$, which is the smallest and best known Pythagorean triple.

Table 6.2 produces the text in decimal system. In this table, we find that the values of $(c/b)^2$ continuously decrease from 1.9834027 to $1.3871604.$ This implies that the values of $c/b = \csc\theta$ continuously decrease, and this in turn shows that θ increases steadily from approximately 45° to 58°. The question that baffles the mind even today how does the Babylonians find these particular triples, including such enormously large one $(13500, 12709, 18541)$. There seems to be only the following plausible explanation: they were not only familiar with the Pythagorean theorem, but knew an algorithm to compute Pythagorean triples, and had enormous computational skills, see Siu [482].

Table 6.2 Numbers in the brackets are wrong

$(c/b)^2$	***b***	***a***	***c***	
1.9834027	120	119	169	1
1.9491585	3456	3367	(11521) 4825	2
1.9188021	4800	4601	6649	3
1.8862478	13500	12709	18541	4
1.8150076	72	65	97	5
1.7851928	360	319	481	6
1.7199836	2700	2291	3541	7
1.6927093	960	799	1249	8
1.6426694	600	(541)481	769	9
1.5861225	6480	4961	8161	10
1.5625000	60	45	75	11

$(c/b)^2$	***b***	***a***	***c***	
1.4894168	2400	1679	2929	12
1.4500173	240	(25921)161	289	13
1.4302388	2700	1771	3229	14
1.3871604	90	56	(53)106	15

Finding Pythagorean triples is one of the earliest problems in the theory of numbers, and certainly, Pythagorean triples are one of the oldest known solutions of the nonlinear Diophantus equation (5.1). In Apastamba it has been recorded that the triplets

$$\left(m, \frac{m^2-1}{2}, \frac{m^2+1}{2}\right), \tag{6.1}$$

where *m* is an odd number, and

$$\left(m, \frac{1}{4}m^2-1, \frac{1}{4}m^2+1\right), \tag{6.2}$$

when *m* is an even number, are Pythagorean triples. Liu Hui in his commentary on the *Jiuzhang Suanshu* included Pythagorean triples and mentioned about right triangles. It is a tradition to assume that Pythagoras himself gave the following partial solution of the equation (5.1),

$$a = 2n+1, \quad b = 2n^2+2n, \quad c = 2n^2+2n+1, \quad n \geq 1. \tag{6.3}$$

He presumably arrived at (6.3) from the relation

$$(2k-1)+(k-1)^2 = k^2 \tag{6.4}$$

and then searching for those *k* for which $2k-1$ is a perfect square, i.e., $2k-1 = m^2$, (since m^2 is odd *m* must be odd). This gives

$$k = \frac{m^2+1}{2} \quad \text{and} \quad k-1 = \frac{m^2-1}{2}.$$

Thus, the relation (6.4) can be written as

$$m^2 + \left(\frac{m^2-1}{2}\right)^2 = \left(\frac{m^2+1}{2}\right)^2$$

from which it is clear that (5.1) is satisfied with

$$a = m, \quad b = \frac{m^2-1}{2}, \quad c = \frac{m^2+1}{2}, \tag{6.5}$$

Finally, in (6.5) letting $m = 2n+1,\ n \geq 1$, we obtain (6.3). Notice that in (6.3) the sum of the long side and hypotenuse is $4n^2+4n+1 = (2n+1)^2$, which is the square of the small side. We also remark that (6.5) is the same as (6.1).

We can directly verify that (6.3) is a solution of (5.1). Indeed, we have

$$c^2 = (2n^2+2n+1)^2 = (2n^2+2n)^2 + 2(2n^2+2n) + 1$$
$$= (2n^2+2n)^2 + (2n+1)^2 = a^2 + b^2.$$

Since $c - b = 1,$ it follows that *b* and *c* are relatively prime, i.e., positive integer that divides both of them is $1,$ and consequently, Pythagorean triples (a, b, c) generated from (6.3) must be primitive. Some of the Pythagorean triples that can be obtained from (6.3) are given in the following table.

n	a	b	c
1	3	4	5
2	5	12	13
3	7	24	25
4	9	40	41
5	11	60	61
6	13	84	85
7	15	112	113
8	17	144	145

It is interesting to note that between the series of larger legs $4, 12, 24, 40, 60, 84, 112, 144, \cdots$ and of hypotenuse $5, 13, 25, 41, 61, 85, 113, 145, \cdots$, there is a fascinating pattern (see Boardman [75])

$$\begin{aligned} 3^2 + 4^2 &= 5^2 \\ 10^2 + 11^2 + 12^2 &= 13^2 + 14^2 \\ 21^2 + 22^2 + 23^2 + 24^2 &= 25^2 + 26^2 + 27^2 \\ 36^2 + 37^2 + 38^2 + 39^2 + 40^2 &= 41^2 + 42^2 + 43^2 + 44^2 \\ \cdots &= \cdots \end{aligned}$$

The following table gives Pythagorean triples obtained from (6.3) by letting $n = 10, 10^2, \cdots, 10^5.$

n	a	b	c
10	21	220	221
10^2	201	20200	20201
10^3	2001	2002000	2002001
10^4	20001	200020000	200020001
10^5	200001	20000200000	20000200001

The aforementioned table gives an obvious pattern, so that if we know one row, then we can continue this table indefinitely. From (6.3) it follows that there are countably infinitely many primitive Pythagorean triples.

Clearly, Pythagoras's solution has the special feature of producing right triangles having the characteristic that the hypotenuse exceeds the larger leg by $1.$ According to Proclus, Plato gave a method for finding Pythagorean triples that combined algebra and geometry. His solution of equation (5.1) is

$$a = 2n, \quad b = n^2 - 1, \quad c = n^2 + 1, \quad n \geq 2. \tag{6.6}$$

For this, it suffices to note that

$$\begin{aligned} c^2 &= (n^2+1)^2 = (n^2-1+2)^2 = (n^2-1)^2 + 4(n^2-1) + 4 \\ &= (n^2-1)^2 + (2n)^2 = b^2 + a^2. \end{aligned}$$

It is interesting to note that for $m = 2n$, (6.2) is the same as (6.6). Some of the Pythagorean triples that can be obtained from (6.6) are given in the following table.

n	a	b	c	(a, b, c)
2	4	3	5	$(3, 4, 5)$
3	6	8	10	$(6, 8, 10)$
4	8	15	17	$(8, 15, 17)$
5	10	24	26	$(10, 24, 26)$
6	12	35	37	$(12, 35, 37)$
7	14	48	50	$(14, 48, 50)$
8	16	63	65	$(16, 63, 65)$
9	18	80	82	$(18, 80, 82)$
10	20	99	101	$(20, 99, 101)$

(For $n = 2$ the Pythagorean triple is written as $(3, 4, 5)$.) From (6.6) it follows that the hypotenuse exceeds one of the legs by 2. Further, for $n = 4$ we have the Pythagorean triple $(8, 15, 17)$, which cannot be obtained from Pythagoras's formula (6.3). Moreover, for $n = 2k + 1, \ k \geq 1$, (6.6) becomes

$$a = 2(2k+1), \quad b = 4k^2 + 4k, \quad c = 4k^2 + 4k + 2, \quad k \geq 1 \tag{6.7}$$

i.e., for odd n, (6.6) does not give primitive triples, and on dividing (6.7) by 2, we find

$$a = 2k + 1, \quad b = 2k^2 + 2k, \quad c = 2k^2 + 2k + 1, \quad k \geq 1.$$

But, this is the same as (6.3). Thus, in conclusion, the primitive triples obtained from (6.6) include those of given by (6.3).

The following interesting table gives Pythagorean triples obtained from (6.6) by letting $n = 2 \times 10, 2 \times 10^2, 2 \times 10^3, 2 \times 10^4$.

a	b	c
40	399	401
400	39999	40001
4000	3999999	4000001
40000	399999999	400000001

We note that (6.6) for $n = k + 1, \ k \geq 1$ becomes
$a = 2(k+1), b = k(k+2), c = k^2 + 2k + 2$. Thus, if $k \geq 1$,

$$\frac{1}{k} + \frac{1}{k+2} = \frac{a}{b},$$

where *a* and *b* are reduced, will yield a primitive Pythagorean triple with $c = \sqrt{a^2 + b^2}$. As an example, for $k = 2$, we have $(1/2) + (1/4) = (3/4)$. Thus, $a = 3, b = 4$, and $c = \sqrt{3^2 + 4^2} = 5$.

We also note that Cairo Papyrus contains three Pythagorean triples $(3, 4, 5), (5, 12, 13)$ and $(20, 21, 29)$.

6.3 The Characterization of Pythagorean Triples

Unfortunately, even (6.6) does not provide all Pythagorean triples, and it was not until Euclid in his *Elements* (Book X, Lemma I, also see Lemma II after Proposition 28) formalized the following statement (fabricated in geometric language): Let *u* and *v* be any two positive integers, with $u > v$, then the three numbers

$$a = u^2 - v^2, \quad b = 2uv, \quad c = u^2 + v^2 \tag{6.8}$$

form a Pythagorean triple. This seems to be the first regressively proved complete integer solution of an indeterminate equation. If in (6.8), $a > b$ we interchange *a* and b. (If in addition *u* and *v* are of opposite parity-one even and the other odd, and they are coprime, then (a, b, c) is a primitive Pythagorean triple.)

It is easy to verify that the numbers $a, b,$ and *c* as given by Eq. (6.8) satisfy the equation $a^2 + b^2 = c^2$:

$$\begin{aligned} a^2 + b^2 &= (u^2 - v^2)^2 + (2uv)^2 \\ &= u^4 - 2u^2v^2 + v^4 + 4u^2v^2 \\ &= u^4 + 2u^2v^2 + v^4 \\ &= (u^2 + v^2)^2 = c^2. \end{aligned}$$

The integers *u* and v, or simply (u, v) in the formula (6.8) are called the generating numbers or *generators* of the triple (a, b, c). Neugebauer claims that this sufficiency part was known to the Babylonians.

From (6.8) the following relations are immediate

$$u = \sqrt{\frac{c+a}{2}}, \quad v = \sqrt{\frac{c-a}{2}}, \quad \frac{u}{v} = \frac{c+a}{b},$$

$$c - b = (u - v)^2, \quad \frac{1}{2}(c - a) = u^2.$$

Diophantus in his *Arithmetica* (de Méziriae made it fully available in Greek and Latin in 1621) also arrived at the solution (6.8) of (5.1) by using the following reasoning. In (5.1), let $b = ka - c$, where *k* is any rational number. Then, it follows that

$$c^2 - a^2 = b^2 = (ka - c)^2 = k^2a^2 - 2kac + c^2,$$

which leads to

$$-a^2 = k^2a^2 - 2kac,$$

or

$$-a = k^2a - 2kc.$$

Thus, we have

$$a = \frac{2k}{k^2+1}c,$$

which gives

$$b = ka - c = \frac{k^2-1}{k^2+1}c.$$

Let $k = u/v,$ with *u* and *v* integers (we can assume that $u > v$), so that

$$a = \frac{2uv}{u^2+v^2}c, \quad b = \frac{u^2-v^2}{u^2+v^2}c.$$

Now we set $c = u^2 + v^2,$ to obtain $a = 2uv,\ b = u^2 - v^2.$

Bhaskara II also gave tentative partial solution of (5.1), which in number theory is considered an exciting result.

The converse, i.e., showing that any Pythagorean triple is necessarily of the form (6.8) is more difficult. The earliest record for some special cases of the proof of the converse can be found in the solutions of Problems 8 and 9 in the book *Arithmetica* of Diophantus. Next, the converse was discussed in the works of Arab mathematicians around the tenth century. The details of Arab's work were available to the well-traveled Fibonacci. It seems the first explicit, rigorous proof of the converse was given in 1738, by C.A. Koerbero (Dickson [164], Vol. 2). In 1901, Kronecker gave the first proof that all positive integer solutions of $a^2 + b^2 = c^2$ are given without duplication by $a = 2uvk,\ b = (u^2 - v^2)k,\ c = (u^2 + v^2)k,$ where u, v and *k* are positive integers such that $u > v,\ u$ and *v* are not both odd, and *u* and *v* are relatively prime. In what follows we shall discuss a simplified and extended version of a known proof (see, e.g., Burton [111]) of Euclid's Proposition and its converse.

Step 1. If (a, b, c) is a primitive Pythagorean triple, then $\gcd(a, b) = 1.$ For this, if $\gcd(a, b) = d$ then $d|c,$ so that *d* is a common divisor of $a, b,$ and $c,$ and consequently (a, b, c) is not a primitive Pythagorean triple.

Step 2. If (a, b, c) is a Pythagorean triple, then $\gcd(a, b) = 1$ implies $\gcd(b, c) = 1$ and $\gcd(a, c) = 1.$ For, if *b* and *c* have a common divisor, say $d \neq 1,$ then *d* is also a divisor of *a* and consequently, *d* is a common divisor of *a* and *b*, which is not possible. Hence, $\gcd(b, c) = 1.$ The proof for $\gcd(a, c) = 1$ is similar.

Step 3.

If (a, b, c) is a primitive solution of (5.1), then exactly one of *a* and *b* must be even, and *c* must be odd. For, if *a* and *b* are both even, then *c* must also be even; and consequently, a, b, c will have a common divisor other than $1.$ If *a* and *b* are both odd, say, $a = 2m + 1, b = 2n + 1,$ then $a^2 + b^2 = 2[2(m^2 + n^2 + m + n) + 1] = 2(2t + 1)$, which is impossible since no perfect square can be of the form $2(2t + 1).$ Thus, exactly one of *a* and *b* must be odd and the other must be even. Also, then $a^2 + b^2$ must be odd, and so *c* must be odd. (By virtue of this step, there exists no primitive Pythagorean triple all of whose numbers a, b, c are prime. There are primitive Pythagorean triples in which *c* and one of *a* or *b* is prime, for example, $(3, 4, 5), (5, 12, 13), (11, 60, 61).$ It is not known if there exist infinitely many such triples. There is no primitive Pythagorean triple whose sum is a prime number.)

Step 4. If p, q, r are integers such that $p^2 = qr,$ and $\gcd(q, r) = 1$, then *q* and *r* must be perfect squares. In fact, writing out the prime factorizations of *q* and $r,$ we have

$$q = q_1^{a_1} q_2^{a_2} \cdots q_s{}^{a_s}, \qquad r = r_1^{b_1} r_2^{b_2} \cdots r_t{}^{b_t}.$$

Since $\gcd(q, r) = 1$, therefore no prime can occur in both the decompositions. Since $p^2 = qr$ and since prime factorization is unique, therefore,

$$p^2 = q_1^{a_1} q_2^{a_2} \cdots q_s{}^{a_s} r_1^{b_1} r_2^{b_2} \cdots r_t{}^{b_t},$$

where $q_1, q_2, \cdots, q_s, r_1, r_2, \cdots, r_t$ are distinct primes. Since p^2 is a perfect square, it is necessary that $a_1, a_2, \cdots, a_s, b_1, b_2, \cdots, b_t$ must be all even, and consequently, *q* and *r* must be perfect squares. Similarly, it follows that if $p^n = qr,$ and $\gcd(q, r) = 1$, then *q* and *r* must be of *n*th power. We note that Step 4 can also be proved by induction on $p.$ For $p = 1$ and $p = 2$ it is trivially true.

(Professeur Tryphon Tournesol discovered an amusing proof of Step 4 while he was in jail for having failed the president's son. The prisoners were put in a long row of cells. At first, all the doors were unlocked, but then the jailor walked by and locked every second door. He walked again and stopped at every third door, locking it if it was unlocked, but unlocking if it was locked. On the next round, he stopped at every fourth door, locking it if it was unlocked, unlocking if it was locked, and so on. Professor Tournesol soon realized that the *q*th cell would be unlocked in the end just in case *q* had odd number of divisors. Now, if *d* divides *q*, then so does $q|d$, and it would seem that the divisors of *q* come in pairs. Unless... what if $d = q|d?,$ thought the professor, then the divisor *d* does not pair off with another, and $d = q|d$ just in case *q* is a square.')

Step 5. If (a, b, c) is a primitive solution of (5.1) and *a* is odd (similar arguments hold if *a* is even), then there must exist positive integers *u* and *v* such that $u > v$ with $\gcd(u, v) = 1,$ and exactly one of *u* and *v* being odd, such that

$a = u^2 - v^2,\ b = 2uv,\ c = u^2 + v^2.$ Since *a* is odd, by Step 3, *b* must be even. Let $b = 2p,$ for some $p.$ From $b^2 = c^2 - a^2,$ we have

$$4p^2 = (c+a)(c-a). \tag{6.9}$$

Now $(c+a)$ and $(c-a)$ are both even (for by Step 3, *a* and *c* are both odd). Let us put $c + a = 2q,\ c - a = 2r,$ i.e., $c = q + r,\ a = q - r.$ Then from (6.9) we have $p^2 = qr.$ Next we shall show that *q* and *r* are relatively prime. For, if *d* is a common divisor of *q* and $r,$ then *d* must also be a common divisor of *c* and $a,$ and therefore also a divisor of $b.$ Thus, a, b, c will have a common divisor $d.$ Since the solution (a, b, c) is a primitive solution, we must have $d = \pm 1,$ i.e., *q* and *r* must be relatively prime. Thus, by Step 4 (from Professor Tournesol's discovery), *q* and *r* must be perfect squares. Let $q = u^2,\ r = v^2$ for some integers *u* and *v*, which are taken to be positive. Then, $c = u^2 + v^2,\ a = u^2 - v^2,\ b^2 = c^2 - a^2 = 4u^2v^2,$ so that $b = 2uv.$ It is clear that $\gcd(u, v) = 1,$ for if $\gcd(u, v) = d,$ then *d* will be a divisor of a, b and $c.$ Also *u* and *v* cannot be both odd or both even, for in either case *c* will be even, and this will contradict the result in Step 3.

Step 6. If $a = u^2 - v^2,\ b = 2uv,\ c = u^2 + v^2$ where *u* and *v* are positive integers such that $u > v,\ \gcd(u, v) = 1$ and exactly one of *u* and *v* is odd, then (a, b, c) is a primitive solution of (5.1). By actual substitution we find that

$$a^2 + b^2 = (u^2 - v^2)^2 + 4u^2v^2 = (u^2 + v^2)^2 = c^2,$$

so that (a, b, c) is a solution. To show that (a, b, c) is a primitive solution, assume to the contrary that *p* is an odd prime that divides both *a* and $c.$ This implies $p | c + a$ and $p | c - a,$ i.e., *p* divides both $2u^2$ and $2v^2.$ Since *p* is odd, it follows that *p* divides both u^2 and $v^2.$ Since *p* is a prime, it must divide both *u* and $v.$ But $\gcd(u, v) = 1$ implies that this is impossible. Thus, the only possible common divisor that *a* and *c* may have must be a power of $2.$ However, this is also not possible because *a* must be odd. Thus, $\gcd(a, c) = 1$ and (a, b, c) is a primitive solution.

Summing the aforementioned steps, we find that (6.8) generates all primitive Pythagorean triples.

- The following proof (see Beauregard and Suryanarayan [55], and Joyce [290]) of the converse of Euclid's statement is short and elementary: If (a, b, c) is a primitive Pythagorean triple, then by Step 1, $\gcd(a, b) = 1,$ so we can always choose *a* to be odd. Now from (5.1) it follows that

$$1 = \left(\frac{c}{b} + \frac{a}{b}\right)\left(\frac{c}{b} - \frac{a}{b}\right).$$

Thus, the two terms on the right are rational and reciprocals of each other. Let the first one be u/v in lowest terms, then the second is $v/u,$ i.e.,

$$\frac{c}{b}+\frac{a}{b}=\frac{u}{v} \quad \text{and} \quad \frac{c}{b}-\frac{a}{b}=\frac{v}{u}.$$

Solving these equations, we get

$$\frac{c}{b}=\frac{u^2+v^2}{2uv} \quad \text{and} \quad \frac{a}{b}=\frac{u^2-v^2}{2uv}. \tag{6.10}$$

Since $\gcd(u,v)=1$, both *u* and *v* cannot be even. If they both are odd, then u^2-v^2 will have 4 as the minimum possible factor, whereas 2*uv* will have 2 as the maximum possible factor, and this will imply that *a* is even, which contradicts our assumption that *a* is odd. Thus, one of *u* and *v* must be odd and the other should be even. Hence, $u^2 \pm v^2$ both must be odd, and obviously $u > v$. In conclusion, both the fractions in (6.10) are fully reduced and hence, lead to Euclid's formula $a=u^2-v^2,\ b=2uv,\ c=u^2+v^2.$

- The Pythagorean triple $(13500, 12709, 18541)$ in row 4, Table 6.2, can be obtained by letting $u=125, v=54$ in (6.8). It is interesting to note that the corresponding *u* and *v* for each row in Table 6.2 have no prime factors other than $2, 3$ and 5 (the prime divisors of the Babylonian scale 60), and $v<60.$ For $u=n+1, v=n,\ n\geq 1$ formula (6.8) reduces to (6.3). Indeed, then we have $u^2-v^2=2n+1, 2uv=2n^2+2n, u^2+v^2=2n^2+2n+1.$ For $u=n, v=1,\ n\geq 2$ formula (6.8) reduces to (6.6). Indeed, then we have $2uv=2n, u^2-v^2=n^2-1, u^2+v^2=n^2+1.$ In general, (6.8) gives infinitely many primitive Pythagorean triples in which the hypotenuse exceeds one of the legs by $2v^2$.

Table 6.3 gives primitive Pythagorean triples with $c\leq 1000,$ sorted by increasing $c.$

Table 6.3 Primitive Pythagorean triples

(3,4,5)	**(5,12,13)**	**(8,15,17)**	**(7,24,25)**
(20,21,29)	(12,35,37)	(9,40,41)	(28,45,53)
(11,60,61)	(16,63,65)	(33,56,65)	(48,55,73)
(13,84,85)	(36,77,85)	(39,80,89)	(65,72,97)
(20,99,101)	(60,91,109)	(15,112,113)	(44,117,125)
(88,105,137)	(17,144,145)	(24,143,145)	(51,140,149)
(85,132,157)	(119,120,169)	(52,165,173)	(19,180,181)
(57, 176,185)	(104,153,185)	(95,168,193)	(28,195,197)
(84,187,205)	(133,156,205)	(21,220,221)	(140,171,221)
(60,221,229)	(105,208,233)	(120,209,241)	(32,255,257)
(23,264,265)	(96,247,265)	(69,260,269)	(115,252,277)
(160,231,281)	(161,240,289)	(68,285,293)	(136,273,305)
(207,224,305)	(25,312,313)	(75,308,317)	(36,323,325)

(3,4,5)	(5,12,13)	(8,15,17)	(7,24,25)
(204,253,325)	(175,288,337)	(180,299,349)	(225,272,353)
(27,364,365)	(76,357,365)	(252,275,373)	(135,352,377)
(152,345,377)	(189,340,389)	(228,325,397)	(40,399,401)
(120,391,409)	(29,420,421)	(87,416,425)	(297,304,425)
(145,408,433)	(84,437,445)	(203,396,445)	(280,351,449)
(168,425,457)	(261,380,461)	(31,480,481)	(319,360,481)
(44,483,485)	(93,476,485)	(132,475,493)	(155,468,493)
(217,456,505)	(336,377,505)	(220,459,509)	(279,440,521)
(92,525,533)	(308,435,533)	(341,420,541)	(33,544,545)
(184,513,545)	(165,532,557)	(276,493,565)	(396,403,565)
(231,520,569)	(48,575,577)	(368,465,593)	(240,551,601)
(35,612,613)	(105,608,617)	(336,527,625)	(100,621,629)
(429,460,629)	(200,609,641)	(315,572,653)	(300,589,661)
(385,552,673)	(52,675,677)	(37,684,685)	(156,667,685)
(111,680,689)	(400,561,689)	(185,672,697)	(455,528,697)
(260,651,701)	(259,660,709)	(333,644,725)	(364,627,725)
(108,725,733)	(216,713,745)	(407,624,745)	(468,595,757)
(39,760,761)	(481,600,769)	(195,748,773)	(56,783,785)
(273,736,785)	(168,775,793)	(432,665,793)	(555,572,797)
(280,759,809)	(429,700,821)	(540,629,829)	(41,840,841)
(116,837,845)	(123,836,845)	(205,828,853)	(232,825,857)
(287,816,865)	(504,703,865)	(348,805,877)	(369,800,881)
(60,899,901)	(451,780,901)	(464,777,905)	(616,663,905)
(43,924,925)	(533,756,925)	(129,920,929)	(215,912,937)
(580,741,941)	(301,900,949)	(420,851,949)	(615,728,953)
(124,957,965)	(387,884,965)	(248,945,977)	(473,864,985)
(696,697,985)	(372,925,997)		

6.4 Properties, Patterns, Extensions, and Problems

After the publication of Euclid's *Elements* (Book X, Proposition 29), hundreds of professional as well as non-professional mathematicians have tried to find properties/patterns of Pythagorean triples, alternatives to Euclid's formula (6.8), different forms of the generators (u, v), and Pythagorean triples with specified properties. This has led to many interesting number-theoretical results, also several innocent looking problems, which are still waiting for their solutions. Dickson [164], in his three-volume history of number theory, has given a 25-page account of what was achieved in the field of Pythagorean triangles during more than two millennia and up to Euler and modern times. The inspiration to this modern development was provided by Fermat, who in his marginal notes stated without proof many theorems involving these integers. Later, these theorems were proved and intensified by great mathematicians such as Euler, Lagrange, Gauss, and Liouville. In the following we present some elementary results to this introductory branch of number theory.

P1. From (6.8) it follows that in a Pythagorean triangle (see Fig. 5.1)

$$\sin\theta = \frac{2uv}{u^2+v^2} = y(\text{say}), \cos\theta = \frac{u^2-v^2}{u^2+v^2} = x(\text{say}), \tan\theta = \frac{2uv}{u^2-v^2}.$$

Further, from the half-angle formula $\tan\theta/2 = \sin\theta/(1+\cos\theta)$, we have $\tan(\theta/2) = v/u$. We also note that $y = (v/u)(x+1)$. Thus if we draw a unit circle in the *xy*-plane with the origin at $(0,0)$, and a straight line from the point $(-1,0)$ with slop v/u, then $P = ((u^2-v^2)/(u^2+v^2), 2uv/(u^2+v^2))$, is the other point of intersection of the line and the circle, see Fig. 6.1.

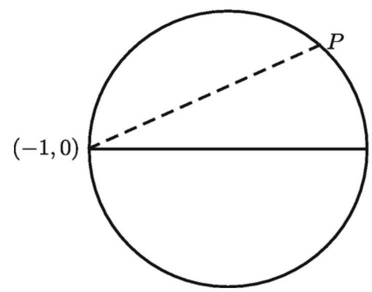

Fig. 6.1 Geometric characterization of Pythagorean triples

P2. From (6.8) it follows that $(c-a)(c-b)/2 = u^2(u-v)^2$, i.e., $(c-a)(c-b)/2$ is always a perfect square. This is only a necessary condition but not a sufficient one, (see Posamentier [417], p. 156). For example, consider the triple $(4,8,12)$ for which $(c-a)(c-b)/2 = 4^2$, but it is not a Pythagorean triple. We also note from this simple observation that $(3,4,7)$ cannot be a Pythagorean triple.

P3. If *P* is the perimeter of a primitive Pythagorean triangle (a,b,c), then *P* is even and $P|ab$. Indeed, we have
$P = a+b+c = u^2-v^2+2uv+u^2+v^2 = 2u(u+v)$, and
$ab = (u^2-v^2)(2uv) = 2u(u+v)v(u-v) = P[v(u-v)]$. Thus, in particular, the perimeter *P* divides $2A$, where $A = (1/2)ab$ is the area of the triangle. Fermat proved that *A* can never be a square number (see P37), later it was shown that it cannot be twice the square number (see Carmichael [117, 118]).

P4. In a primitive Pythagorean triple (a,b,c), either *a* or *b* is divisible by 3. For this, first we note that any number *m* can be written either as $3k, 3k+1$, or

$3k+2.$ But, since $(3k+2)^2 = 9k^2+12k+4 = 3(3k^2+4k+1)+1$ is of the form $3h+1,$ no integer m^2 can be written in the form $3k+2,$ hence all integers squared are of the form $3k,$ or $3k+1.$ Thus, if in (6.8) either u^2 or v^2 happens to be of the form 3*k* (this is to say, if either $3|u^2$ or $3|v^2$), then $3|u$ or $3|v,$ in which case $3|2uv.$ Now assume that both u^2 and v^2 take the form $3k+1;$ to be specific, let $u^2 = 3k+1$ and $v^2 = 3h+1.$ Then, we have $u^2 - v^2 = (3k+1)-(3h+1) = 3(k-h).$ Hence, $3|u^2-v^2.$ In conclusion, in a primitive Pythagorean triple (a,b,c) exactly one of the integers is divisible by $3.$ As an example, see any triple in Table 6.3.

P5. In a primitive Pythagorean triple $(a,b,c),$ either *a* or *b* is divisible by $4.$ In (6.8), *u* and *v* are of opposite parity-one even and the other odd, and hence $4|2uv.$ In conclusion, in a primitive Pythagorean triple (a,b,c) exactly one of the integers is divisible by $4.$ As an example, see any triple in Table 6.3.

P6. In a primitive Pythagorean triple (a,b,c), one side is divisible by $5.$ For this, first we note that any number *m* can be written either as $5k, 5k+1, 5k+2, 5k+3,$ or $5k+4.$ But, m^2 can be written only as $5k, 5k+1,$ or $5k+4.$ Thus, if in (6.8) either u^2 or v^2 happens to be of the form 5*k* (this is to say, if either $5|u^2$ or $5|v^2$), then $5|u$ or $5|v,$ in which case $5|2uv.$ Now assume that both u^2 and v^2 take the form $5k+1;$ to be specific, let $u^2 = 5k+1$ and $v^2 = 5h+1.$ Then, we have $u^2 - v^2 = (3k+1)-(3h+1) = 5(k-h).$ Hence, $5|u^2-v^2.$ Similarly, if $u^2 = 5k+4$ and $v^2 = 5h+4$, then $5|u^2-v^2.$ Finally, if $u^2 = 5k+4$ and $v^2 = 5h+1,$ then $u^2+v^2 = 5(k+h+1),$ and this implies that $5|(u^2+v^2).$ In conclusion, in a primitive Pythagorean triple (a,b,c) exactly one of the integers is divisible by $5.$ As an example, see any triple in Table 6.3.

From P4–P6, it follows that in every primitive Pythagorean triple $(a,b,c),$ the product *ab* is divisible by $12,$ and the product *abc* is divisible by $60.$ The smallest and best known Pythagorean triple $(3,4,5)$ shows that this observation is the best possible, (see MacHale and van den Bosch [354]). Further, out of three divisors $3,4,5$ one of the numbers a,b,c may have any two of these divisors, e.g., $(8,15,17), (7,24,25), (20,21,29),$ or even all three as in $(11,60,61).$ It is not known if there are two distinct Pythagorean triples having the same product. The existence of two such triples corresponds to a nonzero solution of a Diophantine equation.

P7. In a primitive Pythagorean triple (a,b,c), none of the sides can be of the form $4n+2,\ n \geq 1.$ Since *u* and *v* are of different parity $a = u^2 - v^2$ is of the form $4k \pm 1,\ k \geq 1,$ $b = 2uv$ is of the form $4k,\ k \geq 1,$ and $c = u^2+v^2$ is of the form $4k \pm 1,\ k \geq 1.$

P8. From P3 it follows that $P = A,$ i.e., perimeter of a Pythagorean triangle and its area are the same if and only if $2u(u+v) = u(u+v)v(u-v)$, which is the same as $v(u-v) = 2.$ Thus, either (i). $v = 2, u - v = 1,$ or (ii). $v = 1, u - v = 2.$ The case (i) gives $u = 3, v = 2$ and leads to the Pythagorean triple $(5, 12, 13)$, whereas case (ii) implies $u = 3, v = 1$ and gives the Pythagorean triple $(6, 8, 10)$. Thus, there is only one primitive Pythagorean triangle $(5, 12, 13)$ for which $P = A.$

P9. There exist primitive Pythagorean triangles having the same perimeter $P = 2u(u+v)$. The first two primitive Pythagorean triangles with the same perimeter 1716 are $(364, 627, 725)$ and $(195, 748, 773)$. Next, two primitive Pythagorean triangles $(340, 1131, 1181)$ and $(51, 1300, 1301)$ correspond to the same perimeter $2652.$ Other perimeters which have more than one primitive Pythagorean triangles are 3876, 3960, 4290, 5244, 5700, 5720, 6900, 6930, 8004, 8700, 9300, 9690, 10010, 10788, 11088, 12180, 12876, 12920, 13020, 13764, 14280, 15252, 15470, 15540, 15960, 16380, 17220, 17480, 18018, 18060, 18088, 18204, 19092, 19320, 20592, 20868, $\cdots$. The first three primitive Pythagorean triangles with the same perimeter 14280 are $(3255, 5032, 5993), (168, 7055, 7057)$ and $(119, 7080, 7081)$. The next perimeter being 72930 with three primitive Pythagorean triangles $(18480, 24089, 30361), (7905, 32032, 32993),$ and $(2992, 34905, 35033)$. Four primitive Pythagorean triangles having a common perimeter also exist. For a perimeter less than 10^6, there exist only seven quads. The smallest value of the perimeter, for which a quad is possible, is $317460.$

P10. To find Pythagorean triangles with equal areas. Fermat gave a simple method to find pairs of Pythagorean triangles with equal areas. If (a, b, c) is a primitive Pythagorean triple, then the legs of the Pythagorean triangle having $u = c^2, \ v = 2ab$ as generators will be $u^2 - v^2 = c^4 - 4a^2b^2 = (b^2 - a^2)^2, \ 2uv = 4abc^2$, and hypotenuse $u^2 + v^2 = c^4 + 4a^2b^2$. The area of this triangle is $2abc^2(b^2 - a^2)^2$. Also, the area of the Pythagorean triangle having legs $2ca(b^2 - a^2), \ 2cb(b^2 - a^2)$, and hypotenuse $2c^2(b^2 - a^2)$, i.e., $pa, \ pb, \ pc$ where $p = 2c(b^2 - a^2)$ is $2c^2ab(b^2 - a^2)^2$. Taking $a = 3, \ b = 4, \ c = 5$ we find that the Pythagorean triangles $(49, 1200, 1201)$ and $(210, 280, 350)$ have the same area, namely, $29400.$ Notice that the triple $(210, 280, 350)$ is not primitive. The smallest two primitive Pythagorean triples having the same common area 210 are $(20, 21, 29)$ and $(12, 35, 37)$. Some other primitive Pythagorean triples having the same area are $(60, 91, 109), (28, 195, 197)$ with common area $2730;$

$(95, 168, 193), (40, 399, 401)$ with common area $7980;$ and
$(341, 420, 541), (132, 1085, 1093)$ with common area $71610.$

Three Pythagorean triangles having the same area also can be find. It is easy to see that if p, q, r, s are four numbers in arithmetic progression, then the Pythagorean triangles corresponding to generators (i). $u = rs, v = pq;$ (ii). $u = r(r+q), v = p(r-q);$ (iii). $u = q(r+q), v = s(r-q)$ will all have the same area $(r^2s^2 - p^2q^2)pqrs$. For example, let us take $p = 1, q = 2, r = 3, s = 4$. Then, respectively, we have

(A). $u = rs = 12, v = pq = 2, \ a = u^2 - v^2 = 140, b = 2uv = 48, c = u^2 + v^2 = 148;$

(B). $u = r(r+q) = 15, v = p(r-q) = 1, a = u^2 - v^2 = 224, b = 2uv = 30, c = u^2 + v^2 = 226;$

(C). $u = q(r+q) = 10, v = s(r-q) = 4, a = u^2 - v^2 = 84, b = 2uv = 80, c = u^2 + v^2 = 116.$

Three Pythagorean triangles obtained from (A), (B), (C) are $(48, 140, 148)$, $(30, 224, 226), (80, 84, 116)$. They all have a common area $3360.$ Notice that none of these triples is primitive. Another construction for three Pythagorean triangles having the same area was suggested by Beiler [58]: Take three sets of generators as $(u_1 = u^2 + uv + v^2, v_1 = u^2 - v^2)$, $(u_2 = u^2 + uv + v^2, v_2 = 2uv + v^2)$, and $(u_3 = u^2 + 2uv, v_3 = u^2 + uv + v^2)$. Then the right triangle generated by each triple $(u_i^2 - v_i^2, 2u_iv_i, u_i^2 + v_i^2)$ has common area $A = uv(2u+v)(u+2v)(u+v)(u-v)(u^2+uv+v^2)$. In particular, for $u = 2, v = 1$ the three Pythagorean triangles are $(40, 42, 58), (24, 70, 74)$, and $(15, 112, 113)$ and the common area is $840.$ Three primitive Pythagorean triples that have the same area are $(4485, 5852, 7373), (3059, 8580, 9109), (1380, 19019, 19069)$ with the area $13123110.$ This was discovered by Shedd [474] in 1945. Sets of four Pythagorean triangles with equal area are also known, the one having the smallest area is $(111, 6160, 6161), (231, 2960, 2969), (518, 1320, 1418), (280, 2442, 2458)$ with area 341880 (see Beiler [58], p. 127 and Guy [241] pp. 188–190). In general, Fermat proved that for each natural number *n*, there exist *n* Pythagorean triangles with different hypotenuses and the same area.

P11. Pythagorean triangles whose areas consist of a single digit include $(3, 4, 5)$ (area of 6) and $(693, 1924, 2045)$ (area of 666666), (see Wells [531], p. 89).

P12. For $u = 149, \ v = 58$ we get a Pythagorean triple $(17284, 18837, 25565)$, the area of the triangle corresponding to which is 162789354, a number using all the nine digits $1, 2, \cdots, 9.$ Many such triples are known. Another such triple is $(26767, 68544, 73585)$ obtained by letting $u = 224$ and $v = 153$, the corresponding area being $917358624.$

P13. There are Pythagorean triples for which the area of the corresponding triangle is represented by a number using all the ten digits. One such triple is $(6660, 443531, 443581)$ obtained by letting $u = 666$ and $v = 5$. The corresponding area is 1476958230. Another such triple is $(86995, 226548, 242677)$ obtained by putting $u = 406$ and $v = 279$, the corresponding area being 9854271630.

P14. In 1643, Fermat wrote a letter to Mersenne in which he posed the problem of finding a Pythagorean triangle (a, b, c) whose hypotenuse and sum of the legs were squares of integers, i.e., find integers *p* and *q* such that $c = p^2$ and $a + b = q^2$. By using the method of infinite descent, it is found that the values $u = 2150905,\ v = 246792$ give a desired solution. In fact, we find

$$a = u^2 - v^2 = 4565486027761, \quad b = 2uv = 1061652293520$$

and

$$c = u^2 + v^2 = 4687298610289 = 2165017^2,$$
$$a + b = 5627138321281 = 2372159^2.$$

There exist infinitely many such Pythagorean triples, but the aforementioned values are the smallest possible.

P15. Two triples are called *siblings* if they have a common hypotenuse, e.g., $(16, 63, 65)$, $(33, 56, 65)$ and $(13, 84, 85)$, $(36, 77, 85)$ in Table 6.3. If we search among larger hypotenuse, we may find larger sets of siblings, e.g., there are four primitive triples with hypotenuse $1105 : (47, 1104, 1105), (264, 1073, 1105), (576, 943, 1105), (744, 817, 1105)$. The hypotenuse 32045 has eight primitive triples: $(716, 32037, 32045)$, $(2277, 31964, 32045)$, $(6764, 31323, 32045)$, $(8283, 30956, 32045)$, $(15916, 27813, 32045)$, $(17253, 27004, 32045)$, $(21093, 24124, 32045)$, $(22244, 23067, 32045)$. A *twin Pythagorean triple* is a Pythagorean triple (a, b, c) for which two values are consecutive integers, e.g., $(3, 4, 5)$, $(5, 12, 13)$, $(7, 24, 25)$, $(20, 21, 29)$, $(9, 40, 41)$, $(11, 60, 61)$, $(13, 84, 85)$, $(15, 112, 113)$. There are infinitely many twin Pythagorean triples. In the following table (see http://oeis.org/A101903) we present the number of twin Pythagorean triples, denoted as $a(n)$ with hypotenuse less than $10^n,\ n \leq 24$.

n	$a(n)$	n	$a(n)$	n	$a(n)$	n	$a(n)$
1	1	2	7	3	24	4	74
5	228	6	712	7	2243	8	7079
9	22370	10	70722	11	223619	12	707120
13	2236083	14	7071084	15	22360697	16	70710696
17	223606818	18	707106802	19	2236067999	20	7071067836
21	22360679800	22	70710678145	23	223606797778	24	707106781216

P16. Let $N_h(n)$ and $N_p(n)$ denote the number of primitive Pythagorean triangles whose hypotenuses and perimeters do not exceed n, respectively. In 1900, Lehmer proved that

$$\lim_{n\to\infty}\frac{N_h(n)}{n}=\frac{1}{2\pi}\quad\text{and}\quad\lim_{n\to\infty}\frac{N_p(n)}{n}=\frac{\ln(2)}{\pi^2}.$$

From (6.8), hypotenuse $= c = u^2+v^2 < n$ implies that *u* and *v* must lie in the positive quarter of the circle with radius $\sqrt{n}$. Further, since $u > v,\ u, v$ must lie below the line $u = v$, see Fig. 6.1. The area of this segment is $n\pi/8$. Ernesto Cesáro (1859–1906, Italy) in 1880 showed that the probability of $\gcd(u, v) = 1$ is equal to $6/\pi^2$. Further, the probability of both *u* and *v* are not odd conditioned to coprime is $2/3$. Thus, we have

$$N_h(n)\ \simeq\ \frac{n\pi}{8}\times\frac{6}{\pi^2}\times\frac{2}{3}=\frac{n}{2\pi}.$$

Similarly, we can show that $N_p(n) \simeq (n\ln 2)/\pi^2$. From these approximations it follows that $N_h(1000) \simeq 159.15$ and $N_p(1000) \simeq 70.23$. From Table 6.3 actual computation gives $N_h(1000) = 158$ and $N_p(1000) = 71$. In 2002, Benito and Varona [63] proved that the number $N_{a,b}(n)$ of primitive Pythagorean triangles (a, b, c) such that both the legs *a* and *b* do not exceed *n* is

$$N_{a,b}(n)=\frac{4\ln(1+\sqrt{2})}{\pi^2}n+O(\sqrt{n}).$$

On counting we find that the exact value of $N_{a,b}(1000) = 358,$ whereas the aforementioned formula gives $N_{a,b}(1000) \simeq 357.$

P17. Since, $(u+vi)^2 = (u^2-v^2)+i(2uv)$, it follows from (6.8) that the square of any complex number $u + vi$ (where u, v are coprime positive integers with $u > v$) yields the legs of a primitive Pythagorean triangle. Thus, for example $(4+3i)^2 = 7+24i$ gives $a = 7$ and $b = 24,$ and from (5.1) we have $7^2+24^2 = 25^2$, i.e., $(7, 24, 25)$ is a Pythagorean triple.

P18. There are Pythagorean triples (not necessarily primitive) each side of which is a Pythagorean triangular number $t_n = n(n+1)/2,$ (see Sect. 7.2), for example, $(t_{132}, t_{143}, t_{164}) = (8778, 10296, 13530)$. It is not known whether infinitively many such triples exist.

P19. Consider any two integers *u* and *v* such that $u > v > 0$. Then, from the four integers $\{u - v, v, u, u + v\}$ known as *Pythagorean triangle generator* we can always calculate a Pythagorean Triangle. For this, we take the product of the outer two integers, i.e., $(u - v)(u + v) = u^2 - v^2$, which gives one leg, then we take twice the product of the middle two integers, i.e., 2*uv* that gives the another leg, and then take the the sum of squares of the inner two integers, i.e., $u^2 + v^2$, which gives the hypotenuse of the triangle. For example, for $u = 2, v = 1,$ we have $\{1, 1, 2, 3\}$, and from this we can easily calculate the primitive Pythagorean triple $(3, 4, 5)$. If we take $u = 3, v = 1$, then we have $\{2, 1, 3, 4\}$ and we get the triple $(6, 8, 10)$, which is not primitive. Now recall that Fibonacci numbers are generated from the recurrence relation $\mathcal{F}_{k+2} = \mathcal{F}_{k+1} + \mathcal{F}_k,\ \mathcal{F}_1 = 1, \mathcal{F}_2 = 1$. We choose two consecutive Fibonacci numbers $v = \mathcal{F}_{k+1}$ and $u = \mathcal{F}_{k+2}$, and use the recurrence relation to find $u - v = \mathcal{F}_{k+2} - \mathcal{F}_{k+1} = \mathcal{F}_k$ and $u + v = \mathcal{F}_{k+2} + \mathcal{F}_{k+1} = \mathcal{F}_{k+3}$, and hence $\{\mathcal{F}_k, \mathcal{F}_{k+1}, \mathcal{F}_{k+2}, \mathcal{F}_{k+3}\}$ forms a Pythagorean triangle generator. This leads to the relation (see Dujella [172]).

$$(\mathcal{F}_k \mathcal{F}_{k+3})^2 + (2\mathcal{F}_{k+1}\mathcal{F}_{k+2})^2 = (\mathcal{F}_{k+1}^2 + \mathcal{F}_{k+2}^2)^2, \quad k \geq 1. \tag{6.11}$$

Now from a well known identity $\mathcal{F}_{k+1}^2 + \mathcal{F}_{k+2}^2 = \mathcal{F}_{2k+3}$ (see Burton [111]), (6.11) can be better written as

$$(\mathcal{F}_k \mathcal{F}_{k+3})^2 + (2\mathcal{F}_{k+1}\mathcal{F}_{k+2})^2 = \mathcal{F}_{2k+3}^2, \quad k \geq 1. \tag{6.12}$$

As an illustration, for $\{\mathcal{F}_6 = 8, \mathcal{F}_7 = 13, \mathcal{F}_8 = 21, \mathcal{F}_9 = 34\}$ relation (6.11) gives

$$(8 \times 34)^2 + (2 \times 13 \times 21)^2 = (13^2 + 21^2)^2,$$

which is the same as

$$272^2 + 546^2 = (169 + 441)^2 = 610^2 = \mathcal{F}_{15}^2.$$

It does not give a primitive Pythagorean triple $(272, 546, 610)$. However, for $\{\mathcal{F}_5 = 5, \mathcal{F}_6 = 8, \mathcal{F}_7 = 13, \mathcal{F}_8 = 21\}$ relation (6.11) leads to

$$(105)^2 + (208)^2 = 233^2,$$

which gives a primitive Pythagorean triple $(105, 208, 233)$.

From relation (6.12) it is clear that the product $\mathcal{F}_k\mathcal{F}_{k+1}\mathcal{F}_{k+2}\mathcal{F}_{k+3}$ of any four consecutive Fibonacci numbers is equal to the area of a Pythagorean triangle. In conclusion, (6.11), or equivalently (6.12), furnish primitive as well as non-primitive Pythagorean triples. Further, it does not provide all primitive Pythagorean triples. In relation (6.11), Fibonacci numbers can be replaced by Lucas numbers. In fact, for $\{L_5 = 11, L_6 = 18, L_7 = 29, L_8 = 47\}$, we get $(517)^2 + (1044)^2 = (1165)^2$, which gives primitive Pythagorean triple $(517, 1064, 1165)$.

P20. To find Pythagorean triangles with all the three sides consecutive integers. Let *p* be a positive integer. From (6.8), we have either (i). $2uv = p, u^2 - v^2 = p + 1, u^2 + v^2 = p + 2,$ or (ii). $u^2 - v^2 = p, 2uv = p + 1, u^2 + v^2 = p + 2$. Clearly, (i) implies that $(u^2 - v^2) + (u^2 + v^2) = 2u^2 = 2p + 3$, which is impossible. Next, (ii) implies that $2u^2 = 2p + 2,$ i.e., $u^2 = p + 1.$ Also, we have $2uv = p + 1,$ and hence $u^2 = 2uv,$ i.e., $u = 2v.$ Now since *u* and *v* are relatively prime, we must have $u = 2, v = 1,$ and hence $a = u^2 - v^2 = 3, b = 2uv = 4, c = u^2 + v^2 = 5.$ In conclusion, $(3, 4, 5)$ is the only primitive Pythagorean triple with three sides consecutive integers. This fact can also be seen as follows: If $(a, a + 1, a + 2)$ is a Pythagorean triple, then $a^2 + (a + 1)^2 = (a + 2)^2$, which is the same as $a^2 - 2a - 3 = (a - 3)(a + 1) = 0,$ and hence, $a = 3.$ Similarly, it follows that the only Pythagorean triangles with sides in arithmetic progression are those with sides $3n, 4n, 5n, \ n = 1, 2, 3, \cdots$.

P21. To find Pythagorean triangles in which hypotenuse exceeds the larger leg by 1. We shall show that formula (6.3) generates all such Pythagorean triangles. Since (6.8) generates all primitive Pythagorean triangles either (i). $u^2 + v^2 = u^2 - v^2 + 1,$ or (ii). $u^2 + v^2 = 2uv + 1.$ But, (i) leads to $2v^2 = 1$, which is impossible. Now (ii) is the same as $(u - v)^2 = 1$, which implies that $u = v + 1.$ Letting $v = n,$ then $u = n + 1.$ Putting this *u* and *v* in (6.8), we obtain (6.3). Thus, there are such infinite Pythagorean triangles.

In Dudley's book [171], page 127, the following problem is asked: Prove that if the sum of two consecutive integers is a square, then the smaller is a leg, and the larger a hypotenuse, of a Pythagorean triangle. His obvious solution is $k + (k + 1) = m^2$ implies that $k^2 + m^2 = (k + 1)^2$. However, the question is for what integers *k* and $m, \ k + (k + 1) = m^2.$ From P21, we notice that the only choice is $k = 2n^2 + 2n$ and $m = (2n + 1).$

P22. To find Pythagorean triangles with consecutive legs, from (6.8) it is necessary that either (i). $u^2 - v^2 = 2uv + 1,$ or (ii). $2uv = u^2 - v^2 + 1.$ In case (i), it follows that $(u - v)^2 = 2v^2 + 1$, which is the same as $(u - v)^2 - 2v^2 = 1$. Similarly, case (ii) leads to the equation $(u - v)^2 - 2v^2 = -1$. Thus, in case (i) we need to find integer solutions of Pell's equation $x^2 - 2y^2 = 1$, whereas in case (ii) of Pell's equation $x^2 - 2y^2 = -1$. We shall use inductive method to show that all solutions of these equations can be generated by the recurrence equations

$$\begin{aligned} x_{n+1} &= 3x_n + 4y_n \\ y_{n+1} &= 2x_n + 3y_n, \quad n \geq 1. \end{aligned} \tag{6.13}$$

For this, we assume the existence of the minimal solution (known as the fundamental solution) (x_1, y_1) of the concerned equation, and assume that (x_n, y_n) is also a solution, then it follows that

$$x_{n+1}^2 - 2y_{n+1}^2 = (3x_n + 4y_n)^2 - 2(2x_n + 3y_n)^2 = x_n^2 - 2y_n^2,$$

and this shows that (x_{n+1}, y_{n+1}) is also a solution of the equation.

For the equation $x^2 - 2y^2 = 1$ the fundamental/minimal solution (by inspection) is (3,2) and the next three solutions obtained from (6.13) are (17,12), (99,70) and (577,408). Similarly, for the equation $x^2 - 2y^2 = -1$ the minimal solution of (1,1) and the next three solutions generated from (6.13) are (7,5), (41,29), and (239,169). In the following table we use these eight values to record $u(= x + y), v(= y), u^2 - v^2, 2uv, u^2 + v^2$ and the corresponding Pythagorean triples. From the table it is clear that the numbers are increasing very rapidly, but still such triples are infinite.

u	v	$u^2 - v^2$	$2uv$	$u^2 + v^2$	(a, b, c)
2	1	3	4	5	$(3, 4, 5)$
5	2	21	20	29	$(20, 21, 29)$
12	5	119	120	169	$(119, 120, 169)$
29	12	697	696	985	$(696, 697, 985)$
70	29	4059	4060	5741	$(4059, 4060, 5741)$
169	70	23661	23660	33461	$(23660, 23661, 33461)$
408	169	137903	137904	195025	$(137903, 137904, 195025)$
985	408	803761	803760	1136689	$(803760, 803761, 1136689)$

On scanning data in Table 6.3, we notice that triples satisfying $c - b = 1$ are more dense than those satisfying $b - a = 1$. The fact that there are infinitely many Pythagorean triples of this type can also be shown rather easily: If $(a, a + 1, c)$ happens to be a Pythagorean triple, so is $(3a + 2c + 1, 3a + 2c + 2, 4a + 3c + 2)$. Indeed, we have

$$\begin{aligned} (3a+2c+1)^2 + (3a+2c+2)^2 &= 18a^2 + 8c^2 + 5 + 24ac + 18a + 12c \\ &= 9(a^2+(a+1)^2) + 8c^2 + 24ac + 12c - 4 \\ &= 17c^2 + 24ac + 12c - 4 = (4a+3c+2)^2. \end{aligned}$$

Diophantus set the problem of finding a number *p* such that both $10p + 9$ and $5p + 4$ are squares. Letting $10p + 9 = x^2$ and $5p + 4 = y^2$, we get the same equation $x^2 - 2y^2 = 1$, for which the minimal solution (which gives the nonzero solution) is $(17, 12)$. Solving the equations $10p + 9 = 17^2$ and $5p + 4 = 12^2$, we find $p = 28$.

P23. Diophantus as a problem asked to find a Pythagorean triangle in which the hypotenuse minus each of the legs is a cube. His answer is $(40, 96, 104)$, which leads to the required primitive Pythagorean triple $(5, 12, 13)$. From (6.8), we note that we need to find the solutions of the equations $c - a = p^3,\ c - b = q^3$, where *p* and *q* are some positive integers. But since $c - a = 2v^2$ and $c - b = (u - v)^2$ from the equation $c - a = 2v^2 = p^3$ we must have $v = 2$, and now from the equation $c - b = (u - v)^2 = (u - 2)^2 = q^3$, it follows that $u - 2 = k^3$ or $u = k^3 + 2$, where *k* is any positive integer. Thus, the required members of all Pythagorean triples are $a = (k^3 + 2)^2 - 2^2,\ b = 2^2(k^3 + 2),\ c = (k^3 + 2)^2 + 2^2,\ k \geq 1$ with $c - a = 2^3$ and $c - b = (k^2)^3$. For $k = 1, 2, 3$ we get the triples $(5, 12, 13), (40, 96, 104)$, and $(116, 837, 845)$, i.e., we get primitive as well as non-primitive Pythagorean triples.

P24. To find primitive Pythagorean triangles, one of whose legs is a perfect square, we need to solve the equation $(x^2)^2 + b^2 = c^2$, which is the same as $x^4 = (c + b)(c - b)$. We recall that $gcd(c, b) = 1$ and *c* is always odd, so there are two possible cases to consider (i). *b* is even, and (ii). *b* is odd. In case (i), equation $x^4 = (c + b)(c - b)$ holds provided there exist odd integers *p* and *q* such that $p > q$ and $c + b = p^4,\ c - b = q^4$ (see Step 4 above), and hence

$$c = \frac{p^4 + q^4}{2}, \quad b = \frac{p^4 - q^4}{2}, \quad a = x^2 = p^2q^2.$$

We present some desired triples in the following table.

p	q	a	b	c
3	1	$9 = 3^2$	40	41
5	1	$25 = 5^2$	312	313
5	3	$225 = 15^2$	272	353
7	1	$49 = 7^2$	1200	1201

Clearly, there exist infinitely many Pythagorean triples whose odd member is a perfect square.

In case (ii), we can choose $c + b = 2^3s^4,\ c - b = 2t^4$, where *t* is odd. These relations give

$$c = 2^2s^4 + t^4, \quad b = 2^2s^4 - t^4, \quad a = x^2 = 2^2s^2t^2.$$

Using these relation, we compute some Pythagorean triples in the following table.

s	t	a	b	c	(a,b,c)
1	1	$4=2^2$	3	5	$(3,4,5)$
2	1	$16=4^2$	63	65	$(16,63,65)$
3	1	$36=6^2$	323	325	$(36,323,325)$
4	1	$64=8^2$	1023	1025	$(64,1023,1025)$
4	3	$576=24^2$	943	1105	$(576,943,1105)$

Clearly, there exist infinitely many Pythagorean triples whose even member is a perfect square.

P25. To find primitive Pythagorean triangles whose hypotenuse is a perfect square, we need to deal with the equation $a^2+b^2=(z^2)^2$, which is the same as $(a+ib)(a-ib)=z^4$. Now since $\gcd(a,b)=1,$ there exist integers *p* and *q* such that $(p,q)=1$ and $a+ib=(p+iq)^4$ also $a-ib=(p-iq)^4$. Next, comparing the real and imaginary parts, we find

$$a=|p^4+q^4-6p^2q^2| \quad \text{and} \quad b=4pq|p^2-q^2|;$$

here, modulus sign is taken without loss of generality (we need a^2 and b^2). Then, we also have

$$c=(p+iq)^2(p-iq)^2=(p^2+q^2)^2.$$

We present some required triples in the following table.

p	q	a	b	c	(a,b,c)
2	1	7	24	$25=5^2$	$(7,24,25)$
3	2	119	120	$169=13^2$	$(119,120,169)$
4	1	161	240	$289=17^2$	$(161,240,289)$
4	3	527	336	$625=25^2$	$(336,527,625)$
5	1	476	480	$676=26^2$	$(119,120,169)$
5	2	41	840	$841=29^2$	$(41,840,841)$
5	3	644	960	$1156=34^2$	$(161,240,289)$
5	4	1519	720	$1681=41^2$	$(1519,720,1681)$

Note that in the aforementioned table $p=3,q=2$ and $p=5,q=1$ give the same primitive Pythagorean triple. The same holds for $p=4,q=1$ and $p=5,q=3$.

P26. To find primitive Pythagorean triangles $L_p(n)$ whose one leg $n>0$ is given. There are two cases to consider (i). *n* is odd, then $u^2-v^2=n,$ or (ii). *n* is even, then $2uv=n.$ The number of integer solutions of the these equations depends upon the prime factorization of $n.$ Let $n=2^s p_1^{s_1}\cdots p_k^{s_k},$ where $p_1,p_2,\cdots,p_k$ are distinct primes. If $s=0,$ n is odd, and $u>v,$ then *n* can be factored as a product of two relatively prime factors in 2^{k-1} ways. Corresponding to each such way, $u^2 - v^2 = n$ has exactly one solution in positive integers, and hence, there will be 2^{k-1} solutions. If $s = 1$, then *n* is an odd multiple of 2, and the equation $2uv = n$, does not have any solution in integers because 2*uv* is a multiple of 4. If $s \geq 2$ and $u > v$, then the number of ways in which *n* can be factored into a pair of relatively prime factors is $2^{(k+1)-1}$, i.e., 2^k, and therefore, there are 2^k solutions of $2uv = n$ with $u > v$. Summarizing these arguments, we have the following result: If *n* is of the form $4p + 2$, then $L_p(n) = 0$ (see P7). If *n* is not of this form, then $L_p(n) = 2^{q-1}$, where *q* is the number of distinct primes occurring in the prime factorization of n. As examples, consider the cases $n = 60,\ 65,\ 75,\ 86$. Since $60 = 2^2 \cdot 3 \cdot 5$. The number of primes occurring in the factorization is $q = 3$ and so there are 2^{3-1}, i.e., 4 primitive Pythagorean triangles having 60 as a leg. From Table 6.3, these triples are $(11, 60, 61)$, $(60, 91, 109)$, $(60, 221, 229)$, and $(60, 899, 901)$. Since $65 = 5 \cdot 13,\ q = 2$ and so there are $2^{2-1} = 2$ primitive triangles having 65 as leg. These triples are $(65, 72, 97)$ and $(65, 2112, 2113)$. Since $75 = 3 \cdot 5^2,\ q = 2$ and so there are $2^{2-1} = 2$ primitive Pythagorean triangles having 75 as a leg. These triples are $(75, 308, 317)$ and $(75, 2812, 2813)$. Since $86 = 4(21) + 2$, there are no primitive Pythagorean triangles having 86 as a leg.

In a similar way (see Beiler [58]) we can find the number of primitive Pythagorean triangles $H_p(n)$ having *n* as a hypotenuse. We factorize *n* as

$$n = 2^{a_0} \left(p_1^{a_1} \cdots p_s^{a_s}\right) \left(q_1^{b_1} \cdots q_t^{b_t}\right),$$

where $p's$ are of the form $4r - 1$ and $q's$ are of the form $4r + 1$. Then,

$$H_p(n) = \begin{cases} 2^{t-1} & \text{if} \quad s = 0 \quad \text{and} \quad a_0 = 0 \\ 0 & \text{otherwise.} \end{cases}$$

As an example, we consider $65,\ 75,\ 325$, and 389. Since $65 = (4 + 1)(4 \times 3 + 1),\ s = 0, a_0 = 0$ and $t = 2$ so $H_p(65) = 2^{2-1} = 2$. From Table 6.3, these triples are $(16, 63, 65)$ and $(33, 56, 65)$. Since $75 = (4 - 1)(4 + 1)^2,\ s = 1, a_0 = 0$ so $H_p(75) = 0$. Since $325 = (4 + 1)^2(4 \times 3 + 1),\ s = 0, a_0 = 0$ and $t = 2$ so $H_p(325) = 2$. From Table 6.3, these triples are $(36, 323, 325)$ and $(204, 253, 325)$. Since $389 = (4 \times 97 + 1),\ s = 0, a_0 = 0$ and $t = 1$ so $H_p(389) = 1$. From Table 6.3, this triple is $(189, 340, 389)$.

Combining the aforementioned results, we can find the total number of ways in which a given *n* may be either a leg or hypotenuse of a primitive Pythagorean triangle as $T_p(n) = L_p(n) + H_p(n)$. As an example, we consider 325. Since $325 = 5^2 \times 13 = (4 + 1)^2(4 \times 3 + 1)$ it follows that $L_p(325) = 2$

and $H_p(325) = 2,$ and hence $T_p(325) = 2 + 2 = 4.$ Indeed, the triples are $(228, 325, 397), (325, 52812, 52813),$ $(36, 323, 325),$ and $(204, 253, 325).$

If $n = 2^m,\ m \geq 2$, then $L_p(2^m) = 1$ and $H_p(2^m) = 0.$ To find the unique primitive Pythagorean triangle whose one leg is 2^m, from (6.8), it follows that $b = 2uv = 2^m$ or $uv = 2^{m-1}.$ Thus, from Step 4 and the fact that $u > v,$ we have $u = 2^{m-1}$ and $v = 1.$ Hence, the members of the required triple are $2^{2(m-1)} - 1,\ 2^m,\ 2^{2(m-1)} + 1$, which for $p = 2$ and $p = 3,$ respectively, give the primitive Pythagorean triples as $(3, 2^2, 5)$ and $(2^3, 15, 17).$ Now we shall show that there are exactly $m - 2,\ m \geq 3$ non-primitive triangles whose one leg is $2^m.$ Clearly, for a non-primitive triple the members are

$$a = d(u^2 - v^2), \quad b = d(2uv), \quad c = d(u^2 + v^2), \quad d \geq 2.$$

If $2^m = d(u^2 - v^2)$, then from the facts that $u > v,$ and *u* and *v* are of different parity $(u^2 - v^2) \geq 3$ and odd. Thus, it is necessary that $d = 2^m D,\ D \geq 1.$ But, then $2^m = 2^m D(u^2 - v^2)$, which means $1 = D(u^2 - v^2).$ But, it is impossible. Similarly, we can show that $2^m \neq d(u^2 + v^2).$ Thus, the only possibility left is $2^m = d(2uv)$ or $2^{m-1} = d(uv)$, which implies that there exists some $2^k = d,\ 0 \leq k \leq m - 1.$ However, the cases $k = 0$ and $k = m - 1$ can be ruled out. Indeed, for $k = 0,$ it gives $d = 1$, but we have $d \geq 2,$ and the fact that *uv* is even assures that *k* cannot be $m - 1.$ In conclusion, we have $uv = 2^{m-k-1}, 1 \leq k \leq m - 2,$ which implies that $u = 2^{m-k-1},\ v = 1.$ This leads to the following members of the non-primitive triples

$$2^m, \quad 2^k(2^{2(m-k-1)} - 1), \quad 2^k(2^{2(m-k-1)} + 1), \quad k = 1, 2, \cdots, m - 2.$$

For $m = 4$ two non-primitive triples are $(2^4, 30, 34), (12, 2^4, 20)$ and for $m = 5$ three non-primitive triples are $(2^5, 126, 130), (2^5, 60, 68), (24, 2^5, 40).$ This corrects a minor error in the work of Zelator [545, 546] and supplements the conclusion of Problem 5 on page 240 in the book of Burton [111].

P27. In a triangle we can draw a circle touching all three sides, this circle is called in-circle with radius as in-radius denoted as *r* and the center as in-center. The in-radius of a Pythagorean triangle (a, b, c) satisfies (see Burton [111], page 239) the relation $r(a + b + c) = 2\Delta = ab$ (Two formulae for *r* in terms of the sides a, b, c were known to Liu Hui, see his commentary on the *Jiuzhang Suanshu*). In this relation substituting $a = k(u^2 - v^2),\ b = k(2uv),\ c = k(u^2 + v^2),$ (here $k > 0$ is an integer) we get $r = k(u - v)v$, which shows that the in-radius of a Pythagorean triangle is an integer. The number of distinct primitive Pythagorean triangles having a common in-radius *r* depends upon the number of distinct prime divisors of $r.$ If the prime factorization of *r* contains *n* distinct odd primes, then there exist 2^n distinct primitive Pythagorean triangles having a common in-radius *r* (see Robbins [434] and Omland [401]). For example, we consider $u = 985, v = 408, k = 1$ so that $r = 577 \times 408$ whose prime factors are $2^3 \times 3 \times 17 \times 577$, and hence, there exist exactly $8 (= 2^3)$ distinct primitive Pythagorean triangles having in-radius 577×408. One of such triple is $(803760, 803761, 1136689)$.

In addition, the total number $N(r)$ of distinct Pythagorean triangles (not necessarily primitive) having a given in-radius *r* can be determined by writing down the prime factorization of *r*. It is also known that if $r = 2^s p_1^{s_1} p_2^{s_2} \cdots, p_n^{s_n}$, where $p_1, p_2, \cdots, p_n$ are distinct odd primes, then

$$N(r) = (s+1)(2s_1+1)\cdots(2s_n+1).$$

For

$r = 577 \times 408 = 2^3 \times 3 \times 17 \times 577, \;\; N(r) = (3+1)(2+1)(2+1)(2+1) = 108.$

P28. It is possible to construct Pythagorean triples (not necessarily primitive) by factoring *c* into smaller factors, each of which is itself a sum of two squares. For this, we need Diophantus (Brahmagupta-Fibonacci) identities (4.16). For example, we consider the number

$$493 = 17 \times 29 = (1^2 + 4^2)(2^2 + 5^2)$$

for which equalities (4.16) give

$$493 = (1 \times 2 + 4 \times 5)^2 + (1 \times 5 - 4 \times 2) = 22^2 + 3^2 \tag{6.14}$$

and

$$493 = (1 \times 5 + 4 \times 2) + (1 \times 2 - 4 \times 5)^2 = 13^2 + 18^2. \tag{6.15}$$

Now equalities (6.14) and (6.15), again in view of (4.16) give

$$\begin{aligned} 493^2 &= (3^2 + 22^2)(13^2 + 18)^2 \\ &= (3 \times 13 + 22 \times 18)^2 + (3 \times 18 - 22 \times 13)^2 = 435^2 + 232^2 \end{aligned}$$

and

$$\begin{aligned} 493^2 &= (3^2 + 22^2)(13^2 + 18)^2 \\ &= (3 \times 18 + 22 \times 13)^2 + (3 \times 13 - 22 \times 18)^2 = 340^2 + 357^2. \end{aligned}$$

This gives the Pythagorean triples $(232, 435, 493)$ and $(340, 357, 493)$, which are not primitive. In fact, dividing these triples, respectively, by 29 and $17,$ we get the primitive Pythagorean triples $(8, 15, 17)$ and $(20, 21, 29)$.

Notice that we also have

$$\begin{aligned} 493^2 &= 17^2 \times 29^2 = (8^2 + 15^2) \times (20^2 + 21^2) \\ &= (8 \times 20 + 15 \times 21)^2 + (8 \times 21 - 15 \times 20)^2 = 475^2 + 132^2, \end{aligned}$$

and

$$493^2 = (8 \times 21 + 15 \times 20)^2 + (8 \times 20 - 15 \times 21)^2 = 468^2 + 155^2.$$

This gives the primitive Pythagorean triples $(132, 475, 493)$ and $(155, 468, 493)$ (see Table 6.3).

P29. In 1934, Berggren [65] introduced three matrices with the same values in each position but differing in sign $A_i,\ i = 1, 2, 3$, where

$$A_1 = \begin{pmatrix} 1 & 2 & 2 \\ -2 & -1 & -2 \\ 2 & 2 & 3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}, \quad A_3 = \begin{pmatrix} -1 & -2 & -2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}$$

and showed that from a given primitive Pythagorean triple (a_0, b_0, c_0) three new primitive Pythagorean triples $(a_i, b_i, c_i),\ i = 1, 2, 3$ can be generated by

$$(a_i, b_i, c_i) = (a_0, b_0, c_0)A_i, \quad i = 1, 2, 3. \tag{6.16}$$

He also showed that by using these three matrices all primitive Pythagorean triples can be generated from the triple $(a_0, b_0, c_0) = (3, 4, 5)$. A simple calculation shows that with this triple (a_0, b_0, c_0), (6.16) gives $(5, 12, 13)$, $(21, 20, 29)$ which we have agreed to write as $(20, 21, 29)$, and $(15, 8, 17)$ which we have agreed to write as $(8, 15, 17)$. Now if we take $(a_0, b_0, c_0) = (5, 12, 13)$, then (6.16) generates $(7, 24, 25)$, $(48, 55, 73), (28, 45, 53)$. Thus, $(7, 24, 25) = (3, 4, 5)A_1A_1$. In fact, Hall [244] and Roberts [435] prove that (a, b, c) is a primitive Pythagorean triple if and only if $(a, b, c) = (3, 4, 5)U$, where *U* is a finite product of the matrices A_1, A_2, A_3. In other words, the triple $(3, 4, 5)$ is the *parent* of all primitive Pythagorean triples.

P30. In 2008, Price [419] found three new matrices

$$A_1' = \begin{pmatrix} 2 & -2 & -2 \\ 1 & 2 & 1 \\ -1 & 2 & 3 \end{pmatrix}, \quad A_2' = \begin{pmatrix} 2 & 2 & 2 \\ 1 & -2 & -1 \\ 1 & 2 & 3 \end{pmatrix}, \quad A_3' = \begin{pmatrix} 2 & 2 & 2 \\ -1 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$

and showed that like (6.16)

$$(a_i, b_i, c_i) = (a_0, b_0, c_0)A_i', \quad i = 1, 2, 3. \tag{6.17}$$

also produce all primitive Pythagorean triples. However, the three new triples obtained from (6.17) may not be the same as calculated from (6.16). As an example, we find that from $(a_0, b_0, c_0) = (3, 4, 5)$, (6.17) produces the new triples $(5, 12, 13), (8, 15, 17)$, and $(7, 24, 25)$. Further, with $(a_0, b_0, c_0) = (5, 12, 13)$, (6.17) gives $(9, 40, 41), (12, 35, 37)$, and $(11, 60, 61)$.

P31. In 1994, Saunders and Randall [448] establish the following three new matrices

$$B_1 = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

and showed that from a given generator (u, v) of Pythagorean triple three new generators, which preserve coprimeness and opposite parity can be obtained by

$$(U_i, V_i) = (u, v)B_i, \quad i = 1, 2, 3.$$

For this, it suffices to show that each $(U_1, V_1) = (2u - v, u),\ (U_2, V_2) = (2u + v, u), (U_3, V_3) = (u + 2v, v)$ generates Pythagorean triple, i.e., $(a_i = U_i^2 - V_i^2, b_i = 2U_iV_i, c_i = U_i^2 + V_i^2),\ i = 1, 2, 3$ are Pythagorean triples. In particular, for $u = 2, v = 1$ we have $(U_1, V_1) = (3, 2), (U_2, V_2) = (5, 2), (U_3, V_3) = (4, 1)$, and these respectively generate Pythagorean triples $(5, 12, 13), (20, 21, 29), (8, 15, 17)$.

P32. Consider the famous tournament problem, which was posed to Fibonacci by a Master John of Palermo (fl. 1221–1240, Italy), a member of the entourage of the Holy Roman Emperor Frederick II: Find a number *x* such that both $x^2 + 5$ and $x^2 - 5$ are squares of rational numbers, i.e.,

$$x^2 + 5 = a^2 \quad \text{and} \quad x^2 - 5 = b^2. \tag{6.18}$$

We will see that the solution requires Pythagorean triples. We express $x, a,$ and *b* as fractions with a common denominator:

$$x = \frac{x_1}{d}, \quad a = \frac{a_1}{d}, \quad b = \frac{b_1}{d}.$$

Substituting these values in equation (6.18), and multiplying throughout by d^2, we get the equations

$$x_1^2 + 5d^2 = a_1^2, \quad x_1^2 - 5d^2 = b_1^2. \tag{6.19}$$

Subtracting the second equation from the first, we obtain

$$10d^2 = a_1^2 - b_1^2 = (a_1 + b_1)(a_1 - b_1).$$

Since the left-hand side is even, both a_1 and b_1 must be even or odd. In either case, $a_1 - b_1$ is an even integer, say $a_1 - b_1 = 2k,$ from this it follows that $a_1 + b_1 = 5d^2/k.$ Now solving the last two equations, we find

$$a_1 = \frac{5d^2}{2k} + k, \quad b_1 = \frac{5d^2}{2k} - k.$$

Substituting these expressions in (6.19), we get

$$x_1^2 + 5d^2 = \left(\frac{5d^2}{2k} + k\right)^2 = \left(\frac{5d^2}{2k}\right)^2 + 5d^2 + k^2,$$

$$x_1^2 - 5d^2 = \left(\frac{5d^2}{2k} - k\right)^2 = \left(\frac{5d^2}{2k}\right)^2 - 5d^2 + k^2,$$

which on addition yields the single condition

$$\left(\frac{5d^2}{2k}\right)^2 + k^2 = x_1^2.$$

Thus, the three numbers $5d^2/2k,\ k$ and x_1 form a primitive Pythagorean triple and hence can be written as

$$\frac{5d^2}{2k} = u^2 - v^2, \quad k = 2uv, \quad x_1 = u^2 + v^2,$$

where the positive integers *u* and *v* are such that $u > v,\ u$ and *v* are of opposite parity, and they are coprime. Now to eliminate $k,$ we take the product of the first two of these equations, to obtain

$$5d^2 = 4uv(u^2 - v^2).$$

Clearly, we need integers *u* and *v*, which will make the right-hand side of this equation 5 times a perfect square. For this, we set $u = 5,$ so that the condition reduces to

$$d^2 = 4v(5^2 - v^2).$$

Evidently the right-hand side becomes a square when $v = 4:$

$$d^2 = 4 \cdot 4(5^2 - 4^2) = 16 \cdot 9 = 12^2.$$

These values for *u* and *v* lead to

$$x_1 = u^2 + v^2 = 5^2 + 4^2 = 41.$$

Putting these pieces together, we get

$$x = \frac{x_1}{d} = \frac{41}{12}$$

as a solution to Fibonacci's tournament problem.

P33. For (a, b, c) primitive relation (5.3) takes the form

$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2},$$

The only positive integer solutions of this equation are given by

$$a = 2uv(u^2 + v^2), \quad b = (u^4 - v^4), \quad c = 2uv(u^2 - v^2),$$

where *u* and *v* are relatively prime positive integers, one of which is even, and $u > v$. In particular, for $u = 2, v = 1$ we have $1/20^2 + 1/15^2 = 1/12^2$.

P34. Jordanus De Nemore (around 1225–1237, Italy) found integers $x, y,$ and *z* so that $z^2 - y^2 = y^2 - x^2$, i.e., $z^2 + x^2 = 2y^2$, which can be written as

$$\left(\frac{z+x}{2}\right)^2 + \left(\frac{z-x}{2}\right)^2 = y^2.$$

Since $x^2 + z^2$ is even, *x* and *z* must be of the same parity. This shows that $(z + x)/2$ and $(z - x)/2$ are integers, but then this is the same as Pythagorean triple problem and has infinite number of integer solutions, namely from (6.8), $(z - x)/2 = u^2 - v^2, (z + x) = 2uv, y = u^2 + v^2$. Thus, it follows that

$$x = v^2 - u^2 + 2uv, \quad y = u^2 + v^2, \quad z = u^2 - v^2 + 2uv,$$

where $u > v$ and are of different parity. (By expansion it follows that x^2, y^2, z^2 are in arithmetic progression). As an example, for $u = 6, v = 5$ we have $x = 49, y = 61, z = 71$, and hence $71^2 - 61^2 = 61^2 - 49^2$. We also note that $(z - x)/2 = 11, (z + x)/2 = 60$ and we get the Pythagorean triple $(11, 60, 61)$, see Table 6.3. If we begin with the Pythagorean triple $((z - x)/2, (z + x)/2, y) = (3, 4, 5)$, then $x = 1, y = 5, z = 7$, and the equality $7^2 - 5^2 = 5^2 - 1^2$ holds. Similarly, for the Pythagorean triple $((z - x)/2, (z + x)/2, y) = (8, 15, 17)$, we have $x = 7, y = 17, z = 23$ and the relation $23^2 - 17^2 = 17^2 - 7^2$ holds. In conclusion, there is one-to-one correspondence between Pythagorean triples and the solutions of $z^2 - y^2 = y^2 - x^2$. For the same work, Dickson in [164] refers to A. Guibert's (France) problem of 1862.

We can also find integer solutions of the equation $x^2 + 2y^2 = z^2$, where $\gcd(x, y, z) = 1$. Since $2y^2 = z^2 - x^2 = (z + x)(z - x)$, we must have $z + x = 2u^2, z - x = v^2$, which gives $x = (2u^2 - v^2)/2,\ y = uv,\ z = (2u^2 + v^2)/2$, which is better written as

$$x = (2u^2 - v^2), \quad y = 2uv, \quad z = (2u^2 + v^2).$$

If we let $u = 7,\ v = 5$, then $x = 73, y = 70, z = 123$, and this gives $73^2 + 2 \times 70^2 = 123^2$. If we take $u = 6, v = 4$, then $x = 56, y = 48, z = 88$, which on dividing by 8 gives $x = 7, y = 6, z = 11$, and we have the identity $7^2 + 2 \times 6^2 = 11^2$. Finally, if we take $u = 11, v = 6$, then after dividing by 2, we obtain $x = 103, y = 66, z = 139$, and the equality $103^2 + 2 \times 66^2 = 139^2$.

P35. The origin of De Nemore's problem comes from Diophantus' Problem II-19, which states: Find three squares such that the difference between the greatest and the middle has a given ratio $p:1$ to the difference between the middle and the least. If we let nonzero real numbers x, y, z such that $x^2 < y^2 < z^2$, then we need to find the solution of the equation

$$z^2 - y^2 = p(y^2 - x^2), \tag{6.20}$$

where $p > 0.$ It is clear that if x, y, z is a solution of (6.20), then $dx, dy, dz,\ d \neq 0$ is also a solution. Thus, it suffices to consider only integer solutions of (6.20). The case $p = 1$ has been discussed in P34, so we will mainly take up the case when $p \neq 1.$ We rewrite (6.20) as

$$\frac{z+y}{y+x} = p\frac{y-x}{z-y} = k \quad \text{(some nonzero real number)},$$

which leads to the equations

$$z + (1-k)y - kx = 0 \quad \text{and} \quad z - \left(1 + \frac{p}{k}\right) y + \frac{p}{k} x = 0.$$

Solving these equations, we get

$$x = (2k + p - k^2)r, \quad y = (p + k^2)r, \quad z = (2pk + k^2 - p)r, \tag{6.21}$$

where *r* is an arbitrary positive real number, it will be chosen so that $x, y,$ and *z* are integers with $\gcd(x, y, z) = 1.$ Using (6.21) in both sides of (6.20), we find

$$z^2 - y^2 = p(y^2 - x^2) = 4p(k-1)k(k+p)r^2. \tag{6.22}$$

Thus, $x^2 < y^2 < z^2$ holds provided $k > 1.$ If $k = 1$ or $k = 0,$ we have $x = y = z$, which is an empty case. If $0 < k < 1$, then we have $x^2 > y^2 > z^2$, and in such a case we rewrite (6.20) as

$$x^2 - y^2 = \frac{1}{p}(y^2 - z^2). \tag{6.23}$$

If $-p < k < 0$, then once again we have $x^2 < y^2 < z^2$, and hence, (6.20) holds. If $k = -p,$ then we have $x = y = z$, which is again an empty case. Finally, if $k < -p$, then we find $x^2 > y^2 > z^2$, and therefore (6.23) holds.

If $p = 1, r = 1,$ and $k = n \geq 2$ is a natural number, then from (6.21), we find

$$u = 2n, \quad v = n^2 - 1, \quad y = n^2 + 1,$$

which gives Plato's characterization of the Pythagorean triples (6.6). For $k = 3/2, p = 1/4,$ from (6.21) we get $x = (4/4)r, y = (10/4)r, z = (11/4)r,$ so we choose $r = 4,$ to obtain $x = 4, y = 10, z = 11$ and from (6.20) we have the equality $(11^2 - 10^2) = (1/4)(10^2 - 4^2).$ For $k = 1/2, p = 2,$ from (6.21) we get $x = (23/9)r, y = (19/9)r, z = -(5/9)r,$ and we choose $r = 9$ so that

$x = 23, y = 19, z = -5,$ and from (6.23) we get the equality $(23^2 - 19^2) = (1/2)(19^2 - (-5)^2)$. For $k = -1$, (6.21) reduces to $x = (p-3)r, y = (p+1)r, z = (-3p+1)r$. Thus, in particular, for $p = 11/2$ and $r = 2$, we have $x = 5, y = 13, z = -31$, and from (6.20) follows the equality $((-31)^2 - 13^2) = (11/2)(13^2 - 5^2)$. For $k = -(17/3), p = 4$, and $r = 9/5$, (6.21) gives $x = -71, y = 65, z = 31$, and from (6.23) we have the equality $((-71)^2 - 65^2) = (1/4)(65^2 - 31^2)$.

Diophantus gives only one solution for $p = 3$, that is, $x^2 = 25/4, y^2 = 49/4, z^2 = 121/4$ (the common factor $1/4$ can be removed). Unfortunately, this solution cannot be obtained from (6.21). Thus, (6.21) does not generate all solutions of (6.20). Extending Diophantus' method, we assume that $y = x + a, z = x + b, \quad b > a > 0$ and $x > 0$, then the Eq. (6.20) gives

$$\begin{aligned} x &= \frac{b^2 - (p+1)a^2}{2[a(p+1) - b]}, \quad y = x + a = \frac{(b-a)^2 + pa^2}{2[a(p+1) - b]}, \\ z &= x + b = \frac{a(p+1)(2b-a) - b^2}{2[a(p+1) - b]}, \end{aligned} \tag{6.24}$$

where $(b/a) < (p+1) < (b/a)^2$, so that $x > 0$. It follows that

$$(x+b)^2 - (x+a)^2 = p((x+a)^2 - x^2) = \frac{pab(b-a)}{a(p+1) - b} > 0.$$

For $p = 3, a = 2, b = 6$, from (6.24) we get $x = 5, y = 7, z = 11$, i.e., $x^2 = 25, y^2 = 49, z^2 = 121$, which is the same as Diophantus' solution. Similarly, for $p = 5, a = 3, b = 8$, we get $x^2 = 1/4, y^2 = 49/4, z^2 = 289/4$.

P36. For each $n \geq 1$ equation $c^n = a^2 + b^2$ has an infinite number of solutions. The case $n = 1$ has been discussed in Theorem 4.14, also see P28, whereas for $n = 2$ it is Pythagorean relation (5.1). For $n \geq 3$, we begin with an arbitrary *c*, which can be written as the sum of two squares, i.e.,

$$c = p^2 + q^2 = |p + iq|^2,$$

and note that

$$\begin{aligned} c^n &= |(p+iq)^n|^2 = \left| \sum_{k=0}^{n} \binom{n}{k} p^{n-k} (iq)^k \right|^2 \\ &= \left| \mathrm{Re} \left(\sum_{k=0}^{n} \binom{n}{k} p^{n-k} (iq)^k \right) \right|^2 + \left| \mathrm{Im} \left(\sum_{k=0}^{n} \binom{n}{k} p^{n-k} (iq)^k \right) \right|^2. \end{aligned}$$

In particular, we have

$$c^3 = (p^3 - 3pq^2)^2 + (3p^2 q - q^3)^2$$

and

$$c^4 = (p^4 - 6p^2q^2 + q^4)^2 + (4p^3q - 4pq^3)^2.$$

For $c = 5 = 2^2 + 1^2$, the aforementioned relations, respectively, give $5^3 = 2^2 + 11^2$ and $5^4 = 7^2 + 24^2$.

Now we shall show that for each $n \geq 1$ equation $c^2 = a^n + b^2$ has an infinite number of solutions. We assume that *c* and *b* are relatively prime, and rewrite the equation as $a^n = c^2 - b^2 = (c - b)(c + b)$. Then, there exist integers *u* and *v* such that $c - b = u^n$ and $c + b = v^n$. Hence, $c = (u^n + v^n)/2$ and $b = (v^n - u^n)/2$. In particular, if *a* is odd, we can choose $v = a$ and $u = 1$. As a simple example, we consider Pythagorean consecutive triangular numbers $t_n = n(n+1)/2,\ t_{n-1} = (n-1)n/2$ (see Sect. 7.2). Since $t_n^2 - t_{n-1}^2 = n^3$, the equation $c^2 = a^3 + b^2$ has an infinite number of solutions. Similarly, we can show that for each $n \geq 1$ equation $c^2 = a^2 + b^n$ has an infinite number of solutions.

6.5 Construction of Right-Angled Triangles

For the construction of right-angled triangles whose sides are rational numbers or rational cyclic quadrilaterals (all vertices lie on a single circle), Brahmagupta gave the solution (6.8), where *u* and *v* are unequal rational numbers. In particular, for a given rational side a, Brahmagupta's solution is

$$\frac{1}{2}\left(\frac{a^2}{n} - n\right),\ a,\ \frac{1}{2}\left(\frac{a^2}{n} + n\right), \tag{6.25}$$

where *n* is a rational number different from zero. Bhaskara II and Mahavira gave the solution

$$\frac{2ma}{m^2 - 1},\ a,\ \frac{m^2 + 1}{m^2 - 1}a, \tag{6.26}$$

where *m* is any rational number other than ± 1.

The eighth problem in the second book of the *Arithmetica* of Diophantus is to express 16 as a sum of two rational squares. For this, in the identity

$$[a(m^2 + 1)]^2 = (2am)^2 + [a(m^2 - 1)]^2,$$

which follows from (6.26), he used $a = 16/5$ and $m = 1/2$, and obtained $16 = (16/5)^2 + (12/5)^2$.

According to Datta [152] and Puttaswamy [421], Karavindaswami's (nothing seems to be known about him except summary of his mathematical work) solution is

$$\frac{m^2 + 2m}{2m + 2}a,\ a,\ \frac{m^2 + 2m + 2}{2m + 2}a, \tag{6.27}$$

where *m* is any rational number other than -1. It is interesting to note that these solutions can easily be deduced from (6.25). Indeed, for $n = \frac{m-1}{m+1}a$, (6.25) becomes (6.26), whereas for $n = \frac{a}{m+1}$ it reduces to (6.27).

For the given rational hypotenuse *c*, Mahavira constructed a rational right-angled triangle. His solution is

$$\frac{2mnc}{m^2+n^2},\ \frac{m^2-n^2}{m^2+n^2}c,\ c,$$

whereas Bhaskara II gives the solution as

$$\frac{2qc}{q^2+1},\ \frac{q^2-1}{q^2+1}c,\ c,$$

which readily follows from Mahavira's solution by putting $q = m/n$. These solutions were later attributed to the Fibonacci and Viéte, respectively.

6.6 Heronian Triangles

A *Heronian triangle* (a, b, c) (see Carlson [116]) has integer sides whose area is also an integer. Since in a Pythagorean triple at least one of the legs a, b must be even, the area $A = ab/2$ is an integer, every Pythagorean triple is a Heronian triple; however, the converse is not true the Heronian triple $(4, 13, 15)$ has the area 24, but it is not a Pythagorean triple. These triangles are named because such triangles are related to Heron's (Bhaskara I) formula $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = (a+b+c)/2.$ Finding a Heronian triangle is therefore equivalent to solving the Diophantine equation $A^2 = s(s-a)(s-b)(s-c).$ As for Pythagorean triples, if (a, b, c) is a Heronian triple, so is $(na, nb, nc),\ n > 1.$ The Heronian triple (a, b, c) is called primitive provided a, b, c are pairwise relatively prime. There are infinitely many primitive Heronian triples.

Brahmagupta acquired the parametric solution such that every Heronian triangle has sides proportional to

$$a = v(u^2+k^2), \quad b = u(v^2+k^2), \quad c = (u+v)(uv-k^2)$$
$$s = uv(u+v), \quad A = kuv(u+v)(uv-k^2),$$

where $\gcd(u, v, k) = 1,\ uv > k^2 \geq 1,$ and $u \geq v \geq 1.$ The proportionality factor is generally a rational p/q where $q = \gcd(a, b, c)$ reduces the generated Heronian triangle to its primitive and *p* scales up this primitive to the required size. For example, taking $u = 4, v = 2$ and $k = 1$ produces a triangle with $a = 34, b = 20$ and $c = 42,$ which is similar to the $(10, 17, 21)$ primitive Heronian triangle with the proportionality factors $p = 1$ and $q = 2.$ It is known that the perimeter of a Heronian triangle is always an even number, and every primitive Heronian triangle has exactly one even side. The area of a Heronian triangle is always divisible by 6. Further, by Heron's formula, it follows that a triple (a, b, c) with $0 < a < b < c$ is Heronian if $(a^2 + b^2 + c^2)^2 - 2(a^4 + b^4 + c^4)$ is a nonzero perfect square divisible by $16.$

Few primitive Heronian triangles, which are not Pythagorean triples, sorted by increasing areas are

$$\begin{array}{ll} (4,13,15) \text{ with area } 24, & (3,25,26) \text{ with area } 36 \\ (9,10,17) \text{ with area } 36, & (7,15,20) \text{ with area } 42 \\ (6,25,29) \text{ with area } 60, & (11,13,20) \text{ with area } 66 \\ (13,14,15) \text{ with area } 84, & (10,17,21) \text{ with area } 84 \end{array}$$

6.7 Congruent Numbers

A congruent number *n* is a positive integer that is equal to the area of a rational right triangle, i.e., it is a *rational solution* of the equation (5.1), which in addition satisfies the equation $(1/2)ab = A = n.$ Clearly, every Pythagorean triple (primitive as well as non-primitive) gives a congruent number, but in view of P3 and P37 (see Sect. 6.8), *A* and hence *n* cannot be a perfect square. Thus, $2^2, 3^2, 4^2, \cdots$ cannot be congruent numbers. If *n* is a congruent number, i.e., $A = (1/2)ab = n,$ then $m^2 n$ is also a congruent number for any positive integer $m,$ indeed it follows from the facts that

$$\left(\frac{a}{m}\right)^2 + \left(\frac{b}{m}\right)^2 = \left(\frac{c}{m}\right)^2 \quad \text{and} \quad \frac{1}{2}\left(\frac{a}{m}\right)\left(\frac{b}{m}\right) = B$$

imply $A = m^2 n.$ Thus, the main interest is in square-free congruent numbers. Fermat in 1640 proved that there is no right triangle with rational sides and area $1,$ he also showed that 2 and 3 are not congruent numbers. From an Arab manuscript of the tenth century, it is known that 5 and 6 are congruent numbers. In fact, we have

$$\left(\frac{20}{3}\right)^2 + \left(\frac{3}{2}\right)^2 = \left(\frac{41}{6}\right)^2 \quad \text{and} \quad \frac{1}{2}\left(\frac{20}{3}\right)\left(\frac{3}{2}\right) = 5,$$

and the first Pythagorean triple $(3, 4, 5)$ gives the number $6.$ We also have

$$\left(\frac{7}{10}\right)^2 + \left(\frac{120}{7}\right)^2 = \left(\frac{1201}{70}\right)^2 \quad \text{and} \quad \frac{1}{2}\left(\frac{7}{10}\right)\left(\frac{120}{7}\right) = 6,$$

and, see Conrad [137],

$$\left(\frac{1437599}{168140}\right)^2 + \left(\frac{2017680}{1437599}\right)^2 = \left(\frac{2094350404801}{241717895860}\right)^2 \quad \text{and}$$
$$\frac{1}{2}\left(\frac{1437599}{168140}\right)\left(\frac{2017680}{1437599}\right) = 6,$$

i.e., the same congruent number can have several (may be infinite) rational right triangles. The first ten congruent numbers are $5, 6, 7, 13, 14, 15, 20, 21, 22, 23$. Zagier computed the simplest rational right triangle, see Fig. 6.2, for the congruent number 157 (the same has also appeared at several places, e.g., Ann [32], Veljan [518], and Andrew Wiles [534])

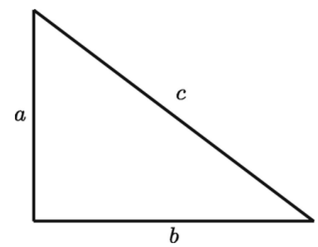

Fig. 6.2 Congruent number 157

where

$$a = \frac{6803298487826435051217540}{411340519227716149383203}, \quad b = \frac{411340519227716149383203}{21666555693714761309610}$$

and

$$c = \frac{224403517704336969924557513090674863160948472041}{8912332268928859588025535178967163570016480830}.$$

In 1983, Jerrold Bates Tunnell (1950–2022, USA) proved the following result (known as Tunnell's Theorem): For a given square-free integer n, define the numbers

$$\begin{aligned}
A_n &= \#\{(x, y, z) \in \mathcal{Z}^3, \quad n = 2x^2 + y^2 + 32z^2\}, \\
B_n &= \#\{(x, y, z) \in \mathcal{Z}^3, \quad n = 2x^2 + y^2 + 8z^2\}, \\
C_n &= \#\{(x, y, z) \in \mathcal{Z}^3, \quad n = 8x^2 + 2y^2 + 64z^2\}, \\
D_n &= \#\{(x, y, z) \in \mathcal{Z}^3, \quad n = 8x^2 + 2y^2 + 16z^2\}.
\end{aligned}$$

Assume that *n* is a congruent number, if *n* is odd, then $2A_n = B_n$ and if *n* is even, then $2C_n = D_n$. For example, for $n = 11$, we have $11 = 2(\pm 1)^2 + (\pm 1)^2 + 8(\pm 1)^2$ and $11 = 2(\pm 1)^2 + (\pm 3)^2 + 8(0)^2$, and hence $B_{11} = 12$. Also, $11 = 2(\pm 1)^2 + (\pm 3)^2 + 32(0)^2$, and hence $A_{11} = 4$. Since $2A_{11} = 2 \times 4 = 8 \neq 12 = B_{11}$ the number $n = 11$ is not congruent. Similarly, for $n = 26$, we have $26 = 8(\pm 1)^2 + 2(\pm 1)^2 + 16(\pm 1)^2$ and $26 = 8(\pm 1)^2 + 2(\pm 3)^2 + 16(0)^2$, and hence $D_{26} = 12$. Also, $26 = 8(\pm 1)^2 + 2(\pm 3)^2 + 64(0)^2$, and hence $C_{26} = 4$. Since $2C_{28} \neq D_{28}$, the number

$n = 28$ is not congruent. To prove the converse of Tunnell's Theorem (known as Tunnell's conjecture), only some progress has been made.

While in 1986, Kramarz [314] has listed all congruent numbers up to less than 2000, an easier technique to decide if a given positive integer *n* is congruent remains an open number-theoretic problem. The relation of this problem to elliptic curves has been studied extensively, see Koblitz [311]. Also, its beautiful equivalent form: *n* is a congruent number, if and only if,

$$\begin{aligned} x^2 + nt^2 &= y^2 \\ x^2 - nt^2 &= z^2 \end{aligned}$$

has solutions, further, if (x, y, z, t) is a solution, then

$$a = \frac{y-z}{t}, \quad b = \frac{y+z}{t}, \quad c = \frac{2x}{t}$$

is only of theoretical interset.

6.8 Fermat's Last Theorem

In the literature Fermat's claim (intellectual curiosity) of 1637 found by his son Samuel, that the equation

$$a^n + b^n = c^n \tag{6.28}$$

has no positive integer solutions for $a, b,$ and *c* if $n > 2$ is known as Fermat's Last Theorem ("last" because it took longer than any other conjecture by Fermat to be proven, finally by Andrew Wiles in 1994, published in 1995 in [533]). Thus, a cube cannot be written as the sum of two smaller cubes, a fourth power cannot be written as the sum of two fourth powers, and so on. In his personal copy of Diophantus's Arithmetica, Fermat just commented that he had discovered a "truly marvelous" proof of this fact, but the margin of the book was too narrow for him to jot it down! It is believed that Fermat himself had a proof for $n = 4,$ and Euler in 1753 (published in 1770) succeeded for the more difficult case of $n = 3,$ (this case can also be settled by using the method of infinite descent, e.g., see Carmichael [117, 118]). Gauss refused to prove Fermat's Last Theorem, remarking tartly that he himself could state a great many theorems that nobody could prove or disprove. In 1820s, Sophie Germain showed that (6.28) has no solution when *abc* is not divisible by *n* for *n* any odd prime less than 100; however, her method did not help to prove the theorem in the case when one of a, b, c is divisible by $n.$ In 1825, Legendre and Dirichlet independently succeeded in proving the theorem for $n = 5$. In 1832, Dirichlet settled the theorem for $n = 14$, and in 1839, Lamé resolved the problem for $n = 7.$ For each of these cases, several alternative proofs were developed later by many prominent mathematicians including Gauss for $n = 3$, but none of these proofs worked for the general case.

A significant contribution toward a proof of Fermat's Last Theorem was made during 1850–1861 by the German mathematician Kummer. Inspired by Gauss's proof for the case $n = 3$ using algebraic integers, Kummer invented the concept of ideal numbers (different from the ideal number 5040 due to Plato), which is destined to play a key role in the development of modern algebra and number theory. Using this, Kummer proved that Fermat's Last Theorem holds when *n* is a prime number of a certain type, known as regular primes (see Bernoulli numbers in Sect. 7.3). The power of Kummer's result is indicated by the fact that the smallest prime that is not regular is 37. Thus, the cases $n = 3, 5, 7, 11, 13, 17, 19, 23, 29,$ and 31 (and many others) are disposed of all at once. In fact, the only primes less than 100 that are not regular are $37, 59$ and 67. Unfortunately, there are still infinitely many primes that are not regular.

In 1850, the Paris Academy offered a prize of a gold medal valued at 3000 francs for a complete solution of Fermat's last theorem. When no proof was forthcoming, even on extension of the terminal date, the medal was awarded to Kummer as the author whose research most merited the prize, even though he had not submitted an entry in the competition. In 1908, a sensational announcement was made that a prize of 100,000 German Marks would be awarded for the complete solution of Fermat's problem. The funds for this prize, which was the largest ever offered in mathematics, were bequeathed by a German industrialist and amateur mathematician Paul Friedrich Wolfskehl (1856–1906) to the "Konigliche Gesellschaft der Wissenschaften in Göttingen" for this purpose. This announcement drew so much attention that during a brief span of three years (1908–1911) over a thousand papers containing supposed solutions reached the committee. But unfortunately, all were wrong. Since then, the number of papers submitted for this prize became so large that they would fill a library. The Committee then very wisely included as one of the conditions that the article be printed, otherwise the number would have been still larger. When Hilbert was asked why he never attempted a proof of the Last Theorem, he replied: "Before beginning I should have to put in three years of intensive study, and I haven't that much time to squander on a probable failure."

In 1983, Gerd Faltings (born 1954, Germany) proved a very decisive result: for $n > 2$ Eq. (6.28) can have at most a finite number of integer solutions. March 1988 issue of Time Magazine reported that the Japanese mathematician Yoichi Miyaoka (born 1949), working at the Max Plank Institute in Bonn, Germany, had discovered the proof of the theorem. However, his announcement turned out to be premature, as a few weeks later holes were found in his argument that could not be repaired. Episodes like this had indeed occurred many times in the three-and-a-half century history of this famous problem. It is said that the Landau had printed post cards, which read, "Dear Sir/Madam: Your attempted proof of Fermat's Theorem has been received and is herewith returned. The mistake is on page —–, line —–." Landau would give them to his students to fill in the missing numbers. From 1977 to 1992 with the help of computers, Fermat's Last Theorem was verified up to $n = 4{,}000{,}000$.

The crowning mathematical achievements of the twentieth century occurred on June 23, 1993, when Andrew Wiles, a Cambridge trained mathematician working at Princeton University, announced the proof of the theorem. He worked on the problem for 8 whole years, often in seclusion and doing nothing else. In developing his solution scheme, Andrew Wiles employed theories from many branches of mathematics: crystalline cohomology, Galöis representation, L-functions, modular forms, deformation theory, Gorenstein rings $\cdots$ and relied on research findings from colleagues in France, Germany, Italy, Japan, Australia, Columbia, Brazil, Russia, the United States $\cdots$. However, soon after Wiles's Cambridge announcement, gaps in his 200-page-long proof (it would be 1000 pages if all details are provided) surfaced. Fortunately this time, with the help from his colleagues, most notably his ex-student Richard Taylor (born 1962, Rngland-USA), Wiles finally filled these gaps after another years's hard work. In 1994, Fermat's Last Theorem is finally resolved. In 1997, Andrew Wiles was awarded long time unclaimed award whose worth then was $40,000. However, now the world awaits a simpler proof.

Fermat's last theorem is a very special case of a central problem in Diophantine analysis. It is required to devise criteria to decide in a finite number of non-tentative steps whether or not a given Diophantine equation is solvable. Fermat's Last Theorem may not seem to be a deeply earth-shattering result. Its importance lies in the fact that it has captured the imagination of some of the most brilliant minds over 350 years and their attempts at solving this conundrum, no matter how incomplete or futile, have led to the development of some of the most important branches of modern mathematics. It is to be noted that Brahmagupta and Bhaskara II had addressed themselves to some of Fermat's problems long before they were thought of in the West and had solved them thoroughly. They have not held a proper place in mathematical history, or received credit for their problems and methods of solution. André Weil in 1984 [530] wrote "What would have been Fermat's astonishment, if some missionary, just back from India, had told him that his problem had been successfully tackled by native mathematicians almost six centuries earlier."

Proving Fermat's Last Theorem for a given exponent $n > 2$ also settles it for any multiple of $n.$ For example, knowing that $a^5 + b^5 = c^5$ is impossible in positive integers also covers the case $n = 10,$ since if we had $x^{10} + y^{10} = z^{10}$, then $a = x^2, b = y^2, c = z^2$ would satisfy the first equation. Similarly, if we had $x^{15} + y^{15} = z^{15}$, then $a = x^3, b = y^3, c = z^3$ will lead to the first equation. Thus, to prove Fermat's Last Theorem in general, it suffices to prove that (6.28) has no positive integer solutions for $n = 4$ and for any value of *n* that is an odd prime. The case $n = 4$ is the only one for which a short proof is known.

For this, first we shall use Fermat's method of infinite descent to show that there are no positive integers a, b and *c* such that $a^4 + b^4 = c^2$. To obtain a contradiction, suppose there are such integers. Let us take such a triple with the product *ab* minimized. Then $\gcd(a, b) = 1.$ Since a^2, b^2 and *c* are the sides of a primitive Pythagorean triangle, exactly one of *a* and *b* is even. Without loss of generality, let us assume that *a* is even. By (6.8), there are positive integers *u* and $v,$ not both odd, with $\gcd(u, v) = 1,$ such that $a^2 = 2uv$ and $b^2 = u^2 - v^2$. Since $v^2 + b^2 = u^2$ and *b* is odd, *v* must be even. Since $(2v, u) = 1$ and $2uv = a^2,$ it follows that 2*v* and *u* are squares. Thus $u = p^2$ for some positive integer *p*. Again by (6.8), there are positive integers *s* and $t,$ not both odd, with $\gcd(s, t) = 1,$ such that $v = 2st$ and $u = s^2 + t^2$

. Since 2*v* is a square, so is $2v/4 = v/2 = st$. Thus, there are positive integers *x* and *y* such that $s = x^2$ and $t = y^2$. The fact that $u = s^2 + t^2$ implies that $x^4 + y^4 = p^2$. Moreover, $(xy)^2 = st = v/2 < 2uv = a^2 \leq (ab)^2$ so that $xy < ab$. But this contradicts the minimality of ab. From this, it immediately follows that $a^4 + b^4 = c^4$ has no solution in positive integers. Indeed, if a_0, b_0, c_0 is a solution of $a^4 + b^4 = c^4$, then a_0, b_0, c_0^2 is a solution of $a^4 + b^4 = c^2$.

Fermat's method of infinite descent also applies to show that there are no positive integers $a, b,$ and *c* such that $a^4 - b^4 = c^2$, (see Burton [111]). Now since none of the equations $x^4 + y^4 = z^4,\ x^4 + y^4 = z^2,\ x^4 = y^4 + z^2$ has a solution, it is clear that in a Pythagorean triple (a, b, c) at most one of the members can be perfect square. However, there exist Pythagorean triples (not necessarily primitive) whose sides if increased by 1 are squares. For example, $(99, 168, 195) = (10^2 - 1, 13^2 - 1, 14^2 - 1)$. It is not known whether infinitively many such triples exist. We can also deduce that the equations $(a^4/p^4) + (b^4/q^4) = c^2$ and $(1/a^4) + (1/b^4) = c^2$ do not have solutions in positive integers.

For further comments on Fermat's Last Theorem, see Cox [142], Dickson [163], Edwards [176], and Ribenboim [431].

P37. In P3 we mentioned that the area $A = (1/2)ab$ of a primitive Pythagorean triangle (a, b, c) can never be a square number. To show this assume that $(1/2)ab = u^2$, which is the same as $2ab = 4u^2$. Adding and subtracting this relation from $a^2 + b^2 = c^2$ gives

$$(a+b)^2 = c^2 + 4u^2 \quad \text{and} \quad (a-b)^2 = c^2 - 4u^2.$$

Multiplying these relations, we find

$$(a^2 - b^2)^2 = c^4 - 16u^4 = c^4 - (2u)^4,$$

which is impossible.

P38. If *n* is an integer greater than $2,\ a$ and *b* are the lengths of the legs of a right triangle, and *c* is the length of the hypotenuse, then $c^n > a^n + b^n$. Indeed, we have $c^n = c^2 c^{n-2} = (a^2 + b^2)c^{n-2}$. Since $c^{n-2} > a^{n-2}$ and $c^{n-2} > b^{n-2}$, it follows that $c^n > a^2(a^{n-2}) + b^2(b^{n-2})$. Consequently, $c^n > a^n + b^n$.

6.9 Pythagorean Quadruple

A tuple of four integers a, b, c and *d* such that $a^2 + b^2 + c^2 = d^2$ is called Pythagorean quadruple, and (a, b, c, d) is called primitive if the greatest common divisor of its numbers is 1. The set of primitive Pythagorean quadruples for which *a* is odd can be generated by (see Carmichael [117, 118] and Spira [493])

$$a = u^2 + v^2 - p^2 - q^2, \quad b = 2(uq + vp), \quad c = 2(vq - up), \quad d = u^2 + v^2 + p^2 + q^2,$$

where u, v, p, q are non-negative integers with greatest common divisor 1 such that $u + v + p + q$ is odd. Thus, all primitive Pythagorean quadruples are characterized by *Lebesgue identity*

$$(u^2 + v^2 - p^2 - q^2)^2 + (2uq + 2vp)^2 + (2vq - 2up)^2 = (u^2 + v^2 + p^2 + q^2)^2.$$

As an example, for $u = 1, v = 3, p = 2, q = 1,$ we have $a = 5, b = 14, c = 2, d = 15$ and hence $(2, 5, 14, 15)$ is a primitive Pythagorean quadruple. Indeed we have the identity $2^2 + 5^2 + 14^2 = 15^2.$ Besides $(2, 5, 14, 15)$, there are 30 more primitive Pythagorean quadruples in which all entries are less than 30 (see https://en.wikipedia.org/wiki/Pythagorean_quadruple)

$$\begin{array}{l}
(1,2,2,3), (2,10,11,15), (4,13,16,21), (2,10,25,27), (2,3,6,7)\\
(1,12,12,17), (8,11,16,21), (2,14,23,27), (1,4,8,9), (8,9,12,17)\\
(3,6,22,23), (7,14,22,27), (4,4,7,9), (1,6,18,19), (3,14,18,23)\\
(10,10,23,27), (2,6,9,11), (6,6,17,19), (6,13,18,23), (3,16,24,29)\\
(6,6,7,11), (6,10,15,19), (9,12,20,25), (11,12,24,29), (3,4,12,13)\\
(4,5,20,21), (12,15,16,25), (12,16,21,29), (4,8,19,21), (2,7,26,27)
\end{array}$$

Quadruple (a, b, c, d) can also be obtained by the simple identities

$$(a^2 - b^2 - c^2)^2 + (2ab)^2 + (2ac)^2 = (a^2 + b^2 + c^2)^2. \tag{6.29}$$

$$(2a^2)^2 + (2ab)^2 + (b^2)^2 = (2a^2 + b^2)^2. \tag{6.30}$$

For example, from (6.29), for $a = 3, b = 2, c = 2$ this identity reduces to $1^2 + 12^2 + 12^2 = 17^2.$ Thus, we get back the quadruple $(1, 12, 12, 17).$ From (6.30), for $a = 5, b = 3,$ we get $50^2 + 30^2 + 9^2 = 59^2$, and hence, we find the quadruple $(9, 30, 50, 59).$

6.10 Generalized Identities

Euler-Aida Ammei (1747–1817, Japan) identity

$$(x_0^2 - x_1^2 - \cdots - x_n^2)^2 + (2x_0x_1)^2 + \cdots + (2x_0x_n)^2 = (x_0^2 + x_1^2 + \cdots + x_n^2)^2 \tag{6.31}$$

implies that the sum of $n + 1$ squares is the square of the sum of $n + 1$ squares. Identity (6.31) for $n = 1, x_0 = u, x_1 = v$ gives (6.8), for $n = 2, x_0 = a, x_1 = b, x_2 = c$ it reduces to (6.29), and for $n = 3, x_0 = a, x_1 = b, x_2 = c, x_3 = d$ it gives quintuples

$$(a^2 - b^2 - c^2 - d^2)^2 + (2ab)^2 + (2ac)^2 + (2ad)^2 = (a^2 + b^2 + c^2 + d^2)^2. \tag{6.32}$$

For example, for $a = 4, b = 3, c = 2, d = 1$ this identity reduces to $2^2 + 24^2 + 16^2 + 8^2 = 30^2.$ Thus, dividing this relation by $2^2,$ we get the quintuple

$(1, 4, 8, 12, 15)$.

Clearly, as in P28, equality (4.20) can be used to find new four-square identities.

Around 1818, Carl Ferdinand Degen (1766–1825, Denmark) discovered the eight-square identity

$$\begin{aligned}
&(a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4 - a_5b_5 - a_6b_6 - a_7b_7 - a_8b_8)^2 \\
&+(a_1b_2 + a_2b_1 + a_3b_4 - a_4b_3 + a_5b_6 - a_6b_5 - a_7b_8 + a_8b_7)^2 \\
&+(a_1b_3 - a_2b_4 + a_3b_1 + a_4b_2 + a_5b_7 + a_6b_8 - a_7b_5 - a_8b_6)^2 \\
&+(a_1b_4 + a_2b_3 - a_3b_2 + a_4b_1 + a_5b_8 - a_6b_7 + a_7b_6 - a_8b_5)^2 \\
&+(a_1b_5 - a_2b_6 - a_3b_7 - a_4b_8 + a_5b_1 + a_6b_2 + a_7b_3 + a_8b_4)^2 \\
&+(a_1b_6 + a_2b_5 - a_3b_8 + a_4b_7 - a_5b_2 + a_6b_1 - a_7b_4 + a_8b_3)^2 \\
&+(a_1b_7 + a_2b_8 + a_3b_5 - a_4b_6 - a_5b_3 + a_6b_4 + a_7b_1 - a_8b_2)^2 \\
&+(a_1b_8 - a_2b_7 + a_3b_6 + a_4b_5 - a_5b_4 - a_6b_3 + a_7b_2 + a_8b_1)^2 \\
&= (a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 + a_7^2 + a_8^2) \\
&\qquad (b_1^2 + b_2^2 + b_3^2 + b_4^2 + b_5^2 + b_6^2 + b_7^2 + b_8^2).
\end{aligned} \tag{6.33}$$

Clearly, for $a_i = b_i,\ i = 1, \cdots, 8$, equality (6.33) is the same as (6.31) with $n = 7$. The equality (6.33) was independently rediscovered by John Thomas Graves (1806–1870, Ireland) in 1843, and Cayley in 1845. In 1898, Adolf Hurwitz (1859–1919, Germany) proved that there is no similar identity for 16 squares or any other number of squares except for 1, 2, 4, and 8.

Ramanujan's identity

$$\begin{aligned}
(3a^2 + 5ab - 5b^2)^3 + (4a^2 - 4ab + 6b^2)^3 + (5a^2 - 5ab - 3b^2)^3 \\
= (6a^2 - 4ab + 4b^2)^3
\end{aligned} \tag{6.34}$$

parameterizes the sum of 3 cubes into a cube, i.e., $x^3 + y^3 + z^3 = c^3$ has infinite number of integral solutions (x, y, z, c) known as Fermat's cubic. The general solution of this Diophantus equation was found by Euler and Viéte. Identity (6.34) for $a = 1, b = 0$ reduces to Euler's equality $3^3 + 4^3 + 5^3 = 6^3 = 216$ (this number is known as Plato's number, geometrical number, or nuptial number [the number of the bride]), whereas for $a = 2, b = 1$ it gives $17^3 + 14^3 + 7^3 = 20^3.$ The sum of 3 cubes into a cube can also be parameterized as (see Hardy and Wright [250])

$$b^3(a^3 + b^3)^3 + a^3(a^3 - 2b^3)^3 + b^3(2a^3 - b^3)^3 = a^3(a^3 + b^3)^3, \tag{6.35}$$

or as

$$a^3(a^3 - b^3)^3 + b^3(a^3 - b^3)^3 + b^3(2a^3 + b^3)^3 = a^3(a^3 + 2b^3)^3. \tag{6.36}$$

For $a = 2, b = 1$ identities (6.35) and (6.36), respectively, reduce to $9^3 + 12^3 + 15^3 = 18^3$ (which is the same as $3^3 + 4^3 + 5^3 = 6^3$) and $14^3 + 7^3 + 17^3 = 20^3$ (which is the same as given by (6.34)). Fermat's cubic $(3, 4, 5, 6)$ and $(1, 6, 8, 9)$ were known to Fibonacci. In 1920, Herbert William

Richmond (1963–1948, England) comprised a set of more than 100 Fermat's cubic; two of these sets are $(25, 38, 87, 90)$ and $(15, 82, 89, 108)$. A simple identity is

$$n^3 + (3n^2 + 2n + 1)^3 + (3n^3 + 3n^2 + 2n)^3 = (3n^3 + 3n^2 + 2n + 1)^3.$$

which immediately gives Fermat's cubic $(1, 6, 8, 9)$, $(2, 17, 40, 41)$, $(3, 34, 114, 115)$.

6.11 Generalizations of Fermat's Last Theorem

In 1769, Euler made the conjecture that

$$x_1^k + x_2^k + \cdots + x_n^k = c^k \tag{6.37}$$

implies $n \geq k$. This conjecture makes an effort to generalize Fermat's Last Theorem, which in fact is a special case, indeed, if $x_1^k + x_2^k = c^k$ then $2 \geq k$. In view of Fermat's Last Theorem and Sect. 6.10, Euler's conjecture holds for $k = 3$; however, it has been disproved for $k = 4$ and $k = 5$, and for $k \geq 6$ the answer is unknown. For the cases $k = 4$ and 5 the known counterexamples are:

$$2682440^4 + 15365639^4 + 18796760^4 = 20615673^4$$
(Noam David Elkies [178] in 1986)

$$95800^4 + 217519^4 + 414560^4 = 422481^4 \quad \text{(Roger Frye [204] in 1988)}$$

$$27^5 + 84^5 + 110^5 + 133^5 = 144^5 \quad \text{(Leon Lander and Thomas Parkin [334] in 1966)}$$

$$55^5 + 3183^5 + 28969^5 + 85282^5 = 85359^5 \quad \text{(Roger Frye in 2004).}$$

The following identities support Euler's conjecture

$$30^4 + 120^4 + 272^4 + 315^4 = 353^4 \quad \text{(R. Norrie in 1911)}$$

$$19^5 + 43^5 + 46^5 + 47^5 + 67^5 = 72^5$$

(Leon Lander, Thomas Parkin, and John Selfridge (LPS) [335] in 1967)

$$7^5 + 43^5 + 57^5 + 80^5 + 100^5 = 107^5 \quad \text{(S. Sastry [447] in 1934)}$$

Sastry also discovered the following parametric quintic identity

$$\begin{aligned}(75b^5 - a^5)^5 + (a^5 + 25b^5)^5 + (a^5 - 25b^5)^5 + (10a^3b^2)^5 + (50ab^4)^5 \\ = (a^5 + 75b^5)^5\end{aligned} \tag{6.38}$$

$$74^6 + 234^6 + 402^6 + 474^6 + 702^6 + 894^6 + 1077^6 = 1141^6 \quad \text{(LPS in 1967)}$$

$$8^6 + 12^6 + 30^6 + 78^6 + 102^6 + 138^6 + 165^6 + 246^6 = 251^6 \quad \text{(LPS in 1967)}$$

$$127^7 + 258^7 + 266^7 + 413^7 + 430^7 + 439^7 + 525^7 = 568^7 \quad \text{(M. Dodrill in 1999)}$$

$$90^8 + 223^8 + 478^8 + 524^8 + 748^8 + 1088^8 + 1190^8 + 1324^8 = 1409^8 \quad \text{(S. Chase in 2000)}$$

For sixth power summations several more identities are available at https://mathworld.wolfram.com/DiophantineEquation6thPowers.html

Euler gave a parametric solution of the equation

$$x_1^3 + x_2^3 = x_3^3 + x_4^3 \tag{6.39}$$

namely,

$$x_1 = 1 - (a - 3b)(a^2 + 3b^2), \quad x_2 = (a + 3b)(a^2 + 3b^2) - 1$$
$$x_3 = (a + 3b) - (a^2 + 3b^2)^2, \quad x_4 = (a^2 + 3b^2)^2 - (a - 3b),$$

where *a* and *b* are any integers. Equation (6.39) has following 10 solutions with sum $< 10^5$ (see Guy [241])

$$\begin{array}{lclcl}
1729 & = & 1^3 + 12^3 & = & 9^3 + 10^3 \\
4104 & = & 2^3 + 16^3 & = & 9^3 + 15^3 \\
13832 & = & 2^3 + 24^3 & = & 18^3 + 20^3 \\
20683 & = & 10^3 + 27^3 & = & 19^3 + 24^3 \\
32832 & = & 4^3 + 32^3 & = & 18^3 + 30^3 \\
39312 & = & 2^3 + 34^3 & = & 15^3 + 33^3 \\
40033 & = & 9^3 + 34^3 & = & 16^3 + 33^3 \\
46683 & = & 3^3 + 36^3 & = & 27^3 + 30^3 \\
64232 & = & 17^3 + 39^3 & = & 26^3 + 36^3 \\
65728 & = & 12^3 + 40^3 & = & 31^3 + 33^3
\end{array}$$

From Euler's parametric solution of (6.39) we cannot find *a* and *b* to obtain the relation $1729 = 1^3 + 12^3 = 9^3 + 10^3$. However, one of Ramanujan's parametrization

$$(a^2 + 7ab - 9b^2)^3 + (2a^2 - 4ab + 12b^2)^3 = (2a^2 + 10b^2)^3 + (a^2 - 9ab - b^2)^3$$

with $a = b = 1$ does give this relation (*Hardy-Ramanujan number*). As we have noted earlier, Hardy and Wright [250] proved that there are numbers that are expressible as the sum of two cubes in *n* ways for any n. For example, the numbers representable in three ways as a sum of two cubes are

$$\begin{array}{lclclcl}
87539319 & = & 167^3 + 436^3 & = & 228^3 + 423^3 & = & 255^3 + 414^3 \\
119824488 & = & 11^3 + 493^3 & = & 90^3 + 492^3 & = & 346^3 + 428^3 \\
143604279 & = & 111^3 + 522^3 & = & 359^3 + 460^3 & = & 408^3 + 423^3 \\
175959000 & = & 70^3 + 560^3 & = & 198^3 + 552^3 & = & 315^3 + 525^3 \\
327763000 & = & 300^3 + 670^3 & = & 339^3 + 661^3 & = & 510^3 + 580^3
\end{array}$$

In 1773, Euler also proved that there exist infinitely many solutions of the equation $x_1^4 + x_2^4 = x_3^4 + x_4^4$, while its smallest solution is (see Dunham [174])

$$59^4 + 158^4 = 133^4 + 134^4 = 635318657.$$

Euler gave the following two solutions

$$12231^4 + 2903^4 = 10381^4 + 10203^4$$

$$2219449^4 + 555617^4 = 1584749^4 + 2061283^4$$

The following equalities are provided by Guy [241].

$$\begin{aligned}
3^6 + 19^6 + 22^6 &= 10^6 + 15^6 + 23^6 \quad \text{(K. Subba Rao in 1934)}\\
36^6 + 37^6 + 67^6 &= 15^6 + 52^6 + 65^6\\
33^6 + 47^6 + 74^6 &= 23^6 + 54^6 + 73^6\\
32^6 + 43^6 + 81^6 &= 3^6 + 55^6 + 80^6\\
37^6 + 50^6 + 81^6 &= 11^6 + 65^6 + 78^6\\
25^6 + 62^6 + 138^6 &= 82^6 + 92^6 + 135^6\\
51^6 + 113^6 + 136^6 &= 40^6 + 125^6 + 129^6\\
71^6 + 92^6 + 147^6 &= 1^6 + 132^6 + 133^6\\
111^6 + 121^6 + 230^6 &= 26^6 + 169^6 + 225^6\\
75^6 + 142^6 + 245^6 &= 14^6 + 163^6 + 243^6
\end{aligned}$$

The amount of effort necessary to find examples/counterexamples for aforementioned type of equalities—even when the effort came from computers—was then astonishing. In view of some of these equalities in 1967, LPS made the following conjecture: If

$$\sum_{i=1}^{n} x_i^k = \sum_{j=1}^{m} y_j^k, \tag{6.40}$$

where $x_i \neq y_j$ are positive integers for all $1 \leq i \leq n$ and $1 \leq j \leq m$, then $m + n \geq k$. Clearly, for $m = 1$, (6.40) is the same as relation (6.37), but then $n + 1 \geq k$.

In 1985, Joseph Oesterló (born 1954, France) and David William Masser (born 1948, England) proposed a generalization of Fermat's Last Theorem, which in the literature is known as *abc conjecture*: For any infinitesimal $\epsilon > 0$, there exists a constant C_ϵ such that for any three relatively prime integers a, b, c satisfying $a + b = c$, the inequality

$$\max(|a|, |b|, |c|) \leq C_\epsilon \prod_{(p|abc)} p^{(1+\epsilon)}$$

holds, where $p|abc$ indicates that the product is over primes *p*, which divide the product *abc*. There are several equivalent forms of this conjecture, and it has been called the most important unsolved problem in Diophantine analysis. In 2012, the Japanese mathematician Shinichi Mochizuki (born 1969) released online a series of papers in which he claimed to have proved the *abc* conjecture. Despite the publication in a peer-reviewed Journal later, his proof has not been accepted as correct in the mainstream mathematical community.

In 1993, Daniel Andrew Beal (born 1952, USA) formulated the following conjecture: The equation $A^x + B^y = C^z$ has no solutions in positive integers and pairwise coprime integers A, B, C if $x, y, z \geq 3.$ Clearly, *Beal's conjecture* is a generalization of Fermat's Last Theorem. Since 1997, Beal has offered a monetary prize for a peer-reviewed proof of this conjecture or a counterexample. The value of the prize has increased several times and is currently $1 million.

6.12 Catalan's and Pillai's Conjectures

In 1844, Catalan combined the squares of integral numbers $\{4, 9, 16, 25, \cdots\}$ with the sequences of cubic numbers $\{8, 27, 64, 125, \cdots\}$ to obtain $\{4, 8, 9, 16, 25, 27, 36, \cdots\}$ and made the conjecture that 8 and 9 are the only numbers that differ by 1 and are both exact powers $8 = 2^3,\ 9 = 3^2$. This conjecture was proved by Preda Mihăilescu (Born 1955) after 158 years, and published 2 years later in [370]. Thus the only solution in natural numbers of the Diophantine equation $x^a - y^b = 1$ for $a, b > 1,\ x, y > 0$ is $x = 3, a = 2,\ y = 2, b = 3.$ In 1931, Pillai conjectured that for fixed positive integers A, B, C the Diophantine equation $Ax^n - By^m = C$ has only finitely many solutions (x, y, m, n) with $(m, n) \neq (2, 2)$. Clearly, Pillai's conjecture is a generalization of Catalan's conjecture. So far for the Pillai's conjecture number of solutions have been calculated only for some particular cases. In fact, from the seventeeth century, Bachet and several others studied the particular cases of the Bachet equation $x^3 - y^2 = k$ where *k* is a given nonzero integer. The origin of this equation goes back to Diophantus's *Arithmetica*.

It was only in 1968 that Alan Baker (1939–2018, England) found a completely general solution, working for any given $k.$ At first, Baker's solution was merely an enormous bound $M(k)$ on the sizes of *x* and $y.$ However, soon after, Baker and other mathematicians, such as Davenport, transformed Baker's insights into a practical method for actually obtaining a solution set for any given $k.$ William John Ellison (1943–2022, England) used Baker's ideas to show, for the first time, that when $k = 28,$ the Bachet equation has only three solutions in positive integers $x = 4, y = 6; x = 8, y = 22; x = 37, y = 225.$ Raymond P. Steiner (1941–2014, USA) used a version of Baker's result, due to Michel Waldschmidt (born 1946, France) to show that when $k = 999,$ the Bachet equation has only six solutions in positive integers $x = 10, y = 1; x = 12, y = 27; x = 40, y = 251; x = 147, y = 1782; x = 174, y = 2295; x = 22480, y = 3370501.$ In his monograph Stolarsky [497] had claimed that the equation $x^3 = y^2 + 999$ could not be solved by "a thousand wise men." Alan Baker was wise man a thousand and one.

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7. Pythagorean Figurative Numbers

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7.1 Introduction and Origin

Pythagoras and his several followers, especially, Hipparchus, Plutarch, Nicomachus, and Theon, portrayed natural numbers in orderly geometrical configuration of points/dots/pebbles and labeled them as *figurative numbers*. From these arrangements they deduced some astonishing number-theoretic results. This was indeed the beginning of number theory and an attempt to relate geometry with arithmetic. Nicomachus in his book (see [394]) collected earlier works of Pythagoreans on natural numbers and presented cubic figurative numbers (solid numbers). Thus, figurate numbers had been studied by the ancient Greeks for polygonal numbers (the first general definition of polygonal numbers was given by Hypsicles and was quoted by Diophantus), pyramidal numbers, and cubes. The connection between regular geometric figures and the corresponding sequences of figurative numbers was profoundly significant in Plato's science, after Plato, for example, in his work *Timaeus*. The study of figurative numbers was further advanced by Diophantus. His main interest was in figurate numbers based on the Platonic solids (tetrahedron, cube, octahedron, dodecahedron, and icosahedron), which he documented in *De solidorum elementis*. However, this treatise was lost and rediscovered only in 1860. Dicuilus wrote *Astronomical Treatise* in Latin about 814–816, which contains a chapter on triangular and square numbers, see Ross and Knott [436].

After Diophantus's work several prominent mathematicians took interest in figurative numbers, the long list includes: Fibonacci, Stifel, Cardano, Faulhaber, Bachet, Descartes, Fermat, Pell. In 1665, Pascal wrote the *Traité du triangle arithmétique, avec quelques autres petits traitez sur la mesme matiére* (Treaty of arithmetic triangle, with a few other small treatise on the same subject), which contains some details of figurate numbers. Work of Euler and Lagrange on figurate numbers opened new avenues in number theory. Octahedral numbers were extensively examined by Friedrich Wilhelm Marpurg (1718–1795, Germany) in 1774 and Georg Simon Klügel (1739–1812, Germany) in 1808. The Pythagoreans could not have anticipated that figurative numbers would engage after 2000 years leading scholars such as Legendre, Gauss, Cauchy, Jacobi, and Sierpinski. In 2011, Michel Marie Deza (1939–2016, Russia–France) and Elena Deza (Russia) in their book [162] have given an extensive information about figurative numbers.

In this chapter, we shall systematically discuss most popular polygonal, centered polygonal, three-dimensional numbers (including pyramidal numbers), and four-dimensional figurative numbers. We shall begin with triangular numbers and end this chapter with pentatope numbers. For each type of polygonal figurative numbers, we shall provide definition in terms of a sequence, possible sketch, explicit formula, possible relations within the class of numbers through simple recurrence relations, properties of these numbers, generating function, sum of first finite numbers, sum of all their inverses, and relations with other type of polygonal figurative numbers. For each other type of figurative numbers mainly we shall furnish definition in terms of a sequence, possible sketch, explicit formula, generating function, sum of first finite numbers, and sum of all their inverses. The study of figurative numbers is interesting in its own sack, and often these numbers occur in real-world situations. We sincerely hope after reading this chapter it will be possible to find new representations, patterns, and relations with other types of popular numbers, which are not discussed here, extensions, and real applications.

7.2 Triangular Numbers t_n

In this arrangement, rows contain $1, 2, 3, 4, \ldots, n$ dots (see Fig. 7.1).

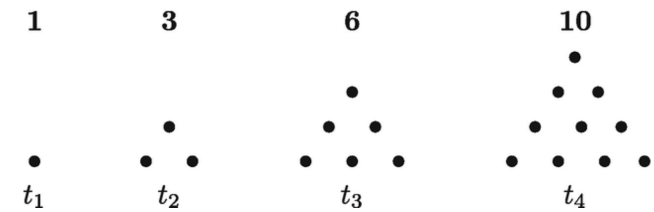

Fig. 7.1 Triangular numbers

From Fig. 7.1, it follows that each new triangular number is obtained from the previous triangular number by adding another row containing one more dot than the previous row added, and hence, t_n is the sum of the first *n* positive integers, i.e.,

$$t_n = t_{n-1} + n = t_{n-2} + (n-1) + n = \cdots = 1 + 2 + 3 + \cdots + (n-1) + n, \quad (7.1)$$

i.e., the differences between successive triangular numbers produce the sequence of natural numbers. To find the sum in (7.1), we shall discuss two methods that are ingenious.

Method 1. Since

$$\begin{array}{ccccccccccccc} t_n & = & 1 & + & 2 & + & 3 & + & \cdots & + & (n-1) & + & n \\ t_n & = & n & + & (n-1) & + & (n-2) & + & \cdots & + & 2 & + & 1. \end{array}$$

An addition of these two arrangements immediately gives

$$2t_n = (n+1) + (n+1) + \cdots + (n+1) = n(n+1)$$

and hence

$$t_n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2} = \frac{1}{2}n^2 + \frac{1}{2}n. \quad (7.2)$$

Thus, it immediately follows that

$t_1 = 1,\ t_2 = 1 + 2 = 3,\ t_3 = 1 + 2 + 3 = 6,\ t_4 = 1 + 2 + 3 + 4 = 10,\ t_5 = 15,\ t_6 = 21,\ t_7 = 28, \cdots .$

This method was first employed by Gauss. The story is his elementary school teacher asked the class to add up the numbers from 1 to 100, expecting to keep them busy for a long time. Young Gauss found the formula (7.2) instantly and wrote down the correct answer 5050.

Method 2. From Fig. 7.2 *Proof without words* of (7.2) is immediate, see Alsina and Nelsen [27]. However, a needless explanation is a "stairstep" configuration made up of one block plus two blocks plus three blocks, etc, replicated it as the shaded section in Fig. 7.2, and fit them together to form an $n \times (n+1)$ rectangular array. Because the rectangle is made of two identical stairsteps (each representing t_n) and the rectangle's area is the product of base and height—that is, $n(n+1)$—then the stairstep's area must be half of the rectangle's, and hence (7.2) holds.

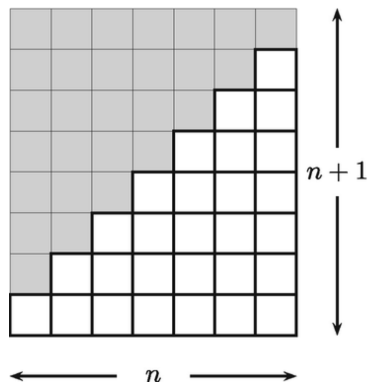

Fig. 7.2 Proof of (7.2) without words

To prove (7.2), the principle of mathematical induction is routinely used. The relation (7.1) is a special case of an arithmetic progression of the finite sequence $\{a_k\},\ k = 0, 1, \cdots, n-1$ where $a_k = a + kd$, or $a_k = a_\ell + (k - \ell)d,\ k \geq \ell \geq 0$, i.e.,

$$S = \sum_{k=0}^{n-1} (a + kd) = a + (a + d) + (a + 2d) + \cdots + (a + (n-1)d). \tag{7.3}$$

For this, following the Method 1, it immediately follows that

$$S = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[a_0 + a_{n-1}]. \tag{7.4}$$

Thus, the mean value of the series is $\bar{S} = S/n = (a_0 + a_{n-1})/2$, which is similar as in discrete uniform distribution. For $a = d = 1$, (7.3) reduces to (7.1), and (7.4) becomes the same as (7.2). From (7.4), it is also clear that

$$\begin{aligned} \sum_{k=m}^{n-1} (a + kd) &= \sum_{k=0}^{n-1} (a + kd) - \sum_{k=0}^{m-1} (a + kd) = \frac{n}{2}[2a + (n-1)d] \\ &- \frac{m}{2}[2a + (m-1)d] = \frac{(n-m)}{2}[2a + (n+m-1)d]. \end{aligned} \tag{7.5}$$

Aryabhata besides giving the formula (7.4) also obtained *n* in terms of S, namely,

$$n = \frac{1}{2}\left[\frac{\sqrt{8Sd + (2a-d)^2} - 2a}{d} + 1\right]. \tag{7.6}$$

He also provided elegant results for the summation of series of squares and cubes. Instead of adding the aforementioned finite arithmetic series $\{a_k\}$, we can multiply its terms, which in terms of Gamma function Γ can be written as

$$a_0 a_1 \cdots a_{n-1} = \prod_{k=0}^{n-1}(a + kd) = d^n \frac{\Gamma\left(\frac{a}{d} + n\right)}{\Gamma\left(\frac{a}{d}\right)}, \tag{7.7}$$

provided a/d is positive.

- The following equalities between triangular numbers can be proved rather easily.

$$\begin{aligned} t_n^2 + t_{n-1}^2 &= t_{n^2} \\ 3t_n + t_{n-1} &= t_{2n} \\ 3t_n + t_{n+1} &= t_{2n+1} \\ 9t_n + 1 &= t_{3n+1} \\ t_n + t_m + nm &= t_{n+m} \\ t_n t_m + t_{n-1}t_{m-1} &= t_{nm}. \end{aligned}$$

- The triangular number t_n solves the *handshake problem* of counting the number of handshakes if each person in a room with $(n+1)$ people shakes hands once with each person. Similarly a fully connected network of $(n+1)$ computing devices requires t_n connections. The triangular number t_n also provides the number of games played by $(n+1)$ teams in a *Round-Robin Tournament* in which each team plays every other team exactly once and no ties are allowed. Further, the triangular number t_n is the number of ordered pairs (x, y), where $1 \le x \le y \le n$. For an $(n+1)$ sided-polygon, the number of diagonals is $(n+1)(n-2)/2 = 2t_n - t_{n+1},\ n \ge 2$. From Fig. 7.1, it follows that the number of line segments between closest pairs of dots in the triangles are $\ell_n = 3t_{n-1} = 3(n-1)n/2$, or recursively, $\ell_n = \ell_{n-1} + 3(n-1),\ \ell_1 = 0$. Thus, for example, $\ell_4 = 18$. A problem of Rudolff reads: I am owed 3240 *florins*. The debtor pays me 1 *florin* the first day, 2 the second day, 3 the third day, and so on. How many days it takes to pay off the debt? (80 days). As we have noted earlier for the Pythagoreans, the fourth triangular number $t_4 = 10$ (decade) tetraktys (see Fig. 2.5 and its connected form Fig. 7.3) was most significant of all. The 36th triangular number is 666 and the 666th triangular number, i.e., t_{666} is 222111. On triangular numbers an interesting article is due to Garge and Shirali [209].

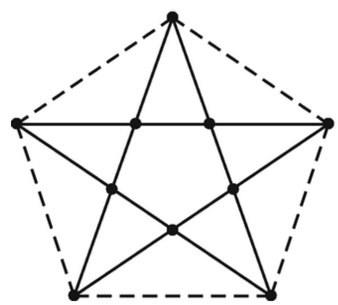

Fig. 7.3 Connected form of tetraktys

Figure 7.3 is actually a connected pentagram, see Fig. 1.1. The *Lute of Pythagoras* is a self-similar geometric figure made from a sequence of such pentagrams, see Price [419].

- No triangular number has as its last digit (unit digit) $2, 4, 7,$ or 9. For this, let $n \equiv k \pmod{10}$, then $(n+1) \equiv (k+1) \pmod{10}$; here $0 \le k \le 9$. Thus, it follows that $t_n = n(n+1)/2 \equiv k(k+1)/2 \pmod{10}$. This relation gives only choices for *k* as $0, 1, 3, 5, 6,$ and 8.
- We shall show that for an integer $k > 1,\ t_n \pmod{k},\ n \ge 1$ repeats every *k* steps if *k* is odd, and every 2*k* steps if *k* is even, i.e., if ℓ is the smallest positive integer such that for all integers *n*

$$\frac{(n+\ell)(n+\ell+1)}{2} \equiv \frac{n(n+1)}{2} \pmod{k}, \tag{7.8}$$

then $\ell = k$ if *k* is odd, and $\ell = 2k$ if *k* is even. For this, note that

$$\frac{(n+\ell)(n+\ell+1)}{2} - \frac{n(n+1)}{2} = n\ell + \frac{\ell(\ell+1)}{2},$$

and hence if (7.8) holds, then

$$n\ell + \frac{\ell(\ell+1)}{2} \equiv 0 \pmod{k}.$$

For $n = k$ and $n = 1$ the above equation respectively gives

$$0 + \frac{\ell(\ell+1)}{2} \equiv 0 \pmod{k} \quad \text{and} \quad \ell + \frac{\ell(\ell+1)}{2} \equiv 0 \pmod{k}.$$

Combining these two relations, we find

$$\ell \equiv 0 \pmod{k}$$

and hence

$$\ell = ck \quad \text{for some positive integer} \quad c. \tag{7.9}$$

Now if *k* is odd, then in view of $(k+1)/2$ is an integer, we have

$$nk + \frac{k(k+1)}{2} \equiv 0 \pmod{k}.$$

This implies that $k \geq \ell$, because ℓ is the smallest integer for which (7.8) holds. But, then from (7.9) it follows that $k = \ell$.

If *k* is even, then $k+1$ is odd, and so $k(k+1)/2 \not\equiv 0 \pmod{k}$. Thus, $\ell \neq k$, but

$$n(2k) + \frac{2k(k+1)}{2} \equiv 0 \pmod{k}$$

and so 2*k* satisfies (7.8). This implies that $2k \geq \ell$, which again from (7.9) gives $\ell = 2k$.

For example, for $t_n \pmod 3$, $n \geq 1$, we have

$$1,0,0, \quad 1,0,0, \quad 1,0,0, \quad 1,0,0,\cdots$$

and for $t_n \pmod 4$, $n \geq 1$,

$$1,3,2,2,3,1,0,0, \quad 1,3,2,2,3,1,0,0,\cdots.$$

- Triangular numbers and binomial coefficients are related by the relation

$$t_n = \binom{n+1}{2} = \binom{n+1}{n-1}.$$

Thus, triangular numbers are associated with Pascal's triangle

$$\begin{array}{c} 1 \\ 1 \quad 1 \\ \mathbf{1} \quad 2 \quad \mathbf{1} \\ 1 \quad \mathbf{3} \quad \mathbf{3} \quad 1 \\ 1 \quad 4 \quad \mathbf{6} \quad 4 \quad 1 \\ 1 \quad 5 \quad \mathbf{10} \quad \mathbf{10} \quad 5 \quad 1 \\ 1 \quad 6 \quad \mathbf{15} \quad 20 \quad \mathbf{15} \quad 6 \quad 1 \\ 1 \quad 7 \quad \mathbf{21} \quad 35 \quad 35 \quad \mathbf{21} \quad 7 \quad 1 \\ 1 \quad 6 \quad \mathbf{28} \quad 56 \quad 70 \quad 56 \quad \mathbf{28} \quad 8 \quad 1 \end{array}$$

- In 1989, Tzanakis and de Weger [514] showed that the only triangular numbers that are the product of three consecutive integers are $t_3 = 6 = 1 \cdot 2 \cdot 3$, $t_{15} = 120 = 4 \cdot 5 \cdot 6$, $t_{20} = 210 = 5 \cdot 6 \cdot 7$, $t_{44} = 990 = 9 \cdot 10 \cdot 11$, $t_{608} = 185136 = 56 \cdot 57 \cdot 58$, and $t_{22736} = 258474216 = 636 \cdot 637 \cdot 638$.
- There are 28 *palindromic triangular numbers* less than 10^{10}, namely, 1, 3, 6, 55, 66, 171, 595, 666, 3003, 5995, 8778, 15051, 66066, 617716, 828828, 1269621, 1680861, 3544453, 5073705, 5676765, 6295926, 351335153, 61477416, 178727871, 1264114621, 1634004361, 5289009825, 6172882716. The largest known palindromic triangular numbers containing only odd digits and even digits are $t_{32850970} = 539593131395935$ and $t_{128127032} = 8208268228628028$. It is known, see Trigg [512], that an infinity of palindromic triangular numbers exist in several different bases, for example, three, five, and nine; however, no infinite sequence of such numbers has been found in base 10.
- Let *m* be a given natural number, then it is *n*th triangular number, i.e., $m = t_n$ if and only if $n = (-1 + \sqrt{1+8m})/2$. This means if and only if $8m+1$ is a perfect square.
- If *n* is a triangular number, then $9n+1,\ 25n+3$, and $49n+6$ are also triangular numbers. This result of 1775 is due to Euler. Indeed, if

$n = t_m$, then $9n+1 = t_{3m+1},\ 25n+3 = t_{5m+2}$, and $49n+6 = t_{7m+3}$. An extension of Euler's result is the identity

$$(2k+1)^2 t_m + t_k = t_{(2k+1)m+k}, \quad k = 1, 2, \cdots$$

i.e.,

$$(2k+1)^2 \cdot \frac{m(m+1)}{2} + \frac{k(k+1)}{2} = \frac{[(2k+1)m+k][(2k+1)m+k+1]}{2}.$$

- From the identity

$$4\left(\frac{x(x+1)}{2} + \frac{y(y+1)}{2}\right) + 1 = (x+y+1)^2 + (x-y)^2$$

  it follows that if *n* is the sum of two triangular numbers, then $4n+1$ is a sum of two squares.

- Differentiating the expansion $(1-x)^{-1} = \sum_{n=0}^{\infty} x^n$ twice, we get

$$\frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)x^{n-2} = \sum_{n=1}^{\infty} (n+1)(n)x^{n-1}, \qquad (7.10)$$

  and hence,

$$\frac{x}{(1-x)^3} = 0x^0 + \sum_{n=1}^{\infty} \frac{(n+1)(n)}{2} x^n = \sum_{n=0}^{\infty} \frac{(n+1)(n)}{2} x^n = \sum_{n=0}^{\infty} t_n x^n.$$

  Hence, $x(1-x)^{-3}$ is the *generating function* of all triangular numbers. In 1995, Neil James Alexander Sloane (born 1939, England–USA) and Simon Plouffe (born 1956, Canada) [483] have shown that

$$\left(1 + 2x + \frac{1}{2}x^2\right) e^x = \sum_{n=0}^{\infty} t_{n+1} \frac{x^n}{n!}.$$

  Also note that

$$(1-x)^3(1-x^2)^3(1-x^3)^3 \cdots = 1 - 3x + 5x^3 - 7x^6 + 9x^{10} \cdots$$
$$= 1 + \sum_{n=1}^{\infty} (-1)^n (2n+1) x^{t_n}.$$

  This curious observation was made by Jacobi.

- To find the sum of the first *n* triangular numbers, we need an expression for $\sum_{k=1}^{n} k^2$ (a general reference for the summation of series is Davis [155]). For this, we begin with Pascal's identity

$$k^3 - (k-1)^3 = 3k^2 - 3k + 1, \quad k \geq 1$$

and hence,

$$(1^3 - 0^3) + (2^3 - 1^3) + (3^3 - 2^3) + \cdots + (n^3 - (n-1)^3) = 3\sum_{k=1}^{n} k^2 - 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1,$$

which in view of (7.2) gives

$$\sum_{k=1}^{n} k^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{2}n - \frac{1}{3}n = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n = \frac{1}{6}n(n+1)(2n+1). \quad (7.11)$$

Archimedes as proposition 10 in his text *On Spirals* stated the formula

$$(n+1)n^2 + (1 + 2 + \cdots + n) = 3(1^2 + 2^2 + \cdots + n^2) \quad (7.12)$$

from which (7.11) is immediate. It is believed that he obtained (7.12) by letting *k* the successive values $1, 2, \cdots, n-1$ in the relation

$$n^2 = [k + (n-k)]^2 = k^2 + 2k(n-k) + (n-k)^2,$$

and adding the resulting $n-1$ equations, together with the identity $2n^2 = 2n^2$, to arrive at

$$(n+1)n^2 = 2(1^2 + 2^2 + \cdots + n^2) + 2[1(n-1) + 2(n-2) + \cdots + (n-1)1]. \quad (7.13)$$

Next, letting $k = 1, 2, \cdots, n$ in the formula

$$k^2 = k + 2[1 + 2 + \cdots + (k-1)]$$

and adding *n* equations to get

$$1^2 + 2^2 + \cdots + n^2 = (1 + 2 + \cdots + n) + 2[1(n-1) + 2(n-2) + \cdots + (n-1)1]. \quad (7.14)$$

From (7.13) and (7.14), the formula (7.12) follows.

Another proof of (7.11) is given by Fibonacci. He begins with the identity

$$k(k+1)(2k+1) = (k-1)k(2k-1) + 6k^2.$$

and takes $k = 1, 2, 3, \cdots, n$ to get the set of equations

$$\begin{aligned} 1 \cdot 2 \cdot 3 &= 6 \cdot 1^2 \\ 2 \cdot 3 \cdot 5 &= 1 \cdot 2 \cdot 3 + 6 \cdot 2^2 \\ 3 \cdot 4 \cdot 7 &= 2 \cdot 3 \cdot 5 + 6 \cdot 3^2 \\ \cdots & \quad \cdots \\ (n-1)n(2n-1) &= (n-2)(n-1)(2n-3) + 6(n-1)^2 \\ n(n+1)(2n+1) &= (n-1)n(2n-1) + 6n^2. \end{aligned}$$

On adding these *n* equations and canceling the common terms, (7.11) follows.

Now from (7.2) and (7.11), we have

$$\sum_{k=1}^{n} t_k = \frac{1}{2}\sum_{k=1}^{n} k^2 + \frac{1}{2}\sum_{k=1}^{n} k = \frac{1}{6}n(n+1)(n+2). \qquad (7.15)$$

Relation (7.15) is due to Aryabhata.

For an alternative proof of (7.15), we note that

$$\begin{aligned} (n+1)t_n - \sum_{k=0}^{n} t_k &= \sum_{k=0}^{n}(t_n - t_k) = (1+2+3+\cdots+n) + (2+3+\cdots+n) \\ &+ (3+\cdots+n) + \cdots + n = 1^2 + 2^2 + \cdots + n^2 = \sum_{k=1}^{n} k^2, \end{aligned}$$

and hence, in view of (7.11), we have

$$\sum_{k=1}^{n} t_k = (n+1)t_n - \sum_{k=1}^{n} k^2 = (n+1)\frac{n(n+1)}{2} - \frac{1}{6}n(n+1)(2n+1) = \frac{1}{6}n(n+1)(n+2).$$

From (7.15) it follows that

$$\sum_{k=m+1}^{n} t_k = \frac{1}{6}(n-m)[(n-m)^2 + 3(n+1)(m+1) - 1],$$

which in particular for $m = 4, n = 7$ gives $t_5 + t_6 + t_7 = 64 = 8^2$, i.e., three successive triangular numbers whose sum is a perfect square. Similarly, we have $t_5 + t_6 + t_7 + t_8 = 10^2$.

From (7.15), we also have $\sum_{k=1}^{n} t_k = (1/3)(n+2)t_n$, which means t_n divides $\sum_{k=1}^{n} t_k$ if $n = 3m-2, \ m = 1, 2, \cdots$.

- The reciprocal of the $(n+1)$-th triangular number is related to the integral
$$\int_0^1 \int_0^1 |x-y|^n dx\, dy = \frac{2}{(n+1)(n+2)} = \frac{1}{t_{n+1}}.$$
- The sum of reciprocals of the first *n* triangular numbers is
$$\sum_{k=1}^{n} \frac{1}{t_k} = \sum_{k=1}^{n} \frac{2}{k(k+1)} = 2\sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right) = 2\left(1 - \frac{1}{n+1}\right), \quad (7.16)$$
and hence
$$\sum_{k=1}^{\infty} \frac{1}{t_k} = 2 \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right) = 2. \quad (7.17)$$
The series $\sum_{n=}^{\infty} 1/[n(n+1)]$ appeared in a 1644 work of Evangelista Torricelli (1608–1647, Italy). Jacob Bernoulli in 1689 summed numerous convergent series, the aforementioned is one of the examples. In the literature, this procedure is now called *telescoping*, also see Lesko [342].
- In 1575 it was observed that $2^{p-1}(2^p-1) = 2^p(2^p-1)/2 = t_{2^p-1}$, i.e., every known even perfect number is also a triangular number. For example, $6 = t_3, 10 = t_4, 28 = t_7, 496 = t_{31}, 8128 = t_{127}$.
- We shall show that for $n > 0$, Fermat number F_n is never a triangular number, i.e., there is no integer *m* that satisfies $2^{2^n}+1 = m(m+1)/2$. This means the discriminant of the equation $m^2+m-2(2^{2^n}+1) = 0$ is not an integer. Suppose to contrary that there exists an integer *p* such that $\sqrt{1+8(2^{2^n}+1)} = p$, but then $2^{2^n+3} = p^2-9 = (p+3)(p-3)$, which implies that there exist integers *r* and *s* such that $p+3 = 2^r$ and $p-3 = 2^s$. Hence, we have $2^r - 2^s = 6$ for which the only solution is $r = 3, s = 1$. This means, $2^{2^n+3} = 2^3 \times 2$, or $2^{2^n} = 2$, which is true only for $n = 0$.
- A *trapezoidal number* is a sum of two ore more consecutive positive integers, greater than 1, i.e.,
$$n + (n+1) + (n+2) + \cdots + (n+(k-1)), \quad n \neq 1,\ k \neq 1.$$

For example, $5, 7, 9, 11, 12, 13, 14, 15, 17, 18, \cdots$ are trapezoidal numbers. Any trapezoidal number is a difference of two non-consecutive triangular numbers, in fact,

$$\begin{aligned} n + (n+1) \ &+ \ \cdots + (n + (k-1)) = (1 + 2 + \cdots + (n + (k-1))) \\ &- \ (1 + 2 + \cdots + (n-1)) = t_{n+(k-1)} - t_{n-1} = k(k + 2n - 1)/2. \end{aligned}$$

A *polite number* is a positive integer that can be represented as the sum of two or more consecutive positive integers. Thus, if such polite representation starts with 1, we obtain a triangular number, otherwise one gets a trapezoidal number. The first few polite numbers are $1, 3, 5, 6, 7, 9, 10, 11, 12, 13$. *Impolite numbers*, i.e., positive integers, which are not polite, are exactly the powers of two. The first few impolite numbers are $1, 2, 4, 8, 16, 32, 64, 128, 256, 512$.

- Using mathematical induction we shall show that

$$\sum_{k=1}^{n} t_k^2 = \frac{1}{30} t_n (3n^3 + 12n^2 + 13n + 2), \quad n \geq 1. \tag{7.18}$$

The initial step, i.e., for $n = 1$ relation (7.18) is obviously true. For the inductive step, we assume (7.18) is true for *n* and need to show that it is also true for $n + 1$. For this, we have

$$\begin{aligned} \sum_{k=1}^{n+1} t_k^2 \ &= \ \frac{1}{30} \cdot \frac{n(n+1)}{2}(3n^3 + 12n^2 + 13n + 2) + \frac{(n+1)^2(n+2)^2}{4} \\ &= \ \frac{1}{30} \cdot \frac{(n+1)}{2}(3n^4 + 27n^3 + 88n^2 + 122n + 60) \\ &= \ \frac{1}{30} \cdot \frac{(n+1)(n+2)}{2}(3n^3 + 21n^2 + 46n + 30) \\ &= \ \frac{1}{30} t_{n+1}[3(n+1)^3 + 12(n+1)^2 + 13(n+1) + 2]. \end{aligned}$$

- We shall find all *square triangular numbers*, i.e., all positive integers *n* and the corresponding *m* so that $n(n+1)/2 = m^2$. This equation can be written as Pell's equation (see P22) $b^2 - 2a^2 = 1$, where $b = 2n + 1$ and $a = 2m$. We note that if $(a_{k-1}, b_{k-1}), \ k \geq 1$ is an integer solution of $b^2 - 2a^2 = \pm 1$, then (a_k, b_k) defined by the recurrence relations

$$a_k = a_{k-1} + b_{k-1}, \quad b_k = 2a_{k-1} + b_{k-1}, \quad k \geq 1 \tag{7.19}$$

satisfy

$$b_k^2 - 2a_k^2 = (2a_{k-1} + b_{k-1})^2 - 2(a_{k-1} + b_{k-1})^2 = -(b_{k-1}^2 - 2a_{k-1}^2),$$

and hence $b^2 - 2a^2 = \mp 1$. From this observation we conclude that if $(a_{k-1}, b_{k-1}),\ k \geq 1$ is an integer solution of $b^2 - 2a^2 = 1$, then so is $(a_{k+1}, b_{k+1}) = (3a_{k-1} + 2b_{k-1}, 4a_{k-1} + 3b_{k-1})$. Since $(a_0, b_0) = (0, 1)$ is a solution of $b^2 - 2a^2 = 1$ (its fundamental solution is $(a, b) = (2, 3)$), it follows that the iterative scheme

$$\begin{aligned} x_k &= 3x_{k-1} + 2y_{k-1} \\ y_k &= 4x_{k-1} + 3y_{k-1}, \quad x_0 = 0,\ y_0 = 1 \end{aligned} \tag{7.20}$$

gives all solutions of $b^2 - 2a^2 = 1$. System (7.20) can be written as

$$\begin{aligned} x_{k+1} &= 6x_k - x_{k-1}, \quad x_0 = 0,\ x_1 = 2 \\ y_{k+1} &= 6y_k - y_{k-1}, \quad y_0 = 1,\ y_1 = 3. \end{aligned} \tag{7.21}$$

Now in (7.21) using the the substitution $x_k = 2m_k,\ y_k = 2n_k + 1$, we get

$$\begin{aligned} m_{k+1} &= 6m_k - m_{k-1}, \quad m_0 = 0,\ m_1 = 1 \\ n_{k+1} &= 6n_k - n_{k-1} + 2, \quad n_0 = 0,\ n_1 = 1 \end{aligned} \tag{7.22}$$

Clearly, (7.22) generates all (infinite) solutions (m_k, n_k) of the equation $n(n+1)/2 = m^2$. First few of these solutions are

$$(1,1), (6,8), (35,49), (204,288), (1189,1681), (6930,9800), \\ (40391,57121), (235416,332928).$$

For $k \geq 1$, explicit solution of the system (7.22) can be computed (for details see Agarwal [10, 18]) rather easily and appears as

$$\begin{aligned} m_k &= \frac{1}{4\sqrt{2}}\left[(3+2\sqrt{2})^k - (3-2\sqrt{2})^k\right] \quad \text{and} \\ n_k &= \frac{1}{4}\left[(3+2\sqrt{2})^k + (3-2\sqrt{2})^k - 2\right]. \end{aligned} \tag{7.23}$$

This result is originally due to Euler, which he obtained in 1730. While compared to the explicit solution (7.23), the computation of (m_k, n_k) from the recurrence relations (7.22) is very simple, the following interesting relation follows from (7.23) by direct substitution

$$m_k^2 - m_{k-1}^2 = m_{2k-1}. \tag{7.24}$$

Hence, the difference between two consecutive square triangular numbers is the square root of another square triangular number. For example, $6^2 - 1^2 = 35,\ 35^2 - 6^2 = 1189,\ 204^2 - 35^2 = 40391.$

Now we note that the system (7.19) can be written as

$$\begin{aligned} a_{n+1} &= 2a_n + a_{n-1}, \quad a_0 = 0,\ a_1 = 1 \\ b_{n+1} &= 2b_n + b_{n-1}, \quad b_0 = 1,\ b_1 = 1 \end{aligned}$$

and its (integer) solution is

$$a_n = \frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^n - (1-\sqrt{2})^n\right], \quad b_n = \frac{1}{2}\left[(1+\sqrt{2})^n + (1-\sqrt{2})^n\right]. \tag{7.25}$$

From this and simple calculations, the following relations follow

$$m_k = a_k b_k, \quad n_{2k} = b_{2k}^2 - 1 = 2(2m_k)^2, \quad n_{2k+1} = b_{2k+1}^2 = 2a_{2k+1}^2 - 1.$$

Thus, $8m_k^2 + 1$ and n_{2k+1} are perfect squares.

It is apparent that if (m_k, n_k) is a solution of $n(n+1)/2 = m^2$, then $((2p+1)m_k, (2p+1)n_k),\ p \geq 1$ is a solution of $n(n+2p+1)/2 = m^2$. Now, if *n* is even, we have

$$\begin{aligned} & t_n t_{n+1} \cdots t_{n+2p} = (n+1)^2 \left(\frac{n+2}{2}\right)^2 (n+3)^2 \left(\frac{n+4}{2}\right)^2 \cdots \\ & (n+2p-1)^2 \left(\frac{n+2p}{2}\right)^2 \left(\frac{n(n+2p+1)}{2}\right), \end{aligned} \tag{7.26}$$

and, when *n* is odd,

$$\begin{aligned} & t_n t_{n+1} \cdots t_{n+2p} = \left(\frac{n+1}{2}\right)^2 (n+2)^2 \left(\frac{n+3}{2}\right)^2 (n+4)^2 \cdots \\ & \left(\frac{n+2p-1}{2}\right)^2 (n+2p)^2 \left(\frac{n(n+2p+1)}{2}\right) \end{aligned} \tag{7.27}$$

and hence, the right side is a perfect square for $n = (2p+1)n_k$. Therefore, the product of $(2p+1)$ consecutive triangular numbers is a perfect square for each $p \geq 1$ and $k \geq 1$. In particular, for $p = k = 2,\ n = 5n_2 = 40$, from (7.26) we have

$$t_{40}t_{41}t_{42}t_{43}t_{44} = (41)^2(21)^2(43)^2(22)^2(30)^2 = (24435180)^2$$

and for $p = k = 3,\ n = 7n_3 = 343$, from (7.27), we find

$$\begin{aligned} t_{343}t_{344}t_{345}t_{346}t_{347}t_{348}t_{349} &= (172)^2(345)^2(173)^2(347)^2(174)^2(349)^2(245)^2 \\ &= (52998536784979800)^2. \end{aligned}$$

Similarly, if *n* is even, we have

$$\begin{aligned} 2t_nt_{n+1}\cdots t_{n+2p-1} = (n+1)^2\left(\frac{n+2}{2}\right)^2(n+3)^2\left(\frac{n+4}{2}\right)^2\cdots \\ \left(\frac{n+2p-2}{2}\right)^2(n+2p-1)^2\left(\frac{n(n+2p)}{2}\right), \end{aligned} \tag{7.28}$$

and hence, the right side is a perfect square for $n = 2pn_k$ (which is always even). Therefore, two times the product of 2*p* consecutive triangular numbers is a perfect square for each $p \geq 1$ and $k \geq 1$. In particular, for $p = k = 2,\ n = 4n_2 = 32$, from (7.28) we have

$$2t_{32}t_{33}t_{34}t_{35} = (33)^2(17)^2(35)^2(24)^2 = (471240)^2$$

and for $p = k = 3,\ n = 6n_3 = 294$, we find

$$\begin{aligned} 2t_{294}t_{295}t_{296}t_{297}t_{298}t_{299} &= (295)^2(148)^2(297)^2(149)^2(299)^2(210)^2 \\ &= (121315678684200)^2. \end{aligned}$$

From the equality

$$\frac{(4n(n+1)(4n(n+1)+1)}{2} = 4\frac{n(n+1)}{2}(2n+1)^2$$

it follows that if the triangular number t_n is square, then $t_{4n(n+1)}$ is also square. Since t_1 is square, it follows that there are infinite number of square triangular numbers. This clever observation was reported in 1662, see Pietenpol et. al. [410]. From this, the first four square triangular numbers, we get are t_1, t_8, t_{288}, and t_{332928}.

- There are infinitely many triangular numbers that are simultaneously expressible as the sum of two cubes and the difference of two cubes. For this, Burton [111] begins with the identity

$$(27k^6)^2 - 1 = (9k^4 - 3k)^3 + (9k^3 - 1)^3 = (9k^4 + 3k)^3 - (9k^3 + 1)^3$$

  and observed that if *k* is odd then this equality can be written as

$$(2n+1)^2 - 1 = (2a)^3 + (2b)^3 = (2c)^3 - (2d)^3,$$

  which is the same as

$$t_n = a^3 + b^3 = c^3 - d^3.$$

  For $k = 1, 3,$ and 5 this gives

$$\begin{aligned} t_{13} &= 3^3 + 4^3 = 6^3 - 5^3 \\ t_{9841} &= (360)^3 + (121)^3 = (369)^3 - (122)^3 \\ t_{210937} &= (2805)^3 + (562)^3 = (2820)^3 - (563)^3. \end{aligned}$$

- From Catalan's conjecture we know that the only solution in natural numbers of the Diophantine equation $x^a - y^b = 1$ for $a, b > 1,\ x, y > 0$ is $x = 3, a = 2,\ y = 2, b = 3.$ Now since $n(n+1)/2 = m^3$ can be written as $(2n+1)^2 - (2m)^3 = 1,$ the only solution of this equation is $2n+1 = 3, 2m = 2,$ i.e., $(1, 1)$ is the only cubic triangular number.
- In 2001, Bennett [64] proved that if $a, b,$ and *n* are positive integers with $n \geq 3,$ then the equation $|ax^n - by^n| = 1,$ possesses at most one solution in positive integers *x* and $y.$ This result is directly applicable to show that for the equation $n(n+1)/2 = m^p,\ p \geq 3$ the only solution is $(1, 1).$ For this, first we note that integers $t, 2t+1$ and $t+1, 2t+1$ are coprime. We also recall that if the product of coprime numbers is a *p*th power, then both are also of *p*the power. Now let *n* be even, i.e., $n = 2t,$ then the equation $n(n+1)/2 = m^p$ is the same as $t(2t+1) = m^p.$ Thus, it follows that $t = x^p$ and $2t+1 = y^p,$ and hence $y^p - 2x^p = 1$, which has only one solution, namely, $x = 0, y = 1$ which gives $t = 0,$ and hence $n = 0$ and so $(0, 0)$ is the solution of $n(n+1)/2 = m^p$, but we are not interested in this solution. Now we assume that *n* is odd, i.e., $n = 2t+1,$ then the

equation $n(n+1)/2 = m^p$ is the same as $(t+1)(2t+1) = m^p$. Thus, we must have $t+1 = x^p$ and $2t+1 = y^p$, which gives $y^p - 2x^p = -1$. The only solution of this equation is $x = y = 1$, and hence, again $t = 0$ and so $(0, 0)$ is the undesirable solution of $n(n+1)/2 = m^p$.

- Startling *generating function* of all square triangular numbers is recorded by Plouffe [412] as

$$\frac{x(1+x)}{(1-x)(x^2-34x+1)} = x + 6^2x^2 + 35^2x^3 + \cdots. \tag{7.29}$$

7.3 Square Numbers S_n

In this arrangement rows as well as columns contain $1, 2, 3, 4, \ldots, n$ dots, (see Fig. 7.4).

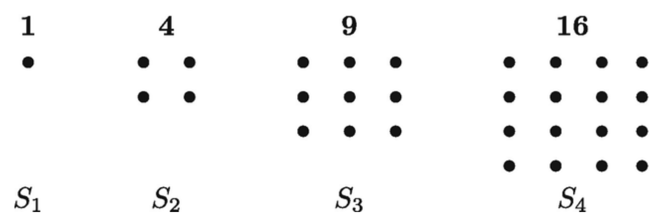

Fig. 7.4 Square numbers

From Fig. 7.4, it is clear that a square made up of $(n+1)$ dots on a side can be divided into a smaller square of side *n* and an *L*–shaped border (a gnomon), which has $(n+1) + n = 2n + 1$ dots (called $(n+1)$*th gnomonic number and denoted as* g_{n+1}), and hence

$$S_{n+1} - S_n = (n+1)^2 - n^2 = (2n+1), \tag{7.30}$$

i.e., the differences between successive nested squares produce the sequence of odd numbers. From (7.30) it follows that

$$(1^2 - 0^2) + (2^2 - 1^2) + (3^2 - 2^2) + \cdots + (n^2 - (n-1)^2) = 1 + 3 + 5 + \cdots + (2n-1)$$

and hence

$$\sum_{k=1}^{n}(2k-1) = 1+3+5+7+\cdots+(2n-1) = n^2 = S_n. \tag{7.31}$$

An alternative proof of (7.31) is as follows

$$\begin{array}{lcccccccccccc} S_n & = & 1 & + & 3 & + & 5 & + & \cdots & + & (2n-3) & + & (2n-1) \\ S_n & = & (2n-1) & + & (2n-3) & + & (2n-5) & + & \cdots & + & 3 & + & 1. \end{array}$$

An addition of these two arrangements immediately gives

$$2S_n = 2n + 2n + \cdots + 2n = 2n^2.$$

Figure 7.5 provides proof of (7.31) without words. Here odd integers —one block, three blocks, five blocks, and so on—arranged in a special way. We begin with a single block in the lower left corner; three shaded blocks surrounded it to form a 2×2 square; five unshaded blocks surround these to form a 3×3 square; with the next seven shaded blocks we have a 4×4 square; and so on. The diagram makes clear that the sum of consecutive odd integers will always yield a (geometric) square.

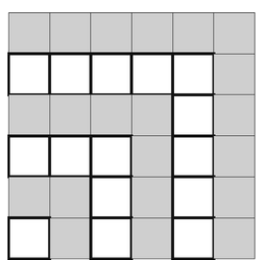

Fig. 7.5 Proof of (7.31) without words

Comparing Figs. 7.1 and 7.4 or Figs. 7.2 and 7.5, it is clear that *n*th square number is equal to the *n*th triangular number increased by its predecessor, i.e.,

$$S_n = t_n + t_{n-1} = n^2. \tag{7.32}$$

Indeed, we have

$$\begin{array}{lcccccccccccc} t_n & = & 1 & + & 2 & + & 3 & + & \cdots & + & (n-1) & + & n \\ t_{n-1} & = & & & 1 & + & 2 & + & \cdots & + & (n-2) & + & (n-1). \end{array}$$

An addition of these two arrangements in view of (7.31) gives

$$t_n + t_{n-1} = 1 + 3 + 5 + \cdots + (2n-1) = n^2 = S_n.$$

Of course, directly from (7.1), (7.2), and (7.32), we also have

$$t_n + t_{n-1} = \frac{n(n+1)}{2} + \frac{(n-1)n}{2} = n^2 = (t_n - t_{n-1})^2 = S_n,$$

or simply from (7.1) and (7.2),
$t_n + t_{n-1} = 2t_{n-1} + n = n(n-1) + n = n^2 = S_n.$

From (7.32), we find the identities

$$\sum_{k=1}^{2n} t_k = (t_2 + t_1) + (t_4 + t_3) + \cdots + (t_{2n} + t_{2n-1}) = 2^2 + 4^2 + \cdots + (2n)^2$$

and

$$\sum_{k=1}^{2n+1} t_k = t_1 + (t_3 + t_2) + (t_5 + t_4) + \cdots + (t_{2n+1} + t_{2n}) = 1^2 + 3^2 + 5^2 + \cdots + (2n+1)^2.$$

It also follows that

$$t_{2n} - 2t_n = \frac{2n(2n+1)}{2} - 2\frac{n(n+1)}{2} = n^2 = S_n. \tag{7.33}$$

We also have equalities

$$t_{9n+4} - t_{3n+1} = [3(2n+1)]^2 = S_{3(2n+1)}, \tag{7.34}$$

$$S_1 - S_2 + S_3 - S_4 + \cdots + (-1)^{n+1} S_n = (-1)^{n+1} t_n, \tag{7.35}$$

and

$$\sum_{k=0}^{n} (t_{2n} + k)^2 = \sum_{k=1}^{n} (4t_n + k)^2, \tag{7.36}$$

which is the same as

$$\sum_{k=0}^{n} (2n^2 + n + k)^2 = \sum_{k=1}^{n} (2n^2 + 2n + k)^2 \quad \text{or} \quad \sum_{k=0}^{n} S_{2n^2+n+k} = \sum_{k=1}^{n} S_{2n^2+2n+k}$$

and, in particular, for $n = 4$ reduces to

$$36^2 + 37^2 + 38^2 + 39^2 + 40^2 = 41^2 + 42^2 + 43^2 + 44^2.$$

The following equality is of exceptional merit

$$S_n + S_{n+1} = S_{n(n+1)+1} - S_{n(n+1)}, \tag{7.37}$$

which, in particular, for $n = 5$ gives $5^2 + 6^2 = 31^2 - 30^2$.

- Relation (7.30) reveals that every odd integer $(2n+1)$ is the difference of two consecutive square numbers S_{n+1} and S_n. Relation (7.32) shows that every square integer n^2 is a sum of two consecutive triangular numbers t_n and t_{n-1}, whereas (7.33) displays it is the difference of 2*n*th and two times *n*th triangular numbers.
- From the equalities

$$8t_n^2 = (n^2+n)^2 + (n^2+n)^2, \quad 8t_n^2 + 1 = (n^2-1)^2 + (n^2+2n)^2,$$
$$8t_n^2 + 2 = (n^2+n-1)^2 + (n^2+n+1)^2$$

  it follows that there are infinite triples of consecutive numbers, which can be written as the sum of two squares.
- From (7.10) it follows that

$$\frac{2x}{(1-x)^3} = \sum_{n=0}^{\infty} n^2 x^n + x \sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} n^2 x^n + x \frac{d}{dx} \frac{1}{(1-x)}$$

  and hence,

$$\frac{2x}{(1-x)^3} - \frac{x}{(1-x)^2} = \frac{x(1+x)}{(1-x)^3} = \sum_{n=0}^{\infty} n^2 x^n. \tag{7.38}$$

  Therefore, $x(x+1)(1-x)^{-3}$ is the *generating function* of all square numbers. From (7.38) it also follows that the generating function for all gnomonic numbers is

$$\frac{x(1+x)}{(1-x)^2} = \sum_{n=1}^{\infty} (2n-1) x^n = \sum_{n=1}^{\infty} g_n x^n.$$

- The sum of the first *n* square numbers is given in (7.11). For the exact sum of the reciprocals of the first *n* square numbers, no formula exists; however, the problem of summing exactly the reciprocals of all square numbers is given in (5.12) has a long history and in the literature, it is known as *the Basel problem* (named after hometown of

Euler). This problem was posed by Pietro Mengoli (1626–1686, Italy) in 1650. After unsuccessful attempts of Bernoulli family (also from Basel), and several others from England, France, and Germany, in 1735, Euler considered the function $\sin x/x,\ x \neq 0$, which has roots at $\pm n\pi,\ n \geq 1.$ Then, he wrote this function in terms of infinite product

$$\frac{\sin x}{x} == 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots = \left(1 - \frac{x^2}{1^2\pi^2}\right)\left(1 - \frac{x^2}{2^2\pi^2}\right)\left(1 - \frac{x^2}{3^2\pi^2}\right)\cdots, \quad (7.39)$$

which on equating the coefficients of x^2, gives

$$\frac{1}{6} = \frac{1}{1^2\pi^2} + \frac{1}{2^2\pi^2} + \frac{1}{3^2\pi^2} + \cdots,$$

and hence,

$$\zeta(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6} \approx 1.6449340668,$$

thus (5.12) holds.

The earlier demonstration of Euler is based on manipulations that were not justified at the time, and it was not until 1741 that he was able to produce a truly rigorous proof. Now in the literature for (5.12) several different elementary proofs have been given, e.g., by Apostol in 1983, Josef Hofbauer (born 1956, Austria) in 2002, Robin Chapman (England) in 2003, Hirschhorn in 2011, Zurab Silagadze (born 1957, Russia) in 2018, and Murty in 2019, see [383]. Basel problem appears in number theory, e.g., if two positive integers are selected at random and independently of each other, then the probability that they are relatively prime is $\left(\sum_{n=1}^{\infty} n^{-2}\right)^{-1}$. The probability that a randomly selected integer is square free is also $\left(\sum_{n=1}^{\infty} n^{-2}\right)^{-1}$. It is interesting to note that $\pi^2/6$ is also the length of the circumference of a circle whose diameter equals the ratio of volume of an ellipsoid to the circumscribed cuboid, also it is the length of the circumference of a circle whose diameter equals the ratio of surface area of a sphere to the circumscribed cube.

Euler also established the following series

$$3\sum_{m=1}^{\infty}\frac{1}{m^2\binom{2m}{m}} = \frac{\pi^2}{6}$$

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots = \frac{\pi^2}{8}$$

$$\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots = \frac{\pi^2}{12}.$$

Later, Euler generalized the Basel problem considerably, in fact, for all positive integers $k = 1, 2, \cdots,$ he established

$$\frac{1}{1^{2k}}+\frac{1}{2^{2k}}+\frac{1}{3^{2k}}+\cdots = \frac{(2\pi)^{2k}(-1)^{n+1}B_{2k}}{2\,(2n)!}, \tag{7.40}$$

where B_k are *Bernoulli Numbers* defined by

$B_0 = 1, B_1 = -1/2, B_2 = 1/6, B_3 = 0, B_4 = -1/30, B_5 = 0, B_6 = 1/42, B_7 = 0, B_8 = -1/30, B_9 = 0, B_{10} = 5/66, B_{11} = 0, B_{12} = -691/2730, \cdots.$

In general, $B_{2k+1} = 0,\ k \geq 1$ and $B_{2k},\ k \geq 1$ are obtained by the recurrence relation

$$\frac{2^{2k}}{1}\binom{2k}{0}B_{2k}+\frac{2^{2k-2}}{3}\binom{2k}{2}B_{2k-2}+\cdots+\frac{1}{2k+1}\binom{2k}{2k}B_0 = 1.$$

In particular, Euler established

$$\frac{1}{1^{26}}+\frac{1}{2^{26}}+\frac{1}{3^{26}}+\cdots = \frac{2^{24}\times 76977927}{27!}\pi^{26}.$$

Seki Kowa (1642–1708, Japan) also independently discovered the Bernoulli numbers, and his result was published posthumously in 1712.

- A prime *p* is said to be *regular* if it divides none of the numerators of $B_1, B_2, \cdots, B_{p-3}$ when these numbers are written in their lowest terms. Otherwise, *p* is said to be *irregular*. The odd regular primes below 100 are
  $3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 41, 43, 47, 53, 61, 71, 73, 79, 83, 89, 97.$
  It is not known if there are an infinite number of regular primes. Since

  $$B_{32} = -\frac{7709321041217}{510},$$

  and $7709321041217 = 37(208360028141)$ it follows that 37 is the smallest irregular prime. Other irregular primes below 100 are $59, 67.$

In 1915, Kaj Løchte Jensen (Denmark) proved that there are infinitely many irregular primes of the form $4n+3.$ In 1954, Leonard Carlitz (1907–1999, USA) gave a simple proof of the weaker result that there are in general infinitely many irregular primes.

- In 1840, Karl Georg Christian von Staudt (1798–1867, Germany) and Thomas Clausen (1801–1885, Denmark) independently showed that

$$B_{2n} = G - \left[\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \cdots\right],$$

where *G* is an integer and $a, b, c, \cdots$ are all the primes *p* such that $2n/(p-1)$ is an integer. For example,
$B_4 = -1/30 = 1 - (1/2 + 1/3 + 1/5)$,
$B_{12} = -691/2730 = 1 - (1/2 + 1/3 + 1/5 + 1/7 + 1/13)$, and
$B_{16} = -3617/510 = -6 - (1/2 + 1/3 + 1/5 + 1/17)$.

- From (5.1), (6.8), and (7.32) the following relations hold

$$S_a + S_b = S_c, \quad (t_a + t_{a-1}) + (t_b + t_{b-1}) = (t_c + t_{c-1}), \quad S_{u^2-v^2} + S_{2uv} = S_{u^2+v^2}.$$

Further, from (6.1) and (6.2), respectively, we have

$$S_m + S_{(m^2-1)/2} = S_{(m^2+1)/2}, \quad (m \text{ odd}).$$

$$S_m + S_{(m^2-4)/4} = S_{(m^2+4)/4}, \quad (m \text{ even}).$$

Similarly, we find the relation

$$S_{2m+3} + S_{2(m+1)(m+2)} = S_{(m+1)^2+(m+2)^2},$$

which is the same as

$$(t_{2m+3} + t_{2m+2}) + 16t_{m+1}^2 = (t_{m+2} + 2t_{m+1} + t_m)^2.$$

- In 1875, Lucas challenged the mathematical community to prove that the only solution of the equation

$$\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1) = m^2$$

with $n > 1$ is when $n = 24$ and $m = 70$. In the literature, this has been termed as the cannonball problem; in fact, it can be visualized as the problem of taking a square arrangement of cannonballs on the ground and building a square pyramid out of them. It was only in 1918, George Neville Watson (1886–1965, England) used elliptic functions to provide correct (filling gaps in earlier attempts) proof of Lucas assertion. Simplified proofs of this result are available, e.g., in Ma [353] and Anglin [31].

- In 1942, Dutch mathematics teacher Albert E. Bosman (1891–1961) used squares to construct the *Pythagoras Tree* which is a plane fractal.

7.4 Rectangular (Oblong, Pronic, Heteromecic) Numbers R_n

In this arrangement rows contain $(n+1)$, whereas columns contain *n* dots, see Fig. 7.6.

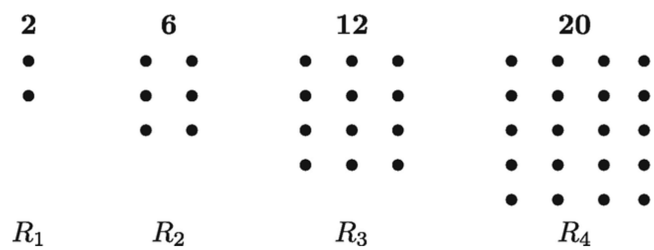

Fig. 7.6 Rectangular numbers

From Fig. 7.6 it is clear that the ratio $(n+1)/n$ of the sides of rectangles depends on $n.$ Further, we have

$$R_n = 2+4+6+8+\cdots+2n = 2(1+2+3+4+\cdots+n) = 2t_n = n(n+1) \quad (7.41)$$

i.e., we add successive even numbers, or two times triangular numbers. It also follows that rectangular number R_{n+1} is made from R_n by adding an *L*–shaped border (a gnomon), with $2(n+1)$ dots, i.e.,

$$R_{n+1} - R_n = 2(n+1), \quad (7.42)$$

i.e., the differences between successive nested rectangular numbers produce the sequence of even numbers.

Thus, the odd numbers generate a limited number of forms, namely squares, while the even ones generate a multiplicity of rectangles, which are not similar. From this the Pythagoreans deduced the following correspondence:

$$\text{odd} \longleftrightarrow \text{limited} \quad \text{and} \quad \text{even} \longleftrightarrow \text{unlimited}.$$

We also have the relations

$$\begin{aligned} R_n + S_n &= 2t_n + (t_n + t_{n-1}) = 3\frac{n(n+1)}{2} + \frac{(n-1)n}{2} = \frac{2n(2n+1)}{2} = t_{2n} \\ R_n - S_n &= 2t_n - (t_n + t_{n-1}) = t_n - t_{n-1} = n \end{aligned} \tag{7.43}$$

$$\begin{aligned} 2R_n + S_n + S_{n+1} &= 6t_n + t_{n-1} + t_{n+1} = (2n+1)^2 \\ S_{2n+1} &= (2n+1)^2 = 4n(n+1) + 1 = 8t_n + 1 = (4t_n+1)^2 - (4t_n)^2 \\ 9S_{2n+1} &= t_{9n+4} - t_{3n+1} \\ R_n &= t_{n+1} + t_{n-1} - 1. \end{aligned} \tag{7.44}$$

From (7.31) and (7.43), it follows that

$$\sum_{k=1}^{2n-1}(-1)^{k+1}t_k = t_1 + (t_3 - t_2) + \cdots + (t_{2n-1} - t_{2n-2}) = 1 + 3 + \cdots + (2n-1) = n^2.$$

- Relation (7.41) shows that the product of two consecutive positive integers n and $(n+1)$ is the same as two times nth triangular numbers. According to historians with this relation, Pythagoreans' enthusiasm was endless. Relation (7.42) reveals that every even integer $2n$ is the difference of two consecutive rectangular numbers R_n and R_{n-1}. Relation (7.43) displays that every positive integer n is the difference of nth and $(n-1)$-th triangular numbers. Relation (7.44) is due to Plutarch), it says an integer n is a triangular number if and only if $8n+1$ is a perfect odd square, also if a triangular number is multiplied by $8,$ and 1 is added, then the result is a square number.
- Let m be a given natural number, then it is nth rectangular number, i.e., $m = R_n$ if and only if $n = (-1 + \sqrt{1+4m})/2.$
- From (7.10) it is clear that $2x(1-x)^{-3}$ is the *generating function* of all rectangular numbers.
- From (7.15)–(7.17) and (7.41) it is clear that

$$\sum_{k=1}^{n} R_k = \frac{1}{3}n(n+1)(n+2), \quad \sum_{k=1}^{n}\frac{1}{R_k} = \left(1 - \frac{1}{n+1}\right), \quad \sum_{k=1}^{\infty}\frac{1}{R_k} = 1. \tag{7.45}$$

- There is no rectangular number that is also a perfect square, in fact, the equation $n(n+1) = m^2$ has no solutions (the product of two consecutive integers cannot be a prefect square).

- To find all *rectangular numbers which are also triangular numbers*, we need to find integer solutions of the equation $n(n+1) = m(m+1)/2$. This equation can be written as Pell's equation $b^2 - 2a^2 = -1$ (its fundamental solution is $(a, b) = (1, 1)$) where $b = 2m + 1$ and $a = 2n + 1.$ For this, corresponding to (7.22) the system is

$$\begin{array}{rcl} m_{k+1} & = & 34m_k - m_{k-1} + 16, \quad m_1 = 3, \; m_2 = 119 \\ n_{k+1} & = & 34n_k - n_{k-1} + 16, \quad n_1 = 2, \; n_2 = 84. \end{array} \tag{7.46}$$

  This system generates all (infinite) solutions (m_k, n_k) of the equation $n(n+1) = m(m+1)/2.$ First few of these solutions are

$$(3,2), (119, 84), (4059, 2870), (137903, 97512), (4684659, 3312554).$$

  For $k \geq 1,$ explicit solution of the system (7.46) can be written as

$$\begin{array}{rcl} m_k & = & \frac{1}{4}\left[(\sqrt{2}+1)^{4k-1} - (\sqrt{2}-1)^{4k-1} - 2\right] \\ n_k & = & \frac{\sqrt{2}}{8}\left[(\sqrt{2}+1)^{4k-1} + (\sqrt{2}-1)^{4k-1} - 2\sqrt{2}\right]. \end{array}$$

- In 1989, Ming [373] has used (7.44) to show that only Fibonacci numbers $1, 3, 21, 55$ are also triangular numbers t_1, t_2, t_6, t_{10}. This conjecture was made by Verner Emil Hoggatt Jr. (1921–1980, USA) in 1971 (also see his book [266]). Similarly, only Lucas numbers, which are also triangular are $1, 3, 5778,$ i.e., t_1, t_2, t_{107}.

7.5 Pentagonal Numbers P_n

The pentagonal numbers are defined by the sequence $1, 5, 12, 22, 35, 51, \cdots,$ i.e., beginning with 5 each number is formed from the previous one in the sequence by adding the next number in the related sequence $4, 7, 10, \cdots, (3n-2).$ Thus, $5 = 1 + 4,\; 12 = 1 + 4 + 7 = 5 + 7,\; 22 = 1 + 4 + 7 + 10 = 12 + 10,$ and so on (see Figs. 7.7 and 7.8).

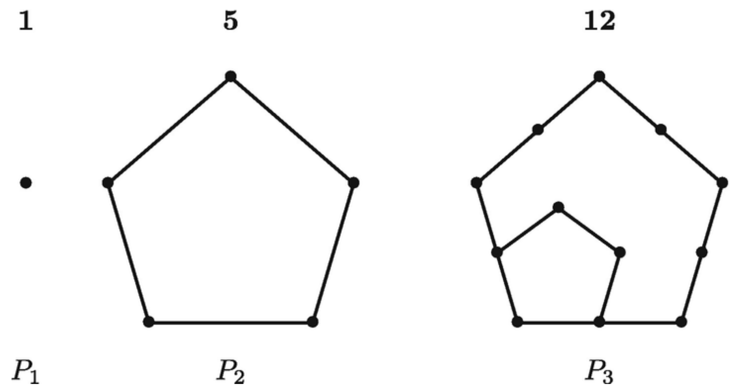

Fig. 7.7 Pentagonal numbers

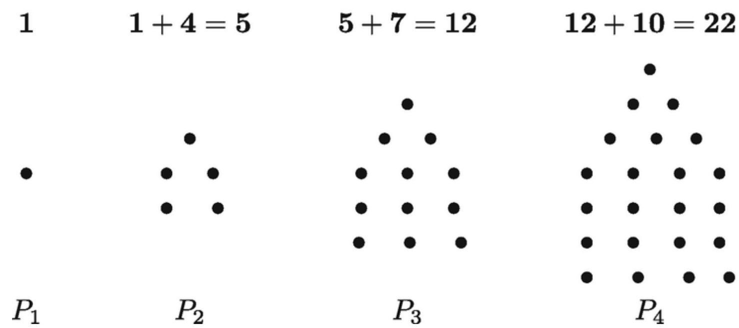

Fig. 7.8 Alternative form of pentagonal numbers

Thus, *n*th pentagonal number is defined as

$$P_n = P_{n-1} + (3n-2) = 1 + 4 + 7 + \cdots + (3n-2). \tag{7.47}$$

Comparing (7.47) with (7.3), we have $a = 1, d = 3$ and hence from (7.4) it follows that

$$P_n = \frac{n}{2}(3n-1) = \frac{1}{3}\frac{(3n-1)(3n)}{2} = \frac{1}{3}t_{3n-1}. \tag{7.48}$$

It is interesting to note that P_n is the sum of *n* integers starting from n, i.e.,

$$P_n = n + (n+1) + (n+2) + \cdots + (2n-1), \tag{7.49}$$

whose sum from (7.4) is the same as in (7.48).

Note that from (7.47), we have

$$\begin{aligned} P_n &= 2P_{n-1} - P_{n-2} - (P_{n-1} - P_{n-2}) + (3n-2) \\ &= 2P_{n-1} - P_{n-2} - (3n-3-2) + (3n-2) = 2P_{n-1} - P_{n-2} + 3. \end{aligned}$$

From (7.32) and (7.48), we also have

$$P_n = \frac{n(n-1)}{2} + n^2 = t_{n-1} + (t_n + t_{n-1}) = t_n + 2t_{n-1} = t_{2n-1} - t_{n-1}. \quad (7.50)$$

- Relation (7.48) shows that pentagonal number P_n is the one-third of the $(3n-1)$-th triangular number, whereas relation (7.50) reveals that it is the sum of *n*th triangular number and two times of $(n-1)$-th triangular number, and it is the difference of $(2n-1)$-th triangular number and $(n-1)$-th triangular number.
- Let *m* be a given natural number, then it is *n*th pentagonal number, i.e., $m = P_n$ if and only if $n = (1 + \sqrt{1+24m})/6$.
- As in (7.38), we have

$$\sum_{n=0}^{\infty} P_n x^n = \frac{3}{2}\frac{x(1+x)}{(1-x)^3} - \frac{1}{2}\frac{x}{(1-x)^2} = \frac{x(2x+1)}{(1-x)^3}$$

  and hence $x(2x+1)(1-x)^{-3}$ is the *generating function* of all pentagonal numbers.
- From (7.2), (7.11) and (7.48), it is easy to find the sum of the first *n* pentagonal numbers

$$\sum_{k=1}^{n} P_k = \frac{1}{2}n^2(n+1). \quad (7.51)$$

- To find the sum of the reciprocals of all pentagonal numbers, we begin with the series

$$f(x) = \sum_{k=1}^{\infty} \frac{2}{k(3k-1)} x^{3k}$$

  and note that

$$f(1) = \sum_{k=1}^{\infty} \frac{2}{k(3k-1)} = \sum_{k=1}^{\infty} \frac{1}{P_k}, \quad f'(x) = 6\sum_{k=1}^{\infty} \frac{1}{(3k-1)} x^{3k-1},$$

$$f''(x) = 6\sum_{k=1}^{\infty} x^{3k-2} = \frac{6x}{1-x^3}.$$

  Now since $f(0) = f'(0) = 0$, we have

$$f(x) = \int_0^x (x-t)\frac{6t}{1-t^3}dt$$

and hence

$$\begin{aligned} f(1) = & \int_0^1 (1-t)\frac{6t}{1-t^3}dt \\ =& 3\left[\int_0^1 \frac{2t+1}{t^2+t+1}dt - \int_0^1 \frac{1}{(t+1/2)^2+(\sqrt{3}/2)^2}dt\right], \end{aligned}$$

which immediately gives

$$\sum_{k=1}^{\infty} \frac{1}{P_k} = 3\ln 3 - \frac{\pi}{\sqrt{3}} \approx 1.4820375018. \tag{7.52}$$

- To find all *square pentagonal numbers*, we need to find integer solutions of the equation $n(3n-1)/2 = m^2$. This equation can be written as Pell's equation $b^2 - 6a^2 = 1$ (its fundamental solution is $(a, b) = (2, 5)$), where $b = 6n - 1$ and $a = 2m$. For this, corresponding to (7.22) the system is

$$\begin{aligned} m_{k+1} &= 98m_k - m_{k-1}, \quad m_1 = 1,\ m_2 = 99 \\ n_{k+1} &= 98n_k - n_{k-1} - 16, \quad n_1 = 1,\ n_2 = 81. \end{aligned} \tag{7.53}$$

This system generates all (infinite) solutions (m_k, n_k) of the equation $n(3n-1)/2 = m^2$. First few of these solutions are

$$(1, 1), (99, 81), (9701, 7921), (950599, 776161), (93149001, 76055841).$$

For $k \geq 1$, explicit solution of the system (7.53) can be written as

$$\begin{aligned} m_k &= \frac{1}{4\times 6^k}\left[(5\sqrt{6}+12)^{2k-1} - (5\sqrt{6}-12)^{2k-1}\right] \\ n_k &= \frac{1}{2\times 6^{k+1/2}}\left[[(5\sqrt{6}+12)^{2k-1} + (5\sqrt{6}-12)^{2k-1}\right] + \frac{1}{6}. \end{aligned}$$

- To find all *pentagonal numbers which are also triangular numbers*, we need to find integer solutions of the equation $n(3n-1)/2 = m(m+1)/2$. This equation can be written as Pell's equation $b^2 - 3a^2 = -2$ (its fundamental solution is $(a, b) = (3, 5)$)

where $b = 6n - 1$ and $a = 2m + 1$. For this, corresponding to (7.22) the system is

$$\begin{aligned} m_{k+1} &= 14m_k - m_{k-1} + 6, \quad m_1 = 1,\ m_2 = 20 \\ n_{k+1} &= 14n_k - n_{k-1} - 2, \quad n_1 = 1,\ n_2 = 12. \end{aligned} \tag{7.54}$$

This system generates all (infinite) solutions (m_k, n_k) of the equation $n(3n-1)/2 = m(m+1)/2$. First few of these solutions are

$$(1,1), (20,12), (285,165), (3976,2296), (55385,31977).$$

For $k \geq 1$, explicit solution of the system (7.54) can be written as

$$\begin{aligned} m_k &= \frac{1}{12}\left[(3+\sqrt{3})(2+\sqrt{3})^{2k-1} + (3-\sqrt{3})(2-\sqrt{3})^{2k-1} - 6\right] \\ n_k &= \frac{1}{12}\left[(1+\sqrt{3})(2+\sqrt{3})^{2k-1} + (1-\sqrt{3})(2-\sqrt{3})^{2k-1} + 2\right]. \end{aligned}$$

- To find all *pentagonal numbers, which are also rectangular numbers*, we need to find integer solutions of the equation $n(3n-1)/2 = m(m+1)$. This equation can be written as Pell's equation $b^2 - 6a^2 = -5$ (its fundamental solution is $(a,b) = (1,1)$) where $b = 6n - 1$ and $a = 2m + 1$. For this, corresponding to (7.22) the system is

$$\begin{aligned} m_{k+1} &= 98m_k - m_{k-1} + 48, \quad m_1 = 3,\ m_2 = 341 \\ n_{k+1} &= 98n_k - n_{k-1} - 16, \quad n_1 = 3,\ n_2 = 279. \end{aligned} \tag{7.55}$$

This system generates all (infinite) solutions (m_k, n_k) of the equation $n(3n-1)/2 = m(m+1)$. First few of these solutions are

$$(3,3), (341,279), (33463,27323), (3279081,2677359), (321316523,262353843).$$

For $k \geq 1$, explicit solution of the system (7.55) can be written as

$$\begin{aligned} m_k &= \frac{1}{24}\left[(6+\sqrt{6})(5+2\sqrt{6})^{2k-1} + (6-\sqrt{6})(5-2\sqrt{6})^{2k-1} - 12\right] \\ n_k &= \frac{1}{12}\left[(\sqrt{6}+1)(5+2\sqrt{6})^{2k-1} + (\sqrt{6}-1)(5-2\sqrt{6})^{2k-1} + 2\right]. \end{aligned}$$

7.6 Hexagonal Numbers H_n

The hexagonal numbers are defined by the sequence $1, 6, 15, 28, 45, \cdots$, i.e., beginning with 6 each number is formed from the previous one in the sequence by adding the next number in the related sequence $5, 9, 13, 17, 21, \cdots, (4n-3)$. Thus, $6 = 1 + 5,\ 15 = 1 + 5 + 9 = 6 + 9,\ 28 = 1 + 5 + 9 + 13 = 15 + 13,$ and so on (see Fig. 7.9).

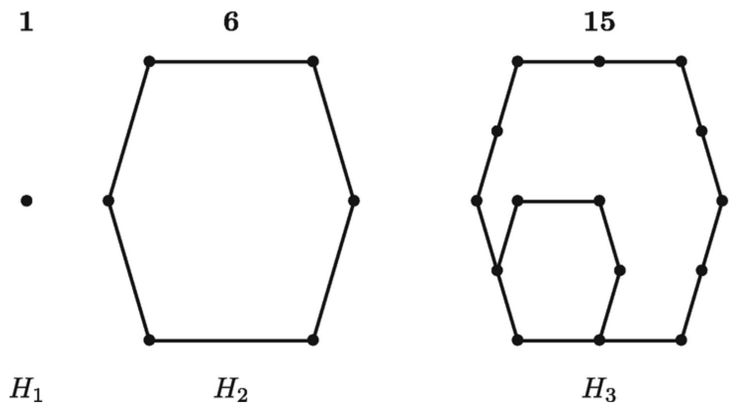

Fig. 7.9 Hexagonal numbers

Thus, *n*th hexagonal number is defined as

$$H_n = H_{n-1} + (4n-3) = 1 + 5 + 9 + 13 + \cdots + (4n-3). \tag{7.56}$$

Comparing (7.56) with (7.3), we have $a = 1, d = 4$ and hence from (7.4) it follows that

$$H_n = \frac{n}{2}(4n-2) = \frac{(2n-1)(2n)}{2} = n(2n-1). \tag{7.57}$$

- From (7.57) it is clear that $H_n = t_{2n-1}$, i.e., alternating triangular numbers are hexagonal numbers.
- Let *m* be a given natural number, then it is *n*th hexagonal number, i.e., $m = H_n$ if and only if $n = (1 + \sqrt{1+8m})/4$.
- As in (7.38), we have

$$\sum_{n=0}^{\infty} H_n x^n = 2\frac{x(1+x)}{(1-x)^3} - \frac{x}{(1-x)^2} = \frac{x(3x+1)}{(1-x)^3}$$

and hence $x(3x+1)(1-x)^{-3}$ is the *generating function* of all hexagonal numbers.

- From (7.2), (7.11), and (7.57) it is easy to find the sum of the first *n* hexagonal numbers
$$\sum_{k=1}^{n} H_k = \frac{1}{6}n(n+1)(4n-1). \tag{7.58}$$

- To find the sum of the reciprocals of all hexagonal numbers, as for pentagonal numbers we begin with the series $f(x) = \sum_{k=1}^{\infty} x^{2n}/[n(2n-1)]$, and get
$$f(1) = \sum_{k=1}^{\infty} \frac{1}{H_k} = 2\int_0^1 \frac{1-t}{1-t^2}dt = 2\ln 2 \ \approx\ 1.3862943611. \tag{7.59}$$

- To find all *square hexagonal numbers*, we need to find integer solutions of the equation $n(2n-1) = m^2$. This equation can be written as Pell's equation $b^2 - 2a^2 = 1$ (its fundamental solution is $(a,b) = (2,3)$), where $b = 4n-1$ and $a = 2m$. For this, corresponding to (7.22) the system is
$$\begin{array}{rcl} m_{k+1} &=& 34m_k - m_{k-1}, \quad m_1 = 1,\ m_2 = 35 \\ n_{k+1} &=& 34n_k - n_{k-1} - 8, \quad n_1 = 1,\ n_2 = 25. \end{array} \tag{7.60}$$

This system generates all (infinite) solutions (m_k, n_k) of the equation $n(2n-1) = m^2$. First few of these solutions are
$$(1,1), (35,25), (1189,841), (40391,28561), (1372105,970225).$$

For $k \geq 1$, explicit solution of the system (7.60) appears as
$$\begin{array}{rcl} m_k = a_{2k-1}b_{2k-1} &=& \frac{1}{4\sqrt{2}}\left[(3+2\sqrt{2})^{2k-1} - (3-2\sqrt{2})^{2k-1}\right] \\ n_k = a_{2k-1}^2 &=& \frac{1}{8}\left[(3+2\sqrt{2})^{2k-1} + (3-2\sqrt{2})^{2k-1} + 2\right]; \end{array}$$

here, a_n and b_n are as in (7.25).

- To find all *hexagonal numbers which are also rectangular numbers*, we need to find integer solutions of the equation $n(2n-1) = m(m+1)$.

This equation can be written as Pell's equation $b^2 - 2a^2 = -1$ (its fundamental solution is $(a,b) = (1,1)$) where $b = 4n - 1$ and $a = 2m + 1$. For this, corresponding to (7.22) the system is

$$\begin{aligned} m_{k+1} &= 34m_k - m_{k-1} + 16, \quad m_1 = 2, \ m_2 = 84 \\ n_{k+1} &= 34n_k - n_{k-1} - 8, \quad n_1 = 2, \ n_2 = 60. \end{aligned} \tag{7.61}$$

This system generates all (infinite) solutions (m_k, n_k) of the equation $n(2n-1) = m(m+1)$. First few of these solutions are

$$(2,2), (84,60), (2870,2030), (97512,68952), (3312554,2342330).$$

For $k \geq 1$, explicit solution of the system (7.61) can be written as

$$\begin{aligned} m_k &= \frac{\sqrt{2}}{8}\left[(\sqrt{2}+1)^{4k-1} + (\sqrt{2}-1)^{4k-1} - 2\sqrt{2}\right] \\ n_k &= \frac{1}{8}\left[(\sqrt{2}+1)^{4k-1} - (\sqrt{2}-1)^{4k-1} + 2\right]. \end{aligned}$$

- To find all *hexagonal numbers which are also pentagonal numbers*, we need to find integer solutions of the equation $n(2n-1) = m(3m-1)/2$. This equation can be written as Pell's equation $b^2 - 3a^2 = -2$ (its fundamental solution is $(a,b) = (1,1)$) where $b = 6m - 1$ and $a = 4n - 1$. For this, corresponding to (7.22) the system is

$$\begin{aligned} m_{k+1} &= 194m_k - m_{k-1} - 32, \quad m_1 = 1, \ m_2 = 165 \\ n_{k+1} &= 194n_k - n_{k-1} - 48, \quad n_1 = 1, \ n_2 = 143. \end{aligned} \tag{7.62}$$

This system generates all (infinite) solutions (m_k, n_k) of the equation $n(2n-1) = m(3m-1)/2$. First few of these solutions are

$$(1,1), (165,143), (31977,27693), (6203341,5372251), (1203416145,1042188953).$$

For $k \geq 1$, explicit solution of the system (7.62) can be written as

$$\begin{aligned} m_k &= \frac{1}{12}\left[(\sqrt{3}-1)(2+\sqrt{3})^{4k-2} - (\sqrt{3}+1)(2-\sqrt{3})^{4k-2} + 2\right] \\ n_k &= \frac{1}{24}\left[(9+5\sqrt{3})(2+\sqrt{3})^{4k-4} + (9-5\sqrt{3})(2-\sqrt{3})^{4k-4} + 6\right]. \end{aligned}$$

7.7 Generalized Pentagonal Numbers (Centered Hexagonal Numbers, Hex Numbers) $(GP)_n$

The generalized pentagonal numbers are defined by the sequence $1, 7, 19, 37, 61, \cdots$, i.e., beginning with 7 each number is formed from the previous one in the sequence by adding the next number in the related sequence $6, 12, 18, \cdots, 6(n-1)$. Thus, $7 = 1 + 6,\ 19 = 1 + 6 + 12 = 7 + 12,\ 37 = 1 + 6 + 12 + 18 = 19 + 18,$ and so on (see Fig. 7.10). These numbers are also called centered hexagonal numbers as these represent hexagons with a dot in the center and all other dots surrounding the center dot in a hexagonal lattice. These numbers have practical applications in materials logistics management, for example, in packing round items into larger round containers, such as Vienna sausages into round cans, or combining individual wire strands into a cable.

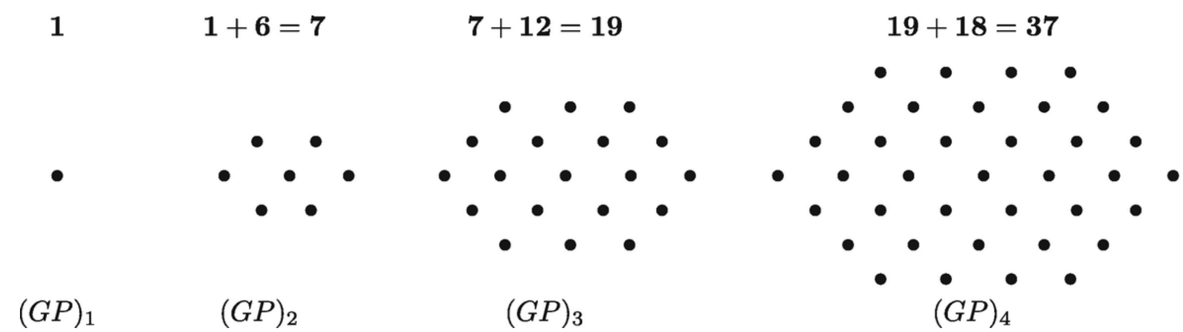

Fig. 7.10 Generalized pentagonal numbers (centered hexagonal numbers)

Thus, *n*th generalized pentagonal number is defined as

$$(GP)_n = (GP)_{n-1} + 6(n-1) = 1 + 6 + 12 + \cdots + 6(n-1) = 1 + 6[1 + 2 + \cdots + (n-1)]. \quad (7.63)$$

Hence, from (7.2) it follows that

$$(GP)_n = 1 + 6\frac{(n-1)n}{2} = 1 + 3n(n-1) = t_1 + 6t_{n-1} = t_n + 4t_{n-1} + t_{n-2}. \quad (7.64)$$

- Incidentally, $(GP)_2 = 7$ occurs in uds (uniform data system) baryon octet, whereas $(GP)_5 = 61$ makes a part of a Chinese checkers board.

- Since $1+3n(n-1) = n^3-(n-1)^3$, generalized pentagonal numbers are differences of two consecutive cubes, so that the $(GP)_n$ are the gnomon of the cubes.
- Clearly, $(2n-1)^2-(GP)_n^2 = n(n-1) = R_{n-1} = 2t_{n-1}$.
- Let *m* be a given natural number, then it is *n*th generalized pentagonal number, i.e., $m = (GP)_n$ if and only if $n = (3+\sqrt{12m-3})/6$.
- From (7.10) and (7.64), we have

$$\sum_{n=0}^{\infty}(GP)_n x^n = \frac{x}{1-x}+\frac{6x^2}{(1-x)^3} = \frac{x(1+4x+x^2)}{(1-x)^3}$$

  and hence $x(1+4x+x^2)(1-x)^{-3}$ is the *generating function* of all generalized pentagonal numbers.
- From (7.15) and (7.64) it is easy to find the sum of the first *n* generalized pentagonal numbers

$$\sum_{k=1}^{n}(GP)_k = n+6\sum_{k=1}^{n}t_{k-1} = n+6\sum_{k=1}^{n-1}t_k = n+(n-1)n(n+1) = n^3. \quad (7.65)$$

- Since from (7.32) and (7.43), we have

$$t_n^2-t_{n-1}^2 = (t_n+t_{n-1})(t_n-t_{n-1}) = n^3$$

  from (7.65) it follows that

$$\sum_{k=1}^{n}(GP)_k = t_n^2-t_{n-1}^2 = n^3. \quad (7.66)$$

  Thus, the equation $c^2 = a^3+b^2$ has an infinite number of integer solutions. In fact, for each $n \geq 1$ equations $c^2 = a^2+b^n$ and $c^2 = a^n+b^2$ have infinite number of solutions (see P36).
- To find the sum of the reciprocals of all generalized pentagonal numbers, we need the following well-known result, e.g., see Andrews et. al. [30], page 536, and Efthimiou [177]

$$\frac{1}{s}+2\sum_{k=1}^{\infty}\frac{1}{k^2+s} = \frac{\pi}{\sqrt{s}}\frac{1+e^{-2\pi\sqrt{s}}}{1-e^{-2\pi\sqrt{s}}}. \quad (7.67)$$

Now from (7.67), we have

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{(GP)_k} &= \sum_{k=1}^{\infty}\frac{1}{3k^2-3k+1} = \frac{4}{3}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2+1/3} \\ &= \frac{4}{3}\left[\sum_{k=1}^{\infty}\frac{1}{k^2+1/3} - \frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^2+1/12}\right] \\ &= \frac{4}{3}\left[\frac{1}{2}\left(\pi\sqrt{3}\frac{1+e^{-2\pi/\sqrt{3}}}{1-e^{-2\pi/\sqrt{3}}}-3\right)-\frac{1}{8}\left(2\pi\sqrt{3}\frac{1+e^{-\pi/\sqrt{3}}}{1-e^{-\pi/\sqrt{3}}}-12\right)\right] \\ &= \frac{\pi}{\sqrt{3}(1-e^{-\pi/\sqrt{3}})}\left[2\frac{1+e^{-2\pi/\sqrt{3}}}{1+e^{-\pi/\sqrt{3}}}-(1+e^{-\pi/\sqrt{3}})\right] \\ &= \tfrac{\pi}{\sqrt{3}}\tfrac{1-e^{-\pi/\sqrt{3}}}{1+e^{-\pi/\sqrt{3}}}\end{aligned}$$

and hence,

$$\sum_{k=1}^{\infty}\frac{1}{(GP)_k} = \frac{\pi}{\sqrt{3}}\tanh\frac{\pi}{2\sqrt{3}} \approx 1.3052841530. \tag{7.68}$$

- To find all *square generalized pentagonal numbers*, we need to find integer solutions of the equation $1+3n(n-1)=m^2$. This equation can be written as Pell's equation $b^2-3a^2=1$ (its fundamental solution is $(a,b)=(1,2)$), where $b=2m$ and $a=2n-1$. For this, corresponding to (7.22) the system is

$$\begin{aligned} m_{k+1} &= 14m_k - m_{k-1}, \quad m_1=1,\ m_2=13 \\ n_{k+1} &= 14n_k - n_{k-1} - 6, \quad n_1=1,\ n_2=8. \end{aligned} \tag{7.69}$$

This system generates all (infinite) solutions (m_k, n_k) of the equation $1+3n(n-1)=m^2$. First few of these solutions are

$$(1,1), (13,8), (181,105), (2521,1456), (35113,20273).$$

For $k \geq 1$, explicit solution of the system (7.69) appears as

$$\begin{aligned} m_k &= \frac{1}{4}\left[(2+\sqrt{3})^{2k-1}+(2-\sqrt{3})^{2k-1}\right] \\ n_k &= \frac{\sqrt{3}}{12}\left[(2+\sqrt{3})^{2k-1}-(2-\sqrt{3})^{2k-1}\right]+\frac{1}{2}. \end{aligned}$$

- To find all *generalized pentagonal numbers which are also triangular numbers*, we need to find integer solutions of the equation $1+3n(n-1)=m(m+1)/2.$ This equation can be written as Pell's equation $b^2-6a^2=3$ (its fundamental solution is $(a,b)=(1,3)$), where $b=2m+1$ and $a=2n-1.$ For this, corresponding to (7.22) the system is

$$\begin{aligned} m_{k+1} &= 10m_k-m_{k-1}+4, \quad m_1=1,\ m_2=13 \\ n_{k+1} &= 10n_k-n_{k-1}-4, \quad n_1=1,\ n_2=6. \end{aligned} \tag{7.70}$$

  This system generates all (infinite) solutions (m_k,n_k) of the equation $1+3n(n-1)=m(m+1)/2.$ First few of these solutions are

$$(1,1),(13,6),(133,55),(1321,540),(13081,5341).$$

  For $k\geq 1,$ explicit solution of the system (7.70) can be written as

$$\begin{aligned} m_k &= \frac{1}{4}\left[(3+\sqrt{6})(5+2\sqrt{6})^{k-1}+(3-\sqrt{6})(5-2\sqrt{6})^{k-1}-2\right] \\ n_k &= \frac{1}{8}\left[(2+\sqrt{6})(5+2\sqrt{6})^{k-1}+(2-\sqrt{6})(5-2\sqrt{6})^{k-1}+4\right]. \end{aligned}$$

- There is no generalized pentagonal number, which is also a rectangular number, in fact, the equation $1+3n(n-1)=m(m+1)$ has no solutions. For this, we note that this equation can be written as Pell's equation $b^2-3a^2=2,$ where $b=2m+1$ and $a=2n-1.$ Now reducing this equation to $\pmod 3$ gives $b^2=2 \pmod 3$, which is impossible since all squares $\pmod 3$ are either 0 or 1 $\pmod 3$.
- To find all *generalized pentagonal numbers which are also pentagonal numbers*, we need to find integer solutions of the equation $1+3n(n-1)=m(3m-1)/2.$ This equation can also be written as Pell's equation $b^2-18a^2=7$ (its fundamental solution is $(a,b)=(1,5)$), where $b=6m-1$ and $a=2n-1.$ For this, corresponding to (7.22) the system is

$$\begin{aligned} m_{k+1} &= 1154m_k-m_{k-1}-192, \quad m_1=1,\ m_2=889 \\ n_{k+1} &= 1154n_k-n_{k-1}-576, \quad n_1=1,\ n_2=629. \end{aligned} \tag{7.71}$$

This system generates all (infinite) solutions (m_k, n_k) of the equation $1 + 3n(n-1) = m(3m-1)/2$. First few of these solutions are

$$(1,1), (889,629), (1025713,725289), (1183671721,836982301),\\ (1365956140129,965876849489).$$

For $k \geq 1$, explicit solution of the system (7.71) can be written as

$$\begin{aligned} m_k &= \frac{1}{10404}\Big[(378879 - 267903\sqrt{2})(577+408\sqrt{2})^k \\ &\quad + (378879 + 267903\sqrt{2})(577-408\sqrt{2})^k + 1734\Big] \\ n_k &= \frac{1}{6936}\Big[(126293\sqrt{2} - 178602)(577+408\sqrt{2})^k \\ &\quad - (126293\sqrt{2} + 178602)(577-408\sqrt{2})^k + 3468\Big]. \end{aligned}$$

- To find all *generalized pentagonal numbers, which are also hexagonal numbers*, we need to find integer solutions of the equation $1 + 3n(n-1) = m(2m-1)$. This equation can also be written as Pell's equation $b^2 - 6a^2 = 3$ (its fundamental solution is $(a,b) = (1,3)$), where $b = 4m-1$ and $a = 2n-1$. For this, corresponding to (7.22) the system is

$$\begin{aligned} m_{k+1} &= 10m_k - m_{k-1} - 2, \quad m_1 = 1,\ m_2 = 7 \\ n_{k+1} &= 10n_k - n_{k-1} - 4, \quad n_1 = 1,\ n_2 = 6. \end{aligned} \tag{7.72}$$

This system generates all (infinite) solutions (m_k, n_k) of the equation $1 + 3n(n-1) = m(2m-1)$. First few of these solutions are

$$(1,1), (7,6), (67,55), (661,540), (6541,5341).$$

For $k \geq 1$, explicit solution of the system (7.72) can be written as

$$\begin{aligned} m_k &= \frac{1}{8}\Big[(3+\sqrt{6})(5+2\sqrt{6})^{k-1} + (3-\sqrt{6})(5-2\sqrt{6})^{k-1} + 2\Big] \\ n_k &= \frac{1}{8}\Big[(2+\sqrt{6})(5+2\sqrt{6})^{k-1} + (2-\sqrt{6})(5-2\sqrt{6})^{k-1} + 4\Big]. \end{aligned}$$

7.8 Heptagonal Numbers (Heptagon Numbers) $(HEP)_n$

These numbers are defined by the sequence $1, 7, 18, 34, 55, 81, \cdots$, i.e., beginning with 7 each number is formed from the previous one in the sequence by adding the next number in the related sequence $6, 11, 16, 21, \cdots, (5n-4)$. Thus, $7 = 1+6,\ 18 = 1+6+11 = 7+11,\ 34 = 1+6+11+16 = 18+16,$ and so on (see Fig. 7.11).

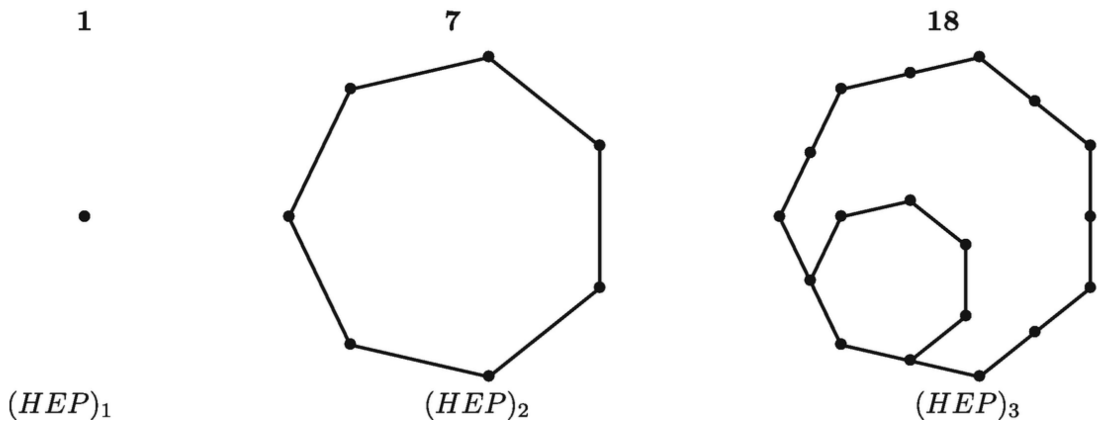

Fig. 7.11 Heptagonal numbers

Thus, *n*th heptagonal number is defined as

$$\begin{aligned}(HEP)_n &= (HEP)_{n-1} + (5n-4) = 1+6+11+16+\cdots+(5n-4)\\ &= 1+(1+5)+(1+2\times 5)+\cdots+(1+(n-1)5).\end{aligned} \tag{7.73}$$

Comparing (76) with (7.3), we have $a = 1, d = 5,$ and hence from (7.4) it follows that

$$(HEP)_n = \frac{n}{2}(5n-3) = \frac{1}{2}n[(n+1)+4(n-1)] = t_n + 4t_{n-1}. \tag{7.74}$$

- For all integers $k \geq 0$ it follows that $(HEP)_{4k+1}$ and $(HEP)_{4k+2}$ are odd, whereas $(HEP)_{4k+3}$ and $(HEP)_{4k+4}$ are even.
- From (7.74) the following equality holds

$$5(HEP)_n + 1 = 5t_n + 20t_{n-1} + 1 = \frac{(5n-2)(5n-1)}{2} = t_{5n-2}.$$

- Let *m* be a given natural number, then it is *n*th heptagonal number, i.e., $m = (HEP)_n$ if and only if $n = (3+\sqrt{9+40m})/10$.
- From (7.10) and (7.74), we have

$$\frac{x(4x+1)}{(1-x)^3} = x + 7x^2 + 18x^3 + 34x^4 + \cdots$$

and hence $x(4x+1)(1-x)^{-3}$ is the *generating function* of all heptagonal numbers.

- In view of (7.15) and (7.74), we have

$$\sum_{k=1}^{n} (HEP)_k = \frac{1}{6} n(n+1)(5n-2). \tag{7.75}$$

- The sum of reciprocals of all heptagonal numbers is (see https://en.wikipedia.org/wiki/Heptagonal_number)

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{(HEP)_k} &= \frac{1}{15}\pi\sqrt{25-10\sqrt{5}} + \frac{2}{3}\ln(5) + \frac{1+\sqrt{5}}{3}\ln\left(\frac{1}{2}\sqrt{10-2\sqrt{5}}\right) \\ &\quad + \frac{1-\sqrt{5}}{3}\ln\left(\frac{1}{2}\sqrt{10+2\sqrt{5}}\right) \\ &\approx 1.3227792531 \end{aligned} \tag{7.76}$$

- To find all *square heptagonal numbers*, we need to find integer solutions of the equation $n(5n-3)/2 = m^2$. This equation can be written as Pell's equation $b^2 - 40a^2 = 9$ (its fundamental solutions are $(a,b) = (1,7),\ (2,13)$, and $(9,57)$), where $b = 10n - 3$ and $a = m$. For $(1,7)$, corresponding to (7.22) the system is

$$\begin{aligned} m_{k+1} &= 1442m_k - m_{k-1}, \quad m_1 = 1,\ m_2 = 1519 \\ n_{k+1} &= 1442n_k - n_{k-1} - 432, \quad n_1 = 1,\ n_2 = 961. \end{aligned} \tag{7.77}$$

This system generates infinite number of solutions (m_k, n_k) of the equation $n(5n-3)/2 = m^2$. First four of these solutions are

$$(1,1), (1519, 961), (2190397, 1385329), (3158550955, 1997643025).$$

For $(2,13)$ recurrence relations remain the same as in (7.77) with $m_1 = 77, m_2 = 111035$ and $n_1 = 49, n_2 = 70225$. This leads to another set of infinite number of solutions (m_k, n_k) of the equation $n(5n-3)/2 = m^2$. First four of these solutions are

$$(77,49),(111035,70225),(160112393,101263969),$$
$$(230881959671,146022572641).$$

For $(9,57)$, also recurrence relations remain the same as in (7.77) with $m_1=9, m_2=12987$ and $n_1=6, n_2=8214.$ This leads to further set of infinite number of solutions (m_k,n_k) of the equation $n(5n-3)/2=m^2.$ First four of these solutions are

$$(9,6),(12987,8214),(18727245,11844150),(27004674303,17079255654).$$

- To find all *heptagonal numbers which are also triangular numbers*, we need to find integer solutions of the equation $n(5n-3)/2=m(m+1)/2.$ This equation can be written as Pell's equation $b^2-5a^2=4$ (its fundamental solutions are $(a,b)=(3,7)$ and $(1,3)$), where $b=10n-3$ and $a=2m+1.$ For $(3,7)$ corresponding to (7.22) the system is

$$\begin{aligned} m_{k+1} &= 322m_k-m_{k-1}+160, \quad m_1=1,\ m_2=493 \\ n_{k+1} &= 322n_k-n_{k-1}-96, \quad n_1=1,\ n_2=221. \end{aligned} \tag{7.78}$$

This system generates infinite number of solutions (m_k,n_k) of the equation $n(5n-3)/2=m(m+1)/2.$ First four of these solutions are

$$(1,1),(493,221),(158905,71065),(51167077,22882613).$$

For $(1,3)$ recurrence relations remain the same as in (7.78) with $m_1=10, m_2=3382$ and $n_1=5, n_2=1513.$ This leads to another set of infinite number of solutions (m_k,n_k) of the equation $n(5n-3)/2=m(m+1)/2.$ First four of these solutions are

$$(10,5),(3382,1513),(1089154,487085),(350704366,156839761).$$

- To find all *heptagonal numbers which are also rectangular numbers*, we need to find integer solutions of the equation $n(5n-3)/2=m(m+1).$ This equation can be written as Pell's equation $b^2-10a^2=-1$ (its fundamental solution is $(a,b)=(1,3)$), where $b=10n-3$ and $a=2m+1.$ For this, corresponding to (7.22) the system is

$$
\begin{aligned}
m_{k+1} &= 1442m_k - m_{k-1} + 720, \quad m_1 = 18,\ m_2 = 26676 \\
n_{k+1} &= 1442n_k - n_{k-1} - 432, \quad n_1 = 12,\ n_2 = 16872.
\end{aligned} \tag{7.79}
$$

This system generates infinite number of solutions (m_k, n_k) of the equation $n(5n-3)/2 = m(m+1)$. First four of these solutions are

(18, 12), (26676, 16872), (38467494, 24328980), (55470100392, 35082371856).

- To find all *heptagonal numbers, which are also pentagonal numbers*, we need to find integer solutions of the equation $n(5n-3)/2 = m(3m-1)/2$. This equation can be written as Pell's equation $b^2 - 15a^2 = 66$ (its fundamental solution is $(a, b) = (-1, 9)$), where $b = 3(10n-3)$ and $a = 6m-1$. For this, corresponding to (7.22) the system is

$$
\begin{aligned}
m_{k+1} &= 62m_k - m_{k-1} - 10, \quad m_1 = 1,\ m_2 = 54 \\
n_{k+1} &= 62n_k - n_{k-1} - 18, \quad n_1 = 1,\ n_2 = 42.
\end{aligned} \tag{7.80}
$$

This system generates infinite number of solutions (m_k, n_k) of the equation $n(5n-3)/2 = m(3m-1)/2$. First few of these solutions are

$$(1, 1), (54, 42), (3337, 2585), (206830, 160210), (12820113, 9930417).$$

- To find all *heptagonal numbers, which are also hexagonal numbers*, we need to find integer solutions of the equation $n(5n-3)/2 = m(2m-1)$. This equation can be written as Pell's equation $b^2 - 5a^2 = 4$ (its fundamental solution is $(a, b) = (-1, 3)$), where $b = 10n-3$ and $a = 4m-1$. For this, corresponding to (7.22) the system is

$$
\begin{aligned}
m_{k+1} &= 322m_k - m_{k-1} - 80, \quad m_1 = 1,\ m_2 = 247 \\
n_{k+1} &= 322n_k - n_{k-1} - 96, \quad n_1 = 1,\ n_2 = 221.
\end{aligned} \tag{7.81}
$$

This system generates infinite number of solutions (m_k, n_k) of the equation $n(5n-3)/2 = m(2m-1)$. First few of these solutions are

$$(1, 1), (247, 221), (79453, 71065), (25583539, 22882613),$$
$$(8237820025, 7368130225).$$

- To find all *heptagonal numbers, which are also generalized pentagonal numbers*, we need to find integer solutions of the equation $n(5n-3)/2 = 1+3m(m-1)$. This equation can be written as Pell's equation $b^2 - 30a^2 = 19$ (its fundamental solution is $(a,b) = (1,7)$), where $b = 10n-3$ and $a = 2m-1$. For this, corresponding to (7.22) the system is

$$\begin{aligned} m_{k+1} &= 22m_k - m_{k-1} - 10, \quad m_1 = 1,\ m_2 = 13 \\ n_{k+1} &= 22n_k - n_{k-1} - 6, \quad n_1 = 1,\ n_2 = 14. \end{aligned} \tag{7.82}$$

  This system generates infinite number of solutions (m_k, n_k) of the equation $n(5n-3)/2 = 1+3m(m-1)$. First few of these solutions are

$$(1,1), (13,14), (275,301), (6027,6602), (132309,144937).$$

7.9 Octagonal Numbers O_n

These numbers are defined by the sequence $1, 8, 21, 40, 65, 96, 133, 176, \cdots$, i.e., beginning with 8 each number is formed from the previous one in the sequence by adding the next number in the related sequence $7, 13, 19, 25, \cdots, (6n-5)$. Thus, $8 = 1+7,\ 21 = 1+7+13 = 8+13,\ 40 = 1+7+13+19 = 21+19$, and so on (see Fig. 7.12).

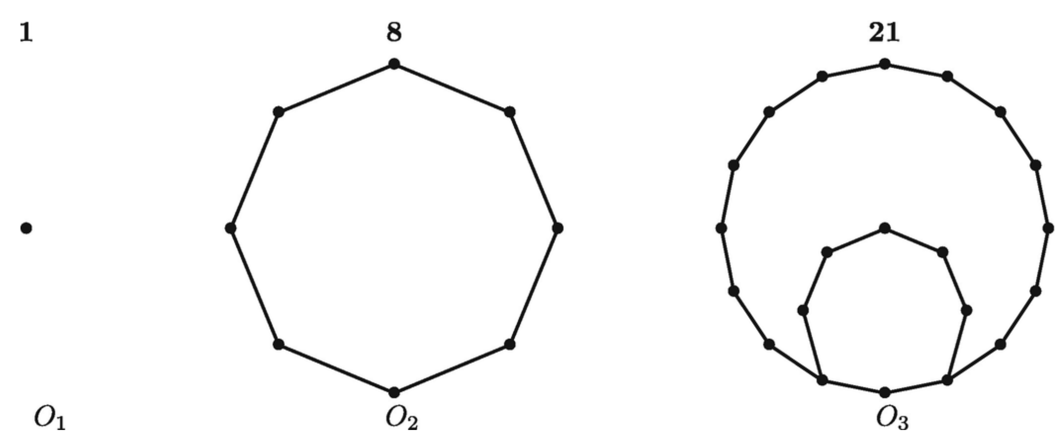

Fig. 7.12 Octagonal numbers

Thus, *n*th octagonal number is defined as

$$O_n = O_{n-1} + (6n-5) = 1 + 7 + 13 + 19 + \cdots + (6n-5)$$
$$= 1 + (1+6) + (1+2\times 6) + \cdots + (1+(n-1)6). \quad (7.83)$$

Comparing (7.83) with (7.3), we have $a = 1, d = 6$, and hence, from (7.4) it follows that

$$O_n = \frac{n}{2}(6n-4) = n(3n-2) = t_n + 5t_{n-1}. \quad (7.84)$$

- For all integers $k \geq 0$, it follows that O_{2k+1} are odd, whereas O_{2k+2} are even (in fact divisible by 4).
- Let *m* be a given natural number, then it is *n*th octagonal number, i.e., $m = O_n$ if and only if $n = (1 + \sqrt{1+3m})/3$.
- From (7.10) and (7.84), we have

$$\frac{x(5x+1)}{(1-x)^3} = x + 8x^2 + 21x^3 + 40x^4 + \cdots$$

  and hence $x(5x+1)(1-x)^{-3}$ is the *generating function* of all octagonal numbers.
- In view of (7.15) and (7.84), we have

$$\sum_{k=1}^{n} O_k = \frac{1}{2}n(n+1)(2n-1). \quad (7.85)$$

- To find the sum of the reciprocals of all octagonal numbers, following Downey [169] we begin with the series

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{k(3k-2)} x^{3k-2}$$

  and note that

$$f(1) = \sum_{k=1}^{\infty} \frac{1}{k(3k-2)} = \sum_{k=1}^{\infty} \frac{1}{O_k}, \quad f'(x) = \sum_{k=1}^{\infty} \frac{1}{k} x^{3k-3} = -\frac{\ln(1-x^3)}{x^3}.$$

  Thus, we have

$$
\begin{aligned}
f(x) &= \int_{0^+}^{x} -\frac{\ln(1-t^3)}{t^3}dt = \frac{\ln(1-x^3)}{2x^2} - \frac{3}{2}\int_{0^+}^{x}\frac{1}{t^3-1}dt \\
&= \frac{\ln(1-x^3)}{2x^2} - \frac{1}{4}\int_{0^+}^{x}\left[\frac{2}{t-1} - \frac{2t+1}{t^2+t+1} - 3\frac{1}{(t+1/2)^2+3/4}\right]dt \\
&\quad \left(\lim_{t\to 0^+}\tfrac{\ln(1-t^3)}{2t^2} = 0\right) \\
&= \frac{\ln(1+x+x^2)}{2x^2} + \frac{\ln(1-x)}{2x^2} - \frac{1}{2}\ln|x-1| + \frac{1}{4}\ln(x^2+x+1) \\
&\quad + \frac{\sqrt{3}}{2}\tan^{-1}\frac{2x+1}{\sqrt{3}} - \frac{\sqrt{3}\pi}{12}.
\end{aligned}
$$

Now since

$$
\lim_{x\to 1^-}\frac{1}{2}\ln(1-x)\left(\frac{1}{x^2}-1\right) = 0
$$

it follows that

$$
\sum_{k=1}^{\infty}\frac{1}{O_k} = \frac{3}{4}\ln 3 + \frac{\sqrt{3}}{12}\pi \;\approx\; 1.2774090576. \tag{7.86}
$$

- To find all *square octagonal numbers*, we need to find integer solutions of the equation $n(3n-2) = m^2$. This equation can be written as Pell's equation $b^2 - 3a^2 = 1$ (its fundamental solution is $(a,b) = (1,2)$), where $b = 3n-1$ and $a = m$. For this, corresponding to (7.22) the system is

$$
\begin{aligned}
m_{k+1} &= 14m_k - m_{k-1}, \quad m_1 = 1,\; m_2 = 15 \\
n_{k+1} &= 14n_k - n_{k-1} - 4, \quad n_1 = 1,\; n_2 = 9.
\end{aligned} \tag{7.87}
$$

  This system generates infinite number of solutions (m_k, n_k) of the equation $n(3n-2) = m^2$. First few of these solutions are

$$
(1,1),(15,9),(209,121),(2911,1681),(40545,23409).
$$

- To find all *octagonal numbers, which are also triangular numbers*, we need to find integer solutions of the equation $n(3n-2) = m(m+1)/2$. This equation can be written as Pell's equation $b^2 - 6a^2 = 10$ (its fundamental solutions are $(a,b) = (1,4)$

and $(3,8))$), where $b=4(3n-1)$ and $a=2m+1.$ For $(1,4)$ corresponding to (7.22) the system is

$$\begin{aligned} m_{k+1} &= 98m_k - m_{k-1} + 48, \quad m_1 = 6,\ m_2 = 638 \\ n_{k+1} &= 98n_k - n_{k-1} - 82, \quad n_1 = 3,\ n_2 = 261. \end{aligned} \tag{7.88}$$

This system generates infinite number of solutions (m_k, n_k) of the equation $n(3n-2) = m(m+1)/2.$ First few of these solutions are

$$(6,3), (638,261), (62566,25543), (6130878,2502921),$$
$$(600763526,245260683).$$

For $(3,8)$ recurrence relations remain the same as in (7.88) with $m_1 = 1, m_2 = 153$ and $n_1 = 1, n_2 = 63.$ This leads to another set of infinite number of solutions (m_k, n_k) of the equation $n(3n-2)/2 = m(m+1)/2.$ First few of these solutions are

$$(1,1), (153,63), (15041,6141), (1473913,601723), (144428481,58962681).$$

- There is no octagonal number which is also a rectangular number, in fact, the equation $n(3n-2) = m(m+1)$ has no solutions. For this, we note that this equation can be written as Pell's equation $b^2 - 3a^2 = 1$ (its fundamental solution is $(a,b) = (1,2)$), where $b = 2(3n-1)$ and $a = 2m+1.$ For this, Pell's equation all solutions can be generated by the system (corresponding to (7.21))

$$\begin{aligned} a_{k+2} &= 4a_{k+1} - a_k, \quad a_1 = 1,\ a_2 = 4 \\ b_{k+2} &= 4b_{k+1} - b_k, \quad b_1 = 2,\ b_2 = 7. \end{aligned} \tag{7.89}$$

Now an explicit solution of the second equation of (7.89) can be written as

$$b_k = \frac{1}{2}[(2+\sqrt{3})^k + (2-\sqrt{3})^k].$$

Next, if $(2+\sqrt{3})^k = s_k + t_k\sqrt{3}$, then $(2-\sqrt{3})^k = s_k - t_k\sqrt{3}$, and hence it follows that $b_k = s_k.$ We note that $s_1 \equiv 2 \pmod 6$ and $s_2 \equiv 1 \pmod 6.$ Thus, from the second equation of (7.89) mathematical induction immediately gives $s_{2\ell-1} \equiv 2 \pmod 6$ and $s_{2\ell} \equiv 1 \pmod 6$ for all $\ell \geq 1.$ In conclusion $b_k = s_k \equiv 1$ or

$2 \pmod 6$. Finally, reducing the relation $b = 2(3n-1)$ to $\pmod 6$ gives $b \equiv -2 \pmod 6$. Hence, in view of $b > 0$, we conclude that $b \neq s_k$ for all integers k, and therefore, the equation $n(3n-2) = m(m+1)$ has no solution.

- To find all *octagonal numbers which are also pentagonal numbers*, we need to find integer solutions of the equation $n(3n-2) = m(3m-1)/2$. This equation can be written as Pell's equation $b^2 - 8a^2 = -7$ (its fundamental solutions are $(a, b) = (1, 1)$ and $(2, 5)$), where $b = 6m - 1$ and $a = 3n - 1$. For $(1, 1)$ corresponding to (7.22) the system is

$$\begin{aligned} m_{k+1} &= 1154m_k - m_{k-1} - 192, \quad m_1 = 1,\ m_2 = 1025 \\ n_{k+1} &= 1154n_k - n_{k-1} - 384, \quad n_1 = 1,\ n_2 = 725. \end{aligned} \tag{7.90}$$

This system generates infinite number of solutions (m_k, n_k) of the equation $n(3n-2) = m(3m-1)/2$. First four of these solutions are

$$(1, 1), (1025, 725), (1182657, 836265), (1364784961, 965048701).$$

For $(2, 5)$, recurrence relations remain the same as in (7.90) with $m_1 = 11, m_2 = 12507$ and $n_1 = 8, n_2 = 8844$. This leads to another set of infinite number of solutions (m_k, n_k) of the equation $n(3n-2) = m(3m-1)/2$. First four of these solutions are

$$(11, 8), (12507, 8844), (14432875, 10205584)(16655525051, 11777234708).$$

- To find all *octagonal numbers which are also hexagonal numbers*, we need to find integer solutions of the equation $n(3n-2) = m(2m-1)$. This equation can be written as Pell's equation $b^2 - 6a^2 = 10$ (its fundamental solutions are $(a, b) = (1, 4)$ and $(3, 8)$), where $b = 4(3n-1)$ and $a = 4m - 1$. For $(3, 8)$ corresponding to (7.22) the system is

$$\begin{aligned} m_{k+1} &= 98m_k - m_{k-1} - 24, \quad m_1 = 1,\ m_2 = 77 \\ n_{k+1} &= 98n_k - n_{k-1} - 32, \quad n_1 = 1,\ n_2 = 63. \end{aligned} \tag{7.91}$$

This system generates infinite number of solutions (m_k, n_k) of the equation $n(3n-2) = m(2m-1)$. First few of these solutions are

$$(1,1), (77,63), (7521,6141), (736957,601723), (72214241,58962681).$$

With $(a,b)=(1,4)$, the system corresponding to (7.21) is

$$\begin{aligned} a_{k+2} &= 10a_{k+1}-a_k, \quad a_1=1,\ a_2=13 \\ b_{k+2} &= 10b_{k+1}-b_k, \quad b_1=4,\ b_2=32. \end{aligned} \tag{7.92}$$

Now note that $a_1 \equiv 1 \pmod 4$ and $a_2 \equiv 1 \pmod 4$. Thus, from the first equation of (7.92) mathematical induction immediately gives $a_{k+2} \equiv 10 \pmod 4 - 1 \pmod 4 \equiv 1 \pmod 4$ for all $k \geq 1$. Next reducing the relation $a = 4m-1$ to $\pmod 4$ gives $a \equiv -1 \pmod 4$. Hence, in view of $b>0$, we conclude that $a \neq a_k$ for all integers *k*, and therefore, the equation $n(3n-2) = m(2m-1)$ has no solution.

- To find all *octagonal numbers, which are also generalized pentagonal numbers*, we need to find integer solutions of the equation $n(3n-2) = 1+3m(m-1)$. This equation can be written as Pell's equation $b^2-a^2=7$, where $b=2(3n-1)$ and $a=3(2m-1)$. For the equation $b^2-a^2=7$ the only meaningful integer solution is $b=4, a=3$ and it gives $(m,n)=(1,1)$.
- To find all *octagonal numbers, which are also heptagonal numbers*, we need to find integer solutions of the equation $n(3n-2) = m(5m-3)/2$. This equation can be written as Pell's equation $b^2-30a^2=-39$ (its fundamental solution is $(a,b)=(2,-9)$), where $b=3(10m-3)$ and $a=2(3n-1)$. For this, corresponding to (7.22) the system is

$$\begin{aligned} m_{k+1} &= 482m_k - m_{k-1} - 144, \quad m_1=1,\ m_2=345 \\ n_{k+1} &= 482n_k - n_{k-1} - 160, \quad n_1=1,\ n_2=315. \end{aligned} \tag{7.93}$$

This system generates infinite number of solutions (m_k, n_k) of the equation $n(3n-2)=m(5m-3)/2$. First few of these solutions are

$$(1,1), (345,315), (166145,151669), (80081401,73103983), \\ (38599068993,35235967977).$$

7.10 Nonagonal Numbers N_n

These numbers are defined by the sequence $1, 9, 24, 46, 75, 111, 154, \cdots$, i.e., beginning with 9 each number is formed from the previous one in the sequence by adding the next number in the related sequence $8, 15, 22, 29, \cdots, (7n-6)$. Thus, $9 = 1+8,\ 24 = 1+8+15 = 9+15,\ 46 = 1+8+15+22 = 24+22,$ and so on (see Fig. 7.13).

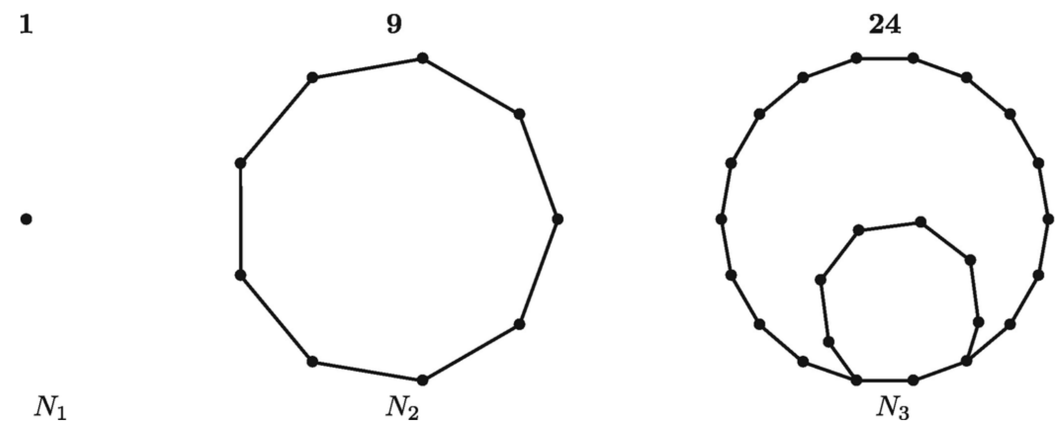

Fig. 7.13 Nonagonal numbers

Thus, *n*th nonagonal number is defined as

$$\begin{aligned} N_n = N_{n-1} + (7n-6) &= 1+8+15+22+\cdots+(7n-6) \\ &= 1+(1+7)+(1+2\times 7)+\cdots+(1+(n-1)7). \end{aligned} \tag{7.94}$$

Comparing (7.94) with (7.3), we have $a = 1, d = 7$, and hence, from (7.4) it follows that

$$N_n = \frac{n}{2}(7n-5) = t_n + 6t_{n-1}. \tag{7.95}$$

- For all integers $k \geq 0$, it follows that N_{4k+1}, N_{4k+2} are odd, whereas N_{4k+3}, N_{4k+4} are even.
- Let *m* be a given natural number, then it is *n*th nonagonal number, i.e., $m = N_n$ if and only if $n = (5 + \sqrt{25+56m})/14$.
- From (7.10) and (7.95), we have

$$\frac{x(6x+1)}{(1-x)^3} = x + 9x^2 + 24x^3 + 46x^4 + \cdots$$

and hence, $x(6x+1)(1-x)^{-3}$ is the *generating function* of all nonagonal numbers.

- In view of (7.15) and (7.95), we have

$$\sum_{k=1}^{n} N_k = \frac{1}{6}n(n+1)(7n-4). \tag{7.96}$$

- The sum of reciprocals of all nonagonal numbers is

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{N_k} = \sum_{k=1}^{\infty}\frac{2}{k(7k-5)} = \sum_{k=1}^{\infty}\left(\frac{14}{5(7k-5)}-\frac{2}{5n}\right)\\ = -\frac{2}{25}\left(5\Psi\left(-\frac{5}{7}\right)-7+5\gamma\right) \approx 1.2433209262;\end{aligned} \tag{7.97}$$

here, $\Psi(x)$ is the *digamma function* defined as the logarithmic derivative of the *gamma function* $\Gamma(x)$, i.e., $\Psi(x)=\Gamma'(x)/\Gamma(x)$, and $\gamma=0.5772156649$ is the *Euler-Mascheroni constant*, after Lorenzo Mascheroni (1750–1800, Italy), who in 1790 calculated γ to 32 (19 correct) decimal places. For several other well-known mathematics constants, see Finch [189].

- To find all *square nonagonal numbers*, we need to find integer solutions of the equation $n(7n-5)/2=m^2$. This equation can be written as Pell's equation $b^2-14a^2=25$ (its fundamental solution are $(a,b)=(2,9)$ and $(6,23)$), where $b=14n-5$ and $a=2m$. For $(2,9)$ corresponding to (7.22) the system is

$$\begin{aligned} m_{k+1} &= 30m_k - m_{k-1}, \quad m_1=1,\ m_2=33\\ n_{k+1} &= 30n_k - n_{k-1}-10, \quad n_1=1,\ n_2=18.\end{aligned} \tag{7.98}$$

This system generates infinite number of solutions (m_k,n_k) of the equation $n(7n-5)/2=m^2$. First few of these solutions are

$$(1,1),(33,18),(989,529),(29637,15842),(888121,474721).$$

For $(6,23)$ recurrence relations remain the same as in (7.98) with $m_1=3, m_2=91$ and $n_1=2, n_2=49$. This leads to another set of infinite number of solutions (m_k,n_k) of the equation $n(7n-5)/2=m^2$. First few of these solutions are

$$(3, 2), (91, 49), (2727, 1458), (81719, 43681), (2448843, 1308962).$$

- To find all *nonagonal numbers, which are also triangular numbers*, we need to find integer solutions of the equation $n(7n-5)/2 = m(m+1)/2$. This equation can be written as Pell's equation $b^2 - 7a^2 = 18$ (its fundamental solutions are $(a, b) = (1, 5), (3, 9)$ and $(7, 19)$), where $b = 14n - 5$ and $a = 2m + 1$. For $(3, 9)$ corresponding to (7.22) the system is

$$\begin{array}{rcl} m_{k+1} & = & 16m_k - m_{k-1} + 7, \quad m_1 = 1, \ m_2 = 25 \\ n_{k+1} & = & 16n_k - n_{k-1} - 5, \quad n_1 = 1, \ n_2 = 10. \end{array} \tag{7.99}$$

  This system generates infinite number of solutions (m_k, n_k) of the equation $n(7n-5)/2 = m(m+1)/2$. First few of these solutions are

$$(1, 1), (25, 10), (406, 154), (6478, 2449), (1032249, 39025).$$

  For $(1, 5)$ and $(7, 19)$ there are no integer solutions.

- There is no nonagonal number that is also a rectangular number, in fact, the equation $n(7n-5)/2 = m(m+1)$ has no solutions.
- To find all *nonagonal numbers, which are also pentagonal numbers*, we need to find integer solutions of the equation $n(7n-5)/2 = m(3m-1)/2$. This equation can be written as Pell's equation $b^2 - 21a^2 = 204$ (its fundamental solutions are $(a, b) = (5, 27)$ and $(125, 573)$), where $b = 3(14n - 5)$ and $a = 6m - 1$. For $(5, 27)$ corresponding to (7.22) the system is

$$\begin{array}{rcl} m_{k+1} & = & 12098m_k - m_{k-1} - 2016, \quad m_1 = 1, \ m_2 = 10981 \\ n_{k+1} & = & 12098n_k - n_{k-1} - 4320, \quad n_1 = 1, \ n_2 = 7189. \end{array} \tag{7.100}$$

  This system generates infinite number of solutions (m_k, n_k) of the equation $n(7n-5)/2 = m(3m-1)/2$. First four of these solutions are

$$(1, 1), (10981, 7189), (132846121, 86968201), \\ (1607172358861, 1052141284189).$$

  For $(125, 573)$ recurrence relations remain the same as in (7.100) with $m_1 = 21, m_2 = 252081$ and $n_1 = 14, n_2 = 165026$. This leads to another set of infinite number of solutions (m_k, n_k) of the equation $n(3n-2) = m(3m-1)/2$. First four of these solutions are

$$(21, 14), (252081, 165026), (3049673901, 1996480214), \\ (36894954600201, 24153417459626).$$

- To find all *nonagonal numbers, which are also hexagonal numbers*, we need to find integer solutions of the equation $n(7n-5)/2 = m(2m-1)$. This equation can be written as Pell's equation $b^2 - 7a^2 = 18$ (its fundamental solutions are $(a, b) = (1, 5), (3, 9)$ and $(7, 19)$), where $b = 14n - 5$ and $a = 4m - 1$. For $(3, 9)$ corresponding to (7.20) the system is

$$\begin{aligned} b_{k+1} &= 8b_k + 21a_k \\ a_{k+1} &= 8a_k + 3b_k, \quad b_1 = 9, \ a_1 = 3. \end{aligned}$$

  This system gives first four integer solutions (m_k, n_k) of the equation $n(7n-5)/2 = m(2m-1)$ rather easily, which appear as $(1, 1), (13, 10), (51625, 39025), (822757, 621946)$. Now the following system generates infinite number of solutions $(m_{2k+1}, n_{2k+1}), \ k \geq 2$

$$\begin{aligned} m_{2k+1} &= 64514m_{2k-1} - m_{2k-3} - 16128, \quad m_1 = 1, \ m_3 = 51625 \\ n_{2k+1} &= 64514n_{2k-1} - n_{2k-3} - 23040, \quad n_1 = 1, \ n_3 = 39025. \end{aligned} \tag{7.101}$$

  Similarly, the following system generates infinite number of solutions $(m_{2k}, n_{2k}), \ k \geq 2$

$$\begin{aligned} m_{2k+2} &= 64514m_{2k} - m_{2k-2} - 16128, \quad m_2 = 13, \ m_4 = 822757 \\ n_{2k+2} &= 64514n_{2k} - n_{2k-2} - 23040, \quad n_2 = 10, \ n_4 = 621946. \end{aligned} \tag{7.102}$$

  The first eight solutions (m_k, n_k) are

$$(1, 1), (13, 10), (51625, 39025), (822757, 621946), (3330519121, 2517635809), \\ (53079328957, 40124201194), (214865110504441, 162422756519761), \\ (3424359827493013, 2588572715184730).$$

  With $(a, b) = (1, 5)$ and $(7, 19)$, there are no integer solutions of the required equation.

- To find all *nonagonal numbers, which are also generalized pentagonal numbers*, we need to find integer solutions of the equation

$n(7n-5)/2 = 1+3m(m-1)$. This equation can be written as Pell's equation $b^2 - 42a^2 = 39$ (its fundamental solutions are $(a,b) = (1,9)$ and $(5,33)$), where $b = 14n-5$ and $a = 2m-1$. For $(1,9)$, corresponding to (7.22) the system is

$$\begin{aligned} m_{k+1} &= 674m_k - m_{k-1} - 336, \quad m_1 = 1,\ m_2 = 403 \\ n_{k+1} &= 674n_k - n_{k-1} - 240, \quad n_1 = 1,\ n_2 = 373. \end{aligned} \tag{7.103}$$

This system generates infinite number of solutions (m_k, n_k) of the equation $n(5n-3)/2 = 1+3m(m-1)$. First few of these solutions are

$$(1,1), (403,373), (271285,251161), (182845351,169281901), \\ (123237494953,114095749873).$$

For $(5,33)$ recurrence relations remain the same as in (7.103) with $m_1 = 66, m_2 = 44148$ and $n_1 = 61, n_2 = 40873$. This leads to another set of infinite number of solutions (m_k, n_k) of the equation $n(7n-5)/2 = 1+3m(m-1)$. First four of these solutions are

$$(66,61), (44148,40873), (29755350,27548101), \\ (20055061416,18567378961).$$

- To find all *nonagonal numbers, which are also heptagonal numbers*, we need to find integer solutions of the equation $n(7n-5)/2 = m(5m-3)/2$. This equation can be written as Pell's equation $b^2 - 35a^2 = 310$ (its fundamental solution is $(a,b) = (7,45)$), where $b = 5(14n-5)$ and $a = 10m-3$. For this, corresponding to (7.22) the system is

$$\begin{aligned} m_{k+1} &= 142m_k - m_{k-1} - 42, \quad m_1 = 1,\ m_2 = 104 \\ n_{k+1} &= 142n_k - n_{k-1} - 50, \quad n_1 = 1,\ n_2 = 88. \end{aligned} \tag{7.104}$$

This system generates infinite number of solutions (m_k, n_k) of the equation $n(7n-5)/2 = m(5m-3)/2$. First few of these solutions are

$$(1,1), (104,88), (14725,12445), (2090804,1767052), \\ (296879401,250908889).$$

- To find all *nonagonal numbers, which are also octagonal numbers*, we need to find integer solutions of the equation $n(7n-5)/2 = m(3m-2)$. This equation can be written as Pell's equation $b^2 - 42a^2 = 57$ (its fundamental solution is $(a,b) = (4,27)$), where $b = 3(14n-5)$ and $a = 2(3m-1)$. For this, corresponding to (7.22) the system is

$$\begin{aligned} m_{k+1} &= 674m_k - m_{k-1} - 224, \quad m_1 = 1,\ m_2 = 459 \\ n_{k+1} &= 674n_k - n_{k-1} - 240, \quad n_1 = 1,\ n_2 = 425. \end{aligned} \tag{7.105}$$

This system generates infinite number of solutions (m_k, n_k) of the equation $n(7n-5)/2 = m(3m-2)$. First few of these solutions are

$$(1,1), (459,425), (309141,286209), (208360351,192904201),$$
$$(140434567209,130017145025).$$

7.11 Decagonal Numbers D_n

These numbers are defined by the sequence $1, 10, 27, 52, 85, 126, 175, \cdots$, i.e., beginning with 10 each number is formed from the previous one in the sequence by adding the next number in the related sequence $9, 17, 25, 33, \cdots, (8n-7)$. Thus, $10 = 1+9,\ 27 = 1+9+17 = 10+17,\ 52 = 1+9+17+25 = 27+25$, and so on (see Fig. 7.14).

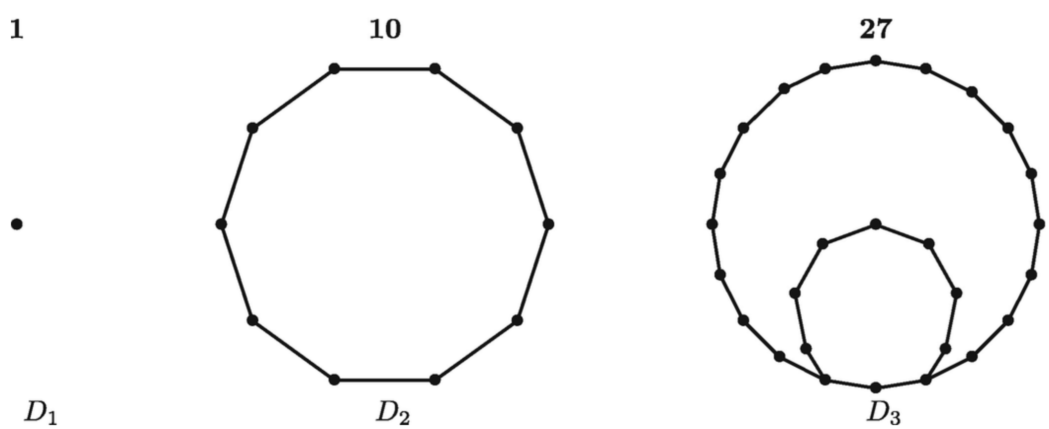

Fig. 7.14 Decagonal numbers

Hence, *n*th decagonal number is defined as

$$D_n = D_{n-1} + (8n-7) = 1 + 9 + 17 + 25 + \cdots + (8n-7)$$
$$= 1 + (1+8) + (1+2\times 8) + \cdots + (1+(n-1)8). \quad (7.106)$$

Comparing (7.106) with (7.3), we have $a = 1, d = 8$, and hence, from (7.4) it follows that

$$D_n = \frac{n}{2}(8n-6) = n(4n-3) = t_n + 7t_{n-1}. \quad (7.107)$$

- For all integers $k \geq 0$, it follows that D_{2k+1} are odd, whereas D_{2k} are even.
- Let *m* be a given natural number, then it is *n*th decagonal number, i.e., $m = D_n$ if and only if $n = (3 + \sqrt{9+16m})/8$.
- From (7.10) and (7.107), we have

  $$\frac{x(7x+1)}{(1-x)^3} = x + 10x^2 + 27x^3 + 52x^4 + \cdots$$

  and hence, $x(7x+1)(1-x)^{-3}$ is the *generating function* of all decagonal numbers.
- In view of (7.15) and (7.107), we have

  $$\sum_{k=1}^{n} D_k = \frac{1}{6}n(n+1)(8n-5). \quad (7.108)$$

- To find the sum of the reciprocals of all decagonal numbers, as in (7.86) we begin with the series

  $$f(x) = \sum_{k=1}^{\infty} \frac{1}{k(4k-3)} x^{4k-3}$$

  and following the same steps Downey [169] obtained

  $$f(1) = \sum_{k=1}^{\infty} \frac{1}{k(4k-3)} = \sum_{k=1}^{\infty} \frac{1}{D_k} = \frac{\pi}{6} + \ln 2 \approx 1.2167459562. \quad (7.109)$$

- To find all *square decagonal numbers*, we need to find integer solutions of the equation $n(4n-3) = m^2$. This equation can be written as Pell's equation $b^2 - a^2 = 9$, where $b = 8n-3$ and

$a = 4m$. For the equation $b^2 - a^2 = 9$ the only meaningful integer solution is $b = 5, a = 4$ and it gives $(m, n) = (1, 1)$.

- To find all *decagonal numbers which are also triangular numbers*, we need to find integer solutions of the equation $n(4n-3) = m(m+1)/2$. This equation can be written as Pell's equation $b^2 - 2a^2 = 7$ (its fundamental solutions are $(a, b) = (1, 3)$ and $(3, 5)$)), where $b = 8n - 3$ and $a = 2m + 1$. For $(3, 5)$ corresponding to (7.22) the system is

$$\begin{aligned} m_{k+1} &= 34m_k - m_{k-1} + 16, \quad m_1 = 1,\ m_2 = 55 \\ n_{k+1} &= 34n_k - n_{k-1} - 12, \quad n_1 = 1,\ n_2 = 20. \end{aligned} \tag{7.110}$$

This system generates infinite number of solutions (m_k, n_k) of the equation $n(4n-3) = m(m+1)/2$. First few of these solutions are

$$(1, 1), (55, 20), (1885, 667), (64051, 22646), (2175865, 769285).$$

For $(1, 3)$ recurrence relations remain the same as in (7.110) with $m_1 = 4, m_2 = 154$ and $n_1 = 2, n_2 = 55$. This leads to another set of infinite number of solutions (m_k, n_k) of the equation $n(4n-3) = m(m+1)/2$. First few of these solutions are

$$(4, 2), (154, 55), (5248, 1856), (178294, 63037)(6056764, 2141390).$$

- There is no decagonal number that is also a rectangular number, in fact, the equation $n(4n-3) = m(m+1)$ has no solutions.
- To find all *decagonal numbers, which are also pentagonal numbers*, we need to find integer solutions of the equation $n(4n-3) = m(3m-1)/2$. This equation can be written as Pell's equation $b^2 - 6a^2 = 75$ (its fundamental solution is $(a, b) = (5, 15)$), where $b = 3(8n-3)$ and $a = 6m - 1$. For $(5, 15)$ corresponding to (7.22) the system is

$$\begin{aligned} m_{k+1} &= 98m_k - m_{k-1} - 16, \quad m_1 = 1,\ m_2 = 91 \\ n_{k+1} &= 98n_k - n_{k-1} - 36, \quad n_1 = 1,\ n_2 = 56. \end{aligned} \tag{7.111}$$

This system generates infinite number of solutions (m_k, n_k) of the equation $n(4n-3) = m(3m-1)/2$. First few of these solutions are

$$(1,1),(91,56),(8901,5451),(872191,534106),(85465801,52336901).$$

- To find all *decagonal numbers, which are also hexagonal numbers*, we need to find integer solutions of the equation $n(4n-3) = m(2m-1)$. This equation can be written as Pell's equation $b^2 - 2a^2 = 7$ (its fundamental solution is $(a,b) = (3,5)$), where $b = 8n-3$ and $a = 4m-1$. For this, corresponding to (7.22) the system is

$$\begin{aligned} m_{k+1} &= 34m_k - m_{k-1} - 8, \quad m_1 = 1,\ m_2 = 28 \\ n_{k+1} &= 34n_k - n_{k-1} - 12, \quad n_1 = 1,\ n_2 = 20. \end{aligned} \tag{7.112}$$

  This system generates infinite number of solutions (m_k, n_k) of the equation $n(4n-3) = m(2m-1)$. First few of these solutions are

$$(1,1),(28,20),(943,667),(32026,22646),(1087933,769285).$$

- To find all *decagonal numbers, which are also generalized pentagonal numbers*, we need to find integer solutions of the equation $n(4n-3) = 1 + 3m(m-1)$. This equation can be written as Pell's equation $b^2 - 12a^2 = 13$ (its fundamental solution is $(a,b) = (1,5)$), where $b = 8n-3$ and $a = 2m-1$. For this, corresponding to (7.22) the system is

$$\begin{aligned} m_{k+1} &= 194m_k - m_{k-1} - 96, \quad m_1 = 1,\ m_2 = 119 \\ n_{k+1} &= 194n_k - n_{k-1} - 72, \quad n_1 = 1,\ n_2 = 103. \end{aligned} \tag{7.113}$$

  This system generates infinite number of solutions (m_k, n_k) of the equation $n(4n-3) = 1 + 3m(m-1)$. First few of these solutions are

$$(1,1),(119,103),(22989,19909),(4459651,3862171),$$
$$(865149209,749241193).$$

- To find all *decagonal numbers, which are also heptagonal numbers*, we need to find integer solutions of the equation $n(4n-3) = m(5m-3)/2$. This equation can be written as Pell's equation $b^2 - 10a^2 = 540$ (its fundamental solution is $(a,b) = (14,50)$), where $b = 10(8n-3)$ and $a = 2(10m-3)$. For this, corresponding to (7.22) the system is

$$\begin{aligned} m_{k+1} &= 1442m_k - m_{k-1} - 432, \quad m_1 = 1,\ m_2 = 1075 \\ n_{k+1} &= 1442n_k - n_{k-1} - 540, \quad n_1 = 1,\ n_2 = 850. \end{aligned} \tag{7.114}$$

This system generates infinite number of solutions (m_k, n_k) of the equation $n(4n-3) = m(5m-3)/2$. First few of these solutions are

$$(1,1), (1075, 850), (1549717, 1225159), (2234690407, 1766677888), (3222422016745, 2547548288797).$$

- To find all *decagonal numbers, which are also octagonal numbers*, we need to find integer solutions of the equation $n(4n-3) = m(3m-2)$. This equation can be written as Pell's equation $b^2 - 3a^2 = 33$ (its fundamental solution is $(a,b) = (8,15)$), where $b = 3(8n-3)$ and $a = 4(3m-1)$. For this, corresponding to (7.22) the system is

$$\begin{aligned} m_{k+1} &= 194m_k - m_{k-1} - 64, \quad m_1 = 1,\ m_2 = 135 \\ n_{k+1} &= 194n_k - n_{k-1} - 72, \quad n_1 = 1,\ n_2 = 117. \end{aligned} \tag{7.115}$$

  This system generates infinite number of solutions (m_k, n_k) of the equation $n(4n-3) = m(3m-2)$. First few of these solutions are

$$(1,1), (135, 117), (26125, 22625), (5068051, 4389061), (983175705, 851455137).$$

- To find all *decagonal numbers, which are also nonagonal numbers*, we need to find integer solutions of the equation $n(4n-3) = m(7m-5)/2$. This equation can be written as Pell's equation $b^2 - 14a^2 = 91$ (its fundamental solution is $(a,b) = (9,35)$), where $b = 7(8n-3)$ and $a = 14m-5$. For this, corresponding to (7.22) the system is

$$\begin{aligned} m_{k+1} &= 898m_k - m_{k-1} - 320, \quad m_1 = 1,\ m_2 = 589 \\ n_{k+1} &= 898n_k - n_{k-1} - 336, \quad n_1 = 1,\ n_2 = 551. \end{aligned} \tag{7.116}$$

  This system generates infinite number of solutions (m_k, n_k) of the equation $n(4n-3) = m(7m-5)/2$. First few of these solutions are

$$(1,1),(589,551),(528601,494461),(474682789,444025091),$$
$$(426264615601,398734036921).$$

7.12 Tetrakaidecagonal Numbers $(TET)_n$

These numbers are defined by the sequence $1, 14, 39, 76,\ 125, \cdots$, i.e., beginning with 14 each number is formed from the previous one in the sequence by adding the next number in the related sequence $13, 25, 37, 49, \cdots, (12n-11)$. Thus, $14 = 1+13,\ 39 = 1+13+25 = 14+25,\ 76 = 1+13+25+37 = 39+37,$ and so on (see Fig. 7.15).

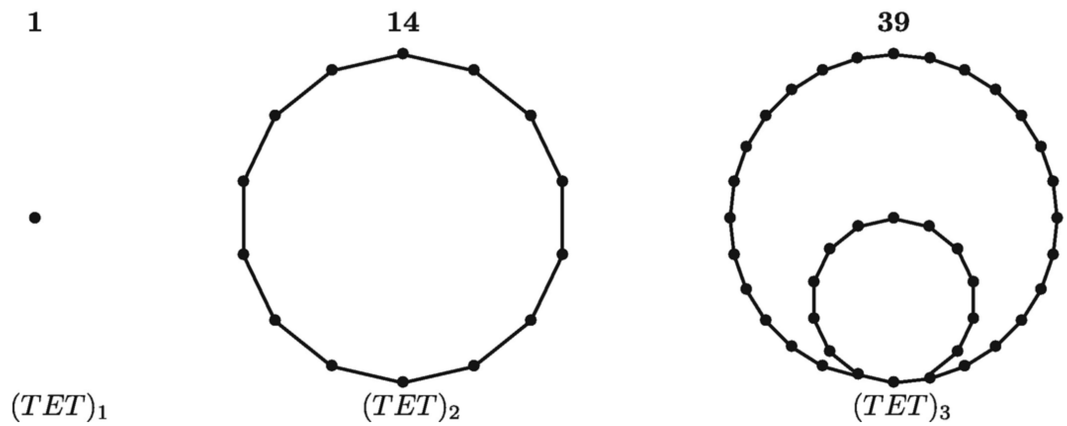

Fig. 7.15 Tetrakaidecagonal numbers

Hence, *n*th tetrakaidecagonal number is defined as

$$\begin{aligned}(TET)_n &= (TET)_{n-1} + (12n-11) = 1+13+25+37+\cdots+(12n-11)\\ &= 1+(1+12)+(1+2\times 12)+\cdots+(1+(n-1)12).\end{aligned} \tag{7.117}$$

Comparing (7.117) with (7.3), we have $a=1, d=12,$ and hence, from (7.4) it follows that

$$(TET)_n = \frac{n}{2}(12n-10) = n(6n-5) = t_n + 11t_{n-1}. \tag{7.118}$$

- For all integers $k \geq 0$, it follows that $(TET)_{2k+1}$ are odd, whereas $(TET)_{2k}$ are even.
- Let *m* be a given natural number, then it is *n*th tetrakaidecagonal number, i.e., $m = (TET)_n$ if and only if $n = (5+\sqrt{25+24m})/12.$

- From (7.10) and (7.118), we have

$$\frac{x(11x+1)}{(1-x)^3} = x + 14x^2 + 39x^3 + 76x^4 + \cdots$$

and hence $x(11x+1)(1-x)^{-3}$ is the *generating function* of all tetrakaidecagonal numbers.

- In view of (7.15) and (7.118), we have

$$\sum_{k=1}^{n} (TET)_k = \frac{1}{2}n(n+1)(4n-3). \qquad (7.119)$$

- To find the sum of the reciprocals of all tetrakaidecagonal numbers, as in (7.86) we begin with the series

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{k(6k-5)} x^{6k-5}$$

and following the same steps Downey [169] obtained

$$f(1) = \sum_{k=1}^{\infty} \frac{1}{k(6k-5)} = \sum_{k=1}^{\infty} \frac{1}{(TET)_k} = \frac{1}{10}(4\ln 2 + 3\ln 3 + \sqrt{3}\pi) \approx 1.1509823681. \qquad (7.120)$$

- To find all *square tetrakaidecagonal numbers*, we need to find integer solutions of the equation $n(6n-5) = m^2$. This equation can be written as Pell's equation $b^2 - 6a^2 = 25$ (its fundamental solution are $(a,b) = (2,7)$ and $(4,11)$), where $b = 12n - 5$ and $a = 2m$. For (2,7) corresponding to (7.22) the system is

$$\begin{aligned} m_{k+1} &= 98m_k - m_{k-1}, \quad m_1 = 1,\ m_2 = 119 \\ n_{k+1} &= 98n_k - n_{k-1} - 40, \quad n_1 = 1,\ n_2 = 49. \end{aligned} \qquad (7.121)$$

This system generates infinite number of solutions (m_k, n_k) of the equation $n(6n-5) = m^2$. First few of these solutions are

$$(1,1), (119,49), (11661,4761), (1142659,466489), (111968921,45711121).$$

For $(4,11)$ recurrence relations remain the same as in (7.121) with $m_1 = 21, m_2 = 2059$ and $n_1 = 2, n_2 = 841$. This leads to another set of infinite number of solutions (m_k, n_k) of the equation $n(6n-5) = m^2$. First few of these solutions are

$$(21, 9), (2059, 841), (201761, 82369), (19770519, 8071281),$$
$$(1937309101, 790903129).$$

- To find all *tetrakaidecagonal numbers which are also triangular numbers*, we need to find integer solutions of the equation $n(6n-5) = m(m+1)/2$. This equation can be written as Pell's equation $b^2 - 3a^2 = 22$ (its fundamental solutions are $(a, b) = (1, 5)$ and $(3, 7)$), where $b = 12n - 5$ and $a = 2m + 1$. For $(3, 7)$ corresponding to (7.22) the system is

$$\begin{aligned} m_{k+1} &= 194m_k - m_{k-1} + 96, \quad m_1 = 1, \ m_2 = 341 \\ n_{k+1} &= 194n_k - n_{k-1} - 80, \quad n_1 = 1, \ n_2 = 99. \end{aligned} \tag{7.122}$$

  This system generates infinite number of solutions (m_k, n_k) of the equation $n(6n-5) = m(m+1)/2$. First few of these solutions are

$$(1, 1), (341, 99), (66249, 19125), (12852061, 3710071),$$
$$(2493233681, 719734569).$$

  For $(1, 5)$ recurrence relations remain the same as in (7.122) with $m_1 = 50, m_2 = 9798$ and $n_1 = 15, n_2 = 2829$. This leads to another set of infinite number of solutions (m_k, n_k) of the equation $n(6n-5) = m^2$. First few of these solutions are

$$(50, 15), (9798, 2829), (1900858, 548731), (368756750, 106450905),$$
$$(71536908738, 20650926759).$$

- There is no tetrakaidecagonal number that is also a rectangular number, in fact, the equation $n(6n-5) = m(m+1)$ has no solutions.
- To find all *tetrakaidecagonal numbers, which are also pentagonal numbers*, we need to find integer solutions of the equation $n(6n-5) = m(3m-1)/2$. This equation can be written as Pell's equation $b^2 - a^2 = 24$, where $b = 12n - 5$ and $a = 6m - 1$. For the equation $b^2 - a^2 = 24$ the only meaningful integer solution is $b = 7, a = 5$ and it gives $(m, n) = (1, 1)$.

- To find all *tetrakaidecagonal numbers, which are also hexagonal numbers*, we need to find integer solutions of the equation $n(6n-5) = m(2m-1)$. This equation can be written as Pell's equation $b^2 - 3a^2 = 22$ (its fundamental solutions are $(a, b) = (1, 5)$ and $(3, 7)$), where $b = 12n - 5$ and $a = 4m - 1$. For $(3, 7)$ corresponding to (7.22) the system is

$$\begin{aligned} m_{k+1} &= 194m_k - m_{k-1} - 48, \quad m_1 = 1, \ m_2 = 171 \\ n_{k+1} &= 194n_k - n_{k-1} - 80, \quad n_1 = 1, \ n_2 = 99. \end{aligned} \tag{7.123}$$

  This system generates infinite number of solutions (m_k, n_k) of the equation $n(6n-5) = m(2m-1)$. First few of these solutions are

$$(1, 1), (171, 99), (33125, 19125), (6426031, 3710071),$$
$$(1246616841, 719734569).$$

  With $(a, b) = (1, 5)$ there are no integer solutions of the required equation.

- To find all *tetrakaidecagonal numbers, which are also generalized pentagonal numbers*, we need to find integer solutions of the equation $n(6n-5) = 1 + 3m(m-1)$. This equation can be written as Pell's equation $b^2 - 18a^2 = 31$ (its fundamental solutions are $(a, b) = (1, 7)$ and $(11, 47)$), where $b = 12n - 5$ and $a = 2m - 1$. For $(1, 7)$, corresponding to (7.22) the system is

$$\begin{aligned} m_{k+1} &= 1154m_k - m_{k-1} - 576, \quad m_1 = 1, \ m_2 = 765 \\ n_{k+1} &= 1154n_k - n_{k-1} - 480, \quad n_1 = 1, \ n_2 = 541. \end{aligned} \tag{7.124}$$

  This system generates infinite number of solutions (m_k, n_k) of the equation $n(6n-5) = 1 + 3m(m-1)$. First few of these solutions are

$$(1, 1), (765, 541), (882233, 623833), (1018095541, 719902261),$$
$$(1174881371505, 830766584881).$$

  For $(11, 47)$ recurrence relations remain the same as in (7.124) with $m_1 = 188, m_2 = 216376$ and $n_1 = 133, n_2 = 151001$. This leads to another set of infinite number of solutions (m_k, n_k) of the equation $n(6n-5) = 1 + 3m(m-1)$. First four of these solutions are

$$(188, 133), (216376, 153001), (249697140, 176562541),$$
$$(288150282608, 203753018833).$$

- To find all *tetrakaidecagonal numbers, which are also heptagonal numbers*, we need to find integer solutions of the equation $n(6n-5) = m(5m-3)/2$. This equation can be written as Pell's equation $b^2 - 15a^2 = 490$ (its fundamental solutions are $(a, b) = (3, 25)$ and $(7, 35)$), where $b = 5(12n-5)$ and $a = 10m-3$. For (7,35), corresponding to (7.22) the system is

$$\begin{aligned} m_{k+1} &= 3842m_k - m_{k-1} - 1152, \quad m_1 = 1,\ m_2 = 3081 \\ n_{k+1} &= 3842n_k - n_{k-1} - 1600, \quad n_1 = 1,\ n_2 = 1989. \end{aligned} \tag{7.125}$$

  This system generates infinite number of solutions (m_k, n_k) of the equation $n(6n-5) = m(5m-3)/2$. First few of these solutions are

$$(1, 1), (3081, 1989), (11836049, 7640137), (45474096025, 29353402765),$$
$$(174711465090849, 112775765781393).$$

  With $(a, b) = (3, 25)$ there are no integer solutions of the required equation.

- To find all *tetrakaidecagonal numbers, which are also octagonal numbers*, we need to find integer solutions of the equation $n(6n-5) = m(3m-2)$. This equation can be written as Pell's equation $b^2 - 2a^2 = 17$ (its fundamental solutions are $(a, b) = (2, 5)$ and $(4, 7)$), where $b = 12n-5$ and $a = 6m-2$. For (4,7), corresponding to (7.22) the system is

$$\begin{aligned} m_{k+1} &= 1154m_k - m_{k-1} - 384, \quad m_1 = 1,\ m_2 = 861 \\ n_{k+1} &= 1154n_k - n_{k-1} - 480, \quad n_1 = 1,\ n_2 = 609. \end{aligned} \tag{7.126}$$

  This system generates infinite number of solutions (m_k, n_k) of the equation $n(6n-5) = m(3m-2)$. First few of these solutions are

$$(1, 1), (861, 609), (993209, 702305), (1146161941, 810458881),$$
$$(1322669886321, 935268845889).$$

With $(a,b)=(2,5)$ there are no integer solutions of the required equation.

- To find all *tetrakaidecagonal numbers which are also nonagonal numbers*, we need to find integer solutions of the equation $n(6n-5)=m(7m-5)/2.$ This equation can be written as Pell's equation $b^2-21a^2=700$ (its fundamental solutions are $(a,b)=(2,28),(5,35)$ and $(9,49)$), where $b=7(12n-5)$ and $a=14m-5.$ For (9,49), corresponding to (7.22) the system is

$$\begin{aligned} m_{k+1} &= 12098m_k - m_{k-1} - 4320, \quad m_1=1,\ m_2=8509 \\ n_{k+1} &= 12098n_k - n_{k-1} - 5040, \quad n_1=1,\ n_2=6499. \end{aligned} \tag{7.127}$$

  This system generates infinite number of solutions (m_k,n_k) of the equation $n(6n-5)=m(7m-5)/2.$ First four of these solutions are

$$(1,1),(8509,6499),(102937561,78619861),\\ (1245338600149,951143066839).$$

  With $(a,b)=(2,28)$ and $(5,35)$ there are no integer solutions of the required equation.

- To find all *tetrakaidecagonal numbers, which are also decagonal numbers*, we need to find integer solutions of the equation $n(6n-5)=m(4m-3)$. This equation can be written as Pell's equation $b^2-6a^2=46$ (its fundamental solutions are $(a,b)=(3,10)$ and $(5,14)$), where $b=2(12n-5)$ and $a=8m-3.$ For (5,14), corresponding to (7.22) the system is

$$\begin{aligned} m_{k+1} &= 98m_k - m_{k-1} - 36, \quad m_1=1,\ m_2=66 \\ n_{k+1} &= 98n_k - n_{k-1} - 40, \quad n_1=1,\ n_2=54. \end{aligned} \tag{7.128}$$

  This system generates infinite number of solutions (m_k,n_k) of the equation $n(6n-5)=m(4m-3)$. First few of these solutions are

$$(1,1),(66,54),(6431,5251),(630136,514504),(61746861,50416101).$$

  With $(a,b)=(3,10)$ there are no integer solutions of the required equation.

7.13 Centered Triangular Numbers $(ct)_n$

These numbers are defined by the sequence $1, 4, 10, 19, 31, 46, 64, 85, 109, \cdots$, i.e., beginning with 4 each number is formed from the previous one in the sequence by adding the next number in the related sequence $3, 6, 9, 12, \cdots, 3(n-1)$. Thus, $4 = 1 + 3,\ 10 = 1 + 3 + 6 = 4 + 6,\ 19 = 1 + 3 + 6 + 9 = 10 + 9,$ and so on (see Fig. 7.16).

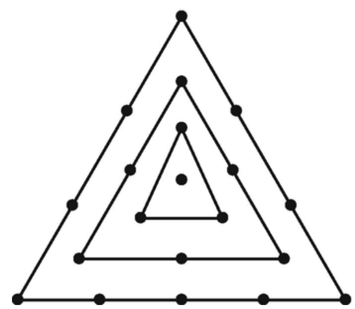

Fig. 7.16 Centered triangular numbers

Hence, *n*th centered triangular number is defined as

$$\begin{aligned}(ct)_n &= (ct)_{n-1} + 3(n-1) = 1 + 3 + 6 + 9 + \cdots + (3n-3) \\ &= 1 + 3(1 + 2 + 3 + \cdots + (n-1)).\end{aligned} \tag{7.129}$$

Thus, from (7.2) it follows that

$$(ct)_n = 1 + 3\frac{(n-1)n}{2} = 1 + 3t_{n-1} = t_n + t_{n-1} + t_{n-2}. \tag{7.130}$$

- Let *m* be a given natural number, then it is *n*th centered triangular number, i.e., $m = (ct)_n$ if and only if $n = (3 + \sqrt{9 + 24(m-1)})/6$.
- From (7.10) and (7.130), we have

  $$\frac{x(x^2 + x + 1)}{(1-x)^3} = x + 4x^2 + 10x^3 + 19x^4 + \cdots$$

  and hence, $x(x^2 + x + 1)(1-x)^{-3}$ is the *generating function* of all centered triangular numbers.
- In view of (7.15) and (7.130), we have

$$\sum_{k=1}^{n}(ct)_k = \frac{1}{2}n(n^2+1). \tag{7.131}$$

- To find the sum of the reciprocals of all centered triangular numbers, we follow as in (7.68), to obtain

$$\sum_{k=1}^{\infty}\frac{1}{(ct)_k} = \frac{2\pi}{\sqrt{15}}\tanh\left(\frac{\pi}{2}\sqrt{\frac{5}{3}}\right) \approx 1.5670651313. \tag{7.132}$$

7.14 Centered Square Numbers $(cS)_n$

These numbers are defined by the sequence $1, 5, 13, 25, 41, 61, 85, 113, \cdots$, i.e., beginning with 5 each number is formed from the previous one in the sequence by adding the next number in the related sequence $4, 8, 12, 16, \cdots, 4(n-1)$. Thus, $5 = 1+4,\ 13 = 1+4+8 = 5+8,\ 25 = 1+4+8+12 = 13+12,$ and so on (see Fig. 7.17).

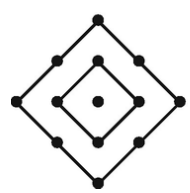

Fig. 7.17 Centered square numbers

Hence, *n*th centered square number is defined as

$$\begin{aligned}(cS)_n =&(cS)_{n-1} + 4(n-1) = 1+4+8+12+\cdots+(4n-4)\\ =&1+4(1+2+3+\cdots+(n-1)).\end{aligned} \tag{7.133}$$

Thus, from (7.2) it follows that

$$\begin{aligned}(cS)_n = 1+4\frac{(n-1)n}{2} = 1+4t_{n-1} = 1+2n^2-2n = n^2+(n-1)^2\\ = S_n + S_{n-1} = t_n + 2t_{n-1} + t_{n-2}.\end{aligned} \tag{7.134}$$

- Let *m* be a given natural number, then it is *n*th centered square number, i.e., $m = (cS)_n$ if and only if $n = (1+\sqrt{2m-1})/2$.

- From (7.10) and (7.134), we have

$$\frac{x(x+1)^2}{(1-x)^3} = x + 5x^2 + 13x^3 + 25x^4 + \cdots$$

  and hence $x(x+1)^2(1-x)^{-3}$ is the *generating function* of all centered square numbers.
- In view of (7.15) and (7.134), we have

$$\sum_{k=1}^{n} (cS)_k = \frac{1}{3} n(2n^2+1). \quad (7.135)$$

- To find the sum of the reciprocals of all centered triangular numbers, we follow as in (7.68), to obtain

$$\sum_{k=1}^{\infty} \frac{1}{(cS)_k} = \frac{\pi}{2} \tanh\left(\frac{\pi}{2}\right) \approx 1.44065952. \quad (7.136)$$

7.15 Centered Pentagonal Numbers $(cP)_n$

These numbers are defined by the sequence $1, 6, 16, 31, 51, 76, 106, 141, 181, \cdots$, i.e., beginning with 6 each number is formed from the previous one in the sequence by adding the next number in the related sequence $5, 10, 15, 20, \cdots, 5(n-1)$. Thus, $6 = 1+5, \; 16 = 1+5+10 = 6+10, \; 31 = 1+5+10+15 = 16+15$, and so on (see Fig. 7.18).

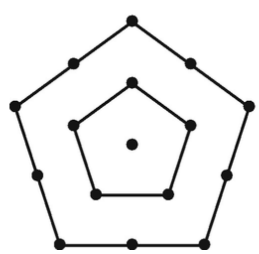

Fig. 7.18 Centered pentagonal numbers

Hence, *n*th centered pentagonal number is defined as

$$(cP)_n = (cP)_{n-1} + 5(n-1) = 1 + 5 + 10 + 15 + \cdots + (5n-5) \\ = 1 + 5(1 + 2 + 3 + \cdots + (n-1)). \tag{7.137}$$

Thus, from (7.2) it follows that

$$(cP)_n = 1 + 5\frac{(n-1)n}{2} = 1 + 5t_{n-1} = t_n + 3t_{n-1} + t_{n-2}. \tag{7.138}$$

- Let *m* be a given natural number, then it is *n*th centered pentagonal number, i.e., $m = (cP)_n$ if and only if $n = (5 + \sqrt{25 + 40(m-1)})/10.$
- From (7.10) and (7.138), we have

  $$\frac{x(x^2 + 3x + 1)}{(1-x)^3} = x + 6x^2 + 16x^3 + 31x^4 + \cdots$$

  and hence $x(x^2 + 3x + 1)(1-x)^{-3}$ is the *generating function* of all centered pentagonal numbers.
- In view of (7.15) and (7.138), we have

  $$\sum_{k=1}^{n} (cP)_k = \frac{1}{6}n(5n^2 + 1). \tag{7.139}$$

- To find the sum of the reciprocals of all centered pentagonal numbers we follow as in (7.68), to obtain

  $$\sum_{k=1}^{\infty} \frac{1}{(cP)_k} = \frac{2\pi}{\sqrt{15}}\tanh\left(\frac{\pi}{2}\sqrt{\frac{3}{5}}\right) \approx 1.36061317. \tag{7.140}$$

7.16 Centered Heptagonal Numbers $(cHEP)_n$

These numbers are defined by the sequence $1, 8, 22, 43,$ $71, 106, 148, 197, 253, \cdots$, i.e., beginning with 8 each number is formed from the previous one in the sequence by adding the next number in the related sequence $7, 14, 21, 28, \cdots, 7(n-1)$. Thus, $8 = 1 + 7,\ \ 22 = 1 + 7 + 14 = 8 + 14,\ \ 43 = 1 + 7 + 14 + 21 = 22 + 21,$ and so on (see Fig. 7.19).

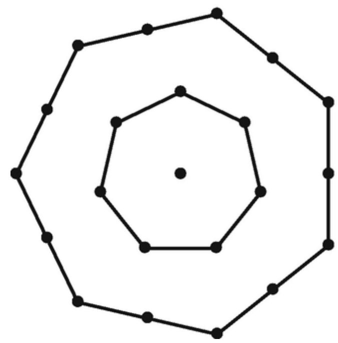

Fig. 7.19 Centered heptagonal numbers

Hence, *n*th centered heptagonal number is defined as

$$\begin{aligned}(cHEP)_n = (cHEP)_{n-1} + 7(n-1) &= 1 + 7 + 14 + 21 + \cdots + (7n-7) \\ &= 1 + 7(1 + 2 + 3 + \cdots + (n-1)).\end{aligned} \tag{7.141}$$

Thus, from (7.2) it follows that

$$(cHEP)_n = 1 + 7\frac{(n-1)n}{2} = 1 + 7t_{n-1} = t_n + 5t_{n-1} + t_{n-2}. \tag{7.142}$$

- Let *m* be a given natural number, then it is *n*th centered heptagonal number, i.e., $m = (cHEP)_n$ if and only if $n = (7 + \sqrt{49 + 56(m-1)})/14$.
- From (7.10) and (7.142), we have

  $$\frac{x(x^2+5x+1)}{(1-x)^3} = x + 8x^2 + 22x^3 + 43x^4 + \cdots$$

  and hence $x(x^2+5x+1)(1-x)^{-3}$ is the *generating function* of all centered heptagonal numbers.
- In view of (7.15) and (7.142), we have

  $$\sum_{k=1}^{n} (cHEP)_k = \frac{1}{6}n(7n^2-1). \tag{7.143}$$

- To find the sum of the reciprocals of all centered heptagonal numbers, we follow as in (7.68), to obtain

  $$\sum_{k=1}^{\infty} \frac{1}{(cHEP)_k} = \frac{2\pi}{\sqrt{7}}\tanh\left(\frac{\pi}{2\sqrt{7}}\right) \approx 1.264723172. \tag{7.144}$$

7.17 Centered Octagonal Numbers $(cO)_n$

These numbers are defined by the sequence $1, 9, 25, 49, 81, 121, 169, 225, 289, 361, \cdots$, i.e., beginning with 9 each number is formed from the previous one in the sequence by adding the next number in the related sequence $8, 16, 24, 32, \cdots, 8(n-1)$. Thus, $9 = 1+8,\ 25 = 1+8+14 = 9+16,\ 49 = 1+8+16+24 = 25+24$, and so on (see Fig. 7.20).

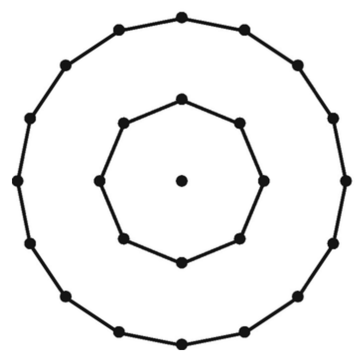

Fig. 7.20 Centered octagonal numbers

Hence, *n*th centered octagonal number is defined as

$$\begin{aligned}(cO)_n = (cO)_{n-1} + 8(n-1) &= 1+8+16+24+\cdots+(8n-8)\\ &= 1+8(1+2+3+\cdots+(n-1)).\end{aligned} \tag{7.145}$$

Thus, from (7.2) it follows that

$$(cO)_n = 1 + 8\frac{(n-1)n}{2} = 1 + 8t_{n-1} = (2n-1)^2 = S_{2n-1} = t_n + 6t_{n-1} + t_{n-2}. \tag{7.146}$$

Hence, the centered octagonal numbers are the same as the odd square numbers.

- Let *m* be a given natural number, then it is *n*th centered octagonal number, i.e., $m = (cO)_n$ if and only if $n = (1+\sqrt{m})/2$.
- From (7.10) and (7.146), we have

$$\frac{x(x^2+6x+1)}{(1-x)^3} = x + 9x^2 + 25x^3 + 49x^4 + \cdots$$

and hence $x(x^2+6x+1)(1-x)^{-3}$ is the *generating function* of all centered octagonal numbers.

- In view of (7.15) and (7.146), we have

$$\sum_{k=1}^{n}(cO)_k = \frac{1}{3}n(4n^2-1). \tag{7.147}$$

- To find the sum of the reciprocals of all centered octagonal numbers we use (5.12), to obtain

$$\sum_{k=1}^{\infty}\frac{1}{(cO)_k} = \sum_{k=1}^{\infty}\frac{1}{(2k-1)^2} = \sum_{k=1}^{\infty}\frac{1}{k^2} - \frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{8} \approx 1.2337005501. \tag{7.148}$$

7.18 Centered Nonagonal Numbers $(cN)_n$

These numbers are defined by the sequence $1, 10, 28,$ $55, 91, 136, 190, 253, 325, \cdots$, i.e., beginning with 10 each number is formed from the previous one in the sequence by adding the next number in the related sequence $9, 18, 27, 36, \cdots, 9(n-1)$. Thus, $10 = 1+9,\ 28 = 1+9+18 = 10+18,\ 55 = 1+9+18+27 = 28+27,$ and so on (see Fig. 7.21).

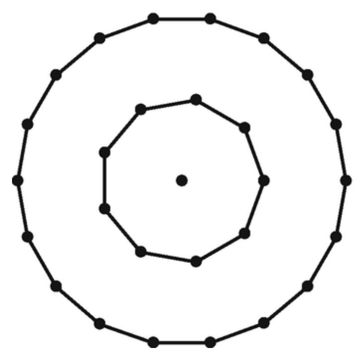

Fig. 7.21 Centered nonagonal numbers

Hence, *n*th centered nonagonal number is defined as

$$\begin{aligned}(cN)_n = (cN)_{n-1} + 9(n-1) &= 1+9+18+27+\cdots+9(n-1)\\ &= 1+9(1+2+3+\cdots+(n-1)).\end{aligned} \tag{7.149}$$

Thus, from (7.2) it follows that

$$(cN)_n = 1 + 9\frac{(n-1)n}{2} = 1 + 9t_{n-1} = \frac{(3n-2)(3n-1)}{2} \qquad (7.150)$$
$$= t_{3n-2} = t_n + 7t_{n-1} + t_{n-2}.$$

- In 1850, Frederick Pollock (1783–1870, England) conjectured that every natural number is the sum of at most 11 centered nonagonal numbers, which has not been proved.
- Let *m* be a given natural number, then it is *n*th centered nonagonal number, i.e., $m = (cN)_n$ if and only if $n = (9 + \sqrt{81 + 72(m-1)})/18.$
- From (7.10) and (7.150), we have
$$\frac{x(x^2+7x+1)}{(1-x)^3} = x + 10x^2 + 28x^3 + 55x^4 + \cdots$$
and hence $x(x^2+7x+1)(1-x)^{-3}$ is the *generating function* of all centered nonagonal numbers.
- In view of (7.15) and (7.150), we have
$$\sum_{k=1}^{n}(cN)_k = \frac{1}{2}n(3n^2-1). \qquad (7.151)$$
- To find the sum of the reciprocals of all centered heptagonal numbers, we follow as in (7.68), to obtain
$$\sum_{k=1}^{\infty}\frac{1}{(cN)_k} = \frac{2\pi}{3}\tan\left(\frac{\pi}{6}\right) = \frac{2\sqrt{3}\pi}{9} \approx 1.2091995762. \qquad (7.152)$$

7.19 Centered Decagonal Numbers $(cD)_n$

These numbers are defined by the sequence $1, 11, 31,$ $61, 101, 151, 211, 281, \cdots$, i.e., beginning with 11 each number is formed from the previous one in the sequence by adding the next number in the related sequence $10, 20, 30, 40, \cdots, 10(n-1).$ Thus, $11 = 1 + 10,\ 31 = 1 + 10 + 20 = 11 + 20,\ 61 = 1 + 10 + 20 + 30 = 31 + 30,$ and so on (see Fig. 7.22).

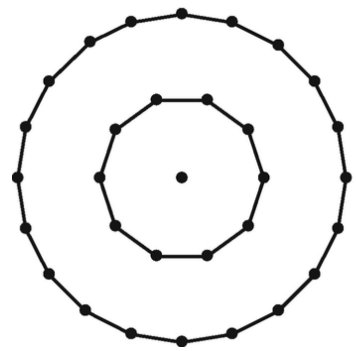

Fig. 7.22 Centered decagonal numbers

Hence, nth centered decagonal number is defined as

$$(cD)_n = (cD)_{n-1} + 10(n-1) = 1 + 10 + 20 + 30 + \cdots + 10(n-1) \\ = 1 + 10(1 + 2 + 3 + \cdots + (n-1)). \tag{7.153}$$

Thus, from (7.2) it follows that

$$(cD)_n = 1 + 10\frac{(n-1)n}{2} = 1 + 10t_{n-1} = 5n^2 - 5n + 1 = t_n + 8t_{n-1} + t_{n-2}. \tag{7.154}$$

- For each $(cD)_n$ the last digit is 1.
- Let m be a given natural number, then it is nth centered decagonal number, i.e., $m = (cD)_n$ if and only if $n = (5 + \sqrt{25 + 20(m-1)})/10$.
- From (7.10) and (7.154), we have

$$\frac{x(x^2 + 8x + 1)}{(1-x)^3} = x + 11x^2 + 31x^3 + 61x^4 + \cdots$$

  and hence $x(x^2 + 8x + 1)(1-x)^{-3}$ is the *generating function* of all centered decagonal numbers.
- In view of (7.15) and (7.154), we have

$$\sum_{k=1}^{n} (cD)_k = \frac{1}{3}n(5n^2 - 2). \tag{7.155}$$

- To find the sum of the reciprocals of all centered heptagonal numbers, we follow as in (7.68), to obtain

$$\sum_{k=1}^{\infty} \frac{1}{(cD)_k} = \frac{\pi}{\sqrt{5}} \tan\left(\frac{\pi}{2\sqrt{5}}\right) \approx 1.189356247. \tag{7.156}$$

7.20 Star Numbers $(ST)_n$

These numbers are defined by the sequence $1, 13, 37, 73, 121, 181, 253, 337, \cdots,$ i.e., beginning with 13 each number is formed from the previous one in the sequence by adding the next number in the related sequence $12, 24, 36, 48, \cdots, 12(n-1).$ Thus, $13 = 1 + 12,\ 37 = 1 + 12 + 24 = 13 + 24,\ 73 = 1 + 12 + 24 + 36 = 37 + 36,$ and so on (see Fig. 7.23).

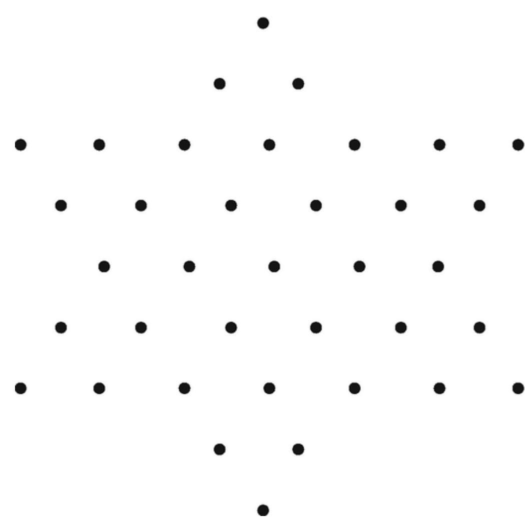

Fig. 7.23 Star number $(ST)_3$

Hence, *n*th star number is defined as

$$\begin{aligned}(ST)_n = (ST)_{n-1} + 12(n-1) &= 1 + 12 + 24 + 36 + \cdots + 12(n-1) \\ &= 1 + 12(1 + 2 + 3 + \cdots + (n-1)).\end{aligned} \tag{7.157}$$

Thus, from (7.2) it follows that

$$(ST)_n = 1 + 12\frac{(n-1)n}{2} = 1 + 12t_{n-1} = 6n^2 - 6n + 1 = t_n + 10t_{n-1} + t_{n-2}. \tag{7.158}$$

- All star numbers are odd. The star number $(ST)_{77} = 35113$ is unique, since its prime factors $13, 37,$ 73 are also consecutive star numbers.

There are infinite number of star numbers, which are also triangular numbers, also square numbers.

- Let *m* be a given natural number, then it is *n*th star number, i.e., $m = (ST)_n$ if and only if $n = (3 + \sqrt{9 + 6(m-1)})/6$.
- From (7.10) and (7.158), we have

$$\frac{x(x^2+10x+1)}{(1-x)^3} = x + 13x^2 + 37x^3 + 73x^4 + \cdots$$

and hence $x(x^2+10x+1)(1-x)^{-3}$ is the *generating function* of all star numbers.

- In view of (7.15) and (7.158), we have

$$\sum_{k=1}^{n} (ST)_k = n(2n^2 - 1). \tag{7.159}$$

- To find the sum of the reciprocals of all-star numbers, we follow as in (7.68), to obtain

$$\sum_{k=1}^{\infty} \frac{1}{(ST)_k} = \frac{\pi}{2\sqrt{3}} \tan\left(\frac{\pi}{2\sqrt{3}}\right) \approx 1.15917332. \tag{7.160}$$

7.21 Centered Tetrakaidecagonal Numbers $(cTET)_n$

These numbers are defined by the sequence $1, 15, 43, 85, 141, \cdots$, i.e., beginning with 15 each number is formed from the previous one in the sequence by adding the next number in the related sequence $14, 28, 42, 56, \cdots, 14(n-1)$. Thus, $15 = 1 + 14,\ 43 = 1 + 14 + 28 = 15 + 28,\ 85 = 1 + 14 + 28 + 42 = 43 + 42,$ and so on (see Fig. 7.24).

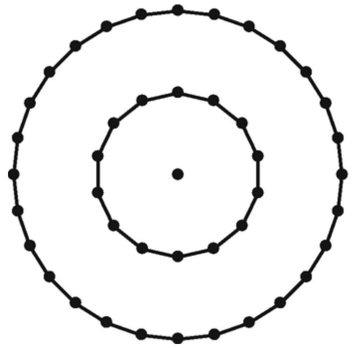

Fig. 7.24 Centered tetrakaidecagonal numbers

Hence, *n*th centered tetrakaidecagonal number is defined as

$$\begin{aligned}(cTET)_n = (cTET)_{n-1} + 14(n-1) &= 1 + 14 + 28 + 42 + \cdots + 14(n-1) \\ &= 1 + 14(1 + 2 + 3 + \cdots + (n-1)).\end{aligned} \quad (7.161)$$

Thus, from (7.2) it follows that

$$(cTET)_n = 1 + 14\frac{(n-1)n}{2} = 1 + 14t_{n-1} = 7n^2 - 7n + 1 = t_n + 12t_{n-1} + t_{n-2}. \quad (7.162)$$

- Each $(cTET)_n$ is odd.
- Let *m* be a given natural number, then it is *n*th centered tetrakaidecagonal number, i.e., $m = (cTET)_n$ if and only if $n = (7 + \sqrt{49 + 28(m-1)})/14.$
- From (7.10) and (7.162), we have

$$\frac{x(x^2 + 12x + 1)}{(1-x)^3} = x + 15x^2 + 43x^3 + 85x^4 + \cdots$$

  and hence $x(x^2 + 12x + 1)(1-x)^{-3}$ is the *generating function* of all centered tetrakaidecagonal numbers.
- In view of (7.15) and (7.162), we have

$$\sum_{k=1}^{n} (cTET)_k = \frac{1}{3}n(7n^2 - 4). \quad (7.163)$$

- To find the sum of the reciprocals of all centered tetrakaidecagonal numbers we follow as in (7.68), to obtain

$$\sum_{k=1}^{\infty} \frac{1}{(cTET)_k} = \frac{\pi}{\sqrt{21}} \tan\left(\frac{\pi}{2}\sqrt{\frac{3}{7}}\right) \approx 1.1372969963. \tag{7.164}$$

7.22 Cubic Numbers C_n

A cubic number can be written as a product of three equal factors of natural numbers. Thus, $1, 8, 27, 64, \cdots, n^3$ are first few cubic numbers (see Fig. 7.25).

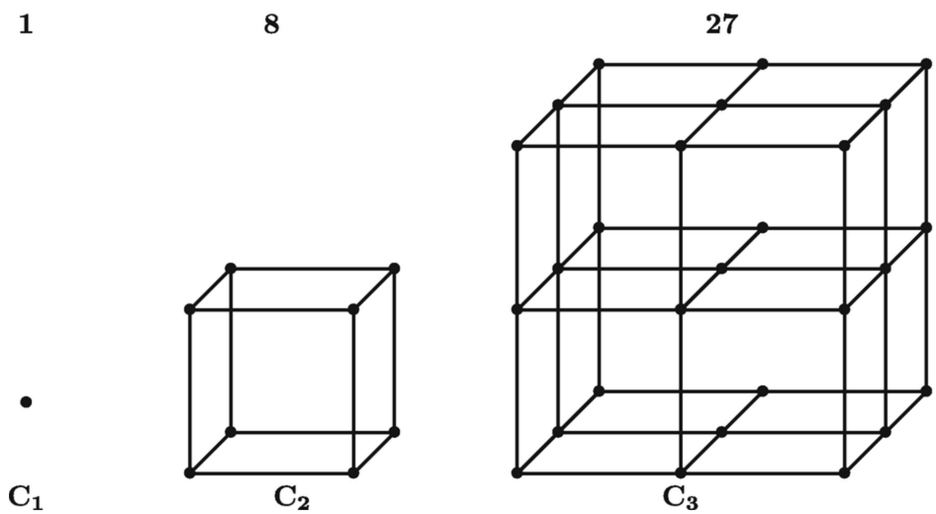

Fig. 7.25 Cubic numbers

- Last digit of a number is the same as the last digit of its cube, except that 2 becomes 8 (and 8 becomes 2) and 3 becomes 7 (and 7 becomes 3).
- Nicomachus considered the following infinite triangle of odd numbers. It is clear that the sum of the numbers in the *n*th row is n^3.

$$\begin{array}{ccccccccccccc}
 & & & & & & & & & 1 & & 1^3\\
 & & & & & & & & 3 & 5 & & 2^3\\
 & & & & & & & 7 & 9 & 11 & & 3^3\\
 & & & & & & 13 & 15 & 17 & 19 & & 4^3\\
 & & & & & 21 & 23 & 25 & 27 & 29 & & 5^3\\
 & & & & 31 & 33 & 35 & 37 & 39 & 41 & & 6^3\\
 & & & 43 & 45 & 47 & 49 & 51 & 53 & 55 & & 7^3\\
 & & 57 & 59 & 61 & 63 & 65 & 67 & 69 & 71 & & 8^3\\
 & 73 & 75 & 77 & 79 & 81 & 83 & 85 & 87 & 89 & & 9^3\\
91 & 93 & 95 & 97 & 99 & 101 & 103 & 105 & 107 & 109 & & 10^3\\
. & . & . & . & . & . & . & . & . & . & . & .
\end{array}$$

In the literature often the aforementioned representation is referred to as Pascal's triangle. Now noting that numbers in each row are consecutive odd, so the general term in view of (7.31) can be written as

$$\begin{aligned}&[k(k-1)+1]+[k(k-1)+3]+[k(k-1)+5]+\cdots+[k(k-1)+(2k-1)]\\ &\qquad = k\times k(k-1)+[1+3+5+\cdots+(2k-1)] = k^3-k^2+k^2 = k^3.\end{aligned}$$

Taking successively $k = 1, 2, 3, \cdots, n$ in the aforementioned relation, adding these *n* equations, and observing that $n(n-1)+(2n-1) = 2n(n+1)/2-1,$ we find

$$1+3+5+7+9+11\cdots+\left[2\frac{n(n+1)}{2}-1\right] = 1^3+2^3+3^3+\cdots+n^3,$$

i.e., the number of terms in the left side are $n(n+1)/2.$ Now again from (7.31) we find that the left-hand side is the same as $[n(n+1)/2]^2$, and hence, it follows that

$$1^3+2^3+3^3+\cdots+n^3 = \left(\frac{n(n+1)}{2}\right)^2 = \frac{1}{4}n^4+\frac{1}{2}n^3+\frac{1}{4}n^2 = t_n^2, \quad (7.165)$$

i.e., a perfect square number. This identity is sometimes called Nicomachus's theorem.

- From (7.66), the relation (7.165) follows immediately. In fact, we have

$$\sum_{k=1}^{n} k^3 = \sum_{k=1}^{n}(t_k^2 - t_{k-1}^2) = t_n^2.$$

- The *generating function* for all cubic numbers is

$$\frac{x(x^2+4x+1)}{(1-x)^4} = x + 8x^2 + 27x^3 + \cdots .$$

- From the relations $(k+1)^2 - k^2 = 2k+1$ and $(k+1)^2 - (k-1)^2 = 4k$, it follows that all odd and multiple of 4 integers can be expressed as the difference of two squares. However, $4k+2$ cannot be expressed as the difference of two squares. Indeed, if $a^2 - b^2 = (a+b)(a-b) = 4k+2,$ then letting $x = a+b,\ y = a-b$ gives $a = (x+y)/2,\ b = (x-y)/2,$ which implies that both *x* and *y* must be of the same parity. If both *x* and *y* are odd (even), then *xy* is odd (multiple of 4), hence in either case, we have a contradiction. Now since cube of any odd (even) integer is odd (multiple of 4), we can conclude that every cube is a difference of two squares. In fact, we have

$$n^3 = \begin{cases} \left(\dfrac{n^3+1}{2}\right)^2 - \left(\dfrac{n^3-1}{2}\right)^2, & n \ \text{odd} \\ \left(\dfrac{n^3+4}{4}\right)^2 - \left(\dfrac{n^3-4}{4}\right)^2, & n \ \text{even.} \end{cases} \quad (7.166)$$

Thus for *n* odd, squares are the difference of consecutive integers, whereas for *n* even, squares are the difference of consecutive odd integers. From (7.165) we also have

$$n^3 = t_n^2 - t_{n-1}^2 = \left(\frac{n(n+1)}{2}\right)^2 - \left(\frac{(n-1)n}{2}\right)^2, \quad n \geq 2. \quad (7.167)$$

Thus, squares are the difference of the integer $n.$ From (7.165), we also have

$$n^3 + (n+1)^3 = t_{n+1}^2 - t_{n-1}^2 = \left(\frac{(n+1)(n+2)}{2}\right)^2 - \left(\frac{(n-1)n}{2}\right)^2, \quad n \geq 2. \quad (7.168)$$

Thus squares are the difference of the integer n. A simple computation from (7.166), (7.167), and (7.168), respectively gives

$$\begin{cases} 3^3 = 14^2 - 13^2,\ 4^3 = 17^2 - 15^2,\ 5^3 = 63^2 - 62^2,\ 6^3 = 55^2 - 53^2, \\ 7^3 = 172^2 - 171^2,\ 8^3 = 129^2 - 127^2,\ 9^3 = 365^2 - 364^2, \\ 10^3 = 251^2 - 249^2 \end{cases}$$

$$\begin{cases} 3^3 = 6^2 - 3^2,\ 4^3 = 10^2 - 6^2,\ 5^3 = 15^2 - 10^2,\ 6^3 = 21^2 - 15^2, \\ 7^3 = 28^2 - 21^2,\ 8^3 = 36^2 - 28^2,\ 9^3 = 45^2 - 36^2,\ 10^3 = 55^2 - 45^2 \end{cases}$$

$$\begin{cases} 2^3 + 3^3 = 6^2 - 1^2,\ 3^3 + 4^3 = 10^2 - 3^2,\ 4^3 + 5^3 = 15^2 - 6^2, \\ 5^3 + 6^3 = 21^2 - 10^2 \\ 6^3 + 7^3 = 28^2 - 15^2,\ 7^3 + 8^3 = 36^2 - 21^2,\ 8^3 + 9^3 = 45^2 - 28^2, \\ 9^3 + 10^3 = 55^2 - 36^2 \end{cases}$$

Clearly, in the conclusion (7.166) appropriately cube can be replaced by any power greater than 3. For example, $5^7 = 39063^2 - 39062^2$ and $6^7 = 69985^2 - 69983^2$.

- There are infinite number of *square cubic numbers*, in fact, $(k^2)^3 = (k^3)^2,\ k = 1, 2, \cdots$.
- Related with cubic numbers there are *centered cubic numbers* $(cC)_n = n^3 + (n-1)^3 = (2n-1)(n^2 - n + 1)$. Thus, $(cC)_n$ is the count of number of points in a body-centered cubic pattern within a cube that has $n+1$ points along each of its edges. First few centered cubic numbers are $1, 9, 35, 91, 189, 341$. Clearly, no centered cubic number is prime. Further, the only centered cube number that is also a square number is 9. The *generating function* for all centered cube numbers is

$$\frac{x(x^3 + 5x^2 + 5x + 1)}{(1-x)^4} = x + 9x^2 + 35x^3 + 91x^4 + \cdots .$$

From (7.165) it follows that

$$\sum_{k=1}^{n} (cC)_k = \frac{1}{2} n^2 (n^2 + 1). \tag{7.169}$$

Further, from (5.12) and (7.67), we find

$$\sum_{k=1}^{\infty} \frac{1}{(cC)_k} = \sum_{k=1}^{\infty} \frac{2}{k^2(k^2+1)} = 2\sum_{k=1}^{\infty} \left(\frac{1}{k^2} - \frac{1}{k^2+1}\right) = \frac{\pi^2}{3} + 1 - \pi \coth \pi \approx 1.1365200388. \quad (7.170)$$

7.23 Tetrahedral Numbers (Triangular Pyramidal Numbers) T_n

These numbers count the number of dots in pyramids built up of triangular numbers. If the base is the triangle of side *n*, then the pyramid is formed by placing similarly situated triangles upon it, each of which has one less in its sides than that which precedes it (see Fig. 7.26).

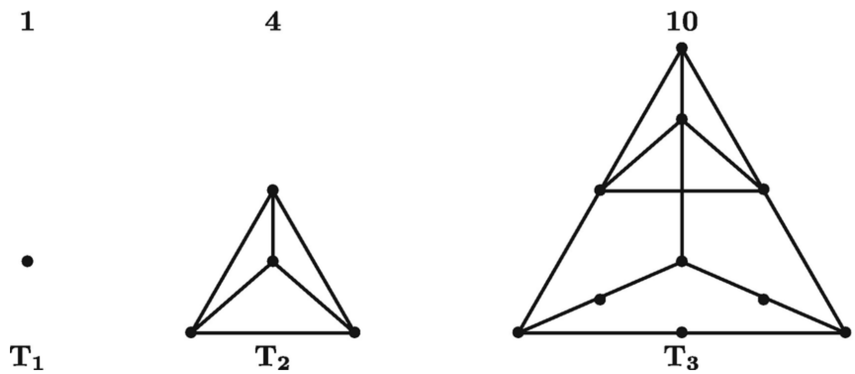

Fig. 7.26 Tetrahedral numbers

In general, the *n*th tetrahedral number T_n is given in terms of the sum of the first *n* triangular numbers, i.e.,

$$T_n = T_{n-1} + t_n = t_1 + t_2 + t_3 + \cdots + t_n,$$

which in view of (7.15) is the same as

$$T_n = \frac{n(n+1)(n+2)}{6} = \frac{(n+2)}{3} t_n. \quad (7.171)$$

Thus, first few tetrahedral numbers are $1, 4, 10, 20, 35, 56, 84, 120, 165.$

- In 1850, Pollock conjectured that every natural number is the sum of at most five tetrahedral numbers, which has not been proved. Tetrahedral numbers are even, except for $T_{4n+1},\ n = 0, 1, 2, \cdots$, which are odd, see Conway and Guy [138]. The only numbers that are simultaneously square and tetrahedral are $T_1 = 1,\ T_2 = 4$, and $T_{48} = 19600$, see Meyl [366].

- The *generating function* for all tetrahedral numbers is

$$\frac{x}{(1-x)^4} = x + 4x^2 + 10x^3 + 20x^4 + \cdots .$$

- From (7.2), (7.11), and (7.165) it follows that

$$\sum_{k=1}^{n} T_k = \frac{1}{6}\sum_{k=1}^{n}(k^3+3k^2+2k) = \frac{1}{24}n(n^3+6n^2+11n+6). \quad (7.172)$$

- To find the sum of the reciprocals of all tetrahedral numbers we follow as in (7.16) and (7.17), to obtain

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{6}{k(k+1)(k+2)} &= \lim_{n\to\infty}\left[3\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)-3\sum_{k=1}^{n}\left(\frac{1}{k+1}-\frac{1}{k+2}\right)\right]\\ &= \lim_{n\to\infty}\left[3\left(1-\frac{1}{n+1}\right)-3\left(\frac{1}{2}-\frac{1}{n+2}\right)\right] = \frac{3}{2}.\end{aligned} \quad (7.173)$$

- As in (7.173), we find

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{6(-1)^{k-1}}{k(k+1)(k+2)} &= 3\sum_{k=1}^{\infty}(-1)^{k-1}\left(\frac{1}{k}-\frac{1}{k+1}\right)\\ &\quad -3\textstyle\sum_{k=1}^{\infty}(-1)^{k-1}\left(\frac{1}{k+1}-\frac{1}{k+2}\right)\\ &= 3\left(1+2\sum_{k=2}^{\infty}\frac{(-1)^{k-1}}{k}\right)-3\left(\frac{1}{2}-2\sum_{k=3}^{\infty}\frac{(-1)^{k-1}}{k}\right)\\ &= 3(1+2\ln 2-2)-3\left(\frac{1}{2}-2\ln 2+1\right)\\ &= 12\ln 2-\tfrac{15}{2} \approx 0.8177661667.\end{aligned} \quad (7.174)$$

- The numbers $T_{3n-2} - 4T_{n-1} = n(23n^2 - 27n + 10)/6$ are called *truncated tetrahedral numbers* and denoted as $(TT)_n$. These numbers are assembled by removing the $(n-1)$th tetrahedral number from each of the four corners from the $(3n-2)$th tetrahedral number. First few of these numbers are $1, 16, 68, 180, 375, 676, 1106$. The *generating function* for all truncated tetrahedral numbers is

$$\frac{x(10x^2+12x+1)}{(1-x)^4} = x \; + 16x^2 + 68x^3 + 180x^4 + \cdots .$$

- The tetrahedral numbers and triangular numbers can be connected to produce the *magic numbers* 2, 8, 20, 28, 50, 82, 126. The magic numbers 2, 8, and 20 are twice the tetrahedral numbers 1, 4, and 10, respectively, while the magic numbers 28, 50, 82, 126 are twice the difference (tetrahedralnumber-triangularnumber), i.e., $2 \cdot (20-6), 2 \cdot (35-10), 2 \cdot (56-15), 2 \cdot (84-21),$ respectively. In physics, magic numbers occur in the shell models of both atomic and nuclear structure. The magic numbers for atoms are 2, 10, 18, 36, 54, and 86, corresponding to the total number of electrons in filled electron shells. The nucleus of an atom that has even numbers of protons and neutrons is found more stable than one with odd numbers of protons and neutrons. If a nucleus has the neutron number $=$ the proton number $=$ one of the magic numbers such as 2, 8, 20, 28, 50, 82, or 126, then it (the nucleus) is specifically found to be stable. This proton number or, equivalently, the neutron number is called the double magic number.

7.24 Square Pyramidal Numbers $(SP)_n$

These numbers count the number of dots in pyramids built up of square numbers. First few square pyramidal numbers are $1, 5, 14, 30, 55, 91, 140.$ In general, the *n*th square pyramidal number $(SP)_n$ is given in terms of the sum of the first *n* square numbers, i.e.,

$$(SP)_n = (SP)_{n-1} + n^2 = 1^2 + 2^2 + \cdots + n^2,$$

which in view of (7.11) and (7.171) is the same as

$$(SP)_n = \frac{n(n+1)(2n+1)}{6} = \frac{2n(2n+2)(2n+1)}{4 \times 6} = \frac{1}{4}T_{2n} = \frac{1}{6}(n+1)t_{2n}. \quad (7.175)$$

- In 1918, Watson proved that besides $1,$ there is only one other number that is both a square and a pyramid number, $4900,$ (as conjectured by Lucas in 1875), the 70th square number and the 24th square pyramidal number, i.e., $S_{70} = (SP)_{24}.$
- The *generating function* for all square pyramidal numbers is

$$\frac{x(x+1)}{(1-x)^4} = x + 5x^2 + 14x^3 + 30x^4 + \cdots .$$

- From (7.2), (7.11), and (7.165) it follows that

$$\sum_{k=1}^{n} (SP)_k = \frac{1}{6}\sum_{k=1}^{n}(2k^3 + 3k^2 + k) = \frac{1}{12} n(n+1)^2(n+2). \tag{7.176}$$

- To find the sum of the reciprocals of all square pyramidal numbers, we note that

$$\begin{aligned}
\sum_{k=1}^{n} \frac{6}{k(k+1)(2k+1)} &= 12\sum_{k=1}^{n}\left(\frac{1}{2k} + \frac{1}{2(k+1)} - \frac{2}{2k+1}\right) \\
&= 12\textstyle\sum_{k=1}^{n}\int_0^1 \left(x^{2k-1} + x^{2k+1} - 2x^{2k}\right) dx \\
&= 12\displaystyle\int_0^1 x(1-x)^2 \left(\sum_{k=0}^{n-1} x^{2k}\right) dx \\
&= 12\textstyle\int_0^1 x(1-x)^2 \frac{1-x^{2n}}{1-x^2} dx \\
&= 12\displaystyle\int_0^1 \frac{x(1-x)}{1+x}(1-x^{2n}) dx.
\end{aligned}$$

Now since

$$\int_0^1 \frac{x(1-x)}{1+x} x^{2n} dx < \int_0^1 x^{2n} dx = \frac{1}{2n+1} \quad \to \quad 0 \quad \text{as} \quad n \to \infty$$

it follows that

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{1}{(SP)_k} &= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{6}{k(k+1)(2k+1)} = 12\int_0^1 \frac{x(1-x)}{1+x} dx \\
&= 12\int_0^1 \left(2 - x - \frac{2}{1+x}\right) dx = 6(3 - 4\ln 2) \approx 1.364467667.
\end{aligned} \tag{7.177}$$

- In 2006, Fearnehough [185] used Nilakanthan's series

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots , \tag{7.178}$$

known in the literature after James Gregory and Leibniz in the aforementioned partial fractions, to obtain

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{6(-1)^{k-1}}{k(k+1)(2k+1)} &= 6\left[\left(\frac{1}{1}+\frac{1}{2}-\frac{4}{3}\right)-\left(\frac{1}{2}+\frac{1}{3}-\frac{4}{5}\right)+\left(\frac{1}{3}+\frac{1}{4}-\frac{4}{7}\right)\right.\\ &\quad \left.-\left(\tfrac{1}{4}+\tfrac{1}{5}-\tfrac{4}{9}\right)+\cdots\right]\\ &= 6\left[1+4\left(-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\cdots\right)\right]\\ &= 6\left[1+4\left(\tfrac{\pi}{4}-1\right)\right]=6(\pi-3).\end{aligned}$$

- The sum of two consecutive square pyramidal numbers, i.e., $(SP)_n+(SP)_{n-1}=n(2n^2+1)/3,\ n\geq 1$ are called *octahedral numbers*, and denoted as $(OH)_n$. The first few octahedral numbers are $1,6,19,44,85,146,\ 231,344.$ These numbers represent the number of spheres in an octahedral formed from close-packed spheres. Descartes initiated the study of octahedral numbers around 1630. In 1850, Pollock conjectured that every positive integer is the sum of at most seven octahedral numbers, which for finitely many numbers have been proved by Brady [101]. The difference between two consecutive octahedral numbers is a centered square number, i.e., $(OH)_n-(OH)_{n-1}=n^2+(n-1)^2=(cS)_n$. The *generating function* for all octahedral numbers is

$$\frac{x(x+1)^2}{(1-x)^4}=x+6x^2+19x^3+44x^4+\cdots\ .$$

From (7.2) and (7.165) it follows that

$$\sum_{k=1}^{n}(OH)_k=\frac{1}{6}n(n+1)(n^2+n+1). \qquad (7.179)$$

Further, as in (7.97), we have

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{(OH)_k} &= \frac{3}{2}\left(2\gamma+\Psi\left(\frac{1}{2}(2-i\sqrt{2})\right)+\Psi\left(\frac{1}{2}(2+i\sqrt{2})\right)\right) \qquad (7.180)\\ &\approx 1.2781850979.\end{aligned}$$

- The sum of two consecutive octahedral numbers, i.e., $(OH)_n + (OH)_{n-1} = (2n-1)(2n^2-2n+3)/3$ is called *centered octahedral number* or *Haüy octahedral numbers* (named after René Just Haüy, 1743–1822, France) and denoted as $(cOH)_n$. The first few centered octahedral numbers are $1, 7, 25, 63, 129, 231, 377.$ The *generating function* for all centered octahedral numbers is

$$\frac{x(x+1)^3}{(1-x)^4} = x + 7x^2 + 25x^3 + 63x^4 + \cdots .$$

  As earlier, we have

$$\sum_{k=1}^{n} (cOH)_k = \frac{1}{3} n^2 (n^2 + 2). \tag{7.181}$$

- From the definitions of $(cC)_n$ and $(SP)_n$, it follows that $(cC)_n + 6(SP)_{n-1} = (2n-1)(2n^2-2n+1).$ These numbers are called *Haüy rhombic dodecahedral numbers*. First few of these numbers are $1, 15, 65, 175, 369, 671.$ These numbers are constructed as a centered cube with a square pyramid appended to each face. The *generating function* for all Haüy rhombic dodecahedral numbers is

$$\frac{x(x+1)(x^2+10x+1)}{(1-x)^4} = x + 15x^2 + 65x^3 + 175x^4 + \cdots .$$

- Haüy also gave construction of another set of numbers involving cubes and odd square numbers, namely,

$$\begin{aligned}(2n-1)^3 + 6[1^2 + 3^2 + \cdots + (2n-3)^2] &= (2n-1)^3 + 2(n-1) \\ &\quad \times (2n-1)(2n-3) \\ &= (2n-1)(8n^2-14n+7).\end{aligned}$$

  First few of these numbers are $1, 33, 185, 553, 1233.$ These numbers are called *Haüy rhombic dodecahedron numbers*. The *generating function* for all of these numbers is

$$\frac{x(1+29x+59x^2-13x^3)}{(1-x)^4} = x + 33x^2 + 185x^3 + 553x^4 + \cdots .$$

- From the definitions of $(OH)_n$ and $(SP)_n$, it follows that $(OH)_{3n-2} - 6(SP)_{n-1} = 16n^3 - 33n^2 + 24n - 6.$ These numbers are called *truncated octahedral numbers*. First few of these numbers are $1, 38, 201, 586, 1289,$ $2406.$ These numbers are obtained by truncating all six vertices of octahedron. The *generating function* for all truncated octahedral numbers is
$$\frac{x(6x^3+55x^2+34x+1)}{(1-x)^4} = x + 38x^2 + 201x^3 + 586x^4 + \cdots .$$

7.25 Pentagonal Pyramidal Numbers $(PP)_n$

These numbers count the number of dots in pyramids built up of pentagonal numbers. First few pentagonal pyramidal numbers are $1, 6, 18, 40, 75, 126, 196, 288.$ In general, the *n*th pentagonal pyramidal number $(PP)_n$ is given in terms of the sum of the first *n* pentagonal numbers, i.e.,

$$(PP)_n = (PP)_{n-1} + \frac{n}{2}(3n-1) = 1 + 5 + 12 + 22 + 35 + \cdots + \frac{n}{2}(3n-1),$$

which in view of (7.51) is the same as

$$(PP)_n = \frac{1}{2}n^2(n+1) = nt_n. \tag{7.182}$$

- The *generating function* for all pentagonal pyramidal numbers is
$$\frac{x(2x+1)}{(1-x)^4} = x + 6x^2 + 18x^3 + 40x^4 + \cdots .$$

- From (7.11) and (7.165) it follows that
$$\sum_{k=1}^{n}(PP)_k = \frac{1}{2}\sum_{k=1}^{n}(k^3+k^2) = \frac{1}{24}n(n+1)(3n^2+7n+2). \tag{7.183}$$

- To find the sum of the reciprocals of all pentagonal pyramidal numbers, from (7.16), (7.17), and (5.12), we have

$$\sum_{k=1}^{\infty} \frac{1}{(PP)_k} = \sum_{k=1}^{\infty} \frac{2}{k^2(k+1)} = 2\sum_{k=1}^{\infty} \frac{1}{k^2} - 2\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right) = \frac{\pi^2}{3} - 2. \quad (7.184)$$

- Similar to (7.174) it follows that

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(PP)_k} = \sum_{k=1}^{\infty} \frac{2(-1)^{k-1}}{k^2(k+1)} = \frac{\pi^2}{6} + 2 - 4\ln 2 \ \approx \ 0.8723453446. \quad (7.185)$$

7.26 Hexagonal Pyramidal Numbers $(HP)_n$

These numbers count the number of dots in pyramids built up of hexagonal numbers. First few pentagonal pyramidal numbers are $1, 7, 22, 50, 95, 161, 252, 372.$ In general, the *n*th hexagonal pyramidal number $(HP)_n$ is given in terms of the sum of the first *n* hexagonal numbers, i.e.,

$$(HP)_n = (HP)_{n-1} + n(2n-1) = 1 + 6 + 15 + 28 + 45 + \cdots + n(2n-1),$$

which in view of (7.58) is the same as

$$(HP)_n = \frac{1}{6}n(n+1)(4n-1) = \frac{1}{3}(4n-1)t_n. \quad (7.186)$$

- The *generating function* for all hexagonal pyramidal numbers is

$$\frac{x(3x+1)}{(1-x)^4} = x + 7x^2 + 22x^3 + 50x^4 + \cdots .$$

- From (7.11) and (7.165) it follows that

$$\sum_{k=1}^{n}(HP)_k = \frac{1}{6}\sum_{k=1}^{n}(4k^3 + 3k^2 - k) = \frac{1}{6}n^2(n+1)(n+2). \quad (7.187)$$

- To find the sum of the reciprocals of all hexagonal pyramidal numbers, we follow as in (7.177) and (7.184), to obtain

$$\sum_{k=1}^{\infty}\frac{1}{(HP)_k} = 6\sum_{k=1}^{\infty}\left(-\frac{1}{k}+\frac{1}{5(k+1)}+\frac{16}{5(4k-1)}\right)$$
$$= \frac{6}{5}(12\ln 2 - 2\pi - 1) \approx 1.2414970314. \tag{7.188}$$

- Similarly, we have

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(HP)_k} = 6\sum_{k=1}^{\infty}(-1)^{k-1}\left(-\frac{1}{k}+\frac{1}{5(k+1)}+\frac{16}{5(4k-1)}\right)$$
$$= \frac{6}{5}\left(1+4\Phi\left(-1,1,\frac{3}{4}\right)\right) - 6\ln 2) \approx 0.8892970462. \tag{7.189}$$

  In (7.189), the function Φ is the Dirichlet beta function defined by the sum

$$\beta(x) = \sum_{n=0}^{\infty}(-1)^n(2n+1)^{-x} = 2^{-x}\Phi(-1,x,1/2).$$

7.27 Generalized Pentagonal Pyramidal Numbers $(GPP)_n$

These numbers count the number of dots in pyramids built up of generalized pentagonal numbers. First few generalized pentagonal pyramidal numbers are $1, 8, 27, 64, 125, 216, 343, 512.$ In general, the *n*th generalized pentagonal pyramidal number $(GPP)_n$ is given in terms of the sum of the first *n* generalized pentagonal numbers, i.e.,

$$(GPP)_n = (GPP)_{n-1} + [1+3n(n-1)] = 1+7+19+37+\cdots+[1+3n(n-1)],$$

which in view of (7.65) is the same as

$$(GPP)_n = n^3. \tag{7.190}$$

Thus, generalized pentagonal pyramidal numbers are the same as cubic numbers.

7.28 Heptagonal Pyramidal Numbers $(HEPP)_n$

These numbers count the number of dots in pyramids built up of heptagonal numbers. First few heptagonal pyramidal numbers are $1, 8, 26, 60, 115, 196, 308, 456$. In general, the *n*th heptagonal pyramidal number $(HEPP)_n$ is given in terms of the sum of the first *n* heptagonal numbers, i.e.,

$$(HEPP)_n = (HEPP)_{n-1} + \frac{n}{2}(5n-3) = 1 + 7 + 18 + 34 + \cdots + \frac{n}{2}(5n-3),$$

which in view of (7.75) is the same as

$$(HEPP)_n = \frac{1}{6}n(n+1)(5n-2) = \frac{1}{3}t_n(5n-2). \tag{7.191}$$

- The *generating function* for all heptagonal pyramidal numbers is

$$\frac{x(4x+1)}{(1-x)^4} = x + 8x^2 + 26x^3 + 60x^4 + \cdots .$$

- From (7.2), (7.11), and (7.165), it follows that

$$\sum_{k=1}^{n}(HEPP)_k = \frac{1}{24}n(n+1)(n+2)(5n-1). \tag{7.192}$$

- The sum of reciprocals of all heptagonal pyramidal numbers appears as

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{(HEPP)_k} &= \frac{15}{14}\left(2\ln 5 - \frac{4}{5} - \pi\sqrt{1-\frac{2\sqrt{5}}{5}}\right.\\ &\quad \left. +(\sqrt{5}+1)\ln\left(\sqrt{\tfrac{5-\sqrt{5}}{2}}\right) - (\sqrt{5}-1)\ln\left(\sqrt{\tfrac{5+\sqrt{5}}{2}}\right)\right)\\ &\approx 1.2072933193.\end{aligned} \tag{7.193}$$

- Similarly, as in (7.189), we have

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(HEPP)_k} = \frac{3}{7}\left(2 + 5\Phi\left(-1, 1, \frac{3}{5}\right) - 9\ln 2\right) \approx 0.9023419344. \tag{7.194}$$

7.29 Octagonal Pyramidal Numbers $(OP)_n$

These numbers count the number of dots in pyramids built up of octagonal numbers. First few octagonal pyramidal numbers are $1, 9, 30, 70, 135, 231, 364, 540.$ In general, the *n*th octagonal pyramidal number $(OP)_n$ is given in terms of the sum of the first *n* octagonal numbers, i.e.,

$$(OP)_n = (OP)_{n-1} + n(3n-2) = 1 + 8 + 21 + 40 + \cdots + n(3n-2),$$

which in view of (7.85) is the same as

$$(OP)_n = \frac{1}{2}n(n+1)(2n-1) = t_n(2n-1). \tag{7.195}$$

- The *generating function* for all octagonal pyramidal numbers is

$$\frac{x(5x+1)}{(1-x)^4} = x + 9x^2 + 30x^3 + 70x^4 + \cdots .$$

- From (7.2), (7.11), and (7.165) it follows that

$$\sum_{k=1}^{n} (OP)_k = \frac{1}{12}n(n+1)(n+2)(3n-1). \tag{7.196}$$

- The sum of reciprocals of all heptagonal pyramidal numbers appears as

$$\sum_{k=1}^{\infty} \frac{1}{(OP)_k} = \frac{2}{3}(4\ln 2 - 1) \approx 1.1817258148. \tag{7.197}$$

- Similarly, we find

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(OP)_k} = \frac{2}{3}(1 + \pi - 4\ln 2) \approx 0.9126692876. \tag{7.198}$$

7.30 Nonagonal Pyramidal Numbers $(NP)_n$

These numbers count the number of dots in pyramids built up of nonagonal numbers. First few nonagonal pyramidal numbers are $1, 10, 34, 80, 155, 266, 420, 624.$ In general, the *n*th nonagonal pyramidal number $(NP)_n$ is given in terms of the sum of the first *n* nonagonal numbers, i.e.,

$$(NP)_n = (NP)_{n-1} + \frac{n}{2}(7n-5) = 1 + 9 + 24 + 46 + \cdots + \frac{n}{2}(7n-5),$$

which in view of (7.96) is the same as

$$(NP)_n = \frac{1}{6}n(n+1)(7n-4) = \frac{1}{3}t_n(7n-4). \tag{7.199}$$

- The *generating function* for all nonagonal pyramidal numbers is

$$\frac{x(6x+1)}{(1-x)^4} = x + 10x^2 + 34x^3 + 80x^4 + \cdots .$$

- From (7.2), (7.11), and (7.165), it follows that

$$\sum_{k=1}^{n}(NP)_k = \frac{1}{24}n(n+1)(n+2)(7n-3). \tag{7.200}$$

- The sum of reciprocals of all heptagonal pyramidal numbers appears as

$$\sum_{k=1}^{\infty}\frac{1}{(NP)_k} = \frac{3}{88}\left(33 - 28\Psi\left(-\frac{4}{7}\right) - 28\gamma\right) \approx 1.6184840638. \tag{7.201}$$

- Similarly, we find

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(NP)_k} = \frac{3}{22}\left(4 + 7\Phi\left(-1, 1, \frac{3}{7}\right) - 15\ln 2\right) \approx 0.9210386965. \tag{7.202}$$

7.31 Decagonal Pyramidal Numbers $(DP)_n$

These numbers count the number of dots in pyramids built up of decagonal numbers. First few decagonal pyramidal numbers are $1, 11, 38, 90, 175, 301, 476, 708.$ In general, the *n*th decagonal pyramidal number $(DP)_n$ is given in terms of the sum of the first *n* decagonal numbers, i.e.,

$$(DP)_n = (DP)_{n-1} + n(4n-3) = 1 + 10 + 27 + 52 + \cdots + n(4n-3),$$

which in view of (7.108) is the same as

$$(DP)_n = \frac{1}{6}n(n+1)(8n-5) = \frac{1}{3}t_n(8n-5). \tag{7.203}$$

- The *generating function* for all decagonal pyramidal numbers is

$$\frac{x(7x+1)}{(1-x)^4} = x + 11x^2 + 38x^3 + 90x^4 + \cdots .$$

- From (7.2), (7.11), and (7.165) it follows that

$$\sum_{k=1}^{n}(DP)_k = \frac{1}{6}n(n+1)(n+2)(2n-1). \tag{7.204}$$

- The sum of reciprocals of all decagonal pyramidal numbers appears as

$$\sum_{k=1}^{\infty}\frac{1}{(DP)_k} = \frac{6}{325}\left(39 - 40\Psi\left(-\frac{5}{8}\right) - 40\gamma\right) \approx 1.1459323453. \tag{7.205}$$

- Similarly, we find

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(DP)_k} = \frac{6}{65}\left(5 + 8\Phi\left(-1, 1, \frac{3}{8}\right) - 18\ln 2\right) \approx 0.9279541642. \tag{7.206}$$

7.32 Tetrakaidecagonal Pyramidal Numbers $(TETP)_n$

These numbers count the number of dots in pyramids built up of tetrakaidecagonal numbers. First few tetrakaidecagonal pyramidal numbers are $1, 15, 54, 130, 255, 441, 700, 1044, 1485$. In general, the *n*th tetrakaidecagonal pyramidal number $(TETP)_n$ is given in terms of the sum of the first *n* tetrakaidecagonal numbers, i.e.,

$$(TETP)_n = (TETP)_{n-1} + n(6n-5) = 1 + 14 + 39 + 76 + \cdots + n(6n-5),$$

which in view of (7.119) is the same as

$$(TETP)_n = \frac{1}{2}n(n+1)(4n-3) = t_n(4n-3). \tag{7.207}$$

- The *generating function* for all tetrakaidecagonal pyramidal numbers is

$$\frac{x(11x+1)}{(1-x)^4} = x + 15x^2 + 54x^3 + 130x^4 + \cdots .$$

- From (7.2), (7.11), and (7.165), it follows that

$$\sum_{k=1}^{n}(TETP)_k = \frac{1}{6}n(n+1)(n+2)(3n-2). \tag{7.208}$$

- The sum of reciprocals of all tetrakaidecagonal pyramidal numbers appears as

$$\sum_{k=1}^{\infty}\frac{1}{(TETP)_k} = \frac{2}{21}(2\pi + 12\ln 2 - 3) \approx 1.1048525213. \tag{7.209}$$

- Similarly, we find

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(TETP)_k} = \frac{2}{21}\left(3 + 4\Phi\left(-1, 1, \frac{1}{4}\right) - 10\ln 2\right) \approx 0.9466758087. \tag{7.210}$$

7.33 Stella Octangula Numbers $(SO)_n$

The word octangula for eight-pointed star was given by Kepler in 1609. Stella octangula numbers count the number of dots in pyramids built up of star numbers. These numbers also arise in a parametric family of instances to the crossed ladders problem in which the lengths and heights of the ladders and the height of their crossing point are all integers. The ratio between the heights of the two ladders is a stella octangula number. First few stella octangula numbers are $1, 14, 51, 124, 245, 426, 679, 1016, 1449.$ In general, the *n*th stella octangula number $(SO)_n$ is given in terms of the sum of the first *n* star numbers, i.e.,

$$(SO)_n = (SO)_{n-1} + [1 + 6n(n-1)] = 1 + 13 + 37 + 73 + \cdots + [1 + 6n(n-1)],$$

which in view of (7.159) is the same as

$$(SO)_n = n(2n^2 - 1) = (OH)_n + 8T_{n-1}. \tag{7.211}$$

- The only known square stella octangula numbers are 1 and $9653449 = 3107^2 = (SO)_{169}$, see Conway and Guy [138].
- The *generating function* for all stella octangula numbers is

$$\frac{x(x^2 + 10x + 1)}{(1-x)^4} = x + 14x^2 + 51x^3 + 1124x^4 + \cdots .$$

- From (7.2) and (7.165) it follows that

$$\sum_{k=1}^{n} (SO)_k = \frac{1}{2} n(n+1)(n^2 + n - 1). \tag{7.212}$$

- The sum of reciprocals of all stella octangula numbers appears as

$$\sum_{k=1}^{\infty} \frac{1}{(SO)_k} = -\frac{1}{2}\left(2\gamma + \Psi\left(-\frac{1}{\sqrt{2}}\right) + \Psi\left(\frac{1}{\sqrt{2}}\right)\right) \approx 1.1114472084. \tag{7.213}$$

- Similarly, we find

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(SO)_k} = \frac{1}{2}\left(\Phi\left(-1, 1, 1 + \frac{1}{\sqrt{2}}\right) + \Phi\left(-1, 1, 1 - \frac{1}{\sqrt{2}}\right) - 2\ln 2\right) \tag{7.214}$$
$$\approx 0.942739143439.$$

7.34 Biquadratic Numbers $(BC)_n$

A biquadratic number can be written as a product of four equal factors of natural numbers. Thus, $1, 16, 81, 256, 625, 1296, 2401, 4096$ are first few biquadratic numbers.

- Last digit of a biquadratic number can only be 0 (in fact 0000), 1, 5 (in fact 0625), or 6.

- The *n*th biquadratic number is the sum of the first *n* Haüy rhombic dodecahedral numbers. Indeed, from (7.2), (7.11), and (7.165) it follows that

$$\sum_{k=1}^{n}(2k-1)(2k^2-2k+1)=n^4.$$

- The *generating function* for all biquadratic numbers is

$$\frac{x(x^3+11x^2+11x+1)}{(1-x)^5}=x+16x^2+81x^3+256x^4+\cdots.$$

- From (7.2), (7.11), (7.165), and the identity

$$(n+1)^5-1=\sum_{k=1}^{n}[(k+1)^5-k^5]=5\sum_{k=1}^{n}k^4+\sum_{k=1}^{n}(10k^3+10k^2+5k+1)$$

it follows that

$$n^5+5n^4+10n^3+10n^2+5n=5\sum_{k=1}^{n}k^4+\frac{5}{2}n^4+\frac{25}{3}n^3+10n^2+\frac{31}{6}n$$

and hence

$$\sum_{k=1}^{n}k^4=\frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)=\frac{1}{5}n^5+\frac{1}{2}n^4+\frac{1}{3}n^3-\frac{1}{30}n. \quad (7.215)$$

- The following identity due to Abu-Ali al-Hassan ibn al-Hasan ibn al-Haitham (965–1039, Iraq) combines the sum of numbers raised to the power of four with different sums of numbers raised to the power of three

$$\sum_{k=1}^{n}k^4=\left(\sum_{k=1}^{n}k^3\right)(n+1)-\sum_{k=1}^{n}\left(\sum_{j=1}^{k}j^3\right). \quad (7.216)$$

- From (7.11), (7.15), (7.165), and (7.215) it follows that

$$\sum_{k=1}^{n}t_k^2=\frac{1}{30}t_n(3n^3+12n^2+13n+2)=\frac{1}{10}\left(\sum_{k=1}^{n}t_k\right)(3n^2+6n+1). \quad (7.217)$$

- To find the sum of reciprocals of all biquadratic numbers, we shall use the derivation of (5.12). First in the two expansions of $(\sin x)/x$, we compare the coefficients of x^4, to get

$$\frac{\pi^4}{5!} = \sum_{\substack{p,q\in\mathcal{N}\\ p\neq q}} \frac{1}{p^2q^2}. \tag{7.218}$$

  Now squaring (5.12), to obtain

$$\sum_{k=1}^{\infty}\frac{1}{k^4} + 2\sum_{\substack{p,q\in\mathcal{N}\\ p\neq q}} \frac{1}{p^2q^2} = \frac{\pi^4}{36},$$

  which in view of (7.218) gives

$$\sum_{k=1}^{\infty}\frac{1}{k^4} = \frac{\pi^4}{36} - 2\frac{\pi^4}{5!} = \frac{\pi^4}{90}. \tag{7.219}$$

- From (7.219) it immediately follows that

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^4} = \frac{7\pi^4}{720}. \tag{7.220}$$

7.35 Pentatope Numbers $(PTOP)_n$

The fifth cell of any row of Pascal's triangle starting with the 5-term row $1\ 4\ 6\ 4\ 1,$ either from left to right or from right to left are defined as pentatope numbers. First few these numbers are $1, 5, 15, 35, 70, 126, 210, 330, 495.$ Thus, the *n*th pentatope number is defined as

$$(PTOP)_n = \binom{n+3}{4} = \frac{1}{24}n(n+1)(n+2)(n+3) = \frac{1}{6}t_nt_{n+2} = \frac{1}{4}(n+3)T_n. \tag{7.221}$$

These numbers can be represented as regular discrete geometric patterns, see Deza [162]. In biochemistry, the pentatope numbers represent the number of possible arrangements of *n* different polypeptide subunits in a tetrameric (tetrahedral) protein.

- Two of every three pentatope numbers are also pentagonal numbers. In fact, the following relations hold

$$(PTOP)_{3n-2} = P_{(3n^2-n)/2} \quad \text{and} \quad (PTOP)_{3n-1} = P_{(3n^2+n)/2}. \tag{7.222}$$

- The *generating function* for all pentatope numbers is

$$\frac{x}{(1-x)^5} = x + 5x^2 + 15x^3 + 35x^4 + \cdots .$$

- From (7.2), (7.11), (7.165), and (7.215) it follows that

$$\sum_{k=1}^{n} (PTOP)_k = \frac{1}{120} n(n+1)(n+2)(n+3)(n+4) = \frac{1}{5}(n+4)(PTOP)_n. \tag{7.223}$$

- As in (7.17), we have

$$\sum_{k=1}^{\infty} \frac{1}{(PTOP)_k} = \sum_{k=1}^{\infty} \left(\frac{8}{k(k+1)(k+2)} - \frac{8}{(k+1)(k+2)(k+3)} \right) = \frac{4}{3}. \tag{7.224}$$

- We also have

$$\sum_{k=1}^{\infty} \frac{(-1)^{n-1}}{(PTOP)_k} = 32 \ln 2 - \frac{64}{3} \approx 0.8473764446. \tag{7.225}$$

7.36 Sums of Powers with Positive Integer Exponents

We shall discuss the sum of the following series

$$S_n(p) = 1^p + 2^p + 3^p + \cdots + n^p = \sum_{k=1}^{n} k^p, \tag{7.226}$$

here $p \geq 1$ is an integer. The cases $p = 1, 2, 3,$ and 4 have already appeared, respectively, in (7.2), (7.11), (7.165), and (7.215). The problem (7.226) has been of interest since antiquity. Among the great scholars who considered particular cases of this problem includes

Aryabhata, Babylonians, Pythagoras, Archimedes, Abu Bakr al-Karaji (953–1029, Iran), al-Haytham, Jyesthadeva (around 1500–1600, India), Harriot, Faulhaber, Fermat, Pascal, and Jacob Bernoulli. In 1631, Faulhaber gave sums for $S_n(p)$ up to $p = 17,$ but he did not give a general formula. In a work entitled *Summae Potestatum* of Jacob Bernoulli, which was published posthumously in 1713, $S_n(p)$ for $1 \leq p \leq 10$ have been evaluated and appear as

$$\begin{aligned} S_n(1) &= \frac{1}{2}n^2 + \frac{1}{2}n = \frac{n(n+1)}{2} \\ S_n(2) &= \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n = \frac{n(n+1)(2n+1)}{6} \end{aligned}$$

$$\begin{aligned} S_n(3) &= \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 = S_n(1)^2 \\ S_n(4) &= \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} \\ S_n(5) &= \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2 = \frac{n^2(n+1)^2(2n^2+2n-1)}{12} \\ S_n(6) &= \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n \\ &= \frac{n(n+1)(2n+1)(3n^4+6n^3-3n+1)}{42} \\ S_n(7) &= \frac{1}{8}n^8 + \frac{1}{2}n^7 + \frac{7}{12}n^6 - \frac{7}{24}n^4 + \frac{1}{12}n^2 \\ S_n(8) &= \frac{1}{9}n^9 + \frac{1}{2}n^8 + \frac{2}{3}n^7 - \frac{7}{15}n^5 + \frac{2}{9}n^3 - \frac{1}{30}n \\ S_n(9) &= \frac{1}{10}n^{10} + \frac{1}{2}n^9 + \frac{3}{4}n^8 - \frac{7}{10}n^6 + \frac{1}{2}n^4 - \frac{1}{12}n^2 \\ S_n(10) &= \frac{1}{11}n^{11} + \frac{1}{2}n^{10} + \frac{5}{6}n^9 - n^7 + n^5 - \frac{1}{2}n^3 + \frac{5}{66}n. \end{aligned}$$

These sums are explicit and could be generalized easily for any $p \geq 1,$

$$S_n(p) = \frac{1}{p+1}\sum_{j=0}^{p}(-1)^j\binom{p+1}{j}B_j\, n^{p+1-j}, \tag{7.227}$$

where B_j are Bernoulli numbers defined in Sect. 7.3. In the literature (7.227) has been called Bernoulli's formula, whereas some authors have named it as Faulhaber's formula.

As an alternative we begin with the binomial theorem (2.4)

$$(k+1)^p = k^p + \sum_{i=0}^{p-1} \binom{p}{i} k^i,$$

which gives

$$\begin{aligned} \sum_{k=1}^{n} (k+1)^p &= \sum_{k=1}^{n} k^p + \sum_{i=0}^{p-1} \binom{p}{i} \sum_{k=1}^{n} k^i \\ &= S_n(p) + \sum_{i=0}^{p-1} \binom{p}{i} S_n(i). \end{aligned}$$

But, since

$$\sum_{k=1}^{n} (k+1)^p = \sum_{k=2}^{n+1} k^p = S_{n+1}(p) - 1 = S_n(p) + (n+1)^p - 1$$

it follows that

$$(n+1)^p - 1 = \sum_{i=0}^{p-1} \binom{p}{i} S_n(i). \tag{7.228}$$

From this we can compute $S_n(p)$ for any *p* recursively.

Ada Augusta King, Countess of Lovelace (1815–1852, England), began from a description of the Bernoulli numbers as the coefficients B_i in the expansion

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + B_2 \frac{x^2}{2} + B_4 \frac{x^4}{4!} + B_6 \frac{x^6}{6!} + \cdots .$$

By a bit of algebraic manipulation using the power series expansion for e^x, Augusta rewrote this equation in the form

$$0 = -\frac{1}{2}\frac{2n-1}{2n+1} + B_2 \left(\frac{2n}{2!}\right) + B_4 \left(\frac{2n(2n-1)(2n-2)}{4!}\right)$$

$$+B_6\left(\frac{2n(2n-1)\cdots(2n-4)}{6!}\right)+\cdots+B_{2n},$$

a form from which the various B_i can be calculated recursively.

If *p* is not an integer, then for $S_n(p)$ only bounds are available, for example,

$$n^{p+1}<(p+1)S_n(p)<(n+1)^{p+1}-1,\quad 0<p<1$$

$$n^{p+1}<(p+1)S_n(p)<(n+1)^p n,\quad p\geq 1$$

$$(n+1)^{p+1}-n^{p+1}<(p+1)[S_n(p)-S_{n-1}(p)]<n^{p+1}-(n-1)^{p+1},\quad -1<p<0$$

$$(p+1)[S_n(p)-1]<n^{p+1}-1<(p+1)S_{n-1}(p),\quad p<-1.$$

These inequalities have been discussed in Agarwal [10], Guo and Feng Qi [231], and Kuang [319].

7.37 Partitions by Polygonal Numbers

Recall that *general polygonal number* can be written as $p_n^r=n[(r-2)n-(r-4)]/2,$ where p_n^r is the *n*th *r*-gonal number. For example, for $r=3$ it gives triangular number, and for $r=4$ gives a square number. Fermat in 1638 claimed that every positive integer is expressible as at most $k,$ *k*-gonal numbers (Fermat's Polygonal Number Theorem). Fermat claimed to have a proof of this result; however, his proof has never been found. In 1750, Euler conjectured that every odd integer can be written as a sum of four squares in such a way that $n=a^2+b^2+c^2+d^2$ and $a+b+c+d=1.$ In 1770. Lagrange proved that every positive integer can be represented as a sum of four squares, known as *four-square theorem* (see Theorem 4.19). For example, the number $1638=4^2+6^2+25^2+31^2=1^2+1^2+6^2+40^2$ has several different partitions, whereas for the number $1536=0^2+16^2+16^2+32^2$ this is the only partition. In 1797–8, Legendre extended the theorem with his *three-square theorem*, by proving that a positive integer can be expressed as the sum of three squares if and only if it is not of the form $4^k(8m+7)$ for integers *k* and *m* (see Theorem 4.18). Later, in 1834, Jacobi gave a formula for the number of ways that a given positive integer *n* can be represented as the sum of four squares.

In 1796, Gauss proved the difficult triangular case (every positive integer is the sum of three or fewer triangular numbers, which was conjectured by Pascal in his treatise of 1665, it was later shown that Gauss result is equivalent to the statement that every number of the form $8n+3$ is a sum of three odd squares, see Theorem 4.18 and for details Duke [173]), commemorating the occasion by writing in his diary the line EΓPHKA! num $= \triangle + \triangle + \triangle,$ and published a proof in his book *Disquisitiones Arithmeticae* of 1798. For this reason, Gauss's result is sometimes known as the Eureka theorem. For example, $16 = 6+10, 25 = 1+3+21, 39 = 3+15+21, 150 = 6+66+78.$ In fact, in view of Theorem 4.18 and the fact $8n+3 \equiv 3 (\bmod 8)$ each such number can be written only as the sum of three odd squares. Thus, it follows that

$$8n+3 = (2a+1)^2 + (2b+1)^2 + (2c+1)^2 = 4a(a+1) + 4b(b+1) + 4c(c+1) + 3$$

and hence

$$n = \frac{a(a+1)}{2} + \frac{b(b+1)}{2} + \frac{c(c+1)}{2} = t_a + t_b + t_c.$$

The full polygonal number theorem was resolved finally in 1813 by Cauchy. In 1872, Lebesgue proved that every positive integer is the sum of a square number (possibly 0^2) and two triangular numbers, and every positive integer is the sum of two square numbers and a triangular number. For further details, see Grosswald [229], Ewell [182, 183], and Guy [242]. The results of this chapter are based on out work in [21].

7.38 Conclusions

Triangular numbers that are believed to have been introduced by Pythagoras himself play dominant role in all types of figurative numbers we have addressed in this chapter. In fact, Eq. (7.1) says natural number *n* is the difference of t_n and $t_{n-1},$ whereas Gauss's Eureka theorem stipulates that *n* can be written as the sum of three triangular numbers. Equation (7.32) shows that square number S_n is the sum of t_n and $t_{n-1}.$ Equation (7.41) shows that square number R_n is 2 times of $t_n.$

Relation (7.48) says pentagonal number P_n is $(1/3)t_{3n-1}$, whereas (7.50) gives $P_n = t_n + 2t_{n-1} = t_{2n-1} - t_{n-1}$. Equation (7.57) informs that hexagonal number H_n is the same as t_{2n-1}. Relation (7.64) says generalized pentagonal number $(GP)_n$ is the same as $t_1 + 6t_{n-1} = t_n + 4t_{n-1} + t_{n-2}$. Equation (7.74) informs that heptagonal number $(HEP)_n$ is the same as $t_n + 4t_{n-1}$. Relation (7.84) declares that octagonal number O_n is equal to $t_n + 5t_{n-1}$. Equation (7.95) implies that nonagonal number N_n is equal to $t_n + 6t_{n-1}$. Relation (7.107) says decagonal number D_n is the same as $t_n + 7t_{n-1}$. Equation (7.118) informs that tetrakaidecagonal number $(TET)_n$ is the same as $t_n + 11t_{n-1}$. Relation (7.130) shows that centered triangular number $(ct)_n$ is the same as $t_n + t_{n-1} + t_{n-2}$, whereas relation (7.134) confirms that centered square number $(cS)_n$ is equal to $t_n + 2t_{n-1} + t_{n-2}$. Equation (7.138) says centered pentagonal number $(cP)_n$ is equal to $t_n + 3t_{n-1} + t_{n-2}$, whereas Eq. (7.142) tells centered heptagonal number $(cHEP)_n$ is the same as $t_n + 5t_{n-1} + t_{n-2}$.

Continuing, relation (7.146) informs centered octagonal number $(cO)_n$ is the same as $t_n + 6t_{n-1} + t_{n-2}$, whereas (7.150) shows centered nonagonal number $(cN)_n$ is the same as $t_n + 7t_{n-1} + t_{n-2}$, and relation (7.154) tells centered decagonal number $(cD)_n$ is the same as $t_n + 8t_{n-1} + t_{n-2}$. Equations (7.158) shows that star number $(ST)_n$ is the same as $t_n + 10t_{n-1} + t_{n-2}$, whereas Eq. (7.162) shows that centered tetrakaidecagonal number $(cTET)_n$ is the same as $t_n + 12t_{n-1} + t_{n-2}$. Relation (7.165) shows that the sum of the first *n* cubic numbers is the same as t_n^2. Equation (7.171) shows that tetrahedral number T_n is the same as $(1/3)(n+2)t_n$. Relation (7.175) says square pyramidal number $(SP)_n$ is the same as $(1/6)(n+1)t_{2n}$. From the definition of octahedral numbers and (7.175), it follows that $(OH)_n = (1/6)(nt_{2n-2} + (n+1)t_{2n})$. From the relation (7.182) it follows that pentagonal pyramidal number $(PP)_n$ is the same as nt_n. Equation (7.186) says hexagonal pyramidal number $(HP)_n$ is the same as $(1/3)(4n-1)t_n$. From Eq. (7.191) it follows that heptagonal pyramidal number $(HEPP)_n$ is the same as $(1/3)(5n-2)t_n$. Equation (7.195) suggests that octagonal pyramidal number $(OP)_n$ is the same as $(2n-1)t_n$. Relation (7.199) tells nonagonal pyramidal number $(NP)_n$ is the same as $(1/3)(7n-4)t_n$. Equation (7.203) informs that decagonal pyramidal number $(DP)_n$ is the same as $(1/3)(8n-5)t_n$, whereas relation (7.207) indicates that tetrakaidecagonal pyramidal number $(TETP)_n$ is the same as $(4n-3)t_n$. Thus, in conclusion almost all figurative numbers we have studied are directly related with triangular numbers.

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8. Pythagorean Irrationality of Numbers

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8.1 Introduction and Origin

Numbers that cannot be expressed as ratios of two integers are called *incommensurable* or *irrational* (not logical or reasonable). The earliest known use of *irrational numbers* is in the Indian *Sulbasutras*. For ritual sacrifices, there was a requirement to construct a square fire altar twice the area of a given square altar, which led to find the value of $\sqrt{2}$ (in the literature, it has been named as Pythagoras number). Ancient Vedic Ascetics also needed the value of π. They were successful in finding reasonable rational approximations of these numbers, keeping in mind the success of ritual sacrifices depending on very precise mathematical accuracy. In Sect. 8.3, we shall see that in Sulbasutras, there is a discussion that these numbers cannot be computed exactly. Thus, the concept of irrationality was implicitly accepted by Indian Brahmins. In fact, it is suggested that the concept of irrationality was unquestioningly accepted by Manava, who believed that the square roots of numbers such as 2 and 61 could not be exactly determined. In 1875, George Frederick William Thibaut (1848–1914, Germany) translated a large portion of the Sulbasutras, which showed that the Indian priests possessed significant mathematical knowledge. Thibaut was a Sanskrit scholar, and his principal objective was to make the mathematical knowledge of the Vedic Indians available to the learned world. After commenting that a good deal of Indian knowledge could be traced back to requirements of ritual, Thibaut adds that these facts have a double interest: In the first place,

they are valuable for the history of the human mind in general. In the second place, they are important for the mental history of India and for answering the question relative to the originality of Indian science. He firmly believed that Hindus had knowledge of irrationality, in particular, of $\sqrt{2}$. In fact, in Apastamba, there is a discussion of the irrationality of π
.

According to Datta [152] and several other Sanskrit scholars such as Leopold von Schröder in 1884 [452] and 1887, Bürk Richard Garbe (1857–1927, Germany) in 1899, Edward Washburn Hopkins (1857–1932, USA) in 1895, and Arthur Anthony Macdonell (1854–1930, born in British-India) in 1900 have claimed that irrationality of $\sqrt{2}$ was first discovered by ancient Hindus. We also find approximations of $\sqrt{2}$ in Babylonians tablets using sexagesimal fractions (see Sect. 8.5). In Greek geometry, two magnitudes *a* and *b* of the same kind were called commensurable if there is another magnitude *c* of the same kind such that both are multiples of *c*, that is, there are numbers *p* and *q* such that $pc = a$ and $qc = b.$ If the two magnitudes are not commensurable, then they are called incommensurable. Pythagoreans essentially believed that all tangible things could be measured and accounted for with rational numbers. However, the hypotenuse of a most obvious right-angled triangle with the same legs led to the number $\sqrt{2}$, which Pythagoreans could not write as a rational number, i.e., $\sqrt{2}$ is incommensurable. According to legend, Hippasus made this discovery at sea, which caused tremendous crisis/confusion/devastation/surprise/shattering effect among the Pythagoreans, for it challenged the adequacy of their basic philosophy that number was the essence of everything. In fact, in the numerical sense, the universe was seen to be irrational. This logical calamity enforced them to maintain the pledge of strict secrecy. To incommensurable numbers, they named as “the unutterable,” (Greeks used the term *logos*, meaning word or speech, for the ratio of two integers, when incommensurable lengths were described as *alogos*, the term carried a double meaning: not a ratio and not to be spoken) as it was a dangerous secret to possess. Since Hippasus uttered the unutterable to an outsider, he was murdered-thrown off a ship to drown at sea by fanatic Pythagoreans, apparently a divine punishment for divulging this impiety, (some historians wrongly claimed that Hippasus had first proof of the existence of irrational numbers, whereas others say he lost his fortune and tried to recoup his losses by teaching the doctrine of irrational numbers, while some have speculated that Hippasus revealed the properties of the dodecahedron and so he was promptly expelled from the community).

In 2007, Borzacchini [99] asserted that Pythagorean music theory is the origin of incommensurability. Anyway, it is hard to keep a secret in science. This revelation/achievement of Pythagoreans, that not all numbers are rational marked, is considered one of the most fundamental discoveries in the entire history of science (it evolved the number concept by filling the gaps that were there between rationales). Plato, realizing the importance of the discovery, thundered: “One who is not aware that the side and the diagonal of a square are incommensurable does not deserve to be called a man.” Historians have also argued that this major discovery also helped in the development of deductive reasoning. Iamblichus tells that after the death of the Master (Pythagoras), there was a split among the disciples of Pythagoras. The “acusmatics” held to the “pure doctrine” and swore by the word of the Master. The “mathematicians,” who, like Hippasus, were convinced of the existence of incommensurable segments, bent their efforts toward making further progress in mathematics. It is understandable that they were interested in exhibiting further pairs of incommensurable segments, and they soon discovered that the diagonal and side of a square have no common measure. Eudoxus placed the doctrine of incommensurables upon a thoroughly sound basis. The irrationality of the square root of two Eudoxus phrased as “a diagonal and a side of a square have no common measure.” He realized that an irrational is known by the rational numbers less than it, and the rational numbers greater than it. This task was done so well that Greek mathematicians made tremendous progress in geometry, and it survived as Book V of Euclid’s *Elements*. It still continues, fresh as ever, after the great arithmetical reconstructions of Dedekind and Weierstrass during the nineteenth century.

Democritus of Abdera (around 460–362 BC, Greece) traveled to Egypt, Persia, Babylon, India, Ethiopia, and throughout Greece. He wrote almost 70 books, in mathematics, he wrote on numbers, geometry, tangencies, mappings, and irrationals. Next, Apollonius wrote a work on the cylindrical helix and another on irrational numbers, which is mentioned by Proclus. Decimal fraction approximations of $\sqrt{2}$ and π appeared during 200–875 AD, in the Jain School of Mathematics. In terms of decimal expansions unlike a rational number, an irrational number never repeats or terminates. In fact, it is only the decimal expansion that immediately shows the difference between rational and irrational numbers. Irrational numbers have also been defined in several other ways, e.g., an irrational number has nonterminating continued fraction (see Sect. 8.16) whereas a rational number has a periodic or repeating expansion, and sequential definition of irrationality based on limiting points of convergent series proposed by Weierstrass, which was extended to classes of equivalent sequences by Heinrich Edward Heine (1821–1881, Germany) in 1872.

From the ninth century, Mediterranean and Arabic mathematicians started treating irrational numbers as algebraic objects and initiated the idea of merging the concepts of number (algebra) and magnitude (geometry) into a more general idea of real numbers. Abu Abd Allah Muhammad ibn Isa al-Mahani (around 820–880, Iran-Iraq) examined and classified quadratic irrationals and cubic irrationals and provided definitions for rational and irrational magnitudes and dealt with them freely but only in geometric terms. Abū Kāmil, Shujā′ ibn Aslam ibn Muammad ibn Shujā′ (850–930, Egypt) was probably the first mathematician who used irrational numbers as coefficients of an algebraic equation and also accepted irrational numbers as solutions of the equation in his *Book of Algebra*, which contains a total of 69 problems. Abu Ja'far al-Khazin (900–971, Iran) provided a meaningful definition of rational and irrational magnitudes. Al-Hashimi (tenth century, Iraq) provided general proofs (rather than geometric demonstrations) for irrational numbers, as he considered multiplication, division, and other arithmetical functions. He also gave a method to prove the existence of irrational numbers.

Al-Baghdadi in his influential book *Treatise on Commensurable and Incommensurable Magnitudes* related the concepts of number and magnitude by establishing a correspondence between numbers and line segments, which continues today. Given a unit magnitude $a,$ each whole number *N* corresponds to an appropriate multiple *Na* of the unit magnitude. Parts of this magnitude, such as $(p/q)a,$ then correspond to parts of a numbers (p/q). Al-Baghdadi considered any magnitude expressible this way as a rational magnitude. He showed that these magnitudes relate to one another as numbers to numbers. Magnitudes that are not parts he considered as irrational numbers. He also attempted to imbed the rational numbers into a number line. Al-Baghdadi also proved a result on the density of irrational magnitudes, namely that between any two rational magnitudes there exist infinitely many irrational magnitudes. In the late nineteenth century, it was proved that between any two real numbers there are infinitely many rational and irrational numbers, further irrational numbers are infinitely more numerous than rational numbers.

To see Al-Baghdadi's geometric interpretation of rational numbers, on a horizontal straight line mark two distinct points O and $A,$ where A is right of O. Now choose the segment OA as a unit of length, and let O and A represent the numbers 0 and 1, respectively. Then the positive and negative integers can be represented by a set of points on the line spaced at unit intervals apart, the positive integers being represented to the right of O and the negative integers to the left of O. The fraction with denominator q may then be represented by the points that divide each of the unit intervals into q equal parts. Thus, each rational number can be represented by a point on the line. In Fig. 8.1, the point P corresponds to the irrational number $\sqrt{2}$, which is between two rational numbers.

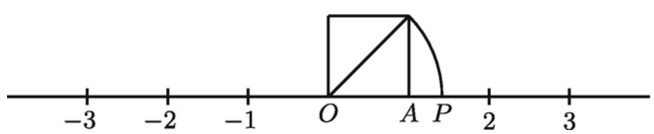

Fig. 8.1 Geometric representation of rational numbers

In 1858, Dedekind, while teaching calculus for the first time at the Polytechnic, came up with the technique now called a *Dedekind cut*, whose history dates back to Eudoxus. He published this in *Stetigkeit und Irrationale Zahlen* (Continuity and Irrational Numbers) in 1872. The central idea of a Dedekind cut is that an irrational number divides the rational numbers into two sets, with all the members of one set (upper) being strictly greater than all the members of the other (lower) set. For example, $\sqrt{2}$ puts all the negative numbers and the numbers whose squares are less than 2 into the lower set and the positive numbers whose squares are greater than 2 into the upper set. Every point on the real line is either a rational or an irrational number. Therefore, on the real line there are no empty locations, gaps, or discontinuities. Dedekind is considered one of the the most responsible for the current definition and understanding of irrational numbers. In current literature, Dedekind cut (also known as Dedekind Property) is stated as follows: Let *A* and *B* be two nonempty subsets of $\mathcal{R}$ such that $A \cup B = \mathcal{R}$ and $x \in A$ and $y \in B$ implies $x < y.$ Then, either *A* has the greatest member, or *B* has the least member.

In what follows, we will correct the speculations that incommensurability of $\sqrt{2}$ was proved by Pythagoras himself (and for all nonsquare integers by Theodorus), by revealing that the first (fully geometric) proof appeared in the *Meno* (Socratic dialog by Plato). Here we will see an infinite process arise in an attempt to understand irrationals. Since then over the period of 2400 years, many different proofs of the irrationality of $\sqrt{2}$ have been given, we will demonstrate a few of these and furnish several algorithms to find its rational approximations. The proof of the irrationality of π had to wait almost two millennia, it was proved only in 1768 by Lambert. In 1683, the number *e* was introduced by Jacob Bernoulli, whose irrationality was proved by Euler in 1748. Thus, the numbers $\sqrt{2},\ \varphi,\ e,$ and π have infinite number of decimal places. Since the invention of computer technology, these numbers have been approximated to trillions of decimal places, we shall report these accomplishments. It is to be noted that such extensive calculations besides human desire to break records have been used to test supercomputers and high-precision multiplication algorithms, the occurrence of the next digit seems to be random, and the statistical distribution expected to be uniform. We list here first 100 decimal digits of these numbers, which are more than sufficient (in fact, not even first twenty) for each and every real-world problem.

$$\begin{aligned}\sqrt{2} &= 1.4142135623, 7309504880, 1688724209, 6980785696, 7187537694,\\ &\quad 8073176679, 7379907324, 7846210703, 8850387534, 3276415727\\ \varphi &= 1.6180339887, 4989484820, 4586834365, 6381177203, 0917980576,\\ &\quad 2862135448, 6227052604, 6281890244, 9707207204, 1893911374\\ e &= 2.7182818284, 5904523536, 0287471352, 6624977572, 4709369995,\\ &\quad 9574966967, 6277240766, 3035354759, 4571382178, 5251664274\\ \pi &= 3.1415926535, 8979323846, 2643383279, 5028841971, 6939937510,\\ &\quad 5820974944, 5923078164, 0628620899, 8628034825, 3421170679\end{aligned}$$

The set of all irrational numbers (positive and negative) we shall denote as $\mathcal{Q}'$. The union of the sets of all rational and irrational numbers make up the set of real numbers $\mathcal{R}$. Thus, this large set contains all decimal representations of numbers terminating, repeating, nonterminating, and nonrepeating. The mysterious objects, irrational numbers achieved respectability and a secure position only in the nineteenth century.

8.2 Properties of Irrational Numbers

A set of numbers is said to be closed under an operation if and only if the operation on two elements of the set produces another element of the set; however, if an element outside the set is obtained, then the set of numbers under that operation is not closed. For example, the set $\mathcal{N}$ is closed under the operations addition and multiplication ($2+3=5,\ 2\cdot 3=6$), but not closed under the operations subtraction and division ($2-3=-1,\ 2/3$), the set $\mathcal{Z}$ is closed under the operations addition, subtraction, and multiplication, but not under division. The set $\mathcal{Q}$ is closed under all four operations. The following examples suggest that the set of irrational numbers is not closed under for any of the four operations. For this, first we note that the addition as well as subtraction of an irrational number *x* and a rational number *y* is irrational. Indeed, if $x \pm y = z$ is rational, then in $x = z \mp y$ left side is irrational, whereas the right side is rational. From this it immediately follows that the golden ratio φ is irrational. Now for the addition it suffices to take $(2-\sqrt{2})+\sqrt{2}=2,$ for subtraction $(2+\sqrt{2})-\sqrt{2}=2,$ for multiplication $\sqrt{2}\cdot\sqrt{8}=\sqrt{16}=4,$ and for division $\sqrt{12}/\sqrt{3}=2.$ The lcm of any two irrational numbers may or may not exist. For example, $\mathrm{lcm}(5\sqrt{3},7\sqrt{3})=35\sqrt{3};$ however, $\mathrm{lcm}(\sqrt{2},\sqrt{3})$ does not exist. For this, a simple argument (but not the proof) is that there is no number *c* such that $c|\sqrt{2}$ and $c|\sqrt{3}$ are integers, otherwise their quotient, $\sqrt{2/3},$ will be a rational.

- Hardy essentially showed that if we take any irrational number, say, $\sqrt{2},\ \ \cos 20^\circ,\ \pi$ or *e* and write these to large decimal places, say, a billion or trillion decimal places, then the number of digits $0,1,2,\cdots,9$ are uniformly randomly distributed, i.e., the frequency with which the digits (0 to 9) appear in the result will tend to the same limit $(1/10)$ as the number of decimal places increases beyond all bounds. In recent years, these digits are being used in applied problems as a random sequence. For details, see Agarwal et. al. [13].

8.3 Approximations of $\sqrt{2}$ and π in Sulbasutras

Sulbasutras (without any proofs) provide remarkable approximations of $\sqrt{2}$ and π (the symbol was first used by Welshman William Jones, 1675–1749, England, in his book [287] of 1706, for the Greek word perimetros [periphery] of a circle with unit diameter), and it became popular after the works of Euler in 1737 and Goldbach in 1742. In three Sulbasutras: Baudhayana, Apastamba, and Katyayana for the approximation of $\sqrt{2}$ the recipe is "increase the measure by its third and this third by its own fourth less the thirty-fourth part of that fourth. This is the value with a special quantity in excess." If we take 1 unit as the dimension of the side of a square, then this in modern terms can be written as

$$\sqrt{2} \simeq 1+\frac{1}{3}+\frac{1}{3\cdot 4}-\frac{1}{3\cdot 4\cdot 34}=\frac{4}{3}+\frac{1}{12}-\frac{1}{408}=\frac{17}{12}-\frac{1}{408}=\frac{577}{408} \tag{8.1}$$

and, similarly, if we take the radius of the circle as 1 unit, then the approximation formula for π is

$$\pi \simeq 18(3-2\sqrt{2})=\left(\frac{6}{2+\sqrt{2}}\right)^2. \tag{8.2}$$

These approximations were used for the construction of altars, particularly, $\sqrt{2}$ in an attempt to construct a square altar twice the area of a given square altar. Datta in his most trusted treatise [152] on Sulbas on page 27 writes "The reference to the sacrificial altars and their construction is found as early as the Rigveda (before 3000 BC). ...It seems that the problem of the squaring of the circle and the theorem of the square of the hypotenuse are as old in India as the time of Rigveda. They might be older still." Approximation (8.1) gives $\sqrt{2} \simeq 1.414215686$, which is correct to five decimal places. Perhaps the approximation (8.1) was used in $\pi \simeq 18(3-2\sqrt{2})$, to obtain $\pi \simeq 105/34 \simeq 3.088235294$. George Joseph in his book [289] mentions about his correspondence with Takao Hayashi (born 1949, Japan) who pointed out that the approximation of $\sqrt{2}$ could also be used for constructing a right-angled triangle and a square. To show (8.1), Datta on pages 193,194, and subsequently by several others, e.g., George Joseph on pages 235,236, have provided the following reasoning which is in line with Sulbasutra's geometry.

Consider two squares, *ABCD* and $PQRS$, each of 1 unit as the side of a square (see Fig. 8.2). Divide *PQRS* into three equal rectangular strips, of which the first two are marked as 1 and $2.$ The third strip is subdivided into three squares, of which the first is marked as 3. The remaining two squares are each divided into four equal strips marked as 4 to 11. These 11 areas are added to the square *ABCD* as shown in Fig. 8.2, to obtain a larger square less a small square at the corner $F.$ The side of the augmented square *AEFG* is

$$1+\frac{1}{3}+\frac{1}{3\cdot 4}.$$

The area of the shaded square is $[1/(3\cdot 4)]^2$, so that the area of the augmented square *AEFG* is greater than the sum of the areas of the original squares, *ABCD* and $PQRS$, by $[1/(3\cdot 4)]^2$.

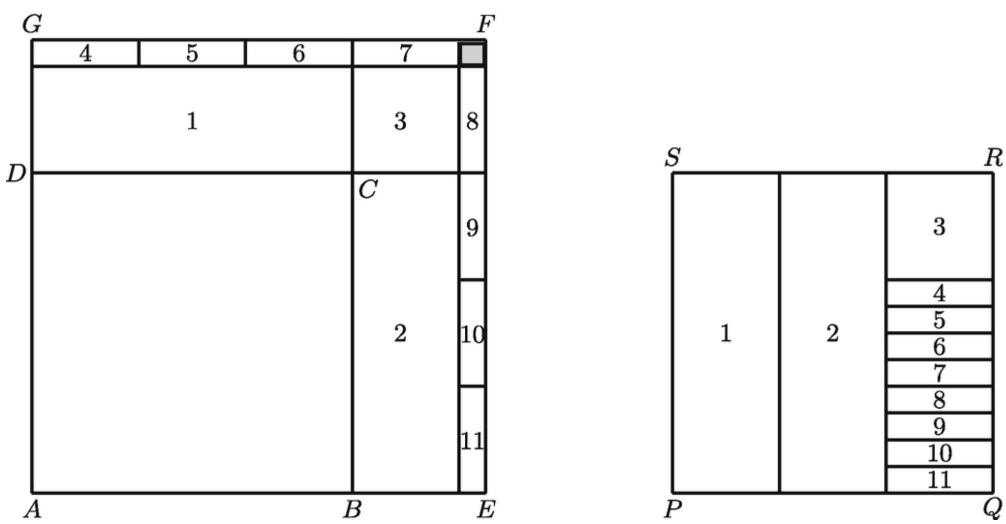

Fig. 8.2 Sulbasutras method for $\sqrt{2}$

Now to make the area of the square *AEFG* approximately equal to the sum of the areas of the original squares *ABCD* and $PQRS$, imagine cutting off two very narrow strips, of width x, from the square $AEFG$, one from the left side and one from the bottom. Then

$$2x\left(1+\frac{1}{3}+\frac{1}{3\cdot 4}\right)-x^2=\left(\frac{1}{3\cdot 4}\right)^2. \tag{8.3}$$

Simplifying the aforementioned expression and ignoring x^2, an insignificantly small quantity, gives

$$x \simeq \frac{1}{3 \cdot 4 \cdot 34}.$$

The diagonal of each of the original squares is $\sqrt{2}$, which can be approximated by the side of the new square as just calculated, i.e., (8.1).

A commentator on the Sulbasutras, Rama (perhaps Rama Chandra) Vajapeyi, who lived in the middle of the fifteenth century AD in India, gave an improved approximation to (8.1) by adding two further terms to the equation, i.e.,

$$\sqrt{2} \simeq 1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34} - \frac{1}{3 \cdot 4 \cdot 34 \cdot 33} + \frac{1}{3 \cdot 4 \cdot 34 \cdot 34} = \frac{647393}{457776}, \quad (8.4)$$

which gives 1.414213502 a value correct to seven decimal places.

In Sulbasutras we also find approximation of $\sqrt{3}$, which can be written as

$$\begin{aligned} \sqrt{3} \simeq 1 + \frac{2}{3} + \frac{1}{3 \cdot 5} - \frac{1}{3 \cdot 5 \cdot 52} &= \frac{5}{3} + \frac{1}{3 \cdot 5} - \frac{1}{3 \cdot 5 \cdot 52} \\ &= \frac{26}{15} - \frac{1}{3 \cdot 5 \cdot 52} = \frac{1351}{780}. \end{aligned} \quad (8.5)$$

Approximation (8.5) gives $\sqrt{3} \simeq 1.732051282$, which is correct to five decimal places. In [Datta, pages 194,195], a geometric construction similar to that of (8.1) for (8.5) is also given. A simple algebraic method to get (8.5) is to take $5/3$ as an approximation of $\sqrt{3}$ and put $\sqrt{3} = (5/3 + x)$, where *x* is unknown. Now square both sides of this expression, neglect x^2, and solve the resulting linear equation for x, to get $x = 1/3 \cdot 5$, thus the new approximation of $\sqrt{3}$ is $26/15$. Repeating this procedure once more, we find $x = -1/3 \cdot 5 \cdot 52$ and the new approximation of $\sqrt{3}$ as $1351/780$.

For (8.1), several other descriptions have been proposed, e.g., Radha Charan Gupta (born 1935, India), in [232] uses linear interpolation to obtain the first two terms of (8.1), he then corrects the two terms to obtain the third term, then correcting the three terms obtaining the fourth term.

In Manava, the following approximate identities have been used to calculate approximate values of $\sqrt{2}$

$$\begin{aligned} 40^2 + 40^2 &\simeq 56^2 \\ 4^2 + 4^2 &\simeq \left(5\frac{2}{3}\right)^2 . \end{aligned} \tag{8.6}$$

The first indentity gives $\sqrt{2} \simeq 7/5 = 1.4,$ whereas the second gives $\sqrt{2} \simeq 17/12 = 1.41666666\cdots$.

- Chuquet in his work *Triparty en la science des nombres* showed that $\sqrt{5} \simeq 2\frac{161}{682}$ and $\sqrt{6} \simeq 2\frac{89}{198}$. Zhu Zaiyu (1536–1611, China) in 1604 wrote a *New Explanation of the Theory of Calculation* in which he derived values of the roots of 2. He was so attracted to $\sqrt{2}$ that he used nine abacuses to compute it to 25—digit accuracy! Friedrich Engel (1861–1941, Germany) proved the following important infinite product formula

$$\sqrt{\frac{q+1}{q-1}} = \prod_{n=0}^{\infty}\left(1 + \frac{1}{q_n}\right),$$

where $q_0 = q \in \mathcal{N},\ q_{n+1} = 2q_n^2 - 1,\ n \geq 0$. For $q = 3$ and $q = 2$ this formula reduces to

$$\sqrt{2} = \left(1 + \frac{1}{3}\right)\left(1 + \frac{1}{17}\right)\left(1 + \frac{1}{577}\right)\left(1 + \frac{1}{665857}\right)\cdots$$

and

$$\sqrt{3} = \left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{7}\right)\left(1 + \frac{1}{97}\right)\left(1 + \frac{1}{18817}\right)\cdots .$$

Another familiar example is the series

$$\frac{1}{\sqrt{2}} = 1 - \frac{1}{2\cdot 1!} + \frac{1\cdot 3}{2^2 2!} - \frac{1\cdot 3\cdot 5}{2^3\cdot 3!} + \frac{1\cdot 3\cdot 5\cdot 7}{2^4\cdot 4!} - \cdots .$$

Its convergence is so slow that to compute $1/\sqrt{2}$ with accuracy of 10^{-5}, we would need to take about 10^{10} terms, which is difficult even with high-speed machines. For an excellent detailed discussion of $\sqrt{2}$ up to 2006, see the book of Flannery [191]. Jerry Bonnell (USA) and Robert Nemiroff (USA) on the website https://apod.nasa.gov/htmltest/gifcity/sqrt2.1mil have posted 1 million digits of $\sqrt{2}$, and in 2009, 5 million digits, see Bonnell and Nemirof [78]. Other records are by Yasumasa Kanada (1949–2020, Japan, life-long "pi digit-hunter," set the record 11 of the past 21 times) in 1997 to 137,438,953,444 decimal places; Shigeru Kondo (born 1959, Japan) in 2010 to 1 trillion decimal places; Alexander Yee (born 1988, China-USA) in 2012 to 2 trillion; Ron Watkins in April, 2016 to 5 trillion, and in June 2016 to 10 trillion.

In Sulbasutras, the priests gave the following procedure for finding a circle whose area was equal to a given square. In the square *ABCD*, let *M* be the intersection of the diagonals (see Fig. 8.3). Draw the circle with *M* as center and *MA* as radius, let *ME* be the radius of the circle perpendicular to the side *AD* and cutting *AD* in $G.$ Let $GN = \frac{1}{3}GE$. Then *MN* is the radius of the desired circle. If $AB = s$ and $2MN = d$, then from the Pythagoras theorem it follows that

$$MN = MG + GN = MG + \frac{1}{3}GE = MG + \frac{1}{3}(ME - MG) = \frac{2}{3}MG + \frac{1}{3}ME$$
$$= \frac{2}{3}\frac{s}{2} + \frac{1}{3}\frac{\sqrt{2}s}{2},$$

and hence

$$MN = \left(\frac{2+\sqrt{2}}{6}\right) s.$$

This gives

$$\pi(MN)^2 \ \simeq \ \pi \left(\frac{2+\sqrt{2}}{6}\right)^2 s^2 = s^2,$$

which leads to

$$\pi \ \simeq \ \left(\frac{6}{2+\sqrt{2}}\right)^2 = 18(3 - 2\sqrt{2}),$$

which is the same as (8.2).

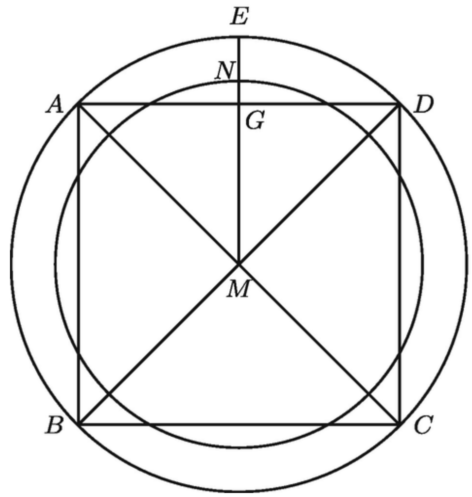

Fig. 8.3 Area of a circle equal to a square

For the converse problem, that of squaring the circle, we are given the following rule: If you wish to turn a circle into a square, divide the diameter into 8 parts, and again one of these 8 parts into 29 parts; of these 29 parts remove 28, and moreover, the sixth part (of the one left) less the eighth part (of the sixth part). The meaning is: side of the required square is

$$\frac{7}{8}+\frac{1}{8\times 29}-\frac{1}{8\times 29\times 6}+\frac{1}{8\times 29\times 6\times 8}=\frac{9785}{11136}$$

times the diameter of given circle. It gives the value of $\pi = 3.088326491$.

All the Sulbasutras contain a method to square the circle. It is an approximate method based on constructing a square of side $13/15$ times the diameter of the given circle as in the Fig. 8.4. This corresponds to taking the value of π as

$$\pi = 4\times(13/15)^2 = 676/225 = 3.00444.$$

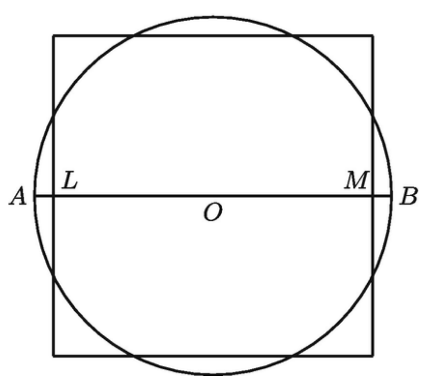

Fig. 8.4 Approximation of π

It is worth noting that many different values of π appear in the Sulbasutras, even several different ones in the same text. This is not surprising that whenever an approximate construction is given some value of π is implied. The authors thought in terms of approximate constructions, not in terms of exact constructions with π but only having an approximate value for it. For example, in Baudhayana Sulbasutra, the different values of π are given as $676/225,\ 900/289,$ and $1156/361.$ In other Sulbasutras the values $2.99,\ 3.00,\ 3.004,\ 3.029,\ 3.047,\ 3.088,\ 3.1141,\ 3.16049$ and 3.2022 can all be found. Particularly, in the Mayana Sulbasutra, see Gupta [237], the value of $\pi \simeq 25/8 \simeq 3.125,$ also see interesting work of Kak [293] and Kulkarni [320].

8.4 Aryabhata's Method for Extracting Square and Cube Roots

Following the traditions of his period, Aryabhata does not provide details to find square and cube roots; however, it has been concluded that his method is based on decimal place-value system, and the equalities $(ab)^2 = (10a+b)^2 = (10a)^2 + [2(10a)+b]b$ and $(ab)^3 = (10a+b)^3 = (10a)^3 + [3(10a)^2 + 3(10a)b + b^2]b.$ An important feature of his method is that it finds each digit of the root successively, from left to right. His method is still taught in schools. We shall summarize his method in simplified terms through the following examples.

To find the square root of 625, we group it in two's from right to left as $6\ \overline{25}.$ Now search largest possible integer *a* such that $a^2 \le 6$, which is obviously $a = 2$. This will be the first digit of the required square root. The next step is to find $6 - a^2 = 6 - 2^2 = 2,$ and with this adjoin $25,$ i.e., $225.$ Now find largest possible integer *b* such that $[2(10a)+b]b = (40+b)b \le 225$, which is obviously $b = 5$. This will be the next digit of the required square root. Since $225 - (45)5 = 0,$ it follows that $\sqrt{625} = 25.$

To find the square root of 474721, we group it in two's from right to left as $\overline{47}\ \overline{47}\ \overline{21}.$ Search largest possible integer *a* such that $a^2 \le 47,$ which is $6.$ Now, we find $47 - a^2 = 47 - 6^2 = 11,$ and with this adjoin

$47,$ i.e., 1147 and find largest possible integer *b* such that $[2(10a)+b]b=(120+b)b\le 1147,$ which is $8.$ Next, we find $1147-128\times 8=1147-1024=123.$ Finally, with this we adjoin $21,$ i.e., 12321 and find largest possible integer *c* such that $[2(10ab)+c]c\le 12321,$ i.e., $(1360+c)c\le 12321,$ which is $9,$ and the equality holds. Thus, $\sqrt{474721}=689.$

- Viéte noted that if one needs to calculate the square root of 2 to a high degree of accuracy, one should add as many zeros as necessary, and calculate the square root of, for example, $20{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000$. That root he shows to be $141{,}421{,}356{,}237{,}309{,}505$, and thus, the square root of 2 is approximately
$$1\frac{41{,}421{,}356{,}237{,}309{,}505}{100{,}000{,}000{,}000{,}000{,}000}.$$

We note that Aryabhata's Method explained above for 625 and $474721,$ combined with Viéte's observation easily computes the same approximation of $\sqrt{2},$ except instead of the last digit $5,$ we get 4; however, if we compute one more digit (which is 8) and then round it, then it is indeed $5.$

To find the cube root of 1728, we group it in three's from right to left as $1\ \overline{728}$. We search largest possible integer *a* such that $a^3\le 1$, which is $1.$ This will be the first digit of the required cube root. Since $1-1^3=0,$ for the next digit we consider 728 and find largest possible integer *b* such that $[3(10a)^2+3(10a)b+b^2]b\le 728,$ which is $2,$ and the equality holds. Thus, $\sqrt[3]{1728}=12.$

To find the cube root of 12977875, we group it in three's from right to left as $12\ \overline{977}\ \overline{875}.$ We search largest possible integer *a* such that $a^3\le 12,$ which is 2. This will be the first digit of the required cube root. Now, we find $12-a^3=12-2^3=4,$ and with this adjoin $977,$ i.e., 4977 and find largest possible integer *b* such that $[3(10a)^2+3(10a)b+b^2]b\le 4977$, which is $3.$ This will be the second digit of the required cube root. Next, we calculate $4977-[3(10a)^2+3(10a)b+b^2]b=4977-4167=810,$ and with this we adjoin $875,$ i.e., $810875.$ Finally, we find largest possible integer *c* such that $[3(10ab)^2+3(10ab)c+c^2]c \le 810875$, which is $5,$ and the equality holds. Thus, $\sqrt[3]{12977875}=235.$

To find the cube root of 961504803, we group it in three's from right to left as $\overline{961}\ \overline{504}\ \overline{803}.$ We search largest possible integer *a* such that $a^3 \le 961$, which is $9.$ This will be the first digit of the required cube root. Now, we find $961-a^3=961-9^3=232,$ and with this adjoin $504,$ i.e., 232504 and find largest possible integer *b* such that $[3(10a)^2+3(10a)b+b^2]b \le 232504,$ which is $8.$ This will be the second digit of the required cube root. Next, we calculate $232504-[3(10a)^2+3(10a)b+b^2]b=232504-212192=20312,$ and with this we adjoin $803,$ i.e., $20312803.$ Finally, we find largest possible integer *c* such that $[3(10ab)^2+3(10ab)c+c^2]c \le 20312803$, which is $7,$ and the equality holds. Thus, $\sqrt[3]{961504803}=987.$

As for the square root, we can add as many zeros as necessary, and calculate the cube root with desired accuracy.

8.5 Babylonians Tablet YBC 7289

There are numerous examples suggesting that Babylonians assembled large number of tables consisting of squares and square roots and cubes and cubic roots. It has been suggested by several historians of mathematics, e.g., Katz in his book [301] that "when square roots are needed in solving problems, the problems are arranged so that the square root is one that is listed in a table and is a rational number. However, where an irrational square root is needed, in particular, for $\sqrt{2},$ the result is generally written as $1;25\ \left(=1\frac{5}{12}\right)$." On a fascinating tablet from Yale Babylonian Collection (YBC) number 7289 (around 1800–1600 BC), there is a scatter diagram of a square with side indicated as 30 and two numbers, see Fig. 8.5,

$$1;24,51,10=\frac{1}{1}+\frac{24}{60}+\frac{51}{60^2}+\frac{10}{60^3}=\frac{30547}{21600}=1.41421296\cdots \qquad (8.7)$$

and

$$42;25,35=42+\frac{25}{60}+\frac{35}{60^2}=\frac{30547}{720}=42.42638888\cdots$$

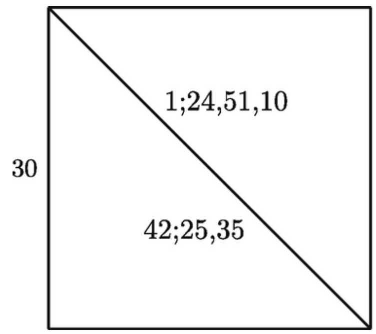

Fig. 8.5 Babylonians tablet YBC 7289

The product of 30 by $1;24,51,10$ is exactly $42;25,35$. Therefore, it is justifiable to presume that the number $42;25,35$ represents the length of the diagonal and the number $1;24,51,10$ is $\sqrt{2}$. This confirms that Babylonians had enormous computational skills. The mathematical significance of this tablet was first recognized by the historians Neugebauer and Sachs. This tablet provides the correct value of $\sqrt{2}$ to six decimal digits. For further details, see Fowler and Robson [193]. The same Babylonian approximation of $\sqrt{2}$ was used later by Ptolemy in his *Almagest*, but he did not mention from where this approximation came, perhaps it was well known by his time. Boyer in his book [100] writes $1;24,51,10 \simeq 1.414222$, which actually corresponds to $1;24,51,12$.

As in Sulbasutras, there is no record how Babylonians obtained the approximations $1;25$ or $1;24,51,10$, of $\sqrt{2}$; however, definitely they must have realized that the exact value of $\sqrt{2}$ cannot be achieved. Thus, the methods that have been suggested by the historians are merely speculative. For example, Katz [301] believes that Babylonians used the algebraic identity $(x+y)^2 = x^2 + 2xy + y^2$, which they might have perceived geometrically. Mathematically, the problem is for a given square of area *N*, we need to find its side $\sqrt{N}$. For this, as a first step we select a *regular number* (evenly dividable of powers of 60) *a* close to, but less than, $\sqrt{N}$ (a good guess). Letting $b = N - a^2$, the next step is to find *c* so that $2ac + c^2$ is as close as possible to b, see Fig. 8.6. If a^2 is "close enough" to N, then c^2 will be small in relation to $2ac$, so *c* can be chosen equal to $b/2a$, that is,

$$\sqrt{N} = \sqrt{a^2 + b} \simeq a + \frac{b}{2a} = a + \frac{N - a^2}{2a} = \frac{1}{2}\left(a + \frac{N}{a}\right). \qquad (8.8)$$

A similar argument shows that if *a* is greater than $\sqrt{N}$, then

$$\sqrt{N} = \sqrt{a^2 - b} \simeq a - \frac{b}{2a} = a - \frac{a^2 - N}{2a} = \frac{1}{2}\left(a + \frac{N}{a}\right). \qquad (8.9)$$

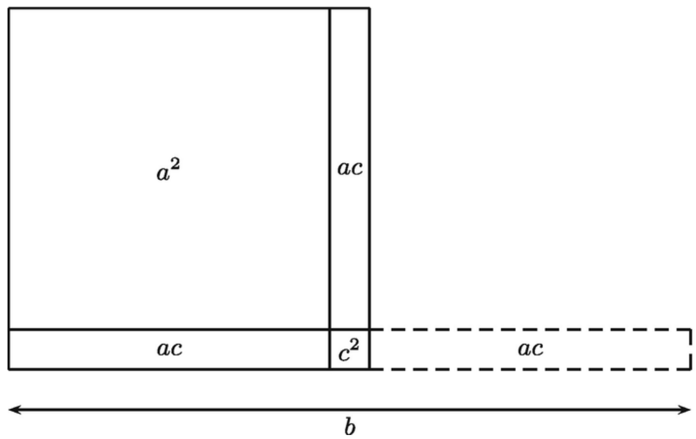

Fig. 8.6 Approximation of $\sqrt{N}$

For $\sqrt{2}$, we begin with $a = 1;20\ (= 4/3) < \sqrt{2}$, (see (8.1)), to obtain $a^2 = 1;46,40\ (= 16/9)$, $b = 0;13,20\ (= 2/9)$ and $b/2a = 0;05\ (= 1/12)$. Thus, from (8.8) it follows that $\sqrt{2} \simeq \sqrt{1;46,40 + 0;13,20} \simeq 1;20 + 0;05 = 1;25\ (= 17/12) > \sqrt{2}$, (see (8.1)). Similarly, if we choose $a = 3/2 > \sqrt{2}$, then (8.9) also gives $\sqrt{2} \simeq 17/12$. Now we choose $a = 17/12$ and apply (8.9), to get

$$\sqrt{2} \simeq \sqrt{\left(\frac{17}{12}\right)^2 - \frac{1}{144}} \simeq \frac{17}{12} - \frac{1/144}{34/12} = \frac{17}{12} - \frac{1}{408} = \frac{577}{408},$$

which is same as (8.1). Thus, we get all steps for $\sqrt{2}$ given in (8.1). Next, since $577/408 > \sqrt{2}$, we again use (8.9), to obtain

$$\sqrt{2} \simeq \frac{665857}{470832} = 1.4142135623746\cdots, \qquad (8.10)$$

which is correct to 11 decimal places. Jinabhadra Gani used (8.8) to obtain

$$\sqrt{58545048750} = 241960\frac{407150}{483920}.$$

Now since $(\sqrt{N} - a)^2 = N + a^2 - 2a\sqrt{N} > 0$, (equality holds only when $\sqrt{N} = a$), it follows that $(a + N/a)/2 > \sqrt{N}$. Thus, when we choose $a < \sqrt{N}$, after applying (8.8), for further improvement we have to proceed to (8.9). Having this in mind, and looking (8.8) and (8.9), we can write the following algorithm to compute $\sqrt{N}$, also see Boyer [100], and Ernst Sondheimer and Alan Rogerson [490]:

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{N}{a_n}\right), \quad n \geq 0 \tag{8.11}$$

where $a_0 > 0$ is any number (greater than or smaller than $\sqrt{N}$), known as the *initial approximation*. Today, algorithm (8.11) is derived by using Newton's method: With appropriate x_0 the iterative scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \geq 0 \tag{8.12}$$

converges quadratically to a root of the general equation $f(x) = 0.$ In our case the equation is $f(x) = x^2 - N = 0.$ For $N = 2$, this is perhaps one of the oldest known algorithms. Historians Neugebauer and Sachs believed that the Babylonians obtained this algorithm for $N = 2$ based on the following principle: Suppose *a* is a guess, which is too small (large), then $2/a$ will be a guess, which is too large (small). Hence, their average $(a + 2/a)/2$ is a better approximation. This assumption that "divide and average" seems to be a general procedure of Babylonians for approximating square roots.

In the literature, the algorithm (8.11) is also known as Heron's method, who gave the first explicit description of the method in his treatise *Metrica*, which was discovered as recently as 1896 in Constantinople in a manuscript form dating from the eleventh or twelfth century. Heron used the formula (8.9), i.e., $\sqrt{N} = \sqrt{a^2 - b} \simeq a - (1/2)b(1/a)$ to calculate the square roots: "Since 720 has not a rational root, we shall make a close approximation to the root in this manner. Since the square nearest to $720 = N$ is $729 = a^2$, having a root $a = 27$, divide 27 into 720, i.e., N/a the result is $N/a = 26\frac{2}{3}$; add $a = 27$, the result is $N/a + a = 53\frac{2}{3}$. Take half of this, i.e., $\frac{1}{2}\left(\frac{N}{a} + a\right) = \frac{1}{2}(a^2 - b + a^2)/a = a - (1/2)b(1/a)$; the result is $26\frac{5}{6}$. Therefore, the square root of 720 will be very nearly $26\frac{5}{6}$. For $26\frac{5}{6}$ multiplied by itself gives $720\frac{1}{36}$; so that the difference is $1/36.$ if we wish to make the difference less than $1/36,$ instead of 729 we shall take the number now found $720\frac{1}{36}$, and by the same method we shall find an approximation differing by much less than $1/36''$. Heron also found approximate square root of 63. The algorithm (8.11) generates a sequence $\{a_n\}$, for which the concept of convergence was not existing even during the time of Heron. For the convergence of the sequence $\{a_n\}$ the following result is well-known, for example, see Agarwal and Hans Agarwal [19]: For the sequence $\{a_n\}$ the following hold

$$\frac{a_n - \sqrt{N}}{a_n + \sqrt{N}} = \left(\frac{a_{n-1} - \sqrt{N}}{a_{n-1} + \sqrt{N}}\right)^2 = \cdots = \left(\frac{a_0 - \sqrt{N}}{a_0 + \sqrt{N}}\right)^{2^n} \tag{8.13}$$

and the fact that $|a_0 - \sqrt{N}|/|a_0 + \sqrt{N}| < 1$. The convergence (quadratic) of this sequence to $\sqrt{N}$ immediately follows from (8.13). From (8.13), we also note that $a_n - \sqrt{N} > 0$ for all $n \geq 1$. It also follows directly from the arithmetic-geometric mean inequality, in fact, for all $n \geq 1$, we have

$$a_n = \frac{1}{2}\left(a_{n-1} + \frac{N}{a_{n-1}}\right) \geq \sqrt{a_{n-1} \cdot \frac{N}{a_{n-1}}} = \sqrt{N},$$

with equality if and only if $a_{n-1}^2 = N$. Hence, the sequence $\{a_n\}$ is bounded below by $\sqrt{N}$. Further, since

$$a_{n+1} - a_n = \frac{1}{2a_n}(N - a_n^2) < 0$$

the sequence $\{a_n\}$ is decreasing. Thus, the sequence $\{a_n\}$, in fact, converges monotonically.

Jöran Friberg (born 1934, Sweden) in his book [198] mentions that Babylonian tablets (such as MS 3051) contain computations of areas of hexagons and heptagons, which involve the approximation of more complicated algebraic numbers. The answer given there leads to the simple approximation $\sqrt{3} \simeq 7/4$. This does not mean they could not have calculated better approximations.

In Table 8.1, we use (8.11) to compute first three iterates for $\sqrt{2}$ and $\sqrt{3}$.

Table 8.1 Monotone convergence

n	$N=2$	$N=2$	$N=3$	$N=3$	$N=3$
0	$\frac{4}{3}$	$\frac{3}{2}$	2	$\frac{5}{3}$	$\frac{3}{2}$
1	$\frac{17}{12}$	$\frac{17}{12}$	$\frac{7}{4}$	$\frac{26}{15}$	$\frac{7}{4}$
2	$\frac{577}{408}$	$\frac{577}{408}$	$\frac{97}{56}$	$\frac{1351}{780}$	$\frac{97}{56}$
3	$\frac{665857}{470832}$	$\frac{665857}{470832}$	$\frac{18817}{10864}$	$\frac{3650401}{2107560}$	$\frac{18817}{10864}$

From this table, it is clear that the algorithm (8.11) gives both Sulbasutras approximations (8.1) and (8.5) of $\sqrt{2}$ and $\sqrt{3}$. It also gives Babylonian approximation $\sqrt{3} \simeq 7/4$. Unfortunately, from (8.11) we cannot get the Babylonians approximation (8.7) of $\sqrt{2}$. In fact, reversing a step in (8.11) leads to the equation

$$\frac{30547}{21600} = \frac{1}{2}\left(x + \frac{2}{x}\right) \quad \text{i.e.} \quad 10800x^2 - 30547x + 21600 = 0,$$

which has only complex roots. Another simple explanation is $30547/21600 < \sqrt{2}$, whereas $a_n > \sqrt{2},\ n \geq 1$. We also note that Boyer in his book [100] has made a false assertion that a_2 with $a_0 = 3/2$ for $\sqrt{2}$ gives (8.7). In conclusion, Babylonians obtained (8.7) by some other unknown technique rather than (8.11), as has been claimed. A probable explanation for (8.7) is that Babylonians from their tables of n^2 and $2n^2,\ n \geq 1$ noticed that $933119209 = (30547)^2 \simeq 2(21600)^2 = 933120000.$

Algorithm (8.11) at *n*-th iteration requires division by a_n, to avoid this we consider the equation $f(x) = (1/x^2) - N = 0$ and apply Newton's method (8.12), to get

$$x_{n+1} = \frac{3}{2}x_n - \frac{N}{2}x_n^3, \quad n \geq 0$$

which converges to $1/\sqrt{N}$. We multiply this by *N* and let $a_n = Nx_n$, to obtain

$$a_{n+1} = \frac{a_n}{2N}(3N - a_n^2), \quad n \geq 0$$

which converges quadratically to $\sqrt{N}$. For $N = 2$ with $a_0 = 3/2$, the aforementioned scheme gives $a_1 = 45/32,\ a_2 = 185355/131072$. These approximations of $\sqrt{2}$ are different from the corresponding entries in Table 8.1.

Problem xviii from the combined Babylonian tablet fragments BM 96957 and VAT 6598 gives two methods for calculating the diagonal *d* of a rectangle with sides of length $a = 40$ and $b = 10$ units. The first leads (in specific numbers) to the approximation

$$d \simeq a + \frac{2ab^2}{3600}, \tag{8.14}$$

and the second method to the approximation is

$$d \simeq a + \frac{b^2}{2a}. \tag{8.15}$$

From Pythagorean theorem $d = \sqrt{40^2 + 10^2} = 41.231056\cdots$. Formulas (8.8), (8.9), (8.14), and (8.15), respectively, give the approximations

$$\begin{aligned}
\sqrt{41^2 + 19} &= 41 + \frac{19}{2 \times 41} = 41.231707\cdots \\
\sqrt{42^2 - 64} &= 42 - \frac{64}{2 \times 42} = 41.238095\cdots \\
\sqrt{40^2 + 10^2} &= 40 + \frac{2 \times 40 \times 10^2}{3600} = 42.222222\cdots \\
\sqrt{40^2 + 10^2} &= 40 + \frac{10^2}{2 \times 40} = 41.25
\end{aligned}$$

A problem similar to one considered in Sect. 5.12(3), also from Cairo Papyrus, which is particularly interesting, in modern terms requires the solution of the system of equations

$$x^2 + y^2 = 225, \quad xy = 60.$$

Again, the scribe's method of solution amounts to adding and subtracting $2xy = 120$ from the equation $x^2 + y^2 = 225$, to get

$$(x + y)^2 = 345, \quad (x - y)^2 = 105$$

or equivalently,

$$x + y = \sqrt{345}, \qquad x - y = \sqrt{105}.$$

And, now employing (8.8), to obtain the approximations

$$x + y = \sqrt{345} = \sqrt{18^2 + 21} \ \simeq \ 18 + \frac{21}{36} = 18 + \frac{1}{2} + \frac{1}{12}$$

and

$$x - y = \sqrt{105} = \sqrt{10^2 + 5} \ \simeq \ 10 + \frac{5}{20} = 10 + \frac{1}{4}.$$

In an old Babylonian tablet (around 2000 BC) found in 1936 in Susa, for the irrational number π, the following expression appears

$$\frac{3}{\pi} = \frac{57}{60} + \frac{36}{(60)^2},$$

which yields $\pi = 3\ 1/8 = 3.125$. Babylonians were also satisfied with $\pi = 3$.

- Bakhshali manuscript to find an approximate root of a non-square number says "In case of a non-square (number), subtract the nearest square number; divide the remainder by twice (the root of that number). Half the square of that (that is, the fraction just obtained) is divided by the sum of the root and the fraction and subtract; (this will be the approximate value of the root) less the square (of the last term)." Thus, if $N = a^2 + b$, then

$$\sqrt{N} = \sqrt{a^2 + b} \ \simeq \ a + \frac{b}{2a} - \frac{(b/2a)^2}{2(a + b/2a)}. \tag{8.16}$$

In fact, to obtain (8.16) both (8.8) and (8.9) are used. Let *a* be the largest integer such that a^2 is less than N, and $N = a^2 + b$. Then, (8.8) gives

$$\sqrt{N} \ \simeq \ a + \frac{b}{2a} \quad \text{and} \quad \left(a + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 = N.$$

Thus, we can use (8.9), to get

$$\sqrt{N} = \sqrt{\left(a + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2} \ \simeq \ a + \frac{b}{2a} - \frac{(b/2a)^2}{2(a + b/2a)}.$$

Since $b = N - a^2$ from (8.16) it follows that

$$\sqrt{N} \simeq a + \frac{N - a^2}{2a} - \frac{(N - a^2)^2}{4a(N + a^2)} = \frac{a^2(a^2 + 6N) + N^2}{4a(a^2 + N)}. \tag{8.17}$$

Now let *a* be the smallest integer such that a^2 is greater than N, and $N = a^2 - b$. Then, (8.9) gives

$$\sqrt{N} = \sqrt{a^2 - b} \simeq a - \frac{b}{2a} \quad \text{and} \quad \left(a - \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 = N.$$

Thus, we can use (8.9) again, to get

$$\sqrt{N} = \sqrt{\left(a - \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2} \simeq a - \frac{b}{2a} - \frac{(b/2a)^2}{2(a - b/2a)}. \tag{8.18}$$

Since $b = a^2 - N$ from (8.1838) it follows that

$$\sqrt{N} \simeq a - \frac{a^2 - N}{2a} - \frac{(a^2 - N)^2}{4a(a^2 + N)} = \frac{a^2(a^2 + 6N) + N^2}{4a(a^2 + N)}. \tag{8.19}$$

Relations (8.17) and (8.19) lead to the algorithm

$$a_{n+1} = \frac{a_n^2(a_n^2 + 6N) + N^2}{4a_n(a_n^2 + N)}, \quad a_0 = a, \quad n \geq 0. \tag{8.20}$$

Clearly, in (8.20) we can take *a* any convenient real number so that a^2 is close to N. Further, from our considerations it is clear that the iterative scheme (8.20) converges quartically. In Table 8.2, we give a few iterates for $N = 2, 3$, and $41, 105, 481$ considered in Bakhshali Manuscript.

Table 8.2 Quartic convergence

n	$N = 2$	$N = 3$	$N = 41$	$N = 105$	$N = 481$
0	1	1	6	10	21
1	$\frac{17}{12}$	$\frac{7}{4}$	$\frac{11833}{1848}$	$\frac{3361}{328}$	$\frac{1698568}{77448}$
2	$\frac{665857}{470832}$	$\frac{18817}{10864}$	A_1	A_2	A_3
0	2	2	7	11	22
1	$\frac{17}{12}$	$\frac{97}{56}$	$\frac{2017}{315}$	$\frac{12737}{1243}$	$\frac{1862441}{84920}$

n	$N=2$	$N=3$	$N=41$	$N=105$	$N=481$
2	$\frac{665857}{470832}$	$\frac{708158977}{408855776}$	A_4	A_5	A_6

In this table

$$A_1 = \frac{1568438544425524193}{244948947747743008}, \qquad A_2 = \frac{1020854854709761}{99625232718112}$$
$$A_3 = \frac{20322223263651344335681}{9266140487240050641}, \qquad A_4 = \frac{1261006858463}{196936184856}$$
$$A_5 = \frac{26318786070520577}{2568450524613787}, \qquad A_6 = \frac{96254287085727658170489761}{4388817717938678567053280}.$$

An immediate extension of (8.12) and (8.20) for any nonlinear equation $f(x) = 0$ is

$$b_n = a_n - \frac{f(a_n)}{f'(a_n)}, \quad a_{n+1} = b_n - \frac{f(b_n)}{2\left(\frac{f(b_n) - f(a_n)}{b_n - a_n}\right) - f'(a_n)}, \quad n \geq 0.$$

For this algorithm and its higher-order extensions and their scope in real-word computation, see Sen et. al. [465].

8.6 Great Pyramid at Gizeh and Rhind Mathematical Papyrus

As we have noted earlier from the dimensions of the Great Pyramid at Gizeh, it is possible to derive the golden ratio φ. Interestingly, it is also possible to drive the number π. In Rhind mathematical papyrus problem number 50 states that a circular field with a diameter of 9 units in area is the same as a square with sides of 8 units, i.e., $\pi\,(9/2)^2 = 8^2$, and hence, $\pi = 4 \times (8/9)^2 = 3.16049\cdots$. For details of Egyptian contribution to quadrature of circle, see the work of Hermann in [260].

8.7 Proofs of the Irrationality of $\sqrt{2}$

Among the many known proofs of the irrationality of $\sqrt{2}$ here we shall give a few that are of historical importance. The following first fully geometric proof appeared in the *Meno* (Socratic dialog by Plato).

Following the Website http://mitp-content-server.mit.edu:18180/books/content/sectbyfn?collid=books_press_0&id=1043&fn=9780262661829_schh_0001.pdf, in the square *ABCD* we use a compass to cut off $AF = AD$ along the diagonal $CA.$ At *F* draw the perpendicular *EF* (see Fig. 8.7). Then the ratio of *CE* to *CF* (hypotenuse to side) will be the same as the ratio of *AC* to *AD*, since the triangles *CDA* and *EFC* are similar. Suppose that *DC* and *CA* were commensurable. Then there would be a segment δ such that both *DC* and *CA* were integral multiples of δ. Since $AF = AD,$ then $CF = CA - AF$ is also a multiple of $\delta.$ Note also that $CF = EF,$ because the sides of triangle *EFC* correspond to the equal sides of triangle $CDA.$ Further, $EF = DE$ because (connecting *A* and *E*) triangles *EDA* and *EFA* are congruent. Thus, $DE = CF$ is a multiple of $\delta.$ Then $CE = CD - DE$ is also a multiple of $\delta.$ Therefore, both the side *CF* and hypotenuse *CE* are multiples of δ, which therefore is a common measure for the diagonal and side of the square of side $CF.$ The process can now be repeated as follows: on *EC* cut off $EG = EF$ and construct *GH* perpendicular to $CG.$ The ratio of hypotenuse to side will still be the same as it was before and hence, the side of the square on *CG* and its diagonal also share δ as a common measure. Because we can keep repeating this process, we will eventually reach a square whose side is less than $\delta,$ contradicting our initial assumption. Therefore, there is no such common measure $\delta.$ The demonstration given here involves the method of infinite descent.

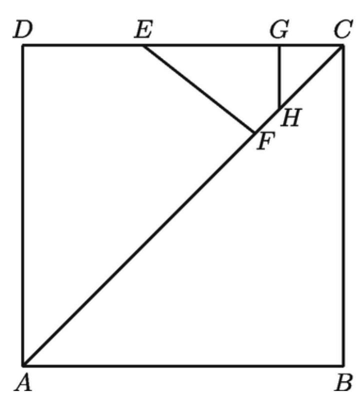

Fig. 8.7 Incommensurability of $\sqrt{2}$ in Meno

- The following inquisitive geometric proof of Tom Mike Apostol (1923–2016, USA) [34] (also for similar proofs see earlier books by Kiselev

[307], and Conway and Guy [138]) is in line with the aforementioned proof. A circular arc with center at the uppermost vertex and radius equal to the vertical leg of the triangle intersects the hypotenuse at a point, from which a perpendicular to the hypotenuse is drawn to the horizontal leg (see Fig. 8.8). Each line segment in the diagram has integer length, and the three segments with double tick marks have equal lengths. (Two of them are tangents to the circle from the same point.) Therefore, the smaller isosceles right triangle with hypotenuse on the horizontal base also has integer sides.

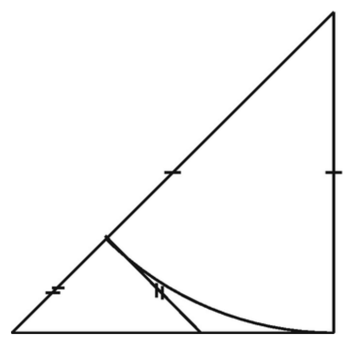

Fig. 8.8 Incommensurability of $\sqrt{2}$ by Apostol

- The first semi-geometric proof of the irrationality of $\sqrt{2}$ is due to Aristotle, which appeared in his *Analytica Priora*. He concludes that if the side and the diagonal are assumed commensurable, then odd numbers are equal to even numbers. For this, he used the method of contradiction: Suppose that the side *EH* and the diagonal $HF,$ see Fig. 8.9, are commensurable, i.e., each can be expressed by the number of times it is measured by their common measure. Now it can be assumed that at least one of these numbers is odd, if not there would be a longer common measure. Then the squares *HEFG* and *ABCD* on the side and diagonal, respectively, represent square numbers. From the Fig. 8.9, it is clear that the area of the latter square is clearly double the former, thus it represents an even square number. Consequently, its side $AB = HF$ is also an even number, and thus, the square *ABCD* is a multiple of four. Finally, since *HEFG* is half of $ABCD,$ it must be a multiple of two, i.e, it is also an even square. Therefore, its side *EH* must also be even. However, this contradicts the original assumption that one of $HF,\ EH$ is odd. In conclusion, the two lines *EH* and *HF* are incommensurable. Thus, Aristotle in number theory succeeded in proving the existence of irrationals.

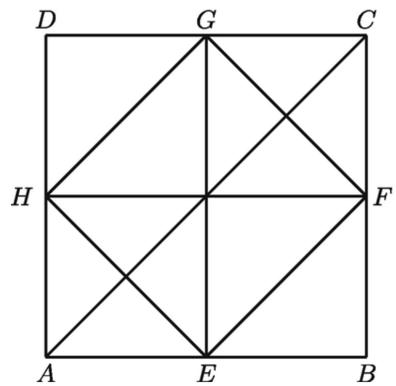

Fig. 8.9 Incommensurability of $\sqrt{2}$ in Analytica Priora

From Fig. 8.9, it is clear that the area of *ABCD* is the same as two times the area of $HEFG.$ This construction is due to Socrates in the *Meno*. In Plato's Republic, Socrates jokes that young people "are as irrational as lines" and hence not yet suited to "rule in the city and be the sovereigns of the greatest things." His joke points to a widely held sense that irrationality in mathematics was a troubling sign of confusion and disorder in the world.

- Euclid's semi-geometrical demonstration by the method of contradiction of the irrationality of $\sqrt{2}$ is given in Book X, Proposition 27. Though it is less perspicuous than the strictly arithmetical proof now, it is more suggestive historically, and more precise than Aristotle's proof. The argument goes as follows: If the diagonal *AC* and side *AB* of the square *ABCD* (see Fig. 8.9) have a common measure, say $\delta,$ then there exist $p, q \in \mathcal{N}$ satisfying $AC = p\delta,\ AB = q\delta.$ The ratio of these segments is

$$\frac{AC}{AB} = \frac{p}{q}. \tag{8.21}$$

In what follows, we can assume that common factors of *p* and *q* have been canceled, i.e., $\gcd(p, q) = 1.$ Thus, at least one of them is odd. Squaring the identity (8.21), we have

$$\frac{(AC)^2}{(AB)^2} = \frac{p^2}{q^2}. \tag{8.22}$$

Now in view of Pythagorean theorem in the triangle $ABC,$ we find $(AC)^2 = 2(AB)^2,$ so that (8.22) is the same as

$$2 = \frac{p^2}{q^2} \quad \text{or} \quad p^2 = 2q^2. \tag{8.23}$$

Now since $2q^2$ is an even integer, p^2 must also be even. But, then *p* is also even, i.e., $p = 2k$. Substituting this in the equation $p^2 = 2q^2$ gives $q^2 = 2k^2$. But then q^2 and hence *q* is also an even number. In conclusion, both *p* and *q* are even, which contradicts our initial assumption that they have no common factor or one of them is odd.

- In the aforementioned proof we can ignore all geometric arguments, and directly proceed to algebraic equation (8.23), where *p* and *q* are in its lowest term, and hence are of different parity. Then, showing that $\sqrt{2}$ is irrational is equivalent to proving that (8.23) is impossible. For this, the Website http://www.cut-the-knot.org/proofs/sq_root.shtml contains 29 proofs. Here we shall discuss a few of them.
- Since $\gcd(p, q) = 1,$ from Corollary 3.3 there exist integers *x* and *y* such that $1 = px + qy.$ Thus, from (8.23) it follows that

$$\sqrt{2} = (\sqrt{2}p)x + (\sqrt{2}q)y = 2qx + py,$$

which leads to a contradiction that $\sqrt{2}$ is an integer.
- Lagrange in his *Lectures on Elementary Mathematics* of 1898 argues that if *p* and *q* are in its lowest terms, then p^2 and q^2 are also in its lowest terms. Since fraction p^2/q^2 is built from the fraction p/q, it cannot be a whole number $2.$ A similar reasoning appeared in 1831 in the work of De Morgan.
- Whittaker and Watson in their book [532] of 1920, and later Gardner [208], and Miklós Laczkovich (born 1948, Hungary) [325] in their books assumed that in $\sqrt{2} = p/q$ the integer *q* is the smallest possible such number. Their main argument is essentially to use the equality $(2q - p)^2 = 2(p - q)^2$ which is true if and only if (8.23) holds. Thus, it follows that

$$2 = \frac{(2q - p)^2}{(p - q)^2} \quad \text{or} \quad \sqrt{2} = \frac{2q - p}{p - q},$$

but $1 < p/q < 2$ implies that $q > p - q > 0.$ This contradicts the minimality of $q.$

It is interesting to note that

$$\sqrt{2} = (\sqrt{2}+1) - 1 = \frac{1}{\sqrt{2}-1} - 1 = \frac{1}{p/q-1} - 1 = \frac{2q-p}{p-1}.$$

- Rademacher and Toeplitz in their book of 1957 [425, Chapter 4] assert that (8.23) implies *p* is even, so *q* must be odd. However, the square of an even number is divisible by 4, which leads to conclude that *q* must be even. Thus, we have Aristotle-type contradiction.
- There is a general consensus that Euclid recognized that irrational numbers simply did not belong in a work based on arithmetic, some authors claim that Euclid in Book X, Proposition 117 uses fundamental theorem of arithmetic (Theorem 4.1) to almost show the impossibility of (8.23), but most of the English translations of *Elements* have only 115 propositions. Von Fritz [202] indicates that the early Greek mathematicians did not explicitly use Theorem 4.1 to prove the irrationality of $\sqrt{2}.$ In fact, on the Website http://people.math.harvard.edu/~mazur/preprints/Eva.Nov.20.pdf, posted in 2005, Mazur claims that the explicit use of Theorem 4.1 is post Gauss. Anyway, in view of Theorem 4.1, *p* and *q* can be factored uniquely into their prime factors, so let $p = p_1p_2\cdots p_r$ and $q = q_1q_2\cdots q_s.$ Putting this back in Eq. (8.23), we get

$$(p_1p_2\cdots p_r)^2 = 2(q_1q_2\cdots q_s)^2,$$

or

$$p_1p_1p_2p_2\cdots p_rp_r = 2q_1q_1q_2q_2\cdots q_sq_s. \tag{8.24}$$

Now among the primes p_i and $q_i,$ the prime 2 may occur (it will occur if either *p* or *q* is even). If it does occur, it must appear an even number of times on the left side of equation (8.24) (since each prime there appears twice), and an odd number of times on the right side (because 2 already appears there once). But, then we have a contradiction: since the factorization into primes is unique, the prime 2 cannot appear an even number of times on one side of the equation and an odd number on the other. Thus, Eq. (8.23) is impossible.
- From the uniqueness of the factorization, one can argue directly that p^2 has even number of prime factors, whereas $2q^2$ has odd number of prime factors, which is absurd.
- Some of the aforementioned illustrations can be extended to prove the result: If $N \in \mathcal{N}$, then $\sqrt{N}$ is a rational number if and only if $\sqrt{N}$ is

an integer. For this, first we model its proof due to Gardner [208]. Clearly, if $\sqrt{N}$ is an integer, then $\sqrt{N}$ is rational. Conversely, we assume that $\sqrt{N}$ is rational, i.e., it can be written as $\sqrt{N} = p/q,$ where $p, q \in \mathcal{N}$ and *q* is the smallest possible such integer. Let $k = [\sqrt{N}].$ Then, it follows that $k < p/q < k+1,$ and therefore $0 < p - kq < q.$ Now note that the equality $(Nq - kp)^2 = N(p - kq)^2$ is true if and only if $p^2 = Nq^2$ holds. Thus,

$$\sqrt{N} = \frac{Nq - kp}{p - kq},$$

but this contradicts the fact that *q* is the smallest.

Now we will apply Theorem 4.1. Again if $\sqrt{N}$ is an integer, then $\sqrt{N}$ is rational. Conversely, we assume that $\sqrt{N}$ is rational, i.e., it can be written as $\sqrt{N} = p/q,$ where $p, q \in \mathcal{N}$ and $\gcd(p, q) = 1.$ Since p/q is not an integer, $q \geq 2.$ Again, we have $p^2 = Nq^2.$ By Theorem 4.1, *q* has a prime factor $m.$ Thus, $m|Nq^2$ and so $m|p^2$, but then $m|p.$ Hence, $m|p$ and $m|q$, which contradicts our assumption that $\gcd(p, q) = 1.$

- Dedekind in his proof assumed that if *N* is not a square of an integer, then there exists a positive integer λ such that $\lambda^2 < N < (\lambda + 1)^2.$ Again, if *N* is rational, then there exist $p, q \in \mathcal{N}$ such that $p^2 - Nq^2 = 0,$ where *q* is the least possible integer possessing the property that its square multiplied by *N* is the square of $p.$ Since $\lambda q < p < (\lambda + 1)q,$ it follows that the integers $s = p - \lambda q$ and $t = Nq - \lambda p$ are positive, and we have $t^2 - Ns^2 = (\lambda^2 - N)(p^2 - Nq^2) = 0$, which contradicts the assumption on $q.$
- On the website https://www.quora.com/If-p-is-a-natural-number-but-not-a-perfect-nth-power-how-does-one-prove-that-the-nth-root-of-p-is-not-rational, Thomas Schürger (2019) has provided a very simple proof of the following general result: The *k*th, $k \in \mathcal{N},\ k \geq 2$ root of a nonnegative integer $N \geq 2$ is rational if and only if *N* is a perfect *k*th power. One direction of this statement is clearly true: the *k*th root of a *k*th power is rational. Let us prove the other direction via proof by contradiction. Let us assume that *N* is not a perfect *k*th power, and

$\sqrt[k]{N}$ is rational, i.e., $\sqrt[k]{N} = p/q$ for some p, q in $\mathcal{N}$ such that p/q is in lowest terms. Since

$$\frac{N}{1} = \frac{p^k}{q^k}$$

and p/q is in lowest terms p^k/q^k is also in lowest terms, and $N/1$ is clearly in lowest terms. It follows that $p^k = N$ and $q^k = 1$, which is a contradiction since we assumed that *N* is not a perfect *k*th power. Hence, $\sqrt[k]{N}$ must be an irrational number.

- Some of the aforementioned arguments need slight modification to prove: If *r* and *s* are distinct primes, then $\sqrt{rs}$ and $\log_r s$ are irrational. For example, to show $\log_r s$ is irrational, we assume contrary, i.e., $\log_r s = p/q,$ where $p, q \in \mathcal{N}.$ We can assume that $\gcd(p, q) = 1.$ Then $r^{p/q} = s$ and so $(r^{p/q})^q = s^q.$ Therefore, $r^p = s^q.$ Since $r|r^p,$ it follows that $r|s^q$ and so $r|s$, which is a contradiction.

8.8 Spiral of Theodorus

From the *dialogues* of Plato, we know that Theodorus demonstrated geometrically that the sides of squares represented by $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}, \sqrt{10}, \sqrt{11},$ $\sqrt{12}, \sqrt{13}, \sqrt{14}, \sqrt{15}$, and $\sqrt{17},$ are incommensurable with a unit length. That is, he showed the irrationality of the square roots of nonsquare integers from 3 to $17,$ "at which point," says Plato, "for some reason he stopped," see Fig. 8.10.

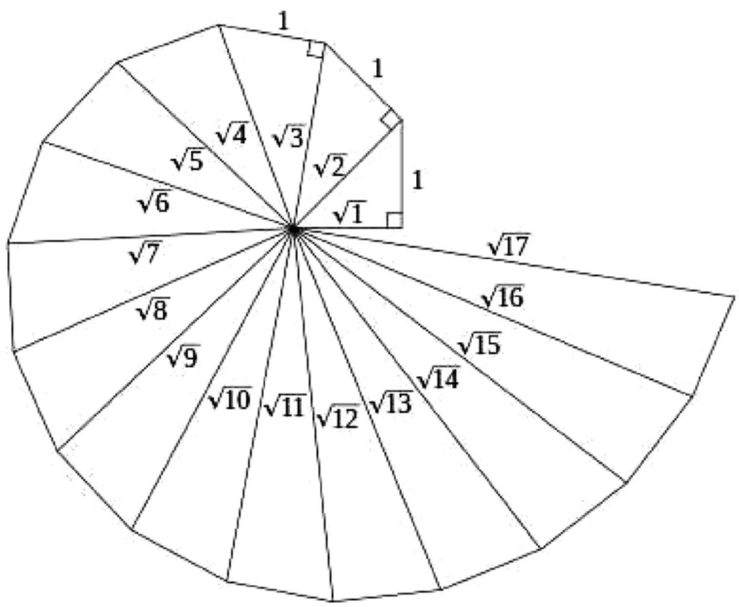

Fig. 8.10 Spiral of Theodorus

It has been speculated that Theodorus constructed his spiral based on right triangles with a common vertex, where in each triangle the side opposite the common vertex has length 1. The hypotenuse of the *n*th triangle then has length $\sqrt{n+1}$, follows immediately by Pythagorean theorem. His spiral also suggest possible reason Theodorus stopped at $\sqrt{17}$: On summing of the vertex angles for the first *n* triangles, we have

$$\tan^{-1}\left(\frac{1}{1}\right)+\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)+\cdots+\tan^{-1}\left(\frac{1}{\sqrt{n}}\right).$$

For $n = 16$ (which gives $\sqrt{17}$) this sum is $351.15°$, while for $n = 17$ the sum is $364.78°$. Thus, for $n > 16$, his spiral started to overlap itself (i.e., cuts the initial axis for the first time) and the drawing became "messy". Theaetetus, who was a pupil of Theodorus and a member of Plato's school in Athens, extended the result, demonstrating that the square root of any nonsquare integer is irrational, and the cube root of any number that is not a perfect cube is irrational. Of course, today, by induction one can draw $\sqrt{n}$ for any n. Also, if *n* is an odd integer, then $\sqrt{n}$ can be represented by the leg of a right triangle whose hypotenuse is $(n+1)/2$ and whose leg is $(n-1)/2$, i.e., $(\sqrt{n})^2 = [(n+1)/2]^2 - [(n-1)/2]^2$. Further, if *n* is an even integer, then $\sqrt{n}$ can be represented by half of the leg of a right triangle whose hypotenuse is $n+1$ and whose other leg is $n-1$, i.e.,

$(2\sqrt{n})^2 = (n+1)^2 - (n-1)^2.$ Plato himself also showed that a rational number could be the sum of two irrationals. In Fig. 8.11, we provide the construction of $\sqrt{5}$ and $\sqrt{6}$ geometrically.

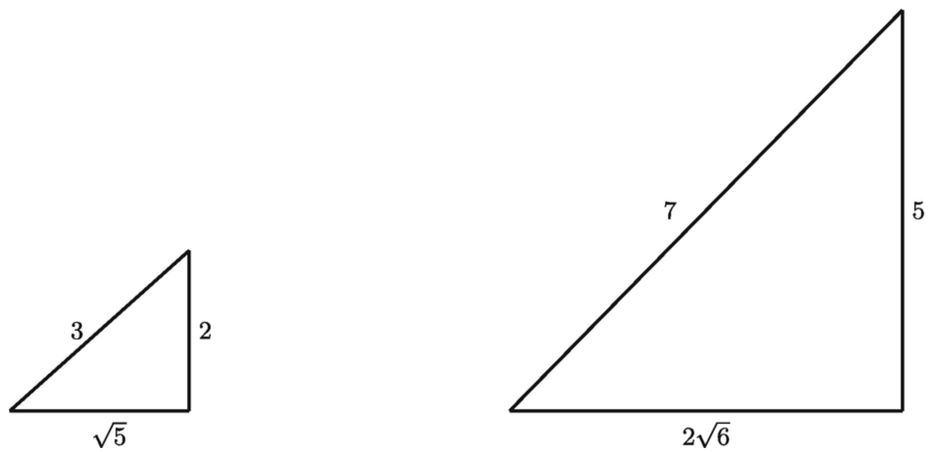

Fig. 8.11 Square roots of 5 and 6

8.9 Chinese Method for Square Root

Liu Hui in his commentary on the *Jiuzhang Suanshu* provided approximation of π as $3.141014,$ and in Chap. 4 *Shao guang* (Short width) suggested algorithms to find square and cube roots of numbers. For square roots the method is a combination of completing squares iteratively, and geometry, i.e., something like Fig. 8.12 always in mind, see Burgos and Beltrán-Pellicer [108], Katz [301], and Yong [543]. We explain the method by considering the problem 12, where square root of 55225 is calculated. We begin with finding the integers a, b, c so that the answer can be written as $100a + 10b + c$. We calculate the largest integer *a* so that $(100a)^2 < 55225$. Clearly, $a = 2$ is the right choice. The difference between the large (given) square (55225) and the square with side $100a = 200,$ i.e. (40000) in Fig. 8.12 is the large gnomon with area $55225 - 40000 = 15225.$ Now if we ignore the outer thin gnomon, then *b* must satisfy $15225 > 2(100a)(10b) = 4000b$, which gives the largest integer $b = 3.$ To verify that the choice $b = 3$ is correct, i.e., when the square on 10*b* included, the area of the large gnomon is still less than $15225,$ it is necessary to check that $2(100a)(10b) + (10b)^2 = 12900 < 15225.$ Since this is true, we can continue to find $c.$ For this, we need

$55225 - 40000 - 30(2 \times 200 + 30) > 2 \times 230c$ or $2325 > 460c.$ An easy check shows that the largest integer which satisfies this is $c = 5.$ Finally, since $(100a + 10b + c)^2 = (200 + 30 + 5)^2 = (235)^2$, the exact square root of 55225 is $235.$

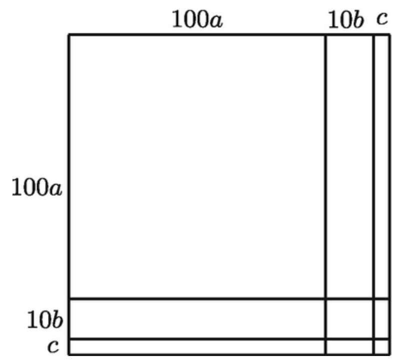

Fig. 8.12 Chinese method for square root

Similar to square roots, having cubes in mind there are examples in *Jiuzhang suanshu* to find cube roots of numbers. For example, it is shown that the cube root of 1860867 is the exact number $123.$ In case, answer is not an exact number, the procedure continues using decimal fractions. Later, Chinese extended their procedure to find roots of polynomial equations up to degree 10. For more details, see Li and Du [345].

8.10 π Before Archimedes

In what follows, chronologically, we shall list the growth and the value of π, which was used for a variety of practical problems before Archimedes.

- To find an approximate value of π, Aryabhata gives the following prescription: Add 4 to 100, multiply by 8 and add to 62,000. This is “approximately” the circumference of a circle whose diameter is 20,000. This means $\pi = 62{,}832/20{,}000 = 3.1416$. It is important to note that Aryabhata used the word asanna (approaching), to mean that not only is this an approximation of $\pi,$ but that the value is irrational. For more details, see [285, 286].
- The earliest Chinese Mathematicians, from the time of Chou-Kong (around 1200 BC) used the approximation $\pi = 3.$ Some of those who used this approximation were mathematicians of considerable attainments in other respects. According to the Chinese mythology, 3 is used because it is the number of the Heavens and the circle.

- In the *Old Testament*, about 950 BC, (I Kings vii.23, and 2 Chronicles iv.2), we find the following verse: "Also, he made a molten sea of ten cubits from brim to brim, round in compass, and five cubits the height thereof; and a line of thirty cubits did compass it round about." This description of the priests' bathing pool in Solomon's Temple seems to indicate that the ancient Jews held that π is $30/10 = 3.$ This value is 5% short of the actual. The Jewish Talmud, which is essentially a commentary on the Old Testament, was published about 500 AD. This shows that the Jews did not pay much attention to geometry. However, debates have raged on for centuries about this verse. According to some, it was just a simple approximation, while others say that "... the diameter perhaps was measured from outside, while the circumference was measured from inside." For other perspective, see Parker [406].
- *Shatapatha Brahmana* means *Priest manual of 100 paths* (around 900 BC) is one of the prose texts describing the Vedic ritual. It survives in two recensions, Madhyandina and Kanva, with the former having the eponymous 100 brahmanas in 14 books, and the latter 104 brahmanas in 17 books. In these books, π is approximated by $339/108 = 3.138888 \cdots$.
- Hippocrates established the formula πr^2 for the area of a circle in terms of its radius. It means that a certain number π exists, and is the same for all circles, although his method does not give the actual numerical value of $\pi.$ In trying to square the circle (unsuccessfully), we have seen in Sect. 5.4, he quadrated two moon-shaped figures.
- Anaxagoras, around 440 BC, while in prison wrote a treatise on the quadrature of the circle.
- Antiphon attempted to find the area of a circle by considering it as the limit of an inscribed regular polygon with an infinite number of sides. Thus, he provided preliminary concept of infinitesimal calculus.
- Bryson of Heraclea (born around 450 BC, Greece) considered the circle squaring problem by comparing the circle to polygons inscribed within it. He wrongly assumed that the area of a circle was the arithmetical mean between circumscribed and inscribed polygons.
- Hippias perhaps realized his *quadratrix* could also be used to square the circle, but failed to prove it.
- Aristophanes (446–386 BC, Greece) in his play *The Birds* makes fun of circle squarer's.

- Plato supposedly obtained for his day a fairly accurate value for $\pi = \sqrt{2} + \sqrt{3} = 3.146 \cdots$.
- Dinostratus used Hippias *quadratrix* to square the circle. For this, he proved *Dinostratus' theorem*. Hippias quadratrix later became known as the *Dinostratus quadratrix* also. However, his demonstration was not accepted by the Greeks as it violated the foundational principles of their mathematics, namely Euclidean tools.

8.11 Archimedes Approximations of π

Archimedes developed a general *method of exhaustion* to approximate the value of π. His method is based on the following arguments: the circumference of a circle lies between the perimeters of the inscribed and circumscribed regular polygons of *n* sides, and as *n* increases, the deviation of the circumference from the two perimeters becomes smaller. If a_n and b_n denote the perimeters of the inscribed and circumscribed regular polygons of *n* sides, and *C* the circumference of the circle, then it is clear that $\{a_n\}$ is an increasing sequence bounded above by C, and $\{b_n\}$ is a decreasing sequence bounded below by C. Both of these sequences converge to the same limit C. For simplicity, we choose a circle with the diameter 1, then from Fig. 8.13 it immediately follows that

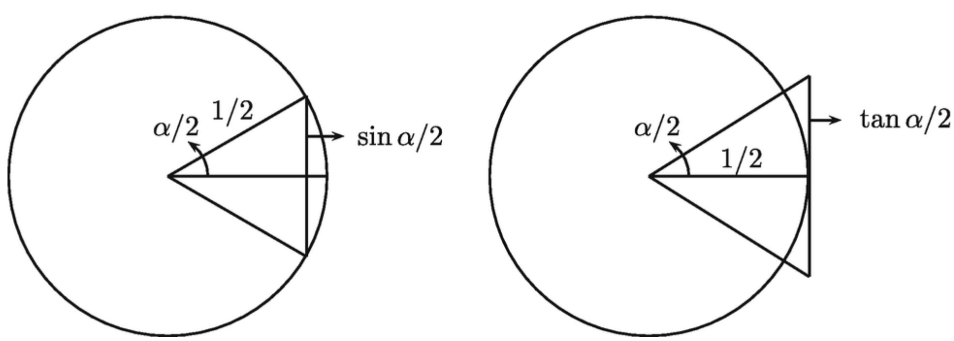

Fig. 8.13 Archimedes approximation of π

$$a_n = n \sin \frac{\pi}{n} \quad \text{and} \quad b_n = n \tan \frac{\pi}{n}. \tag{8.25}$$

It is clear that $\lim_{n\to\infty} a_n = \pi = \lim_{n\to\infty} b_n$. Further, b_{2n} is the harmonic mean of a_n and b_n, and a_{2n} is the geometric mean of a_n and b_{2n}, i.e.,

$$b_{2n} = \frac{2a_n b_n}{a_n + b_n} \quad \text{and} \quad a_{2n} = \sqrt{a_n b_{2n}}. \tag{8.26}$$

From (8.25) for the hexagon, i.e., $n = 6$ it follows that $a_6 = 3,\ b_6 = 2\sqrt{3}$. Then, Archimedes successively took polygons of sides $12, 24, 48,$ and $96,$ used the recursive relations (8.26), and the inequality

$$\frac{265}{153} < \sqrt{3} < \frac{1351}{780}, \tag{8.27}$$

to obtain the bounds

$$3.140845\cdots = 3\frac{10}{71} < \pi < 3\frac{1}{7} = 3.142857\cdots. \tag{8.28}$$

The approximation $22/7$ is often called the *Archimedean value* of π, and it is good for most purposes. If we take the average of the bounds given in (8.28), we obtain $\pi = 3.141851\cdots$. Archimedes' method of computing π by using regular inscribed and circumscribed polygons is known as the *classical method* of computing π. It follows that an inscribed regular polygon of 2^n sides takes up more than $1 - 1/2^{n-1}$ of the area of a circle.

- Archimedes also showed that a curve discovered by Conon of Samos (around 280–220 BC, Greece) could, like Hippias' quadratrix, be used to square the circle. The curve is today called the *Archimedean Spiral*.

8.12 Archimedes Inequality

One of the most frequently debated questions in the history of mathematics is the "puzzling" approximation of $\sqrt{3}$, appeared in his treatise *Measurement of a Circle*, namely, the inequality (8.27), which Archimedes presented without a justification. In fact, it is of paramount interest because the bounds $265/153$ and $1351/780$ are the best rational approximations up to the respective denominators. On the website https://mathpages.com/home/kmath038/kmath038.htm for the inequality (8.27) several reviews which appeared in the popular history of mathematics books have been summarized, for example: Rouse Ball in 1908 "it would seem...that [Archimedes] had some (at present unknown) method of extracting the square root of numbers approximately," Heath in

1921 "the successive solutions in integers of the equations $x^2-3y^2=1$ and $x^2-3y^2=-2$ may have been found...in a similar way to...the Pythagoreans," Bell in 1937, "...he also gave methods for approximating to square roots which show that he anticipated the invention by the Hindus of what amount to periodic continued fractions," Boyer in 1968, "his method for computing square roots was similar to that used by the Babylonians," Morris Kline in 1972, without any explanation claimed that if $N=a^2\pm b$ where a^2 is the rational square nearest to $N,$ larger or smaller, and *b* is the remainder, then the following inequalities can be used to obtain (8.27)

$$a\pm\frac{b}{2a\pm 1}<\sqrt{N}<a\pm\frac{b}{2a}. \tag{8.29}$$

As we have seen the right side bounds of the inequality (8.29) lead to the algorithm (8.11), which indeed gives the upper bound of (8.27) (see Table 8.1, $N=3, a_0=5/3),$ the left side bounds of (8.29) give us two new iterative schemes

$$a_{n+1}=a_n+\frac{N-a_n^2}{2a_n+1}=\frac{a_n^2+a_n+N}{2a_n+1},\quad a_0\le\sqrt{N}<a_0+1,\quad n\ge 0 \tag{8.30}$$

and

$$a_{n+1}=a_n-\frac{a_n^2-N}{2a_n-1}=\frac{a_n^2-a_n+N}{2a_n-1},\quad a_0-1<\sqrt{N}\le a_0,\quad n\ge 0. \tag{8.31}$$

For (8.30), by induction, we shall show that $a_n\le\sqrt{N}<a_n+1$ implies that $a_{n+1}\le\sqrt{N}<a_{n+1}+1,\ n\ge 0.$ For this, it suffices to show that

$$\frac{a_n^2+a_n+N}{2a_n+1}\ \le\sqrt{N}<\frac{a_n^2+a_n+N}{2a_n+1}+1$$

or

$$(a_n-\sqrt{N})(a_n+1-\sqrt{N})\le 0<(a_n-\sqrt{N})^2+(a_n+1-\sqrt{N})+2a_n,$$

which in view of $a_n\le\sqrt{N}<a_n+1$ is obvious. From (8.30), we also have $a_n\le a_{n+1}.$ Thus, the sequence $\{a_n\}$ generated by (8.30) is monotonically increasing, and bounded above, and hence converges to $\sqrt{N}.$

For the sequence $\{a_n\}$ generated by the iterative scheme (8.31) numerical evidence suggests that the convergence is oscillatory. Further, from (8.30) as well as (8.31), we could not get the lower bound of (8.27), see Table 8.3

Table 8.3 Monotone and oscillatory convergence

		Algorithm (8.30)		Algorithm (8.31)
n	$N=2$	$N=3$	$N=2$	$N=3$
0	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	2
1	$\frac{46}{33}$	$\frac{27}{16}$	$\frac{28}{21}$	$\frac{5}{3}$
2	$\frac{5812}{4125}$	$\frac{1929}{1120}$	$\frac{1078}{735}$	$\frac{37}{21}$
3	$\frac{91785094}{64964625}$	$\frac{9644721}{5575360}$	$\frac{1450204}{1044435}$	$\frac{1915}{1113}$

- Again on the website https://mathpages.com/home/kmath038/kmath038.htm, a clever observation is that if *a* is a bound (upper or lower) of $\sqrt{3}$, then $(5a+9)/(3a+5)$ is a closure bound on the opposite side (lower or upper). This suggests the iterative scheme

$$a_{n+1} = \frac{5a_n + 9}{3a_n + 5}, \qquad a_0 = \frac{5}{3}, \qquad n \geq 0. \tag{8.32}$$

Since

$$a_{n+1}^2 - 3 = \left(\frac{5a_n + 9}{3a_n + 5}\right)^2 - 3 = -\frac{2(a_n^2 - 3)}{9a_n^2 + 30a_n + 25} \simeq -\frac{a_n^2 - 3}{51.98\cdots}$$

the error is negated and reduced by a factor of nearly 52 in each iteration. Iterative scheme (8.32) gives

$$a_1 = \frac{26}{15}, \quad a_2 = \frac{265}{153}, \quad a_3 = \frac{1351}{780}, \quad a_4 = \frac{13775}{7953}, \cdots.$$

Thus, a_2 and a_3, respectively, give the lower and upper Archimedes bounds of $\sqrt{3}$.

An immediate extension of the algorithm (8.32) for an arbitrary integer *N* can be written as

$$a_{n+1} = \frac{pa_n + N^2}{Na_n + p}, \quad n \geq 0 \tag{8.33}$$

where p is the smallest (largest) integer so that $p^2 - N^3 > 0 \ (< 0)$, i.e., $p = \lceil N^{3/2} \rceil \ \ (p = \lfloor N^{3/2} \rfloor$). Now, since

$$a_{n+1}^2 - N = \left(\frac{pa_n + N^2}{Na_n + p}\right)^2 - N = \frac{p^2 - N^3}{(Na_n + p)^2}(a_n^2 - N) \tag{8.34}$$

if $p^2 - N^3 > 0$, then in view of $(p^2 - N^3)/(Na_n + p)^2 < 1$, the sequence $\{a_n\}$ generated by (8.33) converges to $\sqrt{N}$ and the convergence is decreasing provided $a_0 > \sqrt{N}$, further from (8.34)

$$a_{n+1}^2 - N \leq \frac{p^2 - N^3}{(N^{3/2} + p)^2}(a_n^2 - N),$$

whereas if $a_0 < \sqrt{N}$ the convergence is increasing and

$$a_{n+1}^2 - N \leq \frac{p^2 - N^3}{(Na_0 + p)^2}(a_n^2 - N).$$

For $N = 2$ and $p = 3$, so that $p^2 - N^3 > 0$, first few iterates are listed below.

$$a_0 = 2, \ \ a_1 = \frac{10}{7}, \ \ a_2 = \frac{58}{41}, \ \ a_3 = \frac{338}{239}, \ \ a_4 = \frac{1970}{1393}$$
$$a_0 = 1, \ \ a_1 = \frac{7}{5}, \quad a_2 = \frac{41}{29}, \ \ a_3 = \frac{239}{169}, \ \ a_4 = \frac{1393}{985}$$

Now we consider the case when $p^2 - N^3 < 0$, i.e., $N^3 - p^2 > 0$. In this case (8.34) is better written as

$$a_{n+1}^2 - N = -\frac{N^3 - p^2}{(Na_n + p)^2}(a_n^2 - N) \tag{8.35}$$

We shall show that $(N^3 - p^2)/(Na_n + p)^2 < 1$. For this, since $a_n \geq 1, \ n \geq 0$ it suffices to show that $(N^3 - p^2)/(N + p)^2 < 1$, which is the same as $N^3 < N^2 + 2p^2 + 2Np$. Now since p is the largest integer such that $N^3 - p^2 > 0$, certainly, $p^2 \geq (N - 1)^3$ which also give $p \geq (N-1)$. Thus, it is adequate to show that $N^3 < N^2 + 2(N-1)^3 + 2N(N-1)$, but it is the same as $0 < (N-1)[(N-1)^2+1]$. In conclusion, the sequence $\{a_n\}$ generated by (8.35) converges, the convergence is clearly oscillatory, and

$$a_{n+1}^2 - N \ \simeq \ -\frac{N^3-p^2}{(N^{3/2}+p)^2}(a_n^2 - N).$$

For $N=3, p=5$ we have $p^2 - N^3 < 0$, and (8.33) reduces to (8.32). We have already employed (8.32) to obtain first few iterates with $a_0 = 5/3 < \sqrt{3}$. Now we compute first few iterates with $a_0 = 2 > \sqrt{3}$.

$$a_1 = \frac{19}{11}, \quad a_2 = \frac{194}{112}, \quad a_3 = \frac{1978}{1142}, \quad a_4 = \frac{20168}{11644}.$$

- On the same website and on the website https://www.mathpages.com/home/kmath190/kmath190.htm following Babylonians' the basic *ladder rule* for generating a sequence of integers to yield the square root of a number *N* the following recurrence relation has been discussed

$$s_n = (2a)s_{n-1} + (N-a^2)s_{n-2}, \quad n \geq 2 \tag{8.36}$$

where *a* is the largest integer such that a^2 is less than N. Letting $q = \sqrt{N} + a$, or $(q-a)^2 = N$, it follows that

$$q^2 = (2a)q + (N-a^2),$$

and hence $s_0 = 1, s_1 = q, \ s_n = q^n, \ n \geq 2$ satisfies (8.36). Now since $q = s_{n-1}/s_{n-2}$ and $q^2 = s_n/s_{n-2}$ from (8.36) it immediately follows that $q = s_{n+1}/s_n, \ n \geq 0$. However, since exactly *q* is unknown, we can begin with arbitrary (initial) integer values of s_0, s_1 and generate the sequence of the ratios $\{s_{n+1}/s_n\}$, which must converge to the solutions of (8.36), namely, $q = \sqrt{N} + a$. Thus, $\{(s_{n+1}/s_n) - a\}$ converges to $\sqrt{N}$. We also note that $\{(N-a^2)(s_n/s_{n+1})\}$ converges to $(N-a^2)/q = (N-a^2)/(\sqrt{N}+a) = \sqrt{N} - a$, and hence $\{a + (N-a^2)(s_n/s_{n+1})\}$ converges to $\sqrt{N}$. Now we shall show that for both the sequences $\{(s_{n+1}/s_n) - a\}$ and

$\{a+(N-a^2)(s_n/s_{n+1})\}$ convergence is oscillatory. For the first sequence it suffices to show that if $(s_{n+1}/s_n)-a>\sqrt{N}$, which is the same as $(s_n/s_{n+1})<1/(\sqrt{N}+a)$, then $(s_{n+2}/s_{n+1})-a<\sqrt{N}$. For this, from (8.36) we have

$$\frac{s_{n+2}}{s_{n+1}}-a=\frac{(2a)s_{n+1}+(N-a^2)s_n}{s_{n+1}}-a=a+(N-a^2)\frac{s_n}{s_{n+1}}$$
$$<a+(N-a^2)\frac{1}{\sqrt{N}+a}=\sqrt{N}.$$

Similarly, for the second sequence it suffices to show that if $a+(N-a^2)(s_n/s_{n+1})>\sqrt{N}$, which is the same as $(s_n/s_{n+1})>1/(\sqrt{N}+a)$, then $a+(N-a^2)(s_{n+1}/s_{n+2})<\sqrt{N}$. But this is the same as proving $(s_{n+2}/s_{n+1})>\sqrt{N}+a$. Now from (8.36) it follows that

$$\frac{s_{n+2}}{s_{n+1}}=2a+(N-a^2)\frac{s_n}{s_{n+1}}>2a+(N-a^2)\frac{1}{(\sqrt{N}+a)}=2a+(\sqrt{N}-a)=(\sqrt{N}+a).$$

For $N=2$ and $N=3$, we need to take $a=1,$ so that the recurrence relation (8.36), respectively, reduces to $s_n=2s_{n-1}+s_{n-2}$ and $s_n=2s_{n-1}+2s_{n-2},\ n\geq 2$. We shall consider these recurrence relations with $s_0=0$ and $s_1=1$, i.e.,

$$s_n=2s_{n-1}+s_{n-2},\quad n\geq 2,\quad s_0=0,\quad s_1=1 \tag{8.37}$$

and

$$s_n=2s_{n-1}+2s_{n-2},\quad n\geq 2,\quad s_0=0,\quad s_1=1. \tag{8.38}$$

Although solutions of (8.37) and (8.38) can be written explicitly as

$$s_n=\frac{1}{2\sqrt{2}}[(1+\sqrt{2})^n-(1-\sqrt{2})^n]\quad \text{and}\quad s_n=\frac{1}{2\sqrt{3}}[(1+\sqrt{3})^n-(1-\sqrt{3})^n]$$

for the computation they are of little help. In Table 8.4, we directly use (8.37) and (8.38) to list successive approximations obtained for $\sqrt{2}$ and $\sqrt{3}$. It is clear that Table 8.4 contains most of the data of Table 8.1, also it includes Archimedes' lower and upper bounds for $\sqrt{3}$, in fact, it is probable that Archimedes used iterative scheme (8.36) to establish the inequality (8.27).

Table 8.4 Babylonian ladder rule

	$N=2$		$N=3$	
n	$(s_{n+1}/s_n)-1$	$1+(s_n/s_{n+1})$	$(s_{n+1}/s_n)-1$	$1+2(s_n/s_{n+1})$
2	$\frac{3^*}{2}$	$\frac{7^*}{5}$	2	$\frac{5}{3}$
3	$\frac{7}{5}$	$\frac{17}{12}$	$\frac{5}{3}$	$\frac{7}{4}$
4	$\frac{17^*}{12}$	$\frac{41^*}{29}$	$\frac{7}{4}$	$\frac{19}{11}$
5	$\frac{41}{29}$	$\frac{99}{70}$	$\frac{19}{11}$	$\frac{26}{15}$
6	$\frac{99^*}{70}$	$\frac{239^*}{169}$	$\frac{26}{15}$	$\frac{71}{41}$
7	$\frac{239}{169}$	$\frac{577}{408}$	$\frac{71}{41}$	$\frac{97}{56}$
8	$\frac{577^*}{408}$	$\frac{1393^*}{985}$	$\frac{97}{56}$	$\frac{265}{153}$
9	$\frac{1393}{985}$	$\frac{3363}{2378}$	$\frac{265}{153}$	$\frac{362}{209}$
10	$\frac{3363^*}{2378}$	$\frac{8119^*}{5741}$	$\frac{362}{209}$	$\frac{989}{571}$
11	$\frac{8119}{5741}$	$\frac{19601}{13860}$	$\frac{989}{571}$	$\frac{1351}{780}$
12	$\frac{19601^*}{13860}$	$\frac{47321^*}{33461}$	$\frac{1351}{780}$	$\frac{3691}{2131}$
13	$\frac{47321}{33461}$	$\frac{114243}{80782}$	$\frac{3691}{2131}$	$\frac{5042}{2911}$
14	$\frac{114243^*}{80782}$	$\frac{275807^*}{195025}$	$\frac{5042}{2911}$	$\frac{13775}{7953}$
15	$\frac{275807}{195025}$	$\frac{665857}{470832}$	$\frac{13775}{7953}$	$\frac{18817}{10864}$
16	$\frac{665857^*}{470832}$	$\frac{1607521^*}{1136689}$	$\frac{18817}{10864}$	$\frac{51409}{29681}$

- Davies in his preprint [154] combined a simple proposition:

$$\text{If} \quad \frac{v}{u} < \frac{y}{x} \quad \text{then} \quad \frac{v}{u} < \frac{v+y}{u+x} < \frac{y}{x}$$

and an argument similar to that of *bisection method* to compute Archimedes lower and upper bounds in (8.27). For this, he assumed a pair of two approximations $\alpha = v/u$ and $\beta = y/x$ of $\sqrt{3}$ such that $\alpha < \sqrt{3} < \beta$. Now calculate $\gamma = (v+y)/(u+x)$ and replace α by γ if $\gamma < \sqrt{3}$, i.e., $(v+y)^2 < 3(u+x)^2$, and replace β by γ if $\gamma > \sqrt{3}$, i.e., $(v+y)^2 > 3(u+x)^2$. This gives an improved pair of approximations. The procedure continues until the desired accuracy is achieved. With $\alpha = 1$ and $\beta = 2$ his first 16 pairs of approximations are

$$\left(\frac{1}{1},\frac{2}{1}\right) \quad \left(\frac{3}{2},\frac{2}{1}\right) \quad \left(\frac{5}{3},\frac{2}{1}\right) \quad \left(\frac{5}{3},\frac{7}{4}\right) \quad \left(\frac{12}{7},\frac{7}{4}\right) \quad \left(\frac{19}{11},\frac{7}{4}\right)$$
$$\left(\frac{19}{11},\frac{26}{15}\right) \quad \left(\frac{45}{26},\frac{26}{15}\right) \quad \left(\frac{71}{41},\frac{26}{15}\right) \quad \left(\frac{71}{41},\frac{97}{56}\right) \quad \left(\frac{168}{97},\frac{97}{56}\right) \quad \left(\frac{265^*}{153},\frac{97}{56}\right)$$
$$\left(\frac{265}{153},\frac{362}{209}\right) \quad \left(\frac{627}{362},\frac{362}{209}\right) \quad \left(\frac{989}{571},\frac{362}{209}\right) \quad \left(\frac{989}{571},\frac{1351^*}{780}\right).$$

While the aforementioned list of pairs of approximations of $\sqrt{3}$ contain lower and upper bounds of Archimedes, an extended algorithm for the computation of $\sqrt{N}$ for an arbitrary integer *N* has no merit.

- For the lower bound on the website https://math.stackexchange.com/questions/894862/archimedes-approximation-of-square-roots, posted in 2015, the secant method has been suggested. Recall from the standard numerical analysis text books, the secant method for finding a simple root a^* of the equation $f(x) = 0$ is

$$a_{n+1} = \frac{a_{n-1}f(a_n) - a_n f(a_{n-1})}{f(a_n) - f(a_{n-1})}, \quad n \geq 1 \tag{8.39}$$

where a_0, a_1 are two initial approximations, one is less than a^* and the other is greater than a^*. For the root a^* the secant method is superlinear, i.e., the rate of convergence is the Golden Number φ. We note that for the equation $f(x) = x^2 - N = 0$ the secant method (8.39) simply reduces to

$$a_{n+1} = \frac{a_{n-1}a_n + N}{a_{n-1} + a_n}, \quad n \geq 1. \tag{8.40}$$

It is interesting to note that if in (8.40), we take $a_{n-1} = a_n$, then it is the same as (8.11). Applying (8.40) with $N = 3,\ a_0 = 5/3$ (which is less than $\sqrt{3}$), and $a_1 = 26/15$ (which is greater than $\sqrt{3}$), see Table

8.1, we immediately get $a_2 = 265/153$, which is the lower bound in (8.27). From (8.40), we also compute $a_3 = 13775/7953 \simeq 1.73205079844$, which is a better lower bound than in (8.27).

- For the lower bound in (8.27) on the website https://hsm.stackexchange.com/questions/771/what-is-so-mysterious-about-archimedes-approximation-of-sqrt-3 posted in 2015, following Babylonian tables are constructed for n^2 and $3n^2, \ n \geq 1$, and it was noticed that $70225 = (265)^2 \simeq 3(153)^2 = 20227$.
- Upper bound in the inequality (8.27) is the same as obtained in Sulbasutras, see (8.5). Unfortunately, historians never found place to write this fact.
- For two positive numbers a, b recall that

$$\min\{a, b\} \leq \mathrm{HM} \leq \mathrm{GM} \leq \mathrm{AM} \leq \max\{a, b\}, \quad \mathrm{GM}^2 = \mathrm{AM} \cdot \mathrm{HM}.$$

Based on the above inequalities, we have the following three algorithms HMA, GMA, and AMA

$$c_{n+1} = \frac{2a_n b_n}{a_n + b_n}, \quad b_{n+1} = \sqrt{a_n b_n}, \quad a_{n+1} = \frac{a_n + b_n}{2}, \quad n \geq 0 \quad (8.41)$$

where a_0, b_0 are positive (initial approximation) numbers. The GMA and AMA first appeared in the works of Lagrange; for their applications to approximate π see the recent monograph of Chan [122]. It is clear that $c_{n+1} \leq b_{n+1} \leq a_{n+1}, \ n \geq 0$. From this, it immediately follows that

$$a_{n+1} - a_n = \frac{a_n + b_n}{2} - a_n = \frac{b_n - a_n}{2} \leq 0,$$
$$b_{n+1} = \sqrt{a_n b_n} \geq \sqrt{b_n b_n} = b_n,$$

and

$$c_{n+1} - c_n = \frac{2a_n b_n}{a_n + b_n} - c_n \geq \frac{2a_n b_n}{2a_n} - c_n = b_n - c_n \geq 0,$$

thus the sequence $\{a_n\}$ is decreasing, the sequence $\{b_n\}$ is increasing, the sequence $\{c_n\}$ is also increasing and $b_n \leq c_{n+1} \leq b_{n+1}$. Thus, $\min\{a_0, b_0\} \leq c_1 \leq c_n \leq b_n \leq a_n \leq a_1 \leq \max\{a_0, b_0\}$. In conclusion all the three sequences $\{c_n\}, \ \{b_n\}, \ \{a_n\}$ converge to the same limit. The convergence of $\{c_n\}$ also follows from the relation $\mathrm{HMA} = \mathrm{GMA}^2/\mathrm{AMA}.$ Now to find $\sqrt{N}$ we let $b_n = N/a_n$ for all $n \geq 0.$ Then HMA, GMA, and AMA, respectively, reduce to

$$c_{n+1} = \frac{2a_nN}{a_n^2+N}, \quad b_{n+1} = \sqrt{N}, \quad a_{n+1} = \frac{1}{2}\left(a_n + \frac{N}{a_n}\right), \quad n \geq 0$$

Here a_0 is some positive rational number. Clearly, AMA is the same as (8.11). We note that the equation $(a+3/a)/2 = 1351/780$ gives $a = 26/15,$ and $(a+3/a)/2 = 26/15$ holds for $a = 5/3.$ Thus, if we employ AMA for $N = 3$ with $a_0 = 5/3$ (which is a reasonable choice, see (8.5)) then a_2 is the same as the upper bound of the inequality (8.27). We further note that the equation $6a/(a^2+3) = 265/153,$ which is the same as $265a^2 - 918a + 795 = 0$ has no rational roots, and hence, lower bound of (8.27) cannot be obtained from HMA for $N = 3.$

- A proof of (8.27) based on very simple inequalities is as follows:

$$\frac{1351}{780} = \frac{1}{15}\left(26 - \frac{1}{52}\right) = \frac{1}{15}\sqrt{26^2 - 1 + \frac{1}{52^2}} > \frac{1}{15}\sqrt{26^2-1} = \sqrt{3}$$

and

$$\frac{265}{153} = \frac{1}{15}\left(26 - \frac{1}{51}\right) = \frac{1}{15}\sqrt{26^2 - 1 - \frac{50}{51^2}} < \frac{1}{15}\sqrt{26^2-1} = \sqrt{3}.$$

8.13 π After Archimedes

The American philosopher and psychologist William James (1842–1910) wrote in 1909 “the thousandth decimal of Pi sleeps there though no one may ever try to compute it.” Thanks to twentieth and twenty-first century, mathematicians and computer scientists, it sleeps no more. In 1889, Hermann Cäsar Hannibal Schubert (1848–1911, Germany) said “there is no practical or scientific value in knowing more than the 17 decimal places used in the foregoing, already somewhat artificial, application,” and according to Arndt and Haenel [40], just 39 decimal places would be enough to compute the circumference of a circle surrounding the known universe to within the radius of a hydrogen atom. Further, an expansion of π to only 47 decimal places would be sufficiently precise to inscribe a circle around the visible universe that doesn’t deviate from perfect circularity by more than the distance across a single proton. The question has been repeatedly asked why so many digits? Perhaps the primary motivation for these computations is the human desire to break records, the extensive calculations involved have been used to test supercomputers and high-precision multiplication algorithms (a stress test for a computer-a kind of "digital cardiogram"), the statistical distribution of the digits, which is expected to be uniform, that is, the frequency with which the digits (0 to 9) appear in the result will tend to the same limit $(1/10)$ as the number of decimal places increases beyond all bounds, and in recent years, these digits are being used in applied problems as a random sequence. It appears experts in the field of π are looking for surprises in the digits of $\pi.$ In this section, we shall provide an update account of the computation of $\pi,$ and realize that Archimedes polygonal method remained unsurpassed until 18 centuries.

- Varahamihira in his work approximated π as $\sqrt{10}.$
- Marcus Vitruvius Pollio (around 75-15 BC, Italy) commonly known as Vitruvius, in his multi-volume work *De Architectura* (On Architecture) described the use of $\sqrt{2}$ progression or ad quadratum technique, which uses geometry to double a square in which the diagonal of the original square is equal to the side of the resulting square. He obtained and used the value $\pi = 3\ 1/8 = 3.125$, which is the same as Babylonians had used 2,000 years earlier. He was the first to describe direct measurement of distances by the revolution of a wheel.
- Brahmagupta used a geometric construction for squaring the circle, which amounts to $\pi = \sqrt{10}.$
- Liu Xin (Liu Hsin) (around 50 BC–23 AD, China), in about 10 BC gave a more accurate calculation of π as $3.1547,$ the exact method he used to reach this figure is unknown. This was first mentioned in the *Sui shu* (387–388, China). He also found the approximations $3.1590,\ 3.1497,$ and $3.1679.$ Liu Xin in 5 AD gave a more accurate calculation of π as $3.1457.$ The method he used to reach this figure is also unknown (see Bai [47]).
- Heron in his *Metrica* refers to an Archimedes work, where he gives the bounds

$$3.14163\cdots = \frac{211875}{67441} < \pi < \frac{197888}{62351} = 3.173774\cdots .$$

Clearly, in the aforementioned right inequality there is a mistake as it is worse than the upper bound $22/7$ found by Archimedes earlier. Heron adds “Since these numbers are inconvenient for measurements, they are reduced to the ratio of the smaller numbers, namely, $22/7''$.

- Zhang Heng (78–139 AD, China), in 125 proposed $\sqrt{10}$ (about 3.1623) for $\pi.$ He also compared the celestial circle to the width (i.e., diameter) of the earth in the proportion of 736 to $232,$ which gives π as $3.1724.$
- Ptolemy, in his *Almagest* used chords of a circle and an inscribed 360-gon, to find approximate value of π in sexagesimal notation, as $3\ 8'\ 30''$, which is the same as $377/120 = 3.141666\cdots$. Eutocius of Ascalon (480–540, Israel) refers to a book *Quick Delivery* by Apollonius, in which he obtained an approximation for π, which was better than known to Archimedes, perhaps the same as $377/120.$
- Wang Fan (228–266, China), in 250, found the rational approximation $142/45$ for $\pi,$ yielding $\pi = 3.155.$
- Liu Hui, in 263, in his *Nine Chapters on the Mathematical Art* used a variation of the Archimedean inscribed regular polygon with 192 sides to approximate π as 3.141014 and later calculated π as 3.14159 by using 3072-sided polygon. He also suggested $157/50 = 3.14$ as a practical approximation.
- Pappus, in 330, in his *Mathematical Collection* for squaring the circle used Dinostratus quadratrix.
- He Chengtian (370–447, China), in 400, gave the approximate value of π as $111035/35329 = 3.142885\cdots$.
- Tsu Ch’ung-chih (Zu Chongzhi) (429–500, China), in 475, with his son used a variation of Archimedes method to find $3.1415926 < \pi < 3.1415927.$ He also obtained a remarkable rational approximation $355/113,$ which yields π correct to six decimal digits. In Chinese this fraction is known as Milü. To compute this accuracy for $\pi,$ he must have taken an inscribed regular 6×2^{12}-gon and performed lengthy calculations. For details see Zha [547]. Note that $\pi = 355/113$ can be obtained from the values of Ptolemy and Archimedes:

$$\frac{355}{113} = \frac{377-22}{120-7}.$$

He declared that $22/7$ is an inaccurate value, whereas $355/113$ is the accurate value of π. We also note that $\pi = 355/113$ can be obtained from the values of Liu Hui and Archimedes. In fact, by using the method of averaging, we have

$$\frac{157 + (9 \times 22)}{50 + (9 \times 7)} = \frac{355}{113}.$$

An integral proof of $\pi > 355/113$ is available in [352].

- Bhaskara II gave several approximations for π. According to him $3927/1250$ is an accurate value, $22/7$ is an inaccurate value, and $\sqrt{10}$ is for ordinary work. The first value may have been taken from Aryabhata. This approximation has also been credited to Liu Hui and Zu Chongzhi. He also gave the value $754/240 = 3.1416$, which is of uncertain origin; however, it is the same as that by Ptolemy.
- Boethius, in 510, declared that circle had been squared in the period since Aristotle's time, but noted that the proof was too long.
- The *Suishu* (History of the Sui Dynasty, China), which was presented to the throne in 656, includes a paragraph on the history of π.
- Al-Khwarizmi, in 800, used $\pi = 22/7$ in algebra, $\pi = \sqrt{10}$ in geometry, and $\pi = 62832/20000 = 3.1416$ in astronomy.
- Mahavira, in 850, used the approximate value of π as $\sqrt{10}$. He also mentions that the approximate volume of a sphere with diameter *d* is $(9/2)(d/2)^3$, i.e., $\pi = 3.375$, and exact volume is $(9/10)(9/2)(d/2)^3$, i.e., $\pi = 3.0375$.
- Franco von Lüttich (around 1015–1083, Belgiam), in about 1040, wrote a treatise on Squaring the Circle, a work of six books, but only preserved in fragments.
- Fibonacci, in 1220, in his treatise *Practica geometriae* used a 96-sided polygon, to obtain the approximate value of π as $864/275 = 3.141818\cdots$.
- Johannes Campanus (around 1220–1296, Italy), in 1260, wrote a Latin edition of Euclid's Elements in 15 books. He used the value of π as $22/7$.
- Zhao Youqin (1271–1335, China), in about 1300, used a regular polygon of 4×2^{12} sides to derive $\pi = 3.1415926$.

- Albert of Saxony, in about 1360, wrote a long treatise *De quadratura circuli* (Question on the Squaring of the Circle) consisting mostly philosophy. He said "following the statement of many philosophers, the ratio of circumference to diameter is exactly $22/7$; of this, there is proof, but a very difficult one."
- Madhava, around 1400, gave many methods for calculating the circumference of a circle. The value of π correct to 13 decimal places is attributed to Madhava. However, the text *Sadratnamala*, usually considered as prior to Madhava, while some researchers have claimed that it was compiled by Madhava, gives the astonishingly accurate value of π correct to 17 decimal places.
- al-Kashi, in 1424 [25], calculated π to 14 decimal places, and later in 1429 to 16 decimal places. For this he used classical polygon method of 6×2^{27} sides.
- George Pürbach (1423–1461, Austria), in 1460, approximated π by the rational $62832/20000$, which is exactly the same as given by Aryabhata.
- Nicholas of Cusa, in 1464, gave the approximations of π as $(3/4)(\sqrt{3}+\sqrt{6})$ and $24\sqrt{21}/35 = 3.142337 \cdots$. He thought this to be the exact value.
- Johann Regiomontanus, in 1464, criticized Nicholas of Cusa's approximations and methods to approximate the value of π and gave the approximation $3.14343.$
- Nilakanthan, about 1500, gave a proof of the arctangent infinite series expansions of Madhava (2.18), and in Sanskrit poetry the series (7.178) which follows from Madhava's series (2.18) when $x = 1.$ He also gave sophisticated explanations of the irrationality of $\pi.$ Since (7.178) is an alternating series, the error committed by stopping at the *n*th term does not exceed $1/(2n+1)$ in absolute value. Thus, to compute $\pi/4$ to eight decimals from (7.178) would require $n > 10^8$ terms. Hence, although it is only of theoretical interest, the expressions on the right are arithmetical, while π arises from geometry. We also note that the series (7.178) can be written as

$$\frac{\pi}{4} = 1 - 2\left(\frac{1}{3\cdot 5} + \frac{1}{7\cdot 9} + \frac{1}{11\cdot 13} + \cdots\right).$$

The following expansion of π is also due to Nilakanthan

$$\pi = 3 + \frac{4}{2 \cdot 3 \cdot 4} - \frac{4}{4 \cdot 5 \cdot 6} + \frac{4}{6 \cdot 7 \cdot 8} - \frac{4}{8 \cdot 9 \cdot 10} + \cdots .$$

This series converges faster than (7.178), which is the same as obtained by Fearnehough by manipulating (7.178) in 2006 (see Chap. 7).

- Leonardo da Vinci, before 1510, briefly worked on squaring the circle, or approximating π.
- Stifel, in 1525, expressed that the quadrature of π is impossible. According to him, "the quadrature of the circle is obtained when the diagonal of the square contains 10 parts of which the diameter of the circle contains 8." Thus, $\pi \simeq 3\ 1/8$
- Albrecht Dürer, in 1525, in his book *Underweysung der Messung mit dem Zirckel und Richtscheyt* provides measurement of lines, areas, and solids by means of Euclidean tools, particularly there is a discussion of squaring the circle.
- Oronce Fińe (1494–1555, France), in 1544, approximated π as $3\ 11/63 = 3.174603 \cdots$. Later, he gave $3\ 2/15 = 3.133333 \cdots$ and, in 1556, $3\ 11/78 = 3.141025 \cdots$.
- Johannes Buteo (1492–1572, France), in 1559, published a book *De quadratura circuli*, which seems to be the first book that accounts the history of π and related problems.
- Valentin Otho (around 1550–1603, Germany), in 1573, came to Wittenberg and proposed to Johannes Praetorius (1537–1616, Germany) the Tsu Ch'ung-chih approximate value of π as $355/113$.
- Tycho Brahe, in 1550, approximated π as $88/\sqrt{785} = 3.140854 \cdots$.
- Sankar Variyar (1500–1560, India), in his book *Kriyakramakari* of 1560 has written, "Multiply 104348 with the diameter of the circle and divide the product by 33215, you will get the value of circumference." From this we can get, $\pi = 104348/33215 = 3.14159265391.$
- Simon Duchesne, in 1583, found $\pi = (39/22)^2 = 3.142561 \cdots$.
- Zhu Zaiyu, in 1584, gave the approximate value of π as $\sqrt{2}/0.45 = 3.142696 \cdots$. Around the same time Xing Yunlu (China) adopted π as 3.1126 and $3.12132034,$ while Chen Jinmo (China) and Fang Yizhi (China), respectively, took as 3.1525 and $52/17.$
- Simon van der Eycke (The Netherland), in 1584, published an incorrect proof of the quadrature of the circle. He approximated π as

$1521/484 = 3.142561\cdots$. In 1585, he gave the value $3.1416055.$

- Adriaen Anthoniszoon (1529–1609, The Netherlands), in 1585, rediscovered the Tsu Ch'ung-chih approximation $355/113$ to π. This was apparently lucky incident, since all he showed was that $377/120 > \pi > 333/106$. He then averaged the numerators and the denominators to obtain the "exact" value of $\pi.$
- Viéte, in his 1593, book *Supplementum geometriae* showed that $3.1415926535 < \pi < 3.1415926537,$ i.e., gave the value of π correct to nine places. For this, he used the classical polygon of $6 \times 2^{16} = 393,216$ sides. He also represented π as an infinite product

$$\begin{aligned}\frac{2}{\pi} &= \cos\frac{\pi}{4}\ \cos\frac{\pi}{8}\ \cos\frac{\pi}{16}\ \cos\frac{\pi}{32}\cdots \\ &= \frac{\sqrt{2}}{2}\frac{\sqrt{(2+\sqrt{2})}}{2}\frac{\sqrt{\left(2+\sqrt{(2+\sqrt{2})}\right)}}{2}\cdots .\end{aligned} \tag{8.42}$$

For this, we note that

$$\sin x = \cos\frac{x}{2}\cdot 2\sin\frac{x}{2} = \cos\frac{x}{2}\cos\frac{x}{2^2}\cdot 2^2\sin\frac{x}{2^2} = \cdots = \left(\prod_{k=1}^{n}\cos\frac{x}{2^k}\right)2^k\sin\frac{x}{2^k}$$

and hence

$$\frac{\sin x}{x} = \left(\prod_{k=1}^{n}\cos\frac{x}{2^k}\right)\frac{\sin x/2^k}{x/2^k},$$

which as $k\to\infty$, and then $x = \pi/2$ gives

$$\frac{2}{\pi} = \cos\frac{\pi}{4}\ \cos\frac{\pi}{8}\ \cos\frac{\pi}{16}\ \cos\frac{\pi}{32}\cdots .$$

Finally, we have

$$\begin{aligned}\cos\frac{\pi}{4} &= \sqrt{\frac{1}{2}\left(1+\cos\frac{\pi}{2}\right)} = \frac{\sqrt{2}}{2}, \\ \cos\frac{\pi}{8} &= \sqrt{\frac{1}{2}\left(1+\cos\frac{\pi}{4}\right)} = \sqrt{\frac{1}{2}\left(1+\frac{\sqrt{2}}{2}\right)} = \frac{1}{2}\sqrt{2+\sqrt{2}},\cdots .\end{aligned}$$

The aforementioned formula (8.42) is one of the milestones in the history of π. The convergence of Vieta's formula was proved by Ferdinand Rudio (1856–1929, Germany-Switzerland) in 1891. It is clear that Vieta's formula cannot be used for the numerical computation of π. In fact, the square roots are much too cumbersome, and the convergence is rather slow. It is clear that if we define $a_1 = \sqrt{1/2}$ and $a_{n+1} = \sqrt{(1 + a_n)/2}$, then (8.42) is the same as $a_1 a_2 a_3 \cdots = 2/\pi$.

- Adrianus van Roomen (1561–1615, Belgium-Germany), in 1593, used the classical method with 2^{30} sides, to approximate π to 15 correct decimal places.
- Joseph Justus Scaliger (1540–1609, France-The Netherlands), in 1594, in his work *Cyclometrica elementa duo* claimed that π is equal to $\sqrt{10}$.
- Ludolph van Ceulen (1539–1610, Germany-The Netherlands), in 1596, wrote a book *Van den Circkel* (On The Circle) in which he published his geometric findings, and the approximate value of π correct to 20 decimal places. For this, he reports that he used classical method with 60×2^{33}, i.e., $515, 396, 075, 520$ sides. This book ends with "Whoever wants to, can come closer." In 1610, in his work *De Arithmetische en Geometrische fondamenten*, which was published posthumously by his wife in 1615, computed π correct to 35 decimal places by using classical method with 2^{62} sides. This computational feat was considered so extraordinary that his widow had all 35 digits of *die Ludolphsche Zahl* (the Ludolphine number) was engraved on his tombstone in St. Peter's churchyard in Leiden. The tombstone was later lost but was restored in 2000 (see Huylebrouck [274]). This was one of the last major attempts to evaluate π by the classical method; thereafter, the techniques of calculus were employed.
- Oliver de Serres (1539–1619, France), around 1600, believed that by weighing a circle and a triangle equal to the equilateral triangle inscribed he had found that the circle was exactly double of the triangle, not being aware that this double is exactly the hexagon inscribed in the same circle. Thus, according to him, $\pi = 3$.
- Willebrord Snell, in 1621, cleverly combined Archimedean method with trigonometry and showed that for each pair of bounds on π given by the classical method, considerably closer bounds can be obtained. By his method, he was able to approximate π to seven places by using just

96 sides, and to van Ceulen's 35 decimal places by using polygons having only 2^{30} sides. The classical method with such polygons yields only two and 15 decimal places.

- Yoshida Mitsuyoshi (1598–1672, Japan), in 1627, used 3.16 for π.
- Christoph (Christophorus) Grienberger (1561–1636, Austria), in 1630, used Snell's refinement to compute π to 39 decimal places. This was the last major attempt to compute π by the Archimedes method.
- *Celiang quanyi* (Complete Explanation of Methods of Planimetry and Stereometry), published in 1635, gives without proof the following bounds
$3.141, 592, 653, 589, 793, 238, 46 < \pi < 3.141, 592, 653, 589, 793, 238, 47,$
i.e., π correct to 19 digits.
- William Oughtred (1575–1660, England), in 1647, designated the ratio of the circumference of a circle to its diameter by π/δ.
- Saint-Vincent, in 1647, in his book *Opus geometricum quadraturae circuli et sectionum coni* proposed at least four methods of squaring the circle, but none of them were implemented. The fallacy in his quadrature was pointed out by Huygens. He also formulated the logarithmic series

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots . \tag{8.43}$$

In 1668, Nicolaus Mercator (Kauffmann) (1620–1687, Germany-France) wrote a treatise entitled *Logarithmo-technica*, and rediscovered the same series.

- Descartes, in 1650, after his death a novel geometric approach to approximate π was found in his papers. His method consisted of doubling the number of sides of regular polygons while keeping the perimeter constant. In modern terms, Descartes' method can be summarized as

$$\pi = \lim_{k\to\infty} 2^k \tan\left(\frac{\pi}{2^k}\right).$$

If we let $a_k = 2^k \tan(\pi/2^k),\ k \geq 2$ then in view of $\tan 2\theta = 2\tan\theta/(1-\tan^2\theta),\ x_k = 1/a_k$ satisfies the relation

$$x_{k+1}(x_{k+1} - x_k) = 2^{-2k-2},$$

and hence

$$x_{k+1} = \frac{1}{2}\left(x_k + \left(x_k^2 + 2^{-2k}\right)^{1/2}\right), \quad k \geq 2, \quad x_2 = 1/4.$$

The sequence $\{x_k\}$ generated by the aforementioned recurrence relation converges to $1/\pi$.

- Huygens, in 1654 [273], for the computation of π, gave the correct proof of Snell's refinement, and using an inscribed polygon of only 60 sides obtained the bounds $3.1415926533 < \pi < 3.1415926538$, for the same accuracy the classical method requires almost 400,000 sides.
- Wallis, in 1655, published his most famous work *Arithmetica infinitorum* in which he established the formula

$$\pi = 2 \cdot \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdot \frac{8}{9} \cdots . \tag{8.44}$$

This formula is a great milestone in the history of π. Like Viéte's formula (8.42), Wallis had found π in the form of an infinite product, but he was the first in history whose infinite sequence involved only rational operations. (Hobbes called *Arithmetica infinitorum* "a scab of symbols," and claimed to have squared the circle). To derive (8.44), we note that $I_n = \int_0^{\pi/2} \sin^n x dx$ satisfies the recurrence relation

$$I_n = \frac{n-1}{n} I_{n-2}. \tag{8.45}$$

Thus, in view of $I_0 = \pi/2$ and $I_1 = 1$, we have

$$I_{2m} = \frac{2m-1}{2m} \cdot \frac{2m-3}{2m-2} \cdots \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

and

$$I_{2m+1} = \frac{2m}{2m+1} \cdot \frac{2m-2}{2m-1} \cdots \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3}.$$

From these relations, a termwise division leads to

$$\frac{\pi}{2} = \left(\frac{2 \cdot 4 \cdot 6 \cdots 2m}{3 \cdot 5 \cdots (2m-1)}\right)^2 \frac{1}{2m+1} \frac{I_{2m}}{I_{2m+1}}.$$

Now, it suffices to show that

$$\lim_{m\to\infty} \frac{I_{2m}}{I_{2m+1}} = 1. \tag{8.46}$$

We know that for all $x \in (0, \pi/2)$ the inequalities $\sin^{2m-1} x > \sin^{2m} x > \sin^{2m+1} x$ hold. Thus, an integration from 0 to $\pi/2$ gives $I_{2m-1} \geq I_{2m} \geq I_{2m+1}$, and hence,

$$\frac{I_{2m-1}}{I_{2m+1}} \geq \frac{I_{2m}}{I_{2m+1}} \geq 1. \tag{8.47}$$

Further, from (8.45) we have

$$\frac{I_{2m-1}}{I_{2m+1}} = \frac{2m+1}{2m},$$

thus, it follows that

$$\lim_{m\to\infty} \frac{I_{2m-1}}{I_{2m+1}} = \lim_{m\to\infty} \frac{2m+1}{2m} = 1. \tag{8.48}$$

Finally, a combination of (8.47) and (8.48) immediately gives (8.46). If we define $a_n = 1 - 1/(2n)^2$, then (8.44) is equivalent to $a_1 a_2 a_3 \cdots = 2/\pi$. We also note that

$$\frac{1}{a_1 a_2 \cdots a_n} = \frac{\pi}{2} + O\left(\frac{1}{n}\right).$$

It is interesting to note that (7.39) with $x = \pi/2$ immediately gives Wallis' formula (8.44).

- Muramatsu Shigekiyo (1608–1695, Japan), in 1663, published *Sanso* (Stack of Mathematics), in which he used classical polygon method of 2^{15} sides to obtain $\pi = 3.14195264877$.
- Newton, in 1665, used 22 terms of the following series to obtain 16 decimal places of π

$$\pi = \frac{3\sqrt{3}}{4} + 24\left(\frac{1}{3 \cdot 2^2} - \frac{1}{5 \cdot 2^5} - \frac{1}{28 \cdot 2^7} - \frac{1}{72 \cdot 2^9} - \cdots\right). \tag{8.49}$$

Later, he wrote “I am ashamed to tell you to how many figures I carried these computations, having no other business at the time.” His result was not published until 1737 (posthumously). However, most of the history books say that to compute π Newton used the series

$$\sin^{-1} x = x + \frac{1 \cdot x^3}{2 \cdot 3} + \frac{1 \cdot 3 \cdot x^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5 \cdot x^7}{2 \cdot 4 \cdot 6 \cdot 7} + \cdots ,$$

which for $x = 1/2$ gives

$$\frac{\pi}{6} = \sin^{-1}\left(\frac{1}{2}\right) = \left(\frac{1}{2} + \frac{1}{2 \cdot 3 \cdot 2^3} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5 \cdot 2^5} + \cdots\right).$$

Using analysis and geometry, the series (8.49) can be obtained as follows: From Fig. 8.14, the equation of the upper half circle is $y = x^{1/2}(1-x)^{1/2}$. Thus, binomial expansion gives

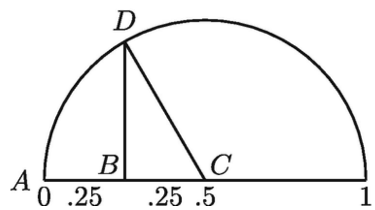

Fig. 8.14 Newton approximation of π

$$y = x^{1/2} - \frac{1}{2}x^{3/2} - \frac{1}{8}x^{5/2} - \frac{1}{16}x^{7/2} - \frac{5}{128}x^{9/2} - \frac{7}{256}x^{11/2} - \cdots .$$

Thus, the area of the sector *ABD* is (integrating the aforementioned series from 0 to $1/4$)

$$\Delta ABD = \frac{1}{3 \cdot 2^2} - \frac{1}{5 \cdot 2^5} - \frac{1}{28 \cdot 2^7} - \frac{1}{72 \cdot 2^9} - \cdots . \tag{8.50}$$

Also, from geometry the area of the sector *ABD* is

$$\begin{aligned}\Delta ABD = \Delta ABCD - \Delta BCD &= \frac{1}{3}\left(\frac{1}{2}\pi\left(\frac{1}{2}\right)^2\right) - \frac{1}{2}\left(\frac{1}{4}\right)\sqrt{\left(\frac{1}{2}\right)^2 - \left(\frac{1}{4}\right)^2} \\ &= \frac{\pi}{24} - \frac{\sqrt{3}}{32}.\end{aligned} \tag{8.51}$$

Equating (8.50) and (8.51), we immediately get (8.49).

- Hobbes, in 1666, approximated π by $3\ 1/5 = 3.2$, which was refuted by Huygens and Wallis. In 1678, he also gave the approximation $\sqrt{10}$.

- James Gregory, in 1667, in his book *Vera circuli et hyperbolae quadratura* showed that the area of a circle can be obtained in the form of an infinite convergent series only, and hence, inferred that the quadrature of the circle was impossible, i.e., π is a transcendental number. However, his attempt, though very interesting, was not regress. Huygens made detailed and rather biased criticisms of it.
- Thomas Shadwell (1642–1692, England), in the mid-1670s, in the popular play "The Virtuoso" satirised the Royal Society, scoffing at its attempts to "square the circle" as futile and impossible.
- Mengoli, in 1672, published on the problem of squaring the circle and provided a proof that Wallis' product (8.44) for π is correct.
- Leibniz, in 1674, for calculating $\pi,$ developed a method without any reference to a circle. He also rediscovered Nilakanthan's series (7.178), whose beauty he described by saying that Lord loves odd numbers.
- Isomura Yoshinori (1640–1710, Japan), in 1684, employed a 2^{17}-sided inscribed polygon to obtain 3.141592664 for $\pi,$ but for some reason he wrote only $\pi = 3.1416.$
- Father Adam Adamad Kochansky (1631–1700, Poland), in 1685 [312], used a new approximate geometric construction for π to obtain

$$\pi \ \simeq \ \sqrt{\frac{40}{3} - 2\sqrt{3}} = 3.141533\cdots .$$

  His method was later quoted in several geometrical textbooks.
- Takebe Katahiro (1664–1739, Japan) also known as Takebe Kenko, in 1690, used polygon (just 1024 sides) approximation and a numerical method, which is essentially equivalent to the Romberg algorithm (rediscovered by Werner Romberg, 1909–2003, Germany, in 1955) to compute π to 41 digits. In 1722, Takebe obtained power series expansion of $(\sin^{-1} x)^2$, 15 years earlier than Euler. Around 1729, essentially the same series was rediscovered by Oyama Shokei (Japan), who used it to find the expansion

$$\begin{aligned}\pi^2 &= 8\left(1 + \sum_{n=1}^{\infty} \frac{2^{n+1}(n!)^2}{(2n+2)!}\right)\\ &= 8\left(1 + \frac{1}{2}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{1\cdot 2}{3\cdot 5} + \frac{1}{4}\cdot\frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7} + \cdots\right).\end{aligned}$$

The aforementioned expansion of π^2 was also given by Yamaji Nushizumi (1704–1772, Japan) around 1765.

- Abraham Sharp (1653–1742, England), in 1699, in the supervision of Edmund Halley (1656–1742, England) realized that by putting $x = 1$ in (2.18) (see (7.178)) we loose the benefit of the powers $x^3, x^5, x^7, \cdots$, which tend to increase the rapidity of convergence for smaller values of x. He substituted $x = 1/\sqrt{3}$ in (2.18), to obtain

$$\frac{\pi}{6} = \frac{1}{\sqrt{3}}\left(1 - \frac{1}{3\cdot 3} + \frac{1}{3^2\cdot 5} - \frac{1}{3^3\cdot 7} + \cdots\right). \tag{8.52}$$

Sharp used (8.52) to calculated π to 72 decimal places out of which 71 digits are correct. In (8.52) the 10th term is $1/(\sqrt{3}\cdot 19\cdot 3^9)$, which is less than 0.00005, and hence, we have at least four places correct after just 9 terms. It is believed that Madhava used the same series to compute the value of π correct to 11 decimal places.

- Seki Takakazu (1642–1708, Japan) also known as Seki Kowa, in 1700, used polygon of 2^{17} sides and Richardson extrapolation (rediscovered by Alexander Craig Aitken, 1895–1967, New Zealand-England) to compute π to 10 digits. Some authors believe that he also used the formula

$$\pi = \lim_{n\to\infty}\frac{4}{n^2}\sum_{j=0}^{n}\sqrt{n^2 - j^2}.$$

- Pirre Jartoux (1669–1720, France) also known with the Chinese name Du Demei, about 1705, claimed that for a circle of diameter *d* the length of the circumference $C = \pi d$ is

$$C = \pi d = 3d\left(1 + \frac{1^2}{3!4} + \frac{1^2\cdot 3^2}{5!4^2} + \frac{1^2\cdot 3^2\cdot 5^2}{7!4^3} + \cdots\right).$$

- John Machin (1680–1752, England), in 1706, computed the value of π to 100 decimal places by using the formula

$$\frac{\pi}{4} = 4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right), \tag{8.53}$$

which in view of (2.18) is the same as

$$\frac{\pi}{4} = 4\left(\frac{1}{5} - \frac{1}{3\cdot 5^3} + \frac{1}{5\cdot 5^5} - \cdots\right) - \left(\frac{1}{239} - \frac{1}{3\cdot 239^3} + \frac{1}{5\cdot 239^5} - \cdots\right). \quad (8.54)$$

To establish (8.53), we let $\tan\theta = 1/5,$ so that

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{5}{12} \quad \text{and} \quad \tan 4\theta = \frac{2\tan 2\theta}{1-\tan^2 2\theta} = \frac{120}{119}.$$

Thus, it follows that

$$\tan\left(4\theta - \frac{\pi}{4}\right) = \frac{\tan 4\theta - 1}{1+\tan 4\theta} = \frac{1}{239}$$

and hence

$$\tan^{-1}\left(\frac{1}{239}\right) = 4\theta - \frac{\pi}{4} = 4\tan^{-1}\left(\frac{1}{5}\right) - \frac{\pi}{4}.$$

The proof of (8.53) also follows by comparing the angles in the identity (the idea originally goes back to Wessel)

$$(5+i)^4(-239+i) = -114244(1+i),$$

i.e.,

$$4\tan^{-1}\left(\frac{1}{5}\right) + \pi - \tan^{-1}\left(\frac{1}{239}\right) = \pi + \tan^{-1} 1.$$

The series (8.54) certainly converges significantly faster than (7.178) and (8.52). In fact, taking six terms of the first series and two terms of the second and paying attention to the remainders and round–off errors, we get the inequalities $3.141592629 < \pi < 3.141592668.$ Thus, the value of π correct to seven decimals is $3.1415926.$

Several other Machin–type formulas are known, e.g.,

$$\begin{aligned}
\frac{\pi}{4} &= 2\tan^{-1}\left(\frac{1}{2}\right) - \tan^{-1}\left(\frac{1}{7}\right) \\
&= 5\tan^{-1}\left(\frac{1}{5}\right) - 3\tan^{-1}\left(\frac{1}{18}\right) - 2\tan^{-1}\left(\frac{1}{57}\right) \\
&= 17\tan^{-1}\left(\frac{1}{22}\right) + 3\tan^{-1}\left(\frac{1}{172}\right) - 2\tan^{-1}\left(\frac{1}{682}\right) - 7\tan^{-1}\left(\frac{1}{5357}\right).
\end{aligned}$$

For a long list of such type of formulas with a discussion of their relative merits in computational work see Lehmer [341] and Hwang [275].

- Mei Gucheng (1680–1763, China) and He Guozong (1687–1766, China), in 1713, in Chapter 15 of *Shu li jing yun* (Collected Basic Principles of Mathematics) gave $\pi = 3.14159265$, which is correct to eight decimal places.
- Thomas Fantet de Lagny (1660–1734, France), in 1719 [327], used the series (8.52) to determine the value of π up to 127 decimal places; however, only 112 are correct.
- The Emperor Kangxi (1654–1722) in 1713 commissioned by imperial order to compile an encyclopedia, which covers all mathematical knowledge, Chinese and Western, available in China. For this more than 100 promising young scholars and a large number of instruments maker were hired. This monumental work *The Shuli Jingyun* (Collected Essential Principles of Mathematics) was completed in 1723. In this compilation the computations begin with the square and the hexagon and are extended to polygons with 4×2^{33} and 6×2^{33} sides, respectively. As a result, a value of 2π composed of 40 digits is derived, but unfortunately only 16 of them are correct.
- Alexander Pope (1688–1744, England), in 1728, in his *Dunciad* mentioned that "The mad Mathesis, now, running round the circle, finds it square." This explains the wild and fruitless attempts of squaring the circle.
- Sieur Malthulon (France), in 1728, offered solutions to squaring the circle and to perpetual motion. He offered 1000 crowns reward in legal form to anyone proving him wrong. Nicoli, who proved him wrong, collected the reward and abandoned it to the Hotel Dieu of Lyons. Later, the courts gave the money to the poor.
- Toshikiyo Kamata (1678–1747, Japan), in 1730, used both the circumscribed and inscribed polygons and gave the bounds $3.1415, 9265, 3589, 7932, 3846,$ $2653, 4166, 7 < \pi < 3.1415, 9265, 3589, 7932, 3846, 2643, 3665, 8$.
- De Moivre, in 1730, in his publication *Miscellanea Analytica* gave the formula for very large n,

$$n! \simeq (2\pi n)^{1/2} e^{-n} n^n,$$

which is known today as *Stirling's formula* after James Stirling (1692–1770, Scotland).

- Puthumana Somyaji (1660–1740, India), in his book *Karanapaddhati* of 1733, has written "Multiply 10000000000 with the circumference of the circle and divide the product by 31415926536, you will get the diameter of the circle. Half of this is radius." From this, the value of π is found to be 3.1415926536.
- Euler, in 1737, in the paper *De variis modis circuli quadraturam numeris proxime exprimendi* derived the formulas

$$\tan^{-1}\left(\frac{1}{p}\right) = \tan^{-1}\left(\frac{1}{p+q}\right) + \tan^{-1}\left(\frac{q}{p^2+pq+1}\right) \tag{8.55}$$

and

$$\tan^{-1}\left(\frac{x}{y}\right) = \tan^{-1}\left(\frac{ax-y}{ay+x}\right) + \tan^{-1}\left(\frac{b-a}{ab+1}\right) + \tan^{-1}\left(\frac{c-b}{cb+1}\right) + \cdots$$

which lead to any number of relations for π; for example, if $x = 1 = y$, and the odd numbers are substituted for $a, b, c, \cdots$, we obtain

$$\frac{\pi}{4} = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{8}\right) + \tan^{-1}\left(\frac{1}{18}\right) + \cdots .$$

The proof of (8.55) immediately follows by comparing the angles in the identity

$$(p+q+i)(p^2+pq+1+iq) = [(p+q)^2+1](p+i).$$

- Matsunaga Yoshisuke (1690–1744, Japan), in 1739, in modern terms used the hypergeometric series

$$F(a,b,c,x) = 1 + \frac{abx}{1!\,c} + \frac{a(a+1)b(b+1)x^2}{2!\,c(c+1)} + \frac{a(a+1)(a+2)b(b+1)(b+2)x^3}{3!\,c(c+1)(c+2)} + \cdots$$

for $a = 1/2,\ b = 1/2,\ c = 3/2,$ and $x = 1/4,$ i.e., the series

$$\pi = 3F\left(\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{1}{4}\right) = 3\left(1+\frac{1^2}{4\cdot 2\cdot 3}+\frac{3^2}{4^2\cdot 2\cdot 3\cdot 4\cdot 5}\right.$$
$$\left.+\frac{3^2\cdot 5^2}{4^3\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}+\cdots\right)$$

to compute π correct to 50 digits. He also gave the following series

$$\pi^2 = 9\left(1+\frac{1^2}{3\cdot 4}+\frac{1^2\cdot 2^2}{3\cdot 4\cdot 5\cdot 6}+\frac{1^2\cdot 2^2\cdot 3^2}{3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}+\cdots\right).$$

- Alexis Claude Clairaut (1713–1765, France), in 1741, used a new approach to show that the area of a circle is equal to the product of the circumference and half the radius. He also used the fact that a circle is a polygon with infinitely many sides. He showed that the area of any regular polygon inscribed in a circle is equal to the perimeter multiplied by half the apothem (a line from the center of a regular polygon at right angles to any of its sides) and then noted that if there are an infinite number of sides, the area, perimeter, and apothem of the polygon become equal to the area, circumference, and radius, respectively of the circle.
- Euler, in 1748, gave the following expansion of π

  $$\pi = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}+\frac{1}{12}-\frac{1}{13}+\cdots,$$

  where the signs are determined by following the rule: If the denominator is 2 or a prime of the form $4m-1,$ the sign is positive; if the denominator is a prime of the form $4m+1,$ the sign is negative; for composite numbers, the sign is equal to the product of signs of its factors. The following curious infinite product was also given by Euler

  $$\frac{\pi}{4} = \frac{3}{4}\cdot\frac{5}{4}\cdot\frac{7}{8}\cdot\frac{11}{12}\cdot\frac{13}{12}\cdot\frac{17}{16}\cdot\frac{19}{20}\cdot\frac{23}{24}\cdot\frac{29}{28}\cdot\frac{31}{32}\cdots,$$

  where the numerators are the odd primes and each denominator is the multiple of four nearest to the numerator.
- Henry Sullamar (England), a real Bedlamite, in 1750, squared the circle by the number of the Beast 666 with seven heads and ten horns. He published periodically every 2 or 3 years some pamphlet in which he endeavored to prop his discovery.
- M. de Causans of the Guards (France), in 1753, cut a circular piece of turf, squared it, and from the result deduced original sin and the Trinity.

He found that the circle was equal to the square in which it is inscribed, i.e., $\pi = 4$. He offered a reward for the detection of any error, and actually deposited 10,000 francs as earnest of 300,000. But the courts did not allow any one to recover.

- Jean Étienne Montucla (1725–1799, France), in 1754, published an anonymous treatise entitled *Histoire des récherches sur la quadrature du cercle* (History research on the quadrature of the circle).
- Euler, in 1755, in his treatise *De relatione inter ternas pluresve quantitates instituenda* (To establish the relationship between three or more quantities), which was published 10 years later, wrote "It appears to be fairly certain that the periphery of a circle constitutes such a peculiar kind of transcendental quantities that it can in no way be compared with other quantities, either roots or other transcendentals." This conjecture haunted mathematicians for 107 years. The following expansion is due to Euler

$$\tan^{-1} x = \frac{y}{x}\left(1 + \frac{2}{3}y + \frac{2\cdot 4}{3\cdot 5}y^2 + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}y^3 + \cdots\right), \qquad (8.56)$$

  where $y = x^2/(1+x^2)$. It converges rapidly.
- Comte of Buffon, in 1760, supposed a number of parallel lines, distance *a* apart, are ruled on a horizontal plane, and suppose a homogeneous uniform rod of length $\ell < a$ is dropped at random onto the plane. Buffon showed that the probability that the rod will fall across one of the lines in the plane is given by $p = (2\ell/\pi a)$. In the literature this problem is known as Buffon's needle problem. This was the earliest problem in geometric probability to be solved. By actually performing this experiment, a large number of times and noting the number of successful cases, we can compute an approximation for π.
- Arima Yoriyuki (1714–1783, Japan), in 1766, found the following rational approximation of π, which is correct to 29 digits

$$\pi = \frac{428224593349304}{136308121570117}.$$

- In 1775, the French Academy of Sciences passed a resolution henceforth not to examine any more solutions of the problem of squaring the circle. In fact, it became necessary to protect its officials against the waste of time and energy involved in examining the efforts of circle squarers. A few years later, the Royal Society in London also banned consideration of any further proofs of squaring the circle. This decision of the Royal Society was described by De Morgan about 100 years later as the official blow to circle-squarers.

- Charles Hutton (1737–1823, England), in 1776, suggested Machin's stratagem in the form

$$\pi = 20 \tan^{-1}\left(\frac{1}{7}\right) + 8 \tan^{-1}\left(\frac{3}{79}\right); \tag{8.57}$$

  however, he didn't carry computations far enough. Euler also developed the formula (8.57).
- Euler, in 1779, used his expansion (8.56) to evaluate right terms of (8.57), to calculate π to 20 decimal places in one hour!
- Franz Xaver Freiherr von Zach (1754–1832, Hungary-France), in 1785, discovered a manuscript by an unknown author in the Radcliffe Library, Oxford, which gives the correct value of π to 152 decimal places.
- Baron Vega, in 1789, used a new series for the arctangent discovered by Euler in 1755, calculated 140 decimal places (126 correct). Vega's result showed that de Lagny's string of digits had a 7 instead of an 8 in the 113th decimal place. His article was published 6 years later, in 1794 [517] (136 correct). Vega retained his record for 52 years until 1841.
- Ajima Naonobu (1732–1798, Japan) also known as Ajima Chokuyen, in 1795, developed a series which in simplified form appears as

$$\begin{aligned}\frac{\pi}{2} = F\left(1, 1, \frac{3}{2}, \frac{1}{2}\right) &= \left(1 + \frac{1!}{3} + \frac{2!}{3 \cdot 5} + \frac{3!}{3 \cdot 5 \cdot 7} + \frac{4!}{3 \cdot 5 \cdot 7 \cdot 9} \cdots\right) \\ &= \sum_{i=0}^{\infty} \frac{i!}{(2i+1)!!} = \sum_{i=0}^{\infty} \frac{(i!)^2 2^i}{(2i+1)!}.\end{aligned}$$

  It is interesting to note that the aforementioned series follows from (7.178) by using an acceleration technique known in the literature as Euler's transform. It can also be derived from the Wallis product formula (8.44).
- Jean-Charles Callet (1744–1799, France), in 1795, gave 154 (152 correct) decimal digits of π.
- Gauss, in 1800, suggested to his teacher Johann Friedrich Pfaff (1765–1825, Germany) to study the sequences $\{x_n\}$ and $\{y_n\}$ generated by the recurrence relations (see Cox [141])

$$x_{n+1} = \frac{1}{2}(x_n + y_n), \quad y_{n+1} = \sqrt{x_{n+1}y_n}, \quad n \geq 0. \tag{8.58}$$

In his reply Pfaff showed that for any positive numbers x_0 and y_0 these sequences converge monotonically to a common limit given by

$$B(x_0, y_0) = \begin{cases} (y_0^2 - x_0^2)^{1/2} / \cos^{-1}(x_0/y_0), & 0 \leq x_0 < y_0 \\ (x_0^2 - y_0^2)^{1/2} / \cosh^{-1}(x_0/y_0), & 0 < y_0 < x_0. \end{cases} \tag{8.59}$$

Pfaff's letter was unpublished. In 1881, Carl Wilhelm Borchardt (1817–1880, Germany) work was published in which he rediscovered this result which now bears his name. For this, it suffices to note that:

1. $\{x_n\}$ and $\{y_n\}$ converge monotonically to the same limit.

2. The ratio $r_n = x_n/y_n$ satisfies $r_{n+1}^2 = (1 + r_n)/2$.

3. If $x_0 < y_0$, let $\theta = \cos^{-1} r_0$. Then, $s_n = 2^n \cos^{-1} r_n = \theta$ and $c_n = 4^n(x_n^2 - y_n^2) = (x_0^2 - y_0^2)$ are independent of n.

4. $$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \frac{2^{-n}|c_n|^{1/2}}{\sin^{-1}\left(2^{-n}|c_n|^{1/2}/y_n\right)} = \lim_{n\to\infty} \frac{|c_n|^{1/2}}{s_n} = \frac{(y_0^2 - x_0^2)^{1/2}}{\theta}.$$

If $y_0 < x_0$, we let $\theta = \cosh^{-1} r_0$, and follow similarly.

Now we let $x_n = 1/a_n, \; y_n = 1/b_n$, then (8.58) and (8.59) take the form

$$a_{n+1} = \frac{2a_nb_n}{a_n + b_n}, \quad b_{n+1} = \sqrt{a_{n+1}b_n}, \quad n \geq 0 \tag{8.60}$$

and

$$A(a_0, b_0) = \frac{1}{B(1/a_0, 1/b_0)} = \begin{cases} a_0b_0(a_0^2 - b_0^2)^{-1/2} \cos^{-1}(b_0/a_0), & a_0 > b_0 \geq 0 \\ a_0b_0(b_0^2 - a_0^2)^{-1/2} \cosh^{-1}(b_0/a_0), & b_0 > a_0 > 0. \end{cases} \tag{8.61}$$

Clearly, the recurrence relations (8.26) as well as (8.41) are different from (8.60). In fact, (8.26) minimize the count of arithmetic operations. In particular, if we let $a_0 = 2\sqrt{3},\ b_0 = 3,$ then (8.60) in view of (8.61) converges to π.

In what follows we let the constant $c = 4^n(x_n^2 - y_n^2) = (x_0^2 - y_0^2)$, then we can uncouple (8.58) and (8.60), respectively, to obtain

$$x_{n+1} = \frac{x_n}{2} + \left(\left(\frac{x_n}{2}\right)^2 - 4^{-n-1}c\right)^{1/2}, \quad y_{n+1}^2 = \frac{y_n^2}{2}\left(1 + \left(1 + 4^{-n}cy_n^{-2}\right)^{1/2}\right) \quad (8.62)$$

and

$$a_{n+1} = \frac{2^{2n+1}}{ca_n}\left(1 - \left(1 - 4^{-n}ca_n^2\right)^{1/2}\right), \quad b_{n+1}^2 = \frac{2^{2n+1}}{c}\left(\left(1 + 4^{-n}cb_n^2\right)^{1/2} - 1\right). \quad (8.63)$$

From (8.62) and (8.63) several known and new recurrence relations can be obtained.

Gauss also developed Machin-type formula

$$\frac{\pi}{4} = 12\tan^{-1}\left(\frac{1}{18}\right) + 8\tan^{-1}\left(\frac{1}{57}\right) - 5\tan^{-1}\left(\frac{1}{239}\right). \quad (8.64)$$

He also estimated the value of π by using lattice theory and considering a lattice inside a large circle, but he did not pursue it further.

- Sakabe Kohan (1759–1824, Japan), in 1810, developed the series

$$\frac{\pi}{4} = 1 - \frac{1}{5} - \frac{1\cdot 4}{5\cdot 7\cdot 9} - \frac{(1\cdot 3)(4\cdot 6)}{5\cdot 7\cdot 9\cdot 11\cdot 13} - \frac{(1\cdot 3\cdot 5)(4\cdot 6\cdot 8)}{5\cdot 7\cdot 9\cdot 11\cdot 13\cdot 15\cdot 17} - \cdots.$$

- Wada Yenzo Nei (1787–1840, Japan) also known as Wada Yasushi, in 1818, developed over one hundred infinite series expressing directly or indirectly π. One of his series can be written as

$$\pi = 2F\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, 1\right) = 2\left(1 + \frac{1^2}{3!} + \frac{1^2\cdot 3^2}{5!} + \frac{1^2\cdot 3^2\cdot 5^2}{7!} + \cdots\right).$$

- Sankar Varman (1774–1839, India), in his book *Sadratnamala* of 1819, has written “if the diameter of a big circle becomes 100000000000000000, then its circumference will be 314159265358979324.” From this, the value of π will be 3.14159265358979324.
- Malacarne (Italy), in 1825, published a geometric construction in Géométrique (Paris), which leads to the value of π less than 3.

- C.G. Specht (Germany), in 1828, published a geometric construction in Crelle's Journal, Volume 3, page 83, which leads to $\pi = 13\sqrt{146}/50 = 3.1415919\cdots$.
- Karl Heinrich Schellbach (1809–1990, Germany), in 1832, began with the relation

$$\frac{\pi i}{2} = \ln(i) = \ln\left(\frac{1+i}{1-i}\right) = \ln(1+i) - \ln(1-i)$$

which was given by Giulio Carlo Fagnano dei Toschi (1682–1766, Italy) in 1750, and used the logarithm expansion (8.43), to obtain

$$\begin{aligned}\frac{\pi i}{2} &= \left(i + \frac{1}{2} - \frac{1}{3}i - \frac{1}{4} + \frac{1}{5}i + \frac{1}{6} - \cdots\right) \\ &\quad - \left(-i + \tfrac{1}{2} + \tfrac{1}{3}i - \tfrac{1}{4} - \tfrac{1}{5}i + \tfrac{1}{6} + \cdots\right) \\ &= 2i - \frac{2}{3}i + \frac{2}{5}i - \cdots ,\end{aligned}$$

which immediately gives Nilakanthan series (7.178). He also considered the relation

$$\begin{aligned}\frac{\pi i}{2} = \ln(i) = \ln\left(\frac{(2+i)(3+i)}{(2-i)(3-i)}\right) &= \left\{\ln\left(1+\frac{1}{2}i\right) - \ln\left(1-\frac{1}{2}i\right)\right\} \\ &+ \left\{\ln\left(1+\frac{1}{3}i\right) - \ln\left(1-\frac{1}{3}i\right)\right\}\end{aligned}$$

and used the expansion (8.43), to obtain, compare to (7.178), a fast converging expansion

$$\frac{\pi}{4} = \left(\frac{1}{2}+\frac{1}{3}\right) - \frac{1}{3}\left(\frac{1}{2^3}+\frac{1}{3^3}\right) + \frac{1}{5}\left(\frac{1}{2^5}+\frac{1}{3^5}\right) - \cdots .$$

- William Baddeley (England), in 1833, in his work *Mechanical quadrature of the circle, London Mechanics' Magazine, August, 1833* writes "From a piece of carefully rolled sheet brass was cut out a circle $1\ 9/10$ inches in diameter, and a square $1\ 7/10$ inches in diameter. On weighing them they were found to be exactly the same weight, which proves that, as each are of the same thickness, the surfaces must also be precisely similar. The rule, therefore, is that the square is to the circle as

17 to 19." We believe for the square it must be the side (not the diameter). Then, it follows that $\pi = 1156/361 = 3.202216\cdots$.

- Joseph LaComme (France), in 1836, at a time when he could neither read nor write—being desirous to ascertain what quantity of stones would be required to prove a circular reservoir he had constructed, consulted a mathematics professor. He was told that it was impossible to determine the full amount, as no one had yet found the exact relation between the circumference of a circle and its diameter. The well-sinker thereupon, full of self-confidence in his powers, applied himself to the celebrated problem and discovered the solution which led to $\pi = 25/8$ by mechanical process. He then taught himself to read and write, and managed to acquire some knowledge of arithmetic by which he verified his mechanical solution. Joseph was honored for his profound discovery with several medals of the first class, bestowed by Parisian societies.
- William Rutherford (1798–1871, England), in 1841 [441], calculated π to 208 places of which 152 were later found to be correct. For this, he employed Euler's formula

$$\frac{\pi}{4} = 4\tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{70}\right) + \tan^{-1}\left(\frac{1}{99}\right)$$

  and Madhava's series expansion (2.18).
- Johann Martin Zacharias Dase (1824–1861, Germany) was a calculating prodigy. At the age of 15, he gave exhibitions in Germany, Austria and England. His extraordinary calculating powers were timed by renowned mathematicians including Gauss. He multiplied $79{,}532{,}853 \times 93{,}758{,}479$ in 54 seconds; two 20-digit numbers in 6 minutes; two 40-digit numbers in 40 minutes; and two 100-digit numbers in 8 hours 45 minutes. In 1840, he made acquaintance with Viennese mathematician L.K. Schulz von Strasznicky (1803–1852), who suggested him to apply his powers to scientific purposes. In 1844, when he was 20, Strasznicky taught him the use of the formula

$$\frac{\pi}{4} = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{8}\right),$$

  and asked him to calculate π . In two months, he carried the approximation to 205 places of decimals, of which 200 are correct. He next calculated a 7-digit logarithm table of the first $1{,}005{,}000$ numbers;

he did this in his off-time from 1844 to 1847, when occupied by the Prussian survey. His next contribution of 2 years was the compilation of hyperbolic table in his spare time which was published by the Austrian Government in 1857. Next he offered to make a table of integer factors of all numbers from 7,000,000 to 10,000,000; for this, on the recommendation of Gauss, the Hamburg Academy of Sciences agreed to assist him financially, but Dase died shortly thereafter in Hamburg. He also had an uncanny sense of quantity. That is, he could just tell, without counting, how many sheep were in a field, or words in a sentence, and so forth, up to about 30.

- Hiromu Hasegawa (1810–1887, Japan) and his father Hiroshi Hasegawa (1782–1838, Japan), in 1844, published many Wasan books. Hiromu developed the series

$$\frac{\pi}{4} = 1 - \frac{1}{2 \cdot 3} - \frac{1}{5 \cdot 8} - \frac{1}{7 \cdot 16} - \frac{5}{9 \cdot 128} - \cdots .$$

This series can be written as

$$\pi = \frac{4}{3} F\left(\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, 1\right).$$

- Clausen, in 1847, used the formula

$$\frac{\pi}{4} = 2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$

to calculate π to 250 decimal places, but only 248 are correct.

- Jacob de Gelder (1765–1848, The Netherlands), in 1849, proposed a geometric construction, which gives π correct to six decimal places. His method is based on the fact that $\pi = 355/113 = 3 + 4^2/(7^2 + 8^2)$.
- Lehmann, in 1853, correctly calculated 261 decimal places of π. For this, he used Euler's formula

$$\frac{\pi}{4} = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right). \tag{8.65}$$

- Rutherford, in 1853, found 440 correct decimal places of π.
- William Shanks (1812–1882, England), in 1853 [472], used Machin's formula (8.53) to calculate π to 607 decimal places. He was assisted by Rutherford in checking first 440 digits.

- Richter, in 1853, published 333 digits (330 correct), and in 1855 (after his death in 1854) 500 decimal places of π.
- Gustav Conrad Bauer (1820–1906, Germany), in 1859, obtained the series

$$\frac{2}{\pi} = 1 - 5\left(\frac{1}{2}\right)^3 + 9\left(\frac{1\cdot 3}{2\cdot 4}\right)^3 - 13\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^3 + \cdots . \qquad (8.66)$$

- James Smith, in 1860, published the value of π as $3\ 1/8$ and argued that it is exact and correct. He attempted to bring it before the British Association for the Advancement of Science. Interestingly, even De Morgan and Rowan Hamilton could not convince him for his mistake.
- Philip H. Vanderweyde, in 1861, published an essay discussing the subject π. He also used several constructions, resulting $\pi = 3.1415926535\cdots$.
- Lawrence Sluter Benson, published about 20 pamphlets on the area of the circle. In 1862, he demonstrated that the area of the circle is equal to $3R^2$, or the arithmetical square between the inscribed and circumscribed squares. According to him $\sqrt{12} = 3.4641016\cdots$ is the ratio between the diameter of a circle and the perimeter of its equivalent square. The ratio between the diameter and circumference, he believed, is not a function of the area of the circle. He accepted the value of $\pi = 3.141592\cdots$.
- S.M. Drach, in 1863, proved that the circumference of a circle can be obtained as follows: From three diameters, deduct $8/1000$ and $7/1{,}000{,}000$ of a diameter, and add 5 percent to the result, i.e.,

$$2\pi = 6 - \frac{16}{1000} - \frac{14}{1{,}000{,}000} + \frac{5}{100}\left(6 - \frac{16}{1000} - \frac{14}{1{,}000{,}000}\right),$$

  which gives $\pi = 3.14159265$.
- Cyrus Pitt Grosvenor (1792–1879, USA), in 1868, described his method in a pamphlet titled *The circle squared*, New York: Square the diameter of the circle; multiply the square by 2; extract the square root of the product; from the root subtract the diameter of the circle; square the remainder; multiply this square by four fifths; subtract the square from the diameter of the circle, i.e.,

$$\frac{\pi D^2}{4} = D^2 - \frac{5}{4}(\sqrt{2D^2} - D)^2 = D^2\left[1 - \frac{5}{4}(\sqrt{2} - 1)^2\right] = D^2(0.785533906\cdots),$$

which gives $\pi = 3.142135\cdots$.

- Benjamin Peirce, in 1870, obtained the formula containing $\pi,\ e$ and *i*, namely, $i^{-i} = e^{\pi/2}$. After proving it in one of his classes on analysis, he said to his students, “Gentlemen, this is surely true, it is absolutely paradoxical, we can’t understand it, and we haven’t the slightest idea what the equation means, but we may be sure that it means something very important.”
- De Morgan, in his book *A Budget of Paradoxes* of 512 pages, which was edited and published by his wife in 1872, is an entertaining text. This book contains the names of 75 writers on π. In this work De Morgan reviewed the works of 42 of these writers, bringing the subject down to 1870. He once remarked that it is easier to square the circle then to get round a mathematician. He was the first to point out that in the decimal expansion of π one should expect each of the 10 digits appear uniformly, i.e., roughly one out of ten digits should be a 4 etc.
- Asaph Hall (1829–1907, USA), in 1872 [245], published the results of an experiment in random sampling that Hall had convinced his friend, Captain O.C. Fox, to perform when Fox was recovering from a wound received at the Second Battle of Bull Run. The experiment was based on Buffon’s needle problem. After throwing his needles eleven hundred times, Fox was able to derive $\pi \simeq 3.14$. This work is considered as a very early documentation use of random sampling, which Nicholas Constantine Metropolis (1915–1999, USA) named as the Monte Carlo method during the Manhattan Project of World War II).
- William Shanks, in 1873–1874 [473], again used Machin’s formula (8.53) to calculate π to 707 decimal places (published in the Proceedings of the Royal Society, London), but only 527 decimal places are correct. For this he used mechanical desk calculator and worked for almost fifteen years. For a long time, this remained the most fabulous piece of calculation ever performed. In the Palais de la Découverte (a science museum in Paris) there is a circular room known as the “pi room.” On its wall are inscribed these 707 digits of π. The digits are large wooden characters attached to the dome-like ceiling.
- Tseng Chi-Hung (died in 1877, China), in 1874, found 100 digits of π in a month. For this he used the formula (8.65).

- John A. Parker (USA), in 1874, in his book *The Quadrature of the Circle. Containing Demonstrations of the Errors of Geometers in Finding Approximations in Use* claimed that $\pi = 20612/6561$ exactly. He exclaims, "all the serial and algebraic formula in the world, or even geometrical demonstration, if it be subjected to any error whatever, cannot overthrow the ratio of circumference to diameter which I have established"! He praises Metius (lived in the sixteenth century) for using the ratio $355/113.$ His book also contains practical questions on the quadrature applied to the astronomical circles.
- Alick Carrick (England), in 1876, proposed in his book *The Secret of the Circle, its Area Ascertained*, the value of π as $3\ 1/7.$
- Pliny Earle Chase (1820–1886, USA), in 1879, in his pamphlet *Approximate Quadrature of the Circle*, used a geometric construction to obtain $\pi = 3.14158499 \cdots$.
- Sylvester Clark Gould (1840–1909, USA), in 1888 [221], compiled the bibliography entitled *What is the Value of Pi*. It contains 100 titles and gives the result of 63 authors. In this work the diagram 16 claims that $\pi = 3\ 3949/27889$ exactly.
- In 1892, a writer announced in the *New York Tribune* the rediscovery of a long-lost secret that gives 3.2 as the exact value of $\pi.$ This announcement caused considerable discussion, and even near the beginning of the twentieth century 3.2 had its advocates as against the value $22/7.$
- Fredrik Carl Mülertz Störmer (1874–1957, Norway), in 1896 [498], gave the following Machin-like formulas for calculating π

$$\frac{\pi}{4} = 44\tan^{-1}\left(\frac{1}{57}\right) + 7\tan^{-1}\left(\frac{1}{239}\right) - 12\tan^{-1}\left(\frac{1}{682}\right) + 24\tan^{-1}\left(\frac{1}{12943}\right) \quad (8.67)$$

and

$$\frac{\pi}{4} = 6\tan^{-1}\left(\frac{1}{8}\right) + 2\tan^{-1}\left(\frac{1}{57}\right) + \tan^{-1}\left(\frac{1}{239}\right). \quad (8.68)$$

- In 1897, in the State of Indiana, the House of Representatives unanimously passed the Bill No. 246 (known as the "π bill") introducing a new mathematical truth "Be it enacted by the General Assembly of the State of Indiana: It has been found that a circular area is to the square on a line equal to the quadrant of the circumference, as the area of an equilateral rectangle is to the square on one side…” ($\pi = 3.2$). The author of the bill was a physician, Edwin J. Goodman (1825–1902), M.D., of Solitude, Posey County, Indiana, and it was introduced in the Indiana House on January 18, 1897, by Mr. Taylor I. Record, Representative from Posey County. Edwin offered this contribution as a free gift for the sole use of the State of Indiana (the others would evidently have to pay royalties). Copies of the bill are preserved in the Archives Division of the Indiana State Library. The bill was sent to the Senate for approval. Fortunately, during the House's debate on the bill, Purdue University Mathematics Professor Clarence Abiathar Waldo (1852–1926) was present. When Professor Waldo informed the Indiana Senate of the “merits” of the bill, the Senate, after some ridicule at the expense of their colleagues, indefinitely postponed voting on the bill and let it die.

- Uhler, in 1900 [515], used Machin's formula (8.53) to compute π to 282 decimal places.
- Mario Lazzarini (Italy), in 1901 [338], performed Buffon's needle experiment. Tossing a needle 3408 times, he obtained the well-known estimate $355/113$ for $\pi.$ Although it is an impressive observation, but suspiciously good (see Badger [46]). In fact, statisticians Maurice George Kendall (1907–1983, England) and Patrick Alfred Pierce Moran (1917–1988, Australia) have commented that one can do better to cut out a large circle and use a tape to measure to find its circumference and diameter. On the same theme of phoney experiments, Norman T. Gridgeman (England), in 1960 [228], pours scorn on Lazzerini and others, created some amusement by using a needle of carefully chosen length $k = 0.7857,$ throwing it twice, and hitting a line once. His estimate for π was thus given by $2 \times 0.7857/\pi = 1/2$ from which he got the highly creditable value of $\pi = 3.1428.$ Of course, he was not being serious!
- Duarte, in 1902, used Machin's formula (8.53) to compute π to 200 decimal places.
- Various mnemonic devices have been given for remembering the decimal digits of π . The most common type of mnemonic is the word-length mnemonic in which the number of letters in each word corresponds to a digit, for example, *How I wish I could calculate pi* (by C. Heckman, in 2005), *May I have a large container of coffee* (by Gardner, in 1959,1966), and *How I want a drink, alcoholic of course, after the*

heavy lectures involving quantum mechanics (by James Jeans), respectively, give π to seven, eight, and fifteen decimal places. Adam C. Orr in *Literary Digest*, vol. 32 (1906), p. 84 published the following poem which gives π to 30 decimal places

Now I, even I, would celebrate
In rhymes inapt, the great
Immortal Syracusan, rivaled nevermore
Who in his wondrous lore,
Passed on before
Left men his guidance,
How to circles mensurate.

Several other such poems not only in English, but almost in every language including Albanian, Bulgarian, Czech, Dutch, French, German, Italian, Latin, Polish, Portuguese, Romanian, Sanskrit, Spanish, and Swedish are known. However, there is a problem with this type of mnemonic, namely, how to represent the digit zero. Fortunately, a zero does not occur in π until the thirty-second place. Several people have come up with ingenious methods of overcoming this, most commonly using a 10-letter word to represent zero. In other cases, a certain piece of punctuation is used to indicate a naught. Michael Keith (born 1955, USA) with such similar understanding, in his work *Circle digits: a self-referential story, Mathematical Intelligencer*, vol.8 (1986), 56–57, wrote an interesting story, which gives first 402 decimals of π.

- Ernest William Hobson (1856–1933, England), in his 1913 book [264] used a geometrical construction to obtain $\pi = 3.14164079\cdots$.
- Ramanujan, in his 1914 paper on "Modulus functions and approximation to π" gave several new innovative empirical formulas and geometrical constructions for approximating π. One of the remarkable formulas for its elegance and inherent mathematical depth is

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801}\sum_{m=0}^{\infty}\frac{(4m)!}{(m!)^4}\frac{(1103+26390m)}{396^{4m}}. \qquad (8.69)$$

It has been used to compute π to a level of accuracy, never attained earlier. Each additional term of the series adds roughly 8 digits. He also developed the series

$$\frac{1}{\pi} = \sum_{m=0}^{\infty} \binom{2m}{m}^3 \frac{42m+5}{2^{12m+4}} \quad \text{and} \quad \frac{2}{\pi} = \sum_{m=0}^{\infty} \frac{(-1)^m(4m+1)[(2m-1)!!]^3}{((2m)!!)^3}.$$

The first series has the property that it can be used to compute the second block of *k* (binary) digits in the decimal expansion of π without calculating the first *k* digits. The second series is the same as (8.66), which Ramanujan included in his first letter on January 31st, 1913 to Hardy. The following mysterious approximation which approximates π to 18 correct decimal places is also due to Ramanujan

$$\pi \simeq \frac{12}{\sqrt{190}} \ln\left((2\sqrt{2}+\sqrt{10})(3+\sqrt{10})\right).$$

- T.M.P. Hughes (England), in his 1914 work *A triangle that gives the area and circumference of any circle, and the diameter of a circle equal in area to any given square, Nature* 93, 110, doi:10.1038/093110a0 uses a geometric construction to obtain $\pi = 3.14159292035\cdots$.
- In March 1928, the University of Minnesota was notified that Gottfried Lenzer (a native of Germany who lived in St. Paul for many years) had bequeathed to the university a series of 60 drawings from 1911 to 1927 and explanatory notes concerning the three classical problems of antiquity. He used a geometrical construction for squaring the circle to obtain $\pi = 3.1378\cdots$.
- Helen Abbot Merrill (1864–1949), in her 1934 book *Mathematical Excursions: Side Trips Along Paths not Generally Traveled in Elementary Courses in Mathematics, Bruce Humphries, Inc., Boston* gave a geometric construction (perhaps by an earlier author) which leads to $\pi = 3.141591953\cdots$.
- Landau, in 1934, in his work defined $\pi/2$ as the value of *x* between 1 and 2 for which $\cos x$ vanishes. One cannot believe this definition was used, at least as an excuse, for a racial attack on Landau. This unleashed an academic dispute which was to end in Landau's dismissal from his chair at Göttingen.
- Carl Theodore Heisel (1852–1937, USA), in 1934, published a book *Mathematical and Geometrical Demonstrations* in which he announced the grand discovery that π was exactly equal to $256/81,$ a value that the Egyptians had used some 4,000 years ago. Substituting this value for calculations of areas and circumferences of circles with diameters $1, 2, \cdots$ up to $9,$ he obtained numbers which showed consistency of

circumference and area, “thereby furnishing incontrovertible proof of the exact truth” of his ratio. He also rejected decimal fractions as inexact (whereas ratios of integers as exact and scientific), and extracted roots of negative numbers thus: $\sqrt{-a} = \sqrt{a}-,\ \sqrt{a-^2} = -a.$, He published this book on his own expense and distributed to colleges and public libraries throughout the United States without charge.

- Miff Butler (USA), in 1934, claimed discovery of a new relationship between π and $e.$ He stated his work to be the first basic mathematical principle ever developed in the USA. He convinced his congressman to read it into the Congressional Record on June 5, 1940.
- Uhler, in 1940, used Machin’s formula (8.53) to compute π to 333 decimal places.
- D.F. Ferguson (England), during 1945–47 [186, 187], used the formula

$$\frac{\pi}{4} = 3\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{1}{20}\right) + \tan^{-1}\left(\frac{1}{1985}\right)$$

to find that his value disagreed with that of William Shanks in the 528th place. In 1946, he approximated π to 620 decimal places, and in January 1947 to 710 decimal places. In the same month William Shanks used Machin’s formula (8.53) to compute 808-place value of $\pi,$ but Ferguson soon found an error in the 723rd place. For all the calculations, he used desk calculator.

- Ferguson and John William Wrench, Jr. (1911–2009, USA), in 1949, used a desk calculator to compute 1120 decimal digits of $\pi.$ This record was broken only by the electronic computers.
- Wrench and L.B. Smith (also attributed to George Reitwiesner et al.), in September 1949 (see [538]), were the first to use an electronic computer ENIAC (Electronic Numerical Integrator and Computer) at the Army Ballistic Research Laboratories in Aberdeen, Maryland, to calculate π to 2037 decimal places. For this they programmed Machin’s formula (8.53). It took 70 hours, a pitifully long time by today’s standards. In this project von Neumann also took part. In 1965, The ENIAC became obsolete, and it was dismembered and moved to the Smithsonian Institution as a museum piece.
- Konrad Knopp (1882–1957, Germany), in 1951 [309], gave the following two expansions of $\pi,$

$$\frac{\pi}{4} = \sum_{k=1}^{\infty} \tan^{-1}\left(\frac{1}{k^2+k+1}\right) \quad \text{and}$$
$$\frac{\pi^2}{16} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}\left(1+\frac{1}{3}+\cdots+\frac{1}{2k+1}\right).$$

- S.C. Nicholson and J. Jeenel (USA), in 1954 [393], programmed NORC (Naval Ordnance Research Calculator) at Dahlgren, Virginia to compute π to 3092 decimals. For this they used Machin's formula (8.53). The run took only 13 minutes.
- John Gurland (1917–1997, USA), in 1956 [240], established that for all positive integers n,

$$\frac{4n+3}{(2n+1)^2}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2 < \pi < \frac{4}{4n+1}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2. \qquad (8.70)$$

- George Eric Felton (1921–2019, England), in 1957, used the Ferranti Pegasus computer to find 10,021 decimal places of π in 33 hours. The program was based on the formula of Samuel Klingenstierna (1698–1765, Sweden)

$$\pi = 32\ \tan^{-1}\left(\frac{1}{10}\right) - 4\tan^{-1}\left(\frac{1}{239}\right) - 16\tan^{-1}\left(\frac{1}{515}\right). \qquad (8.71)$$

  However, a subsequent check revealed that a machine error had occurred, so that "only" 7480 decimal places were correct. The run was therefore repeated in May 1958, but the correction was not published.
- Francois Genuys (France), in January 1958 [210], programmed an IBM 704 at the Paris Data Processing Center. He used Machin's type formula (8.53). It yielded 10,000 decimal places of π in 1 hour and 40 minutes.
- Genuys, in July 1959, programmed an IBM 704 at the Commissariat á l'Energie Atomique in Paris to compute π to $16{,}167$ decimal places. He used Machin's type formula (8.53). It took 4 hours and 30 minutes.
- Daniel Shanks (1917–1996, USA) and William Shanks, in July 1961, used Störmer's formula (8.68) on an IBM 7090 (at the IBM Data Processing Center, New York) to compute π to $100{,}265$ digits, of which the first $100{,}000$ digits were published by photographically reproducing the print-out with $5{,}000$ digits per page. The time required for this computation was 8 hours and 43 minutes. They also checked the

calculations by using Gauss' formula (8.64), which required 4 hours and 22 minutes.

- In 1961, Machin's formula (8.53) was also the basis of a program run on an IBM 7090 at the London Data Center in July 1961, which resulted in 20,000 decimal places and required only 39 minutes running time.
- Jean Guilloud and J. Filliatre, in February 1966, used an IBM 7030 at the Commissariat á l'Energie Atomique in Paris to obtain an approximation of π extending to 250,000 decimal places on a STRETCH computer. For this they used Störmer's and Gauss' formulas (8.68) and (8.64). It took almost 28 hours.
- Guilloud and M. Dichampt, in February 1967, used CDC (Control Data Corporation) 6600 in Paris to approximate π to 500,000 decimal places. For this they used Störmer's and Gauss' formulas (8.68) and (8.64). The computer that churned out half a million digits needed only 26 hours and 40 minutes (plus 1 hour and 30 minutes to convert that final result from binary to decimal notation).
- In 1968, in Putnam Competition the first problem was

$$\pi = \frac{22}{7} - \int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx.$$

  This integral was known to Kurt Mahler (1903–1988, Germany–Australia) in the mid-1960s and has later appeared in several exams. It is also discussed by Borwein, David Harold Bailey (born 1948, USA), and Girgensohn in their book [97] on page 3.
- Guilloud, in 1973, wrote a 400-page book that listed π to the 1 million decimal place.
- Gosper, in 1974 [218], used a refinement of Euler transform on (7.178) to obtain the series

$$\pi = 3 + \frac{1}{60}8 + \frac{1}{60}\frac{2\cdot 3}{7\cdot 8\cdot 3}13 + \frac{1}{60}\frac{2\cdot 3}{7\cdot 8\cdot 3}\frac{3\cdot 5}{10\cdot 11\cdot 3}18 + \frac{1}{60}\frac{2\cdot 3}{7\cdot 8\cdot 3}\frac{3\cdot 5}{10\cdot 11\cdot 3}\frac{4\cdot 7}{13\cdot 14\cdot 3}23 + \cdots .$$

- Guilloud and Martine Bouyer (France), in 1974 [230], used formulas (8.68) and (8.64) on a CDC 7600 to compute π to $1{,}000{,}250$ digits. The run time required for this computation was 23 hours and 18 minutes, of which 1 hour 7 minutes was used to convert the final result from binary to decimal. Results of statistical tests, which generally support the conjecture that π is *simply normal* (In 1909, Borel defined: A real number *a* is simply normal in base *b* if in its representation in base *b* all digits occur, in an asymptotic sense, equally often.) were also performed.

- Louis Comtet (1933–2012, France), in 1974, developed the following Euler's type expansion of π,

$$\frac{\pi^4}{90} = \frac{36}{17}\sum_{m=1}^{\infty}\frac{1}{m^4\binom{2m}{m}}.$$

- Richard Peirce Brent (born 1946, Australia) and Eugene Salamin (USA), in 1976 [102, 103, 444], independently discovered an algorithm that is based on an arithmetic-geometric mean and modifies slightly Gauss–Legendre algorithm. Set $a_0 = 1,\ b_0 = 1/\sqrt{2}$ and $s_0 = 1/2$. For $k = 1, 2, 3, \cdots$ compute

$$\begin{aligned} a_k &= \frac{a_{k-1}+b_{k-1}}{2} \\ b_k &= \sqrt{a_{k-1}b_{k-1}} \\ c_k &= a_k^2 - b_k^2 \\ s_k &= s_{k-1} - 2^k c_k \\ p_k &= \frac{2a_k^2}{s_k}. \end{aligned} \qquad (8.72)$$

Then, p_k converges quadratically to $\pi,$ i.e., each iteration doubles the number of accurate digits. In fact, successive iterations must produce $1, 4, 9, 20, 42, 85, 173, 347$ and 697 correct digits of π. Twenty-fifth iteration must produce 45 million correct decimal digits of π.

- Kazunori Miyoshi (Japan) and Kanada, in 1981, calculated π to $2{,}000{,}036$ significant figures in 137.30 hours on a FACOM M-200 computer. They used Klingenstierna's formula (8.71) and checked their result with Machin's formula (8.53).
- Guilloud, in 1981, computed $2{,}000{,}050$ decimal digits of π.
- Rajan Srinivasan Mahadevan (born 1957, India), in 1981, recited from memory the first $31,811$ digits of π. This secured him a place in the 1984 Guinness Book of World Records, and he has been featured on Larry King Live and Reader's Digest.

- Kikuo Takano (1927–2006, Japan), in 1982, developed the following Machin-like formula for calculating π

$$\frac{\pi}{4} = 12\tan^{-1}\left(\frac{1}{49}\right) + 32\tan^{-1}\left(\frac{1}{57}\right) - 5\tan^{-1}\left(\frac{1}{239}\right) + 12\tan^{-1}\left(\frac{1}{110443}\right). \quad (8.73)$$

- Yoshiaki Tamura (Japan), in 1982, on MELCOM 900II computed $209,7144$ decimal places of $\pi.$ For this he used Brent-Salamin algorithm (8.72).
- Tamura and Kanada in 1982 [505], on HITAC M-280H computed $4,194,288$ decimal places of $\pi.$ For this they used Brent-Salamin algorithm (8.72).
- Tamura and Kanada, in 1982, on HITAC M-280H computed $8,388,576$ decimal places of $\pi.$ For this they used Brent-Salamin algorithm (8.72).
- Kanada, Sayaka Yoshino (Japan) and Yasunori Ushiro (Japan), in 1983, on HITAC M-280H computed $16{,}777{,}206$ decimal places of $\pi.$ For this they used Brent-Salamin algorithm (8.72).
- Kanada, Tamura, Yoshino, and Ushiro, in October 1983 [294], on HITAC S-810/20 computed $10,013,395$ decimal places of $\pi.$ For this they used Brent-Salamin algorithm (8.72). In this work to gather evidence that π is simply normal they also performed statistical analysis. It showed expected behavior. In the first 10 million digits, the frequencies for each 10 digits are 999,440; 999,333; 1,000,306; 999,964; 1,001,093; 1,000,466; 999,337; 1,000,207; 999,814; and 1,000,040. Further, the rate at which the relative frequencies approach $1/10$ agrees with theory. As an example, for the digit 7 relative frequencies in the first $10^i,\ i = 0,1,2,3,4,5,6,7$ digits are $0,\ 0.08,\ 0.095,\ 0.097,\ 0.10025,\ 0.0998,\ 0.1000207,$ which seem to be approaching $1/10$ at rate predicted by the probability theory for random digits, i.e., a speed approximately proportional to $1/\sqrt{n}.$ But this is far from a formal proof of simple normalcy-perhaps for a proof the current mathematics is not sufficiently developed. In spite of the fact that the digits of π pass statistical tests for randomness, π contains some sequences of digits that, to some, may appear non-random, such as Feynman point, which is a sequence of six consecutive 9s that begins at the 762nd decimal place. A number is said to be *normal* if all blocks of digits of the same length occur with equal frequency. Mathematicians

expect π to be normal, so that every pattern possible eventually will occur in the digits of π.

- Jonathan Michael Borwein (1951–2016, Scotland–Canada) and Peter Benjamin Borwein (1953–2020, Scotland–Canada), in 1984 [82], gave the following algorithm. Set $x_0 = \sqrt{2},\ y_0 = 0$, and $\alpha_0 = 2 + \sqrt{2}$. Iterate

$$\begin{aligned} x_{k+1} &= \left(\sqrt{x_k} + 1/\sqrt{x_k}\right)/2 \\ y_{k+1} &= \sqrt{x_k}\left(\frac{1+y_k}{y_k+x_k}\right) \\ \alpha_{k+1} &= \alpha_k y_{k+1}\left(\frac{1+x_{k+1}}{1+y_{k+1}}\right). \end{aligned} \tag{8.74}$$

  Then α_k converges to π quartically. The algorithm is not self-correcting; each iteration must be performed with the desired number of correct digits of π.

- Morris Newman (USA) and Daniel Shanks, in 1984 [392], proved the following: Set

$$\begin{aligned} a &= \frac{1071}{2} + 92\sqrt{34} + \frac{3}{2}\sqrt{255349 + 43792\sqrt{34}} \\ b &= \frac{1533}{2} + 133\sqrt{34} + \frac{1}{2}\sqrt{4817509 + 826196\sqrt{34}} \\ c &= 429 + 304\sqrt{2} + 2\sqrt{92218 + 65208\sqrt{2}} \\ d &= \frac{627}{2} + 221\sqrt{2} + \frac{1}{2}\sqrt{783853 + 554268\sqrt{2}} \end{aligned}$$

  then

$$\left|\pi - \frac{6}{\sqrt{3502}}\ln(2abcd)\right| < 7.4 \times 10^{-82}.$$

- Gosper, in 1985, used Symbolics 3670 and Ramanujan's formula (8.69) to compute π to $17{,}526{,}200$ decimal digits.
- Borwein brothers, in 1985, gave the following algorithm. Set $a_0 = 6 - 4\sqrt{2}$ and $y_0 = \sqrt{2} - 1$. Iterate

$$
\begin{aligned}
y_{k+1} &= \frac{1-(1-y_k^4)^{1/4}}{1+(1-y_k^4)^{1/4}} \\
a_{k+1} &= a_k(1+y_{k+1})^4 - 2^{2k+3}y_{k+1}(1+y_{k+1}+y_{k+1}^2).
\end{aligned} \tag{8.75}
$$

Then, a_k converges quartically to $1/\pi,$ i.e., each iteration approximately quadruples the number of correct digits.

- Berggren and Borwein brothers in their book [67] of 1985 claimed that the following is not an identity but is correct to over 42 billion digits

$$
\left(\frac{1}{10^5}\sum_{n=-\infty}^{\infty} e^{-n^2/10^{10}}\right)^2 \simeq \pi.
$$

- Carl Sagan (1934–1996, USA), in his novel *Contact* of 1986 [442], deals with the theme of contact between humanity and a more technologically advanced, extraterrestrial life form. He suggests that the creator of the universe buried a message deep within the digits of π.
- Bailey, in January 1986 (published in 1988 [49]), used Borwein brothers algorithms (8.74) and (8.75) on CRAY-2 to compute $29{,}360{,}111$ decimal places of π.
- Kanada and Tamura, in September 1986, on HITAC S-810/20 computed $33{,}554{,}414$ decimal places of $\pi.$ For this they used algorithms (8.72) and (8.75).
- Kanada and Tamura, in October 1986, on HITAC S-810/20 computed $67{,}108{,}839$ decimal places of $\pi.$ For this they used algorithm (8.72).
- Kanada, Tamura, Yoshinobu Kubo (Japan) and others, in January 1987, on NEC SX-2 computed $134{,}214{,}700$ decimal places of $\pi.$ For this they used algorithms (8.72) and (8.75).
- Borwein brothers, in 1987, gave the following algorithm. Set $x_0 = 2^{1/2},\ y_1 = 2^{1/4}$ and $p_0 = 2+2^{1/2}.$ Iterate

$$
\begin{aligned}
x_k &= \frac{1}{2}\left(x_{k-1}^{1/2} + x_{k-1}^{-1/2}\right) \\
y_k &= \frac{y_{k-1}x_{k-1}^{1/2} + x_{k-1}^{-1/2}}{y_{k-1}+1} \\
p_k &= p_{k-1}\frac{x_k+1}{y_k+1}.
\end{aligned}
$$

Then, p_k decreases monotonically to π and $|p_k - \pi| \le 10^{-2^{k+1}}$ for $k \ge 4$.

- Hideaki Tomoyori (born 1932, Japan), in 1987, recited π from memory to $40{,}000$ places—taking 17 hours 21 minutes, including breaks totaling 4 hours 15 minutes, at Tsukuba University Club House.
- Kanada, in January 1988 [295], on HITAC S-820/80 computed $201{,}326{,}551$ decimal places of π. For this he used algorithms (8.72) and (8.75).
- Borwein brothers, in 1988 [91], developed the series

$$\frac{1}{\pi} = 12 \sum_{n=0}^{\infty} \frac{(-1)^n (6n)!}{(n!)^3 (3n)!} \frac{(A+Bn)}{C^{n+1/2}},$$

where

$$\begin{aligned} A &= 212175710912\sqrt{61} + 1657145277365 \\ B &= 13773980892672\sqrt{61} + 107578229802750 \\ C &= [5280(236674 + 30303\sqrt{61})]^3. \end{aligned}$$

Each additional term of the series adds roughly 31 digits.

- Dario Castellanos (Venezuela), in 1988 [120, 121], gave the following approximation

$$\pi \simeq \left(\frac{77729}{254}\right)^{1/5} = 3.1415926541 \cdots .$$

- David Volfovich Chudnovsky (born 1947, Ukrainian-USA) and Gregory Volfovich Chudnovsky (born 1952, Ukrainian-USA), in May 1989, on CRAY-2 and IBM 3090/VF computed $480{,}000{,}000$ decimal places of π.
- Chudnovsky brothers, in June 1989, on IBM 3090 computed $535{,}339{,}270$ decimal places of π.
- Kanada and Tamura, in July 1989, on HITAC S-820/80 computed $536, 870, 898$ decimal places of π. For this they used algorithm (8.72).
- Chudnovsky brothers, in August 1989 [130], developed the following rapidly convergent generalized hypergeometric series

$$\frac{1}{\pi} = 12 \sum_{n=0}^{\infty} (-1)^n \frac{(6n)!}{(n!)^3 (3n)!} \frac{13591409 + 545140134n}{(640320^3)^{n+1/2}}. \tag{8.76}$$

Each additional term of the series adds roughly 15 digits. This series is an improved version to that of Ramanujan's (8.69). It was used by them to calculate more than one billion (to be exact $1{,}011{,}196{,}691$) digits on IBM 3090.

- Kanada and Tamura, in November 1989, on HITAC S-820/80 computed $1{,}073{,}740{,}799$ decimal places of $\pi.$ For this they used algorithms (8.72) and (8.75).
- Chudnovsky brothers, in August 1991, used a homemade parallel computer (they called it *m* zero, where *m* stands for machine, and zero for the success) to obtain $2{,}260{,}000{,}000$ decimal places of $\pi.$ For this they used series (8.76).
- David Boll (USA), in 1991 [77], discovered an occurrence of π in the Mandelbrot set fractal.
- Borwein brothers, in 1991, improved on the Brent-Salamin algorithm (8.72). Set $a_0 = 1/3$ and $s_0 = (\sqrt{3}-1)/2.$ Iterate

$$\begin{aligned} r_{k+1} &= \frac{3}{1+2(1-s_k^3)^{1/3}} \\ s_{k+1} &= \frac{r_{k+1}-1}{2} \\ a_{k+1} &= r_{k+1}^2 a_k - 3^k(r_{k+1}^2-1). \end{aligned}$$

Then $1/a_k$ converges cubically to $\pi,$ i.e., each iteration approximately triples the number of correct digits.

Among the several other known iterative schemes, we list the following two, which are easy to implement on a computer: Set $a_0 = 1/2$ and $s_0 = 5(\sqrt{5}-2).$ Iterate

$$\begin{aligned} x_{n+1} &= 5/s_n - 1 \\ y_{n+1} &= (x_{n+1}-1)^2 + 7 \\ z_{n+1} &= \left(\frac{1}{2}x_{n+1}\left(y_{n+1} + \sqrt{y_{n+1}^2 - 4x_{n+1}^3}\right)\right)^{1/5} \\ a_{n+1} &= s_n^2 a_n - 5^n\left(\frac{s_n^2-5}{2} + \sqrt{s_n(s_n^2-2s_n+5)}\right) \\ s_{n+1} &= \frac{25}{(z_{n+1}+x_{n+1}/z_{n+1}+1)^2 s_n}. \end{aligned}$$

Then a_k converges quintically to $1/\pi$, i.e., each iteration approximately quintuples the number of correct digits, and $0 < a_n - 1/\pi < 16 \cdot 5^n \cdot e^{-5^n} \pi$.

Set $a_0 = 1/3,\ r_0 = (\sqrt{3} - 1)/2$ and $s_0 = (1 - r_0^3)^{1/3}$. Iterate

$$\begin{aligned}
t_{n+1} &= 1 + 2r_n \\
u_{n+1} &= (9r_n(1 + r_n + r_n^2))^{1/3} \\
v_{n+1} &= t_{n+1}^2 + t_{n+1}u_{n+1} + u_{n+1}^2 \\
w_{n+1} &= \frac{27(1 + s_n + s_n^2)}{v_{n+1}} \\
a_{n+1} &= w_{n+1}a_n + 3^{2n-1}(1 - w_{n+1}) \\
s_{n+1} &= \frac{(1 - r_n)^3}{(t_{n+1} + 2u_{n+1})v_{n+1}} \\
r_{n+1} &= (1 - s_{n+1}^3)^{1/3}.
\end{aligned}$$

Then a_k converges nonically to $1/\pi$, i.e., each iteration approximately multiplies the number of correct digits by nine.

- Borwein brothers, in 1993, developed the series

$$\frac{\sqrt{-C^3}}{\pi} = \sum_{m=0}^{\infty} \frac{(6m)!}{(3m)!(m!)^3} \frac{A + mB}{C^{3m}},$$

where

$$\begin{aligned}
A &= 63365028312971999585426220 + 28337702140800842046825600\sqrt{5} \\
&\quad +384\sqrt{5}(10891728551171178200467436212395209160385656017 \\
&\quad +4870929086578810225077338534541688721351255040\sqrt{5})^{1/2} \\
B &= 7849910453496627210289749000+3510586678260932028965606400\sqrt{5} \\
&\quad +2515968\sqrt{3110}(6260208323789001636993322654444020882161 \\
&\quad +2799650273060444296577206890718825190235\sqrt{5})^{1/2}
\end{aligned}$$

and

$$\begin{aligned}
C &= -214772995063512240 - 96049403338648032\sqrt{5} \\
&\quad -1296\sqrt{5}(10985234579463550323713318473 \\
&\quad +4912746253692362754607395912\sqrt{5})^{1/2}.
\end{aligned}$$

Each additional term of the series adds approximately 50 digits. However, computation of this series on a computer does not seem to be easy.

- Chudnovsky brothers, in May 1994, used a homemade parallel computer *m* zero to obtain $4{,}044{,}000{,}000$ decimal places of π. For this they used series (8.76).
- Kanada and Daisuke Takahashi (Japan), in June 1995, on HITAC S-3800/480 (dual CPU) computed $3{,}221{,}225{,}466$ decimal places of π. For this they used algorithms (8.72) and (8.75).
- Kanada and Takahashi, in August 1995, on HITAC S-3800/480 (dual CPU) computed $4{,}294{,}967{,}286$ decimal places of π. For this they used algorithms (8.72) and (8.75).
- Kanada and Takahashi, in October 1995, on HITAC S-3800/480 (dual CPU) computed $6{,}442{,}450{,}938$ decimal places of π. For this they used algorithms (8.72) and (8.75).
- Bailey, Peter Borwein and Simon Plouffe, in 1995 (published in 1997 [51]), developed the following formula (known as BBP formula) to compute the *n*th hexadecimal digit (base 16) of π without having the previous $n-1$ digits

$$\pi = \sum_{m=0}^{\infty} \frac{1}{16^m}\left(\frac{4}{8m+1} - \frac{2}{8m+4} - \frac{1}{8m+5} - \frac{1}{8m+6}\right). \tag{8.77}$$

To show the validity of (8.77), for any $k < 8,$ we have

$$\int_0^{1/\sqrt{2}} \frac{x^{k-1}}{1-x^8}dx = \int_0^{1/\sqrt{2}} \sum_{m=0}^{\infty} x^{k-1+8m}dx = \frac{1}{2^{k/2}} \sum_{m=0}^{\infty} \frac{1}{16^m(8m+k)},$$

therefore

$$\begin{aligned}\sum_{m=0}^{\infty} \frac{1}{16^m}\left(\frac{4}{8m+1} - \frac{2}{8m+4} - \frac{1}{8m+5} - \frac{1}{8m+6}\right)\\ = \int_0^{1/\sqrt{2}} \frac{4\sqrt{2} - 8x^3 - 4\sqrt{2}x^4 - 8x^5}{1-x^8}dx.\end{aligned} \tag{8.78}$$

Substituting $u = \sqrt{2}x$ in the equation (8.78), we obtain

$$\int_0^1 \frac{16u-16}{u^4-2u^3+4u-4}du = \int_0^1 \frac{4u}{u^2-2}du - \int_0^1 \frac{4u-8}{u^2-2u+2}du = \pi.$$

The discovery of this formula came as a surprise. For centuries it had been assumed that there was no way to compute the *n*th digit of π without calculating all the preceding $n-1$ digits. Since this discovery, many such formulas for other irrational numbers have been discovered. Such formulas have been called as *spigot algorithms* (see Rabinowitz and Wagon [424]) because, like water dripping from a spigot, they produce digits that are not reused after they are calculated.

- Plouffe, in 1996, discovered an algorithm for the computation of π in any base. Later, he expressed regrets for having shared credit for his discovery of this formula with Bailey and Borwein.
- Chudnovsky brothers, in 1996, used a homemade parallel computer *m* zero to obtain $8{,}000{,}000{,}000$ decimal places of $\pi.$ For this they used series (8.76). They said "we are looking for the appearance of some rules that will distinguish the digits of π from other numbers, i.e., if someone were to give you a million digits from somewhere in $\pi,$ could you tell it was from π? The digits of π form the most nearly perfect random sequence of digits that has ever been discovered. However, each digit appears to be orderly. If a single digit in π were to be changed anywhere between here and infinity, the resulting number would no longer be $\pi,$ it would be garbage. Around the three-hundred-millionth decimal place of $\pi,$ the digits go 88888888-eight eights pop up in a row. Does this mean anything? It appears to be random noise. Later, 10 sixes erupt: 6666666666. What does this mean? Apparently, nothing, only more noise. Somewhere past the half-million mark appears the string 123456789. It is an accident, as it were. We do not have a good, clear, crystallized idea of randomness. It cannot be that π is truly random. Actually, a truly random sequence of numbers has not yet been discovered."
- Gosper, in 1996 [220], posted the following fascinating formula

$$\lim_{n\to\infty} \prod_{m=n}^{2n} \frac{\pi}{2\tan^{-1} m} = 4^{1/\pi} = 1.554682\cdots.$$

For several other formulas involving π see Gosper [219].

- Kanada and Takahashi, in April 1997, on HITACHI SR2201 (1024 CPU) computed $17{,}179{,}869{,}142$ decimal places of $\pi.$ For this they used algorithms (8.72) and (8.75).

- Kanada and Takahashi, in July 1997, on HITACHI SR2201 (1024 CPU) computed $51{,}539{,}600{,}000$ decimal places of π. The computation tool just over 29 hours, at an average rate of nearly $500{,}000$ digits per second. For this they used algorithms (8.72) and (8.75).
- Fabrice Bellard (born 1972, France), in 1997 [62], developed the following formula

$$\pi = \frac{1}{2^6}\sum_{m=0}^{\infty}\frac{(-1)^m}{2^{10m}}\left(-\frac{2^5}{4m+1} - \frac{1}{4m+3} + \frac{2^8}{10m+1} - \frac{2^6}{10m+3} - \frac{2^2}{10m+5} - \frac{2^2}{10m+7} + \frac{1}{10m+9}\right),$$

which can used to compute the *n*th digit of π in base $2.$ It is about 43% faster then (8.77). The following exotic formula is also due to him

$$\pi = \frac{1}{740025}\left[\sum_{m=1}^{\infty}\frac{3P(m)}{\binom{7m}{2m}2^{m-1}} - 20379280\right],$$

where

$$P(m) = -885673181m^5 + 3125347237m^4 - 2942969225m^3 + 1031962795m^2 - 196882274m + 10996648.$$

- Takahashi and Kanada, in 1998 [504] computed 51.5 billion decimal digits of π on distributed memory and parallel processors.
- Kanada and Takahashi, in April 1999, on HITACHI SR8000 (64 of 128 nodes) computed $68{,}719{,}470{,}000$ decimal places of π. For this they used algorithms (8.72) and (8.75).
- Kanada and Takahashi, in September 1999, on HITACHI SR8000/MPP (128 nodes) computed $206{,}158{,}430{,}000,$ i.e., 206 billion decimal places of π. For this they used algorithms (8.72) and (8.75).
- J. Munkhammar (Sweden), in 2000, gave the following formula which is on line with that of Viéte's (8.42)

$$\pi = \lim_{n\to\infty} 2^{n+1}\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}.$$

This as a recurrence relation can be written as $\pi = \lim_{n\to\infty} 2^{n+1} a_n$, where $a_0 = \sqrt{2}$, and

$$a_n = \sqrt{\left(\frac{1}{2}a_{n-1}\right)^2 + \left[1 - \sqrt{1 - \left(\frac{1}{2}a_{n-1}\right)^2}\right]^2} = \sqrt{2 - \sqrt{4 - a_{n-1}^2}}\,.$$

To show this, in Fig. 8.15, we consider a portion of a circle with center *O* and radius $r = 1$. The line segment *AB* of length *a* is a side of a regular inscribed *n*-gon. We bisect *AB* at *D* and draw a radius through *D* out to the circle at C, we generate segment AC, a side of a regular inscribed 2*n*-gon. Let *b* be the length of AC. We will find *b* in terms of a. For this, in $\triangle ADO$ Pythagorean theorem gives

$$1^2 = \left(\frac{1}{2}a\right)^2 + x^2 \implies x = \sqrt{1 - \frac{a^2}{4}}$$

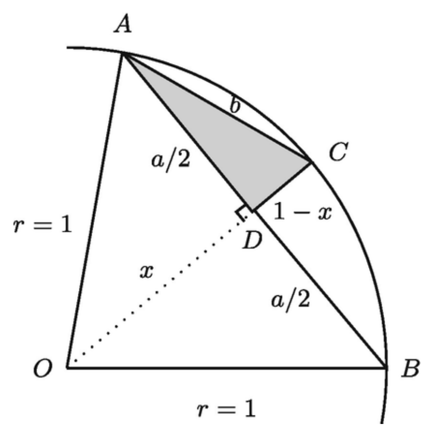

Fig. 8.15 Munkhammar approximation of π

and hence

$$CD = 1 - x = 1 - \sqrt{1 - \frac{a^2}{4}}.$$

Now applying Pythagorean theorem in the triangle ADC, to get

$$b^2 = \left(\frac{1}{2}a\right)^2 + (1-x)^2 = \frac{a^2}{4} + 1 - 2\sqrt{1 - \frac{a^2}{4}} + 1 - \frac{a^2}{4} = 2 - 2\sqrt{1 - \frac{a^2}{4}}$$
$$= 2 - \sqrt{4 - a^2}.$$

Thus, it follows that $b = \sqrt{2 - \sqrt{4 - a^2}}$. Hence, if we denote $a = a_n$ as the side of the 2^{n+2}-gon, then $b = a_{n+1} = \sqrt{2 - \sqrt{4 - a_n^2}}$. Next, since, square has 4 sides and length of each side is $a_0 = \sqrt{2}$, its perimeter is $4a_0$; further since a_n has 2^{n+2} sides, its perimeter is $2^{n+2}a_n$. Finally, since $\pi =$perimeter/diameter, it follows that $\pi \sim 2^{n+1}a_n$. From the geometry it is clear that the sequence $\{a_n\}$ is increasing and convergence to π.

Another closely related formula is

$$\pi = 2 \lim_{n\to\infty} \sum_{m=1}^{n} \sqrt{\left[\sqrt{1 - \left(\frac{m-1}{n}\right)^2} - \sqrt{1 - \left(\frac{m}{n}\right)^2}\right]^2 + \frac{1}{n^2}}.$$

- Robert Palais (USA), in 2001 [403], believed that the notation π is wrongly used right from the beginning. According to him, some suitable symbol (now popular as tau τ) must have been used for 2π. He justifies his claim by giving several formulas where τ appears naturally rather than just π. For some people June 28 is *Tau's Day* and they celebrate.
- Bruce Carl Berndt (born 1938, USA) and Chan in 2001, obtained equivalent form of Ramanujan–Borweins–Chudnovskys series, and found an improved series for $1/\pi$, which yields about 73 or 74 digits of π per term.
- Kanada, in November 2002, used Machin-like formulas (8.67) and (8.73) to compute the value of π to $1{,}241{,}100{,}000{,}000$ decimal places. The calculation took more than 600 hours on 64 nodes of a HITACHI SR8000/MPP supercomputer. The work was done at the Department of Information Science at the University of Tokyo.
- Daniel Tammet (born 1979, England), in 2004, recited $22{,}514$ decimal places of π, scoring the European record. For an audience at the Museum of the History of Science in Oxford, he said these numbers aloud for 5 hours and 9 minutes. Unfortunately, he made his first mistake at position $2{,}965$ and did not correct this error immediately and without outside help, but only after he was told that there was a mistake.
- Stephen K. Lucas (Australia-USA), in 2005 [352], found that

$$\pi = \frac{355}{113} - \int_0^1 \frac{x^8(1-x)^8(25+816x^2)}{3164(1+x^2)}dx.$$

Several other integral formulas of this type are known, here we give the following

$$\pi = \frac{741269838109}{235953517800} - \int_0^1 \frac{x^{16}(1-x)^{16}}{64(1+x^2)}dx,$$

which gives $3.14159265358955 < \pi$. If we substitute $x = 1$ in the denominator of aforementioned integral and note that

$$\int_0^1 \frac{1}{128}x^{16}(1-x)^{16}dx = \frac{1}{2538963567360}$$

then it follows that $\pi < 3.14159265358996.$

- Chao Lu, a chemistry student, at age 23, in November 2005, broke Guiness record by reciting π from memory to 67,890 places. For this he practiced for 4 years. The attempt lasted 24 hrs 4 min and was recorded on 26 video tapes. The attempt was witnessed by 8 officials, Math's professors, and 20 students.
- Kate Bush (born 1958, England), in 2005, in the song π (in her album Aerial) sings the number to its 137th decimal place (though she omits the 79th to 100th decimal places).
- Akira Haraguchi (born 1946, Japan), in October 2006, recited π from memory to 54000 digits in September 2004, 68000 digits in December 2004, 83431 digits in July 2005, and 100000 digits in October 2006. He accomplished the last recitation in $16\ 1/2$–hour in Tokyo. He says – memorization of the digits of π is "the religion of the universe."
- Plouffe, in 2006 [413], found the following curious formula

$$\pi = 72\sum_{k=1}^{\infty}\frac{1}{k(e^{k\pi}-1)} - 96\sum_{k=1}^{\infty}\frac{1}{k(e^{2k\pi}-1)} + 24\sum_{k=1}^{\infty}\frac{1}{k(e^{4k\pi}-1)}.$$

- In 2008, In Midnight (10th episode of the fourth series of British science fiction television series Doctor Who) the character, the businesswoman, Sky Silvestry mimics the speech of The Doctor by repeating the square root of π to 30 decimal places $1.772, 453, 850, 905, 516, 027, 298, 167, 483, 34$.
- Sen and Agarwal, in 2008 (see [459]), suggested four Matlab based procedures, viz., (i) Exhaustive search, (ii) Principal convergents of continued fraction based procedure, (iii) Best rounding procedure for decimal (rational) approximation, and (iv) Continued fraction based

algorithm with intermediate convergents. While the first procedure is exponential–time, the remaining three are polynomial-time. Roughly speaking, they have demonstrated that the absolute best *k*–digit rational approximation of π will be as good as 2*k*–digit decimal approximation of π. The absolute best *k*–digit rational approximation is most desired for error-free computation involving π /any other irrational number.

- Sen, Agarwal, and Shaykhian, in 2008 (see [461]), have demonstrated through numerical experiment using Matlab that π has always scored over φ (golden ratio), as a source of uniformly distributed random numbers, statistically in one dimensional Monte Carlo integration. Whether π fares better than φ for double, triple, and higher-dimensional Monte Carlo integration or not deserves exploration.
- Sen, Agarwal, and Shaykhian, in 2009 (see [463]), compared the four procedures they proposed in (2008) for computing best *k*–digit rational approximations of irrational numbers in terms of quality (error) and cost (complexity). They have stressed on the fact that ultrahigh-speed computing along with abundance of unused computing power allows employing an exponential–time algorithm for most real–world problems. This obviates the need for acquiring and employing the mathematical knowledge involving principal and intermediate convergents computed using a polynomial-time algorithm for practical problems. Since π is the most used irrational number in the physical world, the simple concise Matlab program would do the job wherever π /any other irrational number is involved.
- Sen, Agarwal, and Pavani, in 2009 (see [462]), provided, using Matlab, the best possible rational bounds bracketing π /any irrational number with absolute error and the time complexity involved. Any better bounds are impossible. In these rational bounds, either the lower bound or the upper bound will always be the absolute best rational approximation. The absolute error computed provides the overall error bounds in an error-free computational environment involving π /any other irrational number. The following rational bounds for π where either the lower bound or the upper bound is the best *k*-digit rational approximation were obtained

$$\begin{array}{ccccc} k=1 & k=2 & k=3 & k=4 & k=5 \\ \frac{3}{1}<\pi<\frac{7}{2} & \frac{91}{29}<\pi<\frac{22}{7} & \frac{688}{219}<\pi<\frac{355}{113} & \frac{9918}{3157}<\pi<\frac{1065}{339} & \frac{99733}{31746} \\ & & & & <\pi<\frac{10295}{3277}. \end{array}$$

- Tue N. Vu, in 2009, gave Machin-Type Formula (http://seriesmathstudy.com/sms/machintypetv): For each positive integer n,

$$\frac{\pi}{4}=\tan^{-1}\left(\frac{1}{4+2n}\right)+\tan^{-1}\left(\frac{1}{5+2n}\right)+\sum_{k=0}^{n}\left[\tan^{-1}\left(\frac{1}{2(2+k)^2}\right)+\tan^{-1}\left(\frac{2}{(3+2k)^2}\right)\right].$$

- Cetin Hakimoglu-Brown, in 2009 [105], developed the following expansion

$$\pi=\frac{\sqrt{3}}{6^5}\sum_{k=0}^{\infty}\frac{((4k)!)^2(6k)!}{9^{k+1}(12k)!(2k)!}\left(\frac{127169}{12k+1}-\frac{1070}{12k+5}-\frac{131}{12k+7}+\frac{2}{12k+11}\right),$$

which can be written as

$$\pi=\frac{\sqrt{3}}{1155}\sum_{k=0}^{\infty}\frac{(4k)!(671840k^3+1289936k^2+782458k+150835)}{(72)^{4k+1}(13/12)_k(17/12)_k(19/12)_k(23/12)_k},$$

where $(x)_k=x(x+1)(x+2)\cdots(x+k-1)$ is the pochhammer notation. He also gave the expansion

$$\pi=\frac{1}{2^{10}\sqrt{3}}\sum_{k=0}^{\infty}\frac{1}{\binom{8k}{4k}9^k}\left(\frac{5717}{8k+1}-\frac{413}{8k+3}-\frac{45}{8k+5}+\frac{5}{8k+7}\right).$$

- Takahashi et. al., in 2009, used a massive parallel computer called the T2K Tsukuba System to compute π to $2{,}576{,}980{,}377{,}524$ decimal places in 73 hours 36 minutes. For this they used algorithms (8.72) and (8.75).
- Bellard, in December 2009, used Chudnovsky brothers series (8.76) to compute $2{,}699{,}999{,}990{,}000$, i.e., 2.7 trillion decimal places of π in 131 days. For this, he used a single desktop PC, costing less than $3,000.
- Kondo, in August 2010, used y-cruncher by Alexander Yee, Chudnovsky brothers series (8.76) to compute $5{,}000{,}000{,}000{,}000$, i.e., 5 trillion

decimal places of π in 90 days. For this, they used a server-class machine running dual Intel Xeons, equipped with 96GB of RAM.

- Michael Keith, in 2010 [303], used $10{,}000$ digits of π to establish a new form of constrained writing, where the word-lengths are required to represent the digits of π. His book contains a collection of poetry, short stories, a play, a movie script, crossword puzzles, and other surprises.
- Sen and Agarwal, in 2011 (see [464]), in their monograph systematically organized their work of 2008 and 2009 on π and other irrational numbers. They also included several examples to illustrate the importance of their findings.
- In 2011, during the auction for Nortel's portfolio of valuable technology patents, Google made a series of strange bids based on mathematical and scientific constants, including π.
- Kondo, in October 2011, used y-cruncher by Alexander Yee, Chudnovsky brothers series (8.76) to compute $10{,}000{,}000{,}000{,}050,$ i.e., 10 trillion decimal places of π in 371 days.
- Cristinel Mortici (Romania), in 2011 [378], improved Gurland's bounds (8.70) to $\alpha_n < \pi < \beta_n,\ n \geq 1$ where

$$\alpha_n = \left(\frac{n+\frac{1}{4}}{n^2+\frac{1}{2}n+\frac{3}{32}} + \frac{9}{2048n^5} - \frac{45}{8192n^6}\right)\left(\frac{(2n)!!}{(2n-1)!!}\right)^2$$

  and

$$\beta_n = \left(\frac{n+\frac{1}{4}}{n^2+\frac{1}{2}n+\frac{3}{32}} + \frac{9}{2048n^5}\right)\left(\frac{(2n)!!}{(2n-1)!!}\right)^2.$$

  It follows that

$$\alpha_n = \pi + O\left(\frac{1}{n^6}\right) \quad \text{and} \quad \beta_n = \pi + O\left(\frac{1}{n^5}\right).$$

- Long Lin, in 2013 [346], improved Mortici's bounds to $\lambda_n < \pi < \mu_n,\ n \geq 1$ where

$$\lambda_n = \left(1 + \frac{1}{4n} - \frac{3}{32n^2} + \frac{3}{128n^3} + \frac{3}{2048n^4} - \frac{33}{8192n^5} - \frac{39}{65536n^6}\right) \frac{2}{2n+1}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2$$

and

$$\mu_n = \left(1 + \frac{1}{4n} - \frac{3}{32n^2} + \frac{3}{128n^3} + \frac{3}{2048n^4}\right) \frac{2}{2n+1} \left(\frac{(2n)!!}{(2n-1)!!}\right)^2.$$

It follows that

$$\lambda_n = \pi + O\left(\frac{1}{n^7}\right) \quad \text{and} \quad \mu_n = \pi + O\left(\frac{1}{n^5}\right).$$

He has also obtained the higher-order bounds $\delta_n < \pi < \omega_n, \ n \geq 1$ where

$$\delta_n = \frac{1}{n} \exp\left\{-\frac{1}{4n} + \frac{1}{96n^3} - \frac{1}{320n^5} + \frac{17}{7168n^7} - \frac{31}{9216n^9}\right\} \left(\frac{(2n)!!}{(2n-1)!!}\right)^2$$

and

$$\mu_n = \frac{1}{n} \exp\left\{-\frac{1}{4n} + \frac{1}{96n^3} - \frac{1}{320n^5} + \frac{17}{7168n^7}\right\} \left(\frac{(2n)!!}{(2n-1)!!}\right)^2.$$

It follows that

$$\delta_n = \pi + O\left(\frac{1}{n^{11}}\right) \quad \text{and} \quad \mu_n = \pi + O\left(\frac{1}{n^9}\right).$$

- Kondo, in December 2013, used y-cruncher by Alexander Yee and Chudnovsky brothers series (8.76) to compute $12{,}100{,}000{,}000{,}050,$ i.e., 12 trillion decimal places of π in 94 days.
- Sandon Nash Van Ness (1985–2015, USA), in October 2014, used y-cruncher by Alexander Yee and Chudnovsky brothers series (8.76) $13{,}300{,}000{,}000{,}050,$ i.e., 13.3 trillion decimal places of π in 208 days.
- Peter Trueb (Switzerland-Germany), in November 2016, used y-cruncher by Alexander Yee and Chudnovsky brothers series (8.76) to compute $22{,}459{,}157{,}718{,}361,$ i.e., 22.459 trillion decimal places of π in 94 days.
- Emma Haruka Iwao (born 1986, Japan) a Google cloud developer, in March 2019, used the same software as her successor Peter Trueb to compute $31{,}415{,}926{,}535{,}897$ i.e., 31.4 trillion decimal places of π in 121 days.
- Timothy Mullican (USA), in January 2020, broke all previous records by calculating π to $50{,}000{,}000{,}000{,}000,$ i.e., 50 trillion decimal places of π

in 303 days.

- In August 2021, Team DAViS of the University of Applied Sciences of the Grisons, Switzerland, calculated $62{,}831{,}853{,}071{,}796,$ i.e., 62.83 trillion decimal places of π in 108 days.
- Emma Haruka Iwao, in March 2022, established a new world record of computing $100{,}000{,}000{,}000{,}000,$ i.e., 100 trillion decimal places of π in 158 days.
- In June, 2022, Google cloud announced the largest palindromic prime appearing in the known decimal expansion of π is 9609457639843489367549069.
- In connection of discovering more and more correct digits of π Canadian-American astronomer Simon Newcomb (1835–1909) said: "Ten decimals are sufficient to give the circumference of the earth to the fraction of an inch, and thirty decimals would give the circumference of the whole visible universe to a quantity imperceptible to the most powerful microscope."
- For further results on π see Adamchik and Wagon [3, 4], Adrian [7], Ahmad [22], Akira [24], Almkvist [26], Anderson [29], Anonymous [33], Arndt [39], Assmus [42], Backhouse [45], Bailey [48], Bailey et. al. [50], Beck and Trott [56], Beukers [69], Blatner [73], Bokhari [76], Borwein et. al [81, 83–90, 92–98], Breuer and Zwas [104], Bruins [106], Chan [123], Choong et. al. [126], Chudnovsky Brothers [129], Cohen and Shannon [131], Colzani [133], Dahse [146], Dalzell [147, 148], Datta [151], Delahaye [160], Dixon [167], Engels [179], Eymard and Lafon [184], Fox and Hayes [194], Frisby [200], Fuller [205], Goggins [214], Goldsmith [215], Goodrich [216], Gourdon and Sebah [222], Greenblatt [226], Gupta [233–236, 238, 239], Hata [252, 253], Hayashi et. al. [254, 255], Jami [282], Jesseph [284], Jörg and Haenel [288], Lay-Yong and Tian-Se [337], Mao [359], Eli Maor [360], Matar and Rajagopal [363], Miel [368, 369], Moakes [375, 376], Myers [384], Nakamura [387], Nanjundiah [388], Posamentier and Lehmann [416], Preston [418], Puritz [420], Qian [423], Rajagopal and Aiyar [426], Ramanujan [427], Reitwiesner [430], Roy [438], Salikhov [445], Schepler [449], Schröder [451], Schubert [454], Shanks and Wrench, Jr. [470, 471], da Silva [479], Singmaster [480], Smith [484–486], Snell [488], Stern [495], Stevens [496], Todd [510], Trier [511], Tweddle [513], Volkov [522–524], Wagon [527], Wrench, Jr. [539], Yeo [542], Zebrowski [544], Websites [550–555].

8.14 Theon's Ladder Method for Square Root

Theon constructed two sequences $\{a_k\}$ and $\{b_k\}$ of natural numbers (he called a_k as the *side number* and b_k as the *diagonal number*), which satisfy the recurrence relations (7.19). Since $b_k^2 - 2a_k^2 = -(b_{k-1}^2 - 2a_{k-1}^2)$, if a_{k-1}, b_{k-1} is a solution of $b^2 - 2a^2 = \pm 1$, then a_k, b_k is a solution of $b^2 - 2a^2 = \mp 1$. Thus, it follows that

$$\frac{b_k}{a_k} = \sqrt{2 \pm \frac{1}{a_k^2}}$$

and since $\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = \infty$, we can make $(1/a_k)^2$ arbitrarily small. Hence, $\lim_{k\to\infty}(b_k/a_k) = \sqrt{2}$. In conclusion, if (a_0, b_0) is an integer solution of $b^2 - 2a^2 = \pm 1$ then (7.19) converges to $\sqrt{2}$ and the convergence is oscillatory. From these observations names for a_k as the side number and for b_k as the diagonal number become clear. For more details, see Filep [188], and http://numbers.computation.free.fr/Constants/Sqrt2/sqrt2.html.

Now since for the equation $b^2 - 2a^2 = 1$ the minimal solution is $(a, b) = (2, 3)$, whereas for the equation $b^2 - 2a^2 = -1$ it is $(1, 1)$, from (7.19) with $(a_0, b_0) = (2, 3)$ and $(a_0, b_0) = (1, 1)$, it immediately follows that

$$\begin{aligned} a_{k+1} &= 2a_k + a_{k-1}, \quad a_0 = 2, \quad a_1 = 5 \\ b_{k+1} &= 2b_k + b_{k-1}, \quad b_0 = 3, \quad b_1 = 7, \quad k \geq 1 \end{aligned} \tag{8.79}$$

and

$$\begin{aligned} a_{k+1} &= 2a_k + a_{k-1}, \quad a_0 = 1, \quad a_1 = 2 \\ b_{k+1} &= 2b_k + b_{k-1}, \quad b_0 = 1, \quad b_1 = 3, \quad k \geq 1. \end{aligned} \tag{8.80}$$

We recall that in the construction of Table 8.4 for $N = 2$, we executed the recurrence relation (8.37) to obtain $\{s_k\}$. It can easily be verified that a_k and b_k obtained from (8.80) are connected with s_k by the relations $a_{k-1} = s_k,\ b_{k-1} = s_{k+1} - s_k,\ k \geq 2$, and hence $b_k/a_k,\ k \geq 1$ leads to the second column of Table 8.4. Similarly, a_k and b_k obtained from (8.79) are connected with s_k by the relations

$a_{k-1} = s_{k+1},\ b_{k-1} = s_{k+1} + s_k,\ \ k \geq 2$, and hence $b_k/a_k,\ \ k \geq 1$ leads to the third column of Table 8.4.

Next, from (7.19) first we find the system (7.20), and then with $(x_0, y_0) = (2, 3)$ and $(x_0, y_0) = (1, 1)$, respectively, (as (7.21)), we find

$$\begin{aligned} x_{k+1} &= 6x_k - x_{k-1},\ \ x_0 = 2, \quad x_1 = 12 \\ y_{k+1} &= 6y_k - y_{k-1},\ \ y_0 = 3, \quad y_1 = 17, \quad k \geq 1 \end{aligned} \tag{8.81}$$

and

$$\begin{aligned} x_{k+1} &= 6x_k - x_{k-1},\ \ x_0 = 1, \quad x_1 = 5 \\ y_{k+1} &= 6y_k - y_{k-1},\ \ y_0 = 1, \quad y_1 = 7, \quad k \geq 1. \end{aligned} \tag{8.82}$$

Again, looking at Table 8.4, we find that x_k and y_k obtained from (8.82) are connected with the same s_k by the relations $x_k = s_{2k+2},\ y_k = s_{2k+3} - s_{2k+2},\ \ k \geq 0$, and hence, $y_k/x_k,\ \ k \geq 0$ leads to the second column of Table 8.4 with *, and monotonically decreasing. Similarly, x_k and y_k obtained from (8.81) are connected with s_k by the relations $x_k = s_{2k+1},\ y_k = s_{2k+1} + s_{2k},\ \ k \geq 1$, and hence $y_k/x_k,\ \ k \geq 1$ leads to the third column of Table 8.4 with *, and monotonically increasing.

A generalization of (7.19) for any integer $N \geq 2$ is straightforward. In fact, for the recurrence relations

$$a_k = a_{k-1} + b_{k-1}, \quad b_k = Na_{k-1} + b_{k-1}, \quad k \geq 1 \tag{8.83}$$

we have

$$b_k^2 - Na_k^2 = (1 - N)(b_{k-1}^2 - Na_{k-1}^2),$$

which gives

$$b_k^2 - Na_k^2 = (1 - N)^n(b_0^2 - Na_0^2).$$

Thus, it follows that

$$\frac{b_k^2}{a_k^2} - N = \frac{(-1)^n(N-1)^n}{a_k^2}(b_0^2 - Na_0^2), \quad k \geq 1. \tag{8.84}$$

Now since $\{a_k\}$ is a strictly increasing sequence, and $a_2 = a_1 + b_1 = a_1 + Na_0 + b_0 > N + 1$, the right side of (8.84) tends to zero. This means the sequence $\{b_k/a_k\}$ converges to $\sqrt{N}$, and the convergence is oscillatory. From (8.83) it also follows that

$$\left|\frac{b_k}{a_k}-\sqrt{N}\right| = \frac{(\sqrt{N}-1)}{|b_{k-1}/a_{k-1}+1|}\left|\frac{b_{k-1}}{a_{k-1}}-\sqrt{N}\right| \simeq \left(\frac{\sqrt{N}-1}{\sqrt{N}+1}\right)\left|\frac{b_{k-1}}{a_{k-1}}-\sqrt{N}\right|.$$

In particular, for $N=3$ if we choose fundamental solution of $b^2-3a^2=1$, which is $(a_0,b_0)=(1,2)$ then (8.83) leads to the algorithm

$$\begin{aligned} a_k &= a_{k-1}+b_{k-1} \\ b_k &= 3a_{k-1}+b_{k-1}, \quad k\geq 1, \quad a_0=1, \quad b_0=2. \end{aligned} \tag{8.85}$$

The sequence $\{b_k/a_k\}$ generated from (8.85) gives the fourth column of Table 8.4.

We note that system (8.85) can be written as

$$\begin{aligned} a_{k+1} &= 2a_k+2a_{k-1}, \quad a_0=1, \quad a_1=3 \\ b_{k+1} &= 2b_k+2b_{k-1}, \quad b_0=2, \quad b_1=5, \quad k\geq 1 \end{aligned} \tag{8.86}$$

and its solution is

$$\begin{aligned} a_k &= \frac{2+\sqrt{3}}{2\sqrt{3}}(1+\sqrt{3})^k - \frac{2-\sqrt{3}}{2\sqrt{3}}(1-\sqrt{3})^k \quad \text{and} \\ b_k &= \frac{3+2\sqrt{3}}{2\sqrt{3}}(1+\sqrt{3})^k - \frac{3-2\sqrt{3}}{2\sqrt{3}}(1-\sqrt{3})^k. \end{aligned} \tag{8.87}$$

Again, for $N=3$ if we choose fundamental solution of $b^2-3a^2=-2$ which is $(a_0,b_0)=(3,5)$ then (8.83) leads to the algorithm

$$\begin{aligned} a_k &= a_{k-1}+b_{k-1} \\ b_k &= 3a_{k-1}+b_{k-1}, \quad k\geq 1, \quad a_0=3, \quad b_0=5. \end{aligned} \tag{8.88}$$

The sequence $\{b_k/a_k\}$ generated from (8.88) gives the fifth column of Table 8.4.

Next, we consider the nonlinear recurrence relations

$$\begin{aligned} a_k &= 2a_{k-1}b_{k-1} \\ b_k &= Na_{k-1}^2+b_{k-1}^2, \quad k\geq 1 \end{aligned} \tag{8.89}$$

and note that

$$b_k^2-Na_k^2 = (b_{k-1}^2-Na_{k-1}^2)^2 = \cdots = (b_0^2-Na_0^2)^{2^k}.$$

Thus, if (a_0,b_0) is the fundamental solution (in fact, any integer solution) of $b^2-Na^2=1$, then the sequence $\{b_k/a_k\}$ generated by

(8.89) decreases monotonically to $\sqrt{N}$. From (8.89), we also have

$$\left(\frac{b_k}{a_k}-\sqrt{N}\right)=\frac{1}{2(b_{k-1}/a_{k-1})}\left(\frac{b_{k-1}}{a_{k-1}}-\sqrt{N}\right)^2\simeq\frac{1}{2\sqrt{N}}\left(\frac{b_{k-1}}{a_{k-1}}-\sqrt{N}\right)^2.$$

In Table 8.5, we provide first three iterates to approximate $N=2,3,5,$ and 7 with the corresponding fundamental solutions of $b^2-Na^2=1$ as $(2,3),(1,2),(4,9),$ and $(3,8)$.

Table 8.5 Nonlinear iterates

	$N=2$	$N=3$	$N=5$	$N=7$
n	$(a_0,b_0)=(2,3)$	$(a_0,b_0)=(1,2)$	$(a_0,b_0)=(4,9)$	$(a_0,b_0)=(3,8)$
0	$\frac{3}{2}$	$\frac{2}{1}$	$\frac{9}{4}$	$\frac{8}{3}$
1	$\frac{17}{12}$	$\frac{7}{4}$	$\frac{161}{72}$	$\frac{127}{48}$
2	$\frac{577}{408}$	$\frac{97}{56}$	$\frac{51841}{23184}$	$\frac{32257}{12192}$
3	$\frac{665857}{470832}$	$\frac{18817}{10864}$	$\frac{5374978561}{2403763488}$	$\frac{2081028097}{786554688}$

For $N=2$ and 3 all entries in Table 8.5 are the same as in Table 8.4. Table 8.5 also indicates superiority of the nonlinear algorithm (8.89) compared to all linear algorithms we have discussed above. However, algorithm (8.20) appears to have advantage.

Now we will consider the recurrence relations

$$\begin{aligned} a_k &= (p+q)a_{k-1}+2qb_{k-1} \\ b_k &= 2pa_{k-1}+(p+q)b_{k-1}, \quad k\geq 1 \end{aligned} \tag{8.90}$$

where $p\neq q$ and $a_0,\ b_0$ are positive integers. For (8.90) it follows that

$$\left(b_k^2-\frac{p}{q}a_k^2\right)=(p-q)^2\left(b_{k-1}^2-\frac{p}{q}a_{k-1}^2\right)=\cdots=(p-q)^{2^k}\left(b_0^2-\frac{p}{q}a_0^2\right),$$

which is the same as

$$\left(\frac{b_k^2}{a_k^2}-\frac{p}{q}\right)=\frac{(p-q)^{2^k}}{a_k^2}\left(b_0^2-\frac{p}{q}a_0^2\right).$$

Since $a_k\geq(p+q)a_{k-1}$ implies $a_k\geq(p+q)^ka_0,$ we find

$$\left|\frac{b_k^2}{a_k^2}-\frac{p}{q}\right| \le \left(\frac{p-q}{p+q}\right)^{2k}\left|\frac{b_0^2}{a_0^2}-\frac{p}{q}\right|.$$

Thus the sequence $\{b_k/a_k\}$ generated by (8.90) converges to $\sqrt{p/q}$, furher if $b_0/a_0 > \sqrt{p/q}$ $(b_0/a_0 < \sqrt{p/q})$ the convergence is monotonically decreasing (increasing). For $p = 11, q = 5$ we list first few terms of $\{b_k/a_k\}$.

$$\frac{b_0}{a_0}=\frac{1}{1} \quad \frac{b_1}{a_1}=\frac{38}{26} \quad \frac{b_2}{a_2}=\frac{1180}{796} \quad \frac{b_3}{a_3}=\frac{36392}{24536} \quad \frac{b_4}{a_4}=\frac{1122064}{756496}$$

$$\frac{b_0}{a_0}=\frac{3}{2} \quad \frac{b_1}{a_1}=\frac{92}{62} \quad \frac{b_2}{a_2}=\frac{2836}{1912} \quad \frac{b_3}{a_3}=\frac{87440}{58952} \quad \frac{b_4}{a_4}=\frac{2695984}{1817632}$$

8.15 Approximations of *e*

As we have noted earlier, Euler in 1748, used the expansion (3.6) to obtain a numerical value of *e* to 23 decimal places. Other records are by William Shanks in 1853 to 137 digits and in 1871 to 205 digits; von Neumann to 2010 in 1949; Shanks and Wrench to 100265 in 1961; Bonnell and Nemiroff to 10 million in 1994; Patrick Demichel (USA) to 18 million in 1997; Birger Seifert (Germany) to twenty million in 1997; Demichel to 50 million in 1997; Sebastian Wedeniwski (born 1971, Germany-Japan) to 200 million in 1999, and more than 800 million later same year; Xavier Gourdon (France) to one and quatre billion in 1999; Gourdon and Kondo to 2 billion in 2000 and 12 billion eight hundred later same year; and Kondo and Alexander Yee to one trillion in 2010. The following rational bounds for *e* where either the lower bound or the upper bound is the best *k*-digit rational approximation are obtained in Sen et. al. [462]

$$\begin{array}{ccccc} k=1 & k=2 & k=3 & k=4 & k=5 \\ \frac{8}{3}<e<\frac{3}{1} & \frac{19}{7}<e<\frac{87}{32} & \frac{878}{323}<e<\frac{193}{71} & \frac{2721}{1001}<e<\frac{8620}{3539} & \frac{75117}{27634}<e<\frac{49171}{18089}. \end{array}$$

We also remark that employing several different algorithms, massive details about the approximations of *e* have been given in Sen and Agarwal [464].

8.16 Continued Fractions

A finite equation of the type

$$r_k = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{\ddots\, b_{k-1} + \cfrac{a_k}{b_k}}}} \tag{8.91}$$

where a_i, b_i are positive integers (functions) except possibly b_0 is called the *k*-th convergent r_k of a continued fraction (also known as *k*-story number). Here we allow *k* to tend to infinity to have an infinite continued fraction. A simple manipulation of Euclid's Algorithm (Theorem 3.3), which is mainly used to find gcd of the ratio of two geometric magnitudes leads to a finite (infinite) continued fraction provided the ratio is rational (irrational). There is sufficient evidence that Aryabhata used continued fraction to solve a linear indeterminate equation. In Greek and Arab mathematical literature, there are fragments of continued fractions. Fibonacci introduced a type of continued fraction. We meet with several algorithms for $\sqrt{13}$ and $\sqrt{18}$ similar to current forms of continued fractions in the works of Bombelli and Cataldi, respectively. Wallis in his treatise *Arithmetica infinitorum* after presenting (8.44) writes that Brouncker expanded $\triangle = 4/\pi$ in an infinite continued fraction

$$\triangle = 1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \cfrac{81}{2 + \cdots}}}}} \tag{8.92}$$

and pointed out how the partial fractions are successively larger and smaller than $\triangle$, and the process converges to $\triangle$. While Brouncker was not kind enough to provide details of his expansion, he undertook some calculations to verify formula (8.44), and showed that $3.141592653569 \cdots < \pi < 3.141592653696 \cdots$, which is very satisfactory. Neither of the expressions (8.44), and (8.92); however, later has served for an extensive calculation of π.

Wallis in his book *Opera Mathematica* of 1695 detailed basic facts and properties of continued fractions (this term is also coined by him). Later Huygens, Euler, Lambert, Lagrange, and Perron enriched the theory and applications of continued fractions to the extent that it became a subject in its own right. Especially, Euler showed that every rational number can be expressed as a (unique) terminating continued fraction, and consequently, every other infinite continued fraction is irrational. Continued fractions play dominate role in finding best rational approximations of irrational numbers. For (8.91), Alfred Pringsheim (1850–1941, Germany) in short wrote as

$$r_k = b_0 + \frac{a_1|}{|b_1} + \frac{a_2|}{|b_2} + \frac{a_3|}{|b_3} + \cdots + \frac{a_k|}{|b_k}, \tag{8.93}$$

and when all $a_i = 1,\ i \geq 1$ Gauss wrote as

$$r_k = [b_0; b_1, b_2, b_3, \cdots, b_k] \tag{8.94}$$

where a_i, b_i are integers (functions) determined from a given irrational number (function). The representation (8.94) is called a simple continued fraction.

It is well known, for example, see Agarwal [10], and Sen and Agarwal [464] that r_k can be written as $r_k = u_k/v_k$, where the numerator u_k and the denominator v_k satisfy the recurrence relations

$$\begin{aligned} u_k &= b_k u_{k-1} + a_k u_{k-2}, \quad u_0 = b_0, \quad u_1 = b_0 b_1 + a_1 \\ v_k &= b_k v_{k-1} + a_k v_{k-2}, \quad v_0 = 1, \quad v_1 = b_1, \quad k = 2, 3, \cdots . \end{aligned} \tag{8.95}$$

A remarkable known property is that if the irrational number $x = [b_0; b_1, b_2, b_3, \cdots]$ has convergents $r_k = u_k/v_k$, then

$$\frac{1}{v_k(v_k + v_{k+1})} < \left| x - \frac{u_k}{v_k} \right| < \frac{1}{v_k v_{k+1}} \leq \frac{1}{v_k^2}. \tag{8.96}$$

Now we shall use the algorithm (8.95) to find rational approximations of a given positive number $\sqrt{N}$. For this, again we assume that *a* to be an initial guess of $\sqrt{N}$ so that $N = a^2 + r$. Since $N - a^2 = (\sqrt{N} + a)(\sqrt{N} - a) = r$, it follows that

$$\sqrt{N} = a + \frac{r}{\sqrt{N}+a} = a + \frac{r}{a+\left(a+\frac{r}{\sqrt{N}+a}\right)} = a + \frac{r}{2a+\dfrac{r}{\sqrt{N}+a}},$$

and the process continues.

In particular, for $N = 2$ and $a = 1$, we have $r = 1$ and $r_k = [1; 2, 2, 2, \cdots, 2]$. Thus, (8.95) reduces to

$$\begin{aligned} u_k &= 2u_{k-1} + u_{k-2}, \qquad u_0 = 1, \quad u_1 = 3 \\ v_k &= 2v_{k-1} + v_{k-2}, \qquad v_0 = 1, \quad v_1 = 2. \end{aligned} \tag{8.97}$$

which is the same as (8.79) with $a_k = v_{k+1}, b_k = u_{k+1}$, and (8.80) with $a_k = v_k, b_k = u_k$ for all $k \geq 0$.

Similarly, for $N = 3$ and $a = 1$, we have $r = 2$ and

$$r_k = 1 + \frac{2|}{|2} + \frac{2|}{|2} + \frac{2|}{|2} + \cdots + \frac{2|}{|2}.$$

Thus (8.95) reduces to

$$\begin{aligned} u_k &= 2u_{k-1} + 2u_{k-2}, \qquad u_0 = 1, \quad u_1 = 4 \\ v_k &= 2v_{k-1} + 2v_{k-2}, \qquad v_0 = 1, \quad v_1 = 2, \end{aligned}$$

which is the same as (8.86) with $2a_k = v_{k+1}, 2b_k = u_{k+1}$.

For the golden ratio $\varphi = (\sqrt{5}+1)/2$, the continued fraction is

$$\varphi = [1; 1, 1, 1, 1, 1, 1, \cdots].$$

It immediately follows from $\varphi = 1 + 1/\varphi = 1 + 1/(1+\varphi) = \cdots$. In terms of Fibonacci numbers the *k*-th convergent of the golden ratio is $r_k = \mathcal{F}_{k+2}/\mathcal{F}_{k+1},\ k \geq 0$. Indeed, it suffices to show that $r_0 = \mathcal{F}_2/\mathcal{F}_1 = 1/1 = 1,\ r_1 = \mathcal{F}_3/\mathcal{F}_2 = 2/1 = 2 = 1 + 1/1$. Now assuming $r_k = \mathcal{F}_{k+2}/\mathcal{F}_{k+1}$, then since $r_{k+1} = 1 + 1/r_k = 1 + \mathcal{F}_{k+1}/\mathcal{F}_{k+2} = (\mathcal{F}_{k+2} + \mathcal{F}_{k+1})/\mathcal{F}_{k+2} = \mathcal{F}_{k+3}/\mathcal{F}_{k+2}$.

For the number π the following continued fractions are known:

- From (8.92) it follows that

$$\pi = \frac{4|}{|1} + \frac{1^2|}{|2} + \frac{3^2|}{|2} + \frac{5^2|}{|2} + \frac{7^2|}{|2} + \cdots .$$

- Lambert around 1770 gave the following simple continued fraction,

$$\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, \cdots];$$

however, it does not show any obvious pattern. Some convergents of this continued fraction are

$$\frac{3}{1}, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{103993}{33102}, \frac{104348}{33215}, \frac{208341}{66317}, \frac{312689}{99532}, \frac{833719}{265381}, \frac{1146408}{364913}, \frac{4272943}{1360120}.$$

- From (7.178) it follows that

$$\pi = \frac{4|}{|1} + \frac{1^2|}{|3} + \frac{2^2|}{|5} + \frac{3^2|}{|7} + \frac{4^2|}{|9} + \cdots .$$

- In 1971, Choong, Daykin and Rathbone [127] used $100,000$ digits of Daniel Shanks and William Shanks of 1961 to generate the first $21,230$ partial quotients of the continued fraction expansion of π.
- In 1999, Lange [336] developed the following continued fraction of π,

$$\pi = 3 + \frac{1^2|}{|6} + \frac{3^2|}{|6} + \frac{5^2|}{|6} + \frac{7^2|}{|6} + \cdots .$$

For the number *e* the following continued fractions are known:

- In 1737, Euler gave the simple continued fraction representation of e,

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \cdots] = \left[2; \overline{1, 2n, 1}\right]_{n=1}^{\infty}, \tag{8.98}$$

i.e., the pattern repeats indefinitely with a period of 3 except that 2 is added to one of the terms in each cycle. Some convergents of this continued fraction are
$2, 3, 8/3, 11/4, 19/7, 87/32, 106/39, 193/71, 1264/465, 1457/536.$

- Hubert Stanley Wall (1902–1971, USA) in his book *Analytic Theory of Continued Fractions* of 1948, gave the following beautiful representation

$$e = 2 + \frac{1|}{|1} + \frac{1|}{|2} + \frac{2|}{|3} + \frac{3|}{|4} + \cdots .$$

- For e^2 the simple continued fraction representation is

$$e^2 = \left[7; \overline{3n+2, 1, 1, 3n+3, 12n+18}\right]_{n=0}^{\infty}.$$

- The simple continued fraction for Euler–Mascheroni constant γ is

$$\gamma = [0; 1, 1, 2, 1, 2, 1, 4, 3, 13, 5, 1,\ 1, 8, 1, 2, 4, 1, 1, 40, 1, 11, 3, 7, 1, 7, 1, 1, 5, 1, 49, 4, 1, 65, \cdots].$$

  The first few convergents of this continued fraction are are $1, 1/2, 3/5, 4/7, 11/19,$ $15/26, 71/123, 228/395, 3035/5258, 15403/26685$. Weisstein in 2011 computed 970258158 terms of the continued fraction of gamma, and in 2013, 4851382841 terms. On May 13, 2023, Jordan Ranous and Kevin O'Brien have computed γ to 700 billion decimal places. It is not known whether γ is rational or irrational. The question about its being transcendental also is an open problem.

- The simple continued fraction expansion of $\zeta(2)$ is

$$[1; 1, 1, 1, 4, 2, 4, 7, 1, 4, 2, 3, 4, 10, 1, 2, 1, 1, 1, 15, 1, 3, 6, 1, 1, 2, 1, 1, 1, 2, 2, 3, 1, 3, 1, 1, 5, 1, 2, 2, 1, 1, 6, 27, 20, 3, 97, \cdots].$$

- The simple continued fraction expansion of $\zeta(3)$ is

$$[1; 4, 1, 18, 1, 1, 1, 4, 1, 9, 9, 2, 1, 1, 1, 2, 7, 1, 1, 7, 11, 1, 1, 1, 3, 1, 6, 1, 30, 1, 4, 1, 1, 4, 1, 3, 1, 2, 7, 1, 3, 1, 2, 2, 1, \cdots].$$

8.17 Irrationality of *e* and e^2

Euler, in 1737, showed that both *e* and e^2 are irrational and gave several continued fractions for e. His proof of irrationality for *e* is based on the infinite continued fractional representation (8.98). He indicated that the irrationality of *e* is of *different kind*, which led to transcendental numbers. Here we provide a most admired elementary proof of 1815 due to Fourier, also see Agarwal et. al. [16]. From (3.6), we have

$$2 = 1 + \frac{1}{1!} < e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots < 1 + \left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots\right) = 3.$$

Now supposed to contrary that $e = p/q,$ where *p* and *q* are integers and $q > 1$. Thus, we have $e = p/q = \sum_{n=0}^{\infty} 1/n!$, which is the same as

$$p(q-1)! = q!\sum_{k=0}^{q}\frac{1}{k!} + q!\sum_{k=q+1}^{\infty}\frac{1}{k!}.$$

Now, we observe that

$$0 < p(q-1)! - q!\sum_{k=0}^{q}\frac{1}{k!} \;=\; \frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \cdots$$
$$< \frac{1}{q+1} + \frac{1}{(q+1)^2} + \cdots = \frac{1}{q} < 1.$$

But, $p(q-1)! - q!\sum_{k=0}^{q} 1/k!$ is a positive integer. Thus, *e* is irrational.

The aforementioned method to prove the irrationality of e^2 does not work. In 1998, Aigner and Ziegler in their book [23] provided details about the irrationality of e^t for any nonzero rational $t.$ Their approach is based on the work of Hermite. For a fixed $n \geq 1,$ they begin with the function $f(x) = x^n(1-x)^n/n!.$ It follows that (i) $f(x) = (1/n!)\sum_{i=n}^{i=2n} c_i x^i,$ where c_i are integers; (ii) for $0 < x < 1,\ 0 < f(x) < 1/n!;$ (iii) for $n \leq k \leq 2n,\ f^{(k)}(0) = (k!/n!)c_k,\ f^{(k)}(1) = (-1)^k f^{(k)}(0)$ are integers, and $f^{(k)}(x) = 0$ for $k \geq 2n+1.$ Now it suffices to show that for a positive integer $r,\ e^r$ is irrational, in fact, if $e^{r/s}$ is rational, then $(e^{r/s})^s = e^r$ will also be rational. Suppose that $e^r = a/b,$ where the integers $a, b > 0,$ and choose *n* sufficiently large so that $n! > ar^{2n+1}.$ We define

$$F(x) = r^{2n}f(x) - r^{2n-1}f'(x) + r^{2n-2}f''(x) \mp \cdots + f^{(2n)}(x),$$

which in view of (iii) is the same as

$$F(x) = r^{2n}f(x) - r^{2n-1}f'(x) + r^{2n-2}f''(x) \mp \cdots .$$

For the polynomial $F(x)$ it follows that $F'(x) = -rF(x) + r^{2n+1}f(x),$ and hence,

$$(e^{rx}F(x))' = re^{rx}F(x) + e^{rx}F'(x) = r^{2n+1}e^{rx}f(x).$$

Thus, we have

$$N = b\int_0^1 r^{2n+1}e^{rx}f(x)dx = b\left[e^{rx}F(x)\right]_0^1 = aF(1) - bF(0),$$

which in view of (iii) is an integer. Now from (ii), we have

$$0 < N = b\int_0^1 r^{2n+1}e^{rx}f(x)dx < br^{2n+1}e^r\frac{1}{n!} = \frac{ar^{2n+1}}{n!} < 1,$$

and hence *N* cannot be an integer. Thus, our assumption that e^r is rational is false.

Lambert generalized Euler's method to show that continued fractions of e^x and $\tan x$ are irrational if *x* is a nonzero rational. His following continued fractions of e^x and $\tan x$ of 1761 [331] are of great historical importance

$$e^x = 1 + \frac{x|}{|1} + \frac{-1x|}{|(2+x)} + \frac{-2x|}{|(3+x)} + \frac{-3x|}{|(4+x)} + \cdots$$

and

$$\tan x = \frac{x|}{|1} + \frac{-x^2|}{|3} + \frac{-x^2|}{|5} + \frac{-x^2|}{|7} + \cdots . \tag{8.99}$$

8.18 Irrationality of π and π^2

To prove the irrationality of π, in 1768, Lambert substituted $x = \pi/4$ in (8.99), so that the left side of (8.99) is simply one. Then he assumed that there exist integers *p* and *q* such that $\pi/4 = p/q$, i.e., $\pi/4$ is rational and then showed that the right side of (8.99) is irrational. The complete Lambert's proof is available on the website https://math.stackexchange.com/questions/895611/lamberts-original-proof-that-pi-is-irrational. After Lambert's proof, several prominent mathematicians gave alternative proofs to prove the irrationality of π, for example, Legendre, in his *Elements de Géometrie* (1794) used a slightly modified version of Lambert's argument to prove the irrationality of π more rigorously, and also gave a proof that π^2 is irrational. He writes: "It is probable that the number π is not even contained among the algebraic irrationalities, i.e., that it cannot be the root of an algebraic equation with a finite number of terms, whose coefficients are rational. But, it seems to be very difficult to prove this strictly." In 1873, Hermite proved by contradiction that π^2 is irrational, from which the irrationality of π follows immediately. In 1945, Dame Mary Lucy Cartwright (1900–1998, England) set as an example in an exam at the Cambridge University a new proof of the irrationality of π

(the origin of the proof is not yet known). In 1947, Niven [395] gave half page proof (also see his book [396]). Bourbaki in spirit followed Niven's proof in 1949, and Laczkovich in 1997 [324]. For details, see the website https://en.wikipedia.org/wiki/Proof_that_pi_is_irrational. Here we shall provide proof from Bourbaki.

For each natural number *q* and each nonnegative integer $n,$ let

$$A_n(q) = q^n \int_0^\pi \frac{x^n(\pi - x)^n}{n!} \sin x dx.$$

Since $A_n(q)$ is the integral of a function that is defined on $[0, \pi],$ takes the value 0 at the lower and upper limits and positive in $(0, \pi),$ $A_n(q) > 0.$ Further, since $x(\pi - x) \le (\pi/2)^2,$ we have

$$A_n(q) \le \pi q^n \frac{1}{n!} \left(\frac{\pi}{2}\right)^{2n} = \pi \frac{(q\pi^2/4)^n}{n!}$$

and hence $A_n(q) < 1$ for sufficiently large $n.$ On the other hand, recursive integration by parts leads to the fact that, if *p* and *q* are natural numbers such that $\pi = p/q$ and *f* is the polynomial function from $[0, \pi]$ to $\mathcal{R}$ defined by $f(x) = x^n(p - qx)^n/n!,$ then

$$\begin{aligned} A_n(q) = \int_0^\pi f(x)\sin(x)dx &= [-f(x)\cos x]_{x=0}^{x=\pi} - \left[-f'(x)\sin x\right]_{x=0}^{x=\pi} \\ &\quad + \cdots \pm \left[f^{(2n)}(x)\cos x\right]_{x=0}^{x=\pi} \\ &\quad \pm \int_0^\pi f^{(2n+1)}(x)\cos x\, dx. \end{aligned}$$

Since *f* is a polynomial of degree $2n,$ the last term is $0.$ Now since each function $f^{(k)},\ 0 \le k \le 2n$ as well as $\sin x$ and $\cos x$ take integer values at 0 and $\pi,$ this shows that $A_n(q)$ is an integer. Since it is greater than $0,$ it must be a positive integer. But we have seen that $A_n(q) < 1$ if *n* is sufficiently large. This contradiction shows that $\pi = p/q$ is impossible.

To prove π^2 is irrational, we again follow the book of Aigner and Ziegler [23] and the proof of the irrationality e^2 in Sect. 8.17. Suppose that $\pi^2 = a/b,$ where the integers $a, b > 0.$ We consider the polynomial

$$F(x) = b^n\left(\pi^{2n} f(x) - \pi^{2n-2} f^{(2)}(x) + \pi^{2n-4} f^{(4)}(x) \mp \cdots\right),$$

which satisfies $F''(x) = -\pi^2 F(x) + b^n \pi^{2n+2} f(x)$. From (iii) it is clear that $F(0)$ and $F(1)$ are integers. Now we have

$$\begin{aligned}(F'(x)\sin\pi x - \pi F(x)\cos\pi x)' &= (F''(x) + \pi^2 F(x))\sin\pi x \\ &= b^n \pi^{2n+2} f(x)\sin\pi x = \pi^2 a^n f(x)\sin\pi x.\end{aligned}$$

Thus, we obtain

$$N = \pi \int_0^1 a^n f(x)\sin\pi x\, dx = \left[\frac{1}{\pi}F'(x)\sin\pi x - F(x)\cos\pi x\right]_0^1 = F(0) + F(1),$$

which is an integer. From the definition of *N* it is clear that it is positive. Now we choose *n* so large that $\pi a^n/n! < 1$, then from (ii) it follows that

$$0 < N = \pi \int_0^1 a^n f(x)\sin\pi x\, dx < \pi a^n \frac{1}{n!} < 1,$$

which is a contradiction.

An excellent survey about the irrationality of *e* and π has been documented in Nagell [385]. In [382], Ram Murty and Kumar Murty gave a general result for the irrationality of numbers, and as a consequence concluded that π^2 is irrational, (hence so also is π); $\ln r$ is irrational for every rational $r > 0,\ r \neq 1$; $e^r,\ \sin r,\ \cos r,\ \cosh r,\ \sinh r$ are irrational for every nonzero rational r.

8.19 Irrationality of $\zeta(2)$ and $\zeta(3)$

From the irrationality of π^2 it is immediate that $\zeta(2)$ is irrational. In 1979, Roger Apéry (1916–1994, France) published an unexpected proof of the irrationality of $\zeta(3)$. His success was based on the highly non-obvious rapidly converging series representation

$$\zeta(3) = \frac{5}{2}\sum_{k=1}^{\infty}(-)^{k-1}\frac{k!^2}{(2k)!k^3} = \frac{5}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3\binom{2n}{n}}. \qquad (8.100)$$

In the literature $\zeta(3)$ is known as Apéry constant. In July 2020, Seungmin Kim (Korea) has computed the value of $\zeta(3)$ to 1 trillion 200 billion and 100 decimal places. Besides (8.100), several other representations of $\zeta(3)$ are known, e.g., in 1772, Euler showed that

$$\zeta(3) = \frac{\pi^2}{7}\left(1 - 4\sum_{k=1}^{\infty}\frac{\zeta(2k)}{2^{2k}(2k+1)(2k+2)}\right);$$

Ramanujan developed

$$\zeta(3) = \frac{7}{180}\pi^3 - 2\sum_{k=1}^{\infty}\frac{1}{k^3(e^{2\pi k}-1)};$$

in 1979, Frits Beukers (born 1953, Turkey-The Netherlands) gave

$$\zeta(3) = -\frac{1}{2}\int_0^1\int_0^1\frac{\ln(xy)}{1-xy}dx\,dy = -\int_0^1\int_0^1\frac{\ln(1-xy)}{xy}dx\,dy;$$

in 2018, Silagadze obtained

$$\zeta(3) = \int_0^1\int_0^1\int_0^1\frac{dx\,dy\,dz}{1-xyz} \qquad \text{and} \qquad \zeta(3) = \frac{1}{7}\int_0^{\pi}\frac{x(\pi-x)}{\sin x}dx.$$

8.20 Transcendental Numbers

A number that is not a solution of any polynomial equation with integer coefficients is called *transcendental number* (in particular, this means that it cannot be expressed as a finite number of addition, subtraction, multiplication, division, and extraction of root of integers). The irrationality of *e* and π, which is equivalent to the fact these numbers are not roots of any linear equation of the form $ax+b=0$, whose coefficients are integers, had been proved by Euler and Lambert. In 1844, Liouville showed that *e* is not a root of any quadratic equation with integral coefficients. This led him to conjecture that *e* does not satisfy any polynomial equation $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0$ with integral coefficients, i.e., *e* is transcendental. In 1851, Liouville showed, by using continued fractions, that there are an infinite number of transcendental numbers, a result which had previously been suspected by Euler in 1737 and Legendre in 1794, but had not been proved. He produced the first examples of real numbers that are provably transcendent. One of these is

$$\sum_{n=1}^{\infty}\frac{1}{10^{n!}} = \frac{1}{10^1} + \frac{1}{10^2} + \frac{1}{10^6} + \cdots = 0.11000100\cdots.$$

His methods led to extensive further research.

In 1874, Cantor in his seminal paper, *Über eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen* (On a Characteristic Property of All Real Algebraic Numbers) gave an existential proof to show that there are as many transcendental numbers (although subset of irrationals) as there are real numbers, which are uncountable. Cantor's work established the ubiquity of transcendental numbers. In fact, the discovery of transcendentals, the establishment of the fact that they are far richer in extent and variety than the irrationals of algebra, that they comprise some of the most fundamental magnitudes of modern mathematics—all this showed definitely that the powerful machinery of algebra had failed just where the elementary tools of rational arithmetic had failed 2000 years earlier. Both failures were due to the same source: algebra, like rational arithmetic, dealt with finite processes only.

Recall that two sets *A* and *B* are *equivalent*, or have the same *cardinal number*, provided there is a one-to-one onto mapping $f : A \to B.$ The equivalence of these sets is written as $A \sim B.$ The following properties of uncountable sets are fundamental.

(U1). If $A = B \cup C,$ where *B* is an arbitrary infinite set and *C* is at most countable (finite or countable), then $A \sim B.$

(U2). If *A* is an uncountable infinite set and *B* is its finite or countable subset then $A \backslash B \sim A.$

(U3). If $A \subset B$ and *A* is uncountable, then *B* is uncountable, i.e., uncountable remains the same, even when the uncountable is taken out of it.

- To prove the set of all real numbers contained in the interval $(0, 1)$ is uncountable, Cantor used the method of contradiction and the *diagonalization argument*. He assumed to contrary that there is a one-to-one onto mapping *f* from $\mathcal{N}$ onto the interval $(0, 1).$ Thus these numbers can be written as $f(1), f(2), \cdots$. Now each number $f(n)$ has an infinite decimal expansion, so these numbers can be written as

$$\begin{aligned} f(1) &= 0.a_{11}a_{12}a_{13}\cdots \\ f(2) &= 0.a_{21}a_{22}a_{23}\cdots \\ f(3) &= 0.a_{31}a_{32}a_{33}\cdots \\ &\vdots \end{aligned}$$

where each $a_{ij} \in \{0, 1, \cdots, 9\}$. Here some numbers such as $1/4 = 0.25000\cdots = 0.24999\cdots$ have more than one representation, but this will not create any problem. Now construct a number $y = 0.b_1b_2b_3\cdots$ where

$$b_n = \begin{cases} 4 & \text{if} \quad a_{nn} \neq 4 \\ 3 & \text{if} \quad a_{nn} = 4, \quad n \in \mathcal{N}. \end{cases}$$

This number *y* is obviously in the interval $(0, 1)$. But, *y* is not one of the numbers $f(n)$, because it differs from $f(n)$ at the *n*th decimal place. (Since none of the digits in *y* are 0 or 9, it is also not one of the numbers with two representations.) This contradiction confirms that numbers in the interval $(0, 1)$ are uncountable.

From the aforementioned demonstration and (U3) it is clear that the set of all real numbers contained in the interval $[0, 1]$ is uncountable. Now from the transformation $y = a + (b - a)x$, which maps the interval $[0, 1]$ to $[a, b]$, it follows that any interval $[a, b]$ is uncountable. Next from (U2) it is immediate that half-open as well as open intervals are uncountable. Now from (U3), or from the equivalence $(-\pi/2, \pi/2) \sim \mathcal{R}$ it follows that $\mathcal{R}$ is uncountable. Finally, from (U2) it is clear that the set of transcendental numbers in any interval is uncountable. Thus, if we randomly pick a real number, it is almost certain to be an irrational number.

A set *A* is said to have the *power of continuum* (cardinality), denoted as earlier in Sect. 2.8 by $\mathfrak{c}$ if it is equivalent to the set of real numbers contained in the interval $[0, 1]$. It is clear that any interval closed, open, half–open, finite or infinite has the power $\mathfrak{c}$. Also, the power of the set of transcendental numbers in any interval is $\mathfrak{c}$.

- A finite or countable union of disjoint sets each of power $\mathfrak{c}$ is itself of power $\mathfrak{c}$. For this, let $A = \cup_n A_n$, where the union of disjoint sets is finite or countable, and each of the sets A_n has the power $\mathfrak{c}$. For each *n* we consider the interval $[n-1, n)$. Since every interval has the power of the continuum, it follows that $A_n \sim [n-1, n)$. Therefore, $A \sim \cup_n [n-1, n)$, and hence $A \sim [0, m)$ or $A \sim [0, \infty)$ according as the union is finite or countable. Thus, in either case *A* is equivalent to an interval, and hence its power is $\mathfrak{c}$.

- Let the elements of a set *A* be specified by a finite number of parameters each of which can independently take on any value belonging to a set of power $\mathfrak{c}$. Then, the set *A* is also of power $\mathfrak{c}$. If in this result, we identify the coordinates *x* and *y* of the *xy*–plane as parameters, then it immediately follows that the set of all points in the square $0 \le x, y \le 1$, as well as the set of all points in the *xy*–plane is of power $\mathfrak{c}$. This means that it is possible to set up a one-to-one onto mapping between the points of the square $0 \le x, y \le 1$ and the interval $[0, 1]$.
- If $A = \cup_x A_x$, where *x* runs through a set of power $\mathfrak{c}$ and each of the sets A_x is of power $\mathfrak{c}$, then the set *A* is of power c.

Now to compare the sizes of infinite sets we note that for any two given sets *A* and *B* with cardinal numbers *a* and b, (which are just symbols), only three possibilities can arise:

1. *A* is equivalent to some subset of B, and *B* is equivalent to some subset of A. In this case $A \sim B$ (Felix Bernstein's Theorem), and we have $a = b$.

2. *A* is equivalent to some subset of B, but *B* is not equivalent to any subset of A. In this case $a < b$.

3. *B* is equivalent to some subset of A, but *A* is not equivalent to any subset of B. In this case $a > b$.

This ordering of the cardinal numbers immediately allows us to conclude that $\aleph_0 < \mathfrak{c}$. The question whether there exist uncountable sets of cardinality less than $\mathfrak{c}$, is the *continuum hypothesis* discussed in Sect. 2.8. The following properties of the cardinal numbers $\aleph_0$, $\mathfrak{c}$ can be proved: (a). $\aleph_0 \cdot \aleph_0 = \aleph_0$. (b). $\mathfrak{c} + \aleph_0 = \mathfrak{c}$. (c). $\mathfrak{c} + \mathfrak{c} = \mathfrak{c}$. (d). $\mathfrak{c} \cdot \mathfrak{c} = \mathfrak{c}$. (e). $2^{\aleph_0} = \mathfrak{c}$. The cardinality of the set of all continuous functions $f : \mathcal{R} \to \mathcal{R}^2$ is $2^{\aleph_0}$. This follows from the fact that rationals are dense, i.e., any open set will contain some rational, so they are everywhere in the real line, and curves are continuous. This is equal to the cardinality of the set of all geometric points, not greater than it.

Now we shall show that there are uncountable sets of cardinality greater than $\mathfrak{c}$. For this, first we shall show that for any set A, cardinality of $A <$ cardinality of the power set $P(A)$. The function $f : A \to P(A)$ defined by $f(x) = \{x\}$ is clearly one-to-one, so cardinality of $A \leq$ cardinality of $P(A)$. To show that cardinality of $A \neq$ cardinality of $P(A)$, we shall show that no function $g : A \to P(A)$ can be onto. Clearly, for each $x \in A$, $g(x)$ is a subset of A. Now for $x \in A$ either $x \in g(x)$, or $x \notin g(x)$. Let $B = \{x \in A : x \notin g(x)\}$. Since, $B \subseteq A$ it follows that $B \in P(A)$. If g were onto, then $B = g(y)$ for some $y \in A$. Now either $y \in B$ or $y \notin B$. But, we shall show that both possibilities lead to contradictions. If $y \in B$, then $y \notin g(y)$ by the definition of B. But, $g(y) = B$, so $y \notin B$. On the other hand, if $y \notin B$, then $y \notin g(y)$, which implies that $y \in B$. From this observation it follows that cardinality of $\mathcal{N} = \aleph_0 <$ cardinality of $\mathcal{P}(\mathcal{N}) = 2^{\aleph_0} = \mathfrak{c} <$ cardinality of $\mathcal{P}(\mathcal{P}(\mathcal{N})) < \cdots$. Thus, we have an infinite sequence of transfinite cardinals each larger than the one preceding.

- Let $I = [0, 1]$. Remove the open middle third segment $(1/3, 2/3)$ and let A_1 be the set that remains, i.e., $A_1 = [0, 1/3] \cup [2/3, 1]$. Then, remove the open middle third segment from each of the two parts of A_1 and call the remaining set A_2. Thus, $A_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$. Continue in this manner, i.e., given A_k, remove the open middle third segment from each of the closed segments whose union is A_k, and call the remaining set A_{k+1}. Clearly, $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$ and that for each $k \in \mathcal{N}$, A_k is the union of 2^k closed intervals each of length 3^{-k}. The set $C = \cap_{k=1}^{\infty} A_k$ was discovered in 1874 by Henry John Stephen Smith (1826–1883, Ireland), introduced by Cantor in 1883, and in the literature it is called the *Cantor set*. The cardinality of Cantor's set is the continuum, i.e., $\mathfrak{c}$.

8.21 Transcendence of *e*

Hermite in 1873 proved the conjecture of Liouville and affirmed that *e* is indeed a transcendental number. Here we shall follow an elegant demonstration of Richard Schwartz (born 1966, USA) to prove the transcendence of $e.$ For contrary, assume that *e* is algebraic, i.e., satisfies the polynomial equation with integer coefficients

$$\sum_{k=0}^{n} c_k e^k = 0, \quad c_0 \neq 0, \quad \max_{0\le k\le n} |c_k| < n.$$

Here the degree of the polynomial may be less than $n.$ Consider the functions

$$F = \sum_{i=0}^{\infty} f^{(i)}, \qquad f(x) = \frac{x^{p-1}(1-x)^p(2-x)^p\cdots(n-x)^p}{(p-1)!}.$$

Here the integer $p > n$ will be chosen later. Clearly, *f* is a Hermite polynomial, and so for *F* the sum is finite. We need the following three steps:

(a). We write *f* as $g \times h,$ where

$$g(x) = \frac{x^{p-1}}{(p-1)!} \quad \text{and} \quad h(x) = (1-x)^p(2-x)^p\cdots(n-x)^p.$$

Since $g^{(p-1)}(0) = 1$ and otherwise $g^{(i)}(0) = 0,$ from the formula $f^{(i)} = \sum_{j=0}^{i}\binom{i}{j} g^{(j)} h^{(i-j)}$ it follows that

$$\begin{aligned} F(0) &= \sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{i}{j} g^{(j)}(0) h^{(i-j)}(0) = \sum_{i=p-1}^{\infty}\binom{i}{p-1} h^{(i-(p-1))}(0) \\ &= h(0) + p(\cdots) = (n!)^p + p(\cdots) \ \in \ \mathcal{Z} - p\mathcal{Z}. \end{aligned}$$

Hence, in view of $0 < |c_0| < n$ it follows that $c_0 F(0) \in \mathcal{Z} - p\mathcal{Z}.$

(b). Now we write *f* as $g \times h,$ where

$$g(x) = \frac{(x-k)^p}{(p-1)!} \ \text{ and } \ h(x) = \frac{x^{p-1}(1-x)^p(2-x)^p\cdots(n-x)^p}{(k-x)^p}, \quad 1 \le k \le n.$$

Since $g^{(p)}(k) = p$ and $g^{(i)}(k) = 0$ otherwise, we have

$$F(k) = p \times \sum_{i=p}^{\infty} \binom{i}{p} h^{(i-p)}(k) \ \in \ p\mathcal{Z}.$$

(c). Let $\phi(x) = e^{-x}F(x)$, so that

$$\phi'(x) = -e^{-x}(F(x) - F'(x)) = -e^{-x}\left(\sum_{i=0}^{\infty} f^{(i)}(x) - \sum_{i=1}^{\infty} f^{(i)}(x)\right)$$
$$= -e^{-x}f(x).$$

Here we have used the fact that for *F* the sum is finite. Thus, we have $|\phi'(x)| \leq |f(x)|$ for $x \geq 0$. Now in view of the mean value theorem, it follows that for all $1 \leq k \leq n$,

$$|F(k) - e^k F(0)| = |e^k||\phi(k) - \phi(0)| \leq ke^k \max_{[0,k]} |\phi'|$$
$$\leq ne^n \max_{[0,n]} |f| \leq \frac{e^n (n^{n+2})^p}{(p-1)!} < \frac{1}{n^2}.$$

The last inequality follows for $p (\geq n)$ sufficiently large.

Next since from (a) and (b), respectively, we have $c_0 F(0) \in \mathcal{Z} - p\mathcal{Z}$, and for each $1 \leq k \leq n, \ F(k) \in p\mathcal{Z}$, it follows that $\sum_{k=0}^{n} c_k F(k) \in \mathcal{Z} - p\mathcal{Z}$, and therefore is a nonzero integer. The required contradiction now follows from the following successive equalities and inequalities, and (c),

$$\begin{aligned} 1 \ &\leq \ \left|\sum_{k=0}^{n} c_k F(k)\right| = \left|\sum_{k=0}^{n} c_k F(k) - 0 \times F(0)\right| \\ &= \left|\sum_{k=0}^{n} c_k F(k) - \left(\sum_{k=0}^{n} c_k e^k\right) \times F(0)\right| \\ &= \left|\sum_{k=0}^{n} c_k (F(k) - e^k F(0))\right| < n \sum_{k=0}^{n} |F(k) - e^k F(0)| \\ &= \ n \sum_{k=1}^{n} |F(k) - e^k F(0)| < \frac{n^2}{n^2} = 1. \end{aligned}$$

8.22 Transcendence of π

Using a slight extension of Hermite's method of proving the transcendence of $e,$ in 1882 [347], Carl Louis Ferdinand von Lindemann (1852–1939, Germany) succeeded in proving the transcendence of $\pi.$ For this, first he proved that e^a is transcendental if *a* is a non-zero algebraic number (possibly complex). This result in the literature is known as Weak Hermite-Lindemann-Weierstrass Theorem (WHLWT). Now suppose that π is algebraic, i.e., π is a root of a polynomial with rational coefficients, say, $P(x).$ Then, $\phi(x) = P(ix)P(-ix)$ is also a polynomial with rational coefficients and $\phi(i\pi) = 0,$ i.e., $i\pi$ is also algebraic. But then WHLWT implies that $e^{i\pi}$ is transcendental; however, it contradicts Euler's identity $e^{i\pi} = -1,$ and hence π is transcendental. His proof was 13 pages long. Lindemann also developed a method of solving equations of any degree using transcendental functions. Most astonishingly, he supervised the doctoral theses of Hilbert, Minkowski, and Arnold Johannes Wilhelm Sommerfeld (1868–1951, Germany). They laid the future foundation of mathematics. In 1885, Weierstrass simplified the proof of Lindemann's theorem, and it was further simplified in later years by renowned mathematicians Stieltjes, Hurwitz, Hilbert, and others. In a lecture given in 1886, Kronecker complimented Lindemann on a beautiful proof but, he claimed, one that proved nothing since transcendental numbers do not exist. The proof of the transcendence of π definitely settled the age-old problem of squaring the circle by a ruler-and-compass construction is impossible. Nonetheless, there are still some amateur mathematicians who do not understand the significance of this result, and futilely look for techniques to square the circle. • An important property of a transcendental number, say, *s* is that s^t for any nonzero rational *t* is also transcendental. Indeed, if $s^{a/b}$ is algebraic $r,$ then $s^a = r^b,$ but this implies that *s* is a solution of an algebraic equation $x^a - r^b = 0,$ and hence algebraic, a contradiction. In particular, from (7.40) it follows that each $\zeta(2n),\ n \geq 1$ is transcendental. The irrationality of $\zeta(2n+1),\ n \geq 2$ is expected but not yet established. In this direction an advanced result was obtained by Wadim Zudilin (Russia-The Netherlands) in 2001, which confirms that at least one of $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational. • A real number *x* is called a Liouville number if for every positive integer $n,$ there exist integers *p* and *q* with $q > 1$ and such that

$$0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^n}.$$

A Liouville number can thus be approximated “quite closely” by a sequence of rational numbers. In 1844, Liouville showed that all Liouville numbers are transcendental, whereas in 1953, Mahler showed that π is not a Liouville number, and Baker in [52] reported that *e* is also not a Liouville number.

8.23 More About Transcendental Numbers

While the existence of transcendental numbers has been proved to be uncountable, only for very few numbers their transcendence (one by one) has been established. As it stands, even to prove irrationality of a number no general method exists, proving transcendence (or otherwise) of a number is considered as life’s great achievement. In 1900, Hilbert posed a question about transcendental numbers, Hilbert’s seventh problem: If *a* is an algebraic number that is not zero or one, and *b* is an irrational algebraic number, is a^b necessarily transcendental? The affirmative answer was provided in 1934 by Alexander Osipovich Gelfond (1906–1968, Russia) and followed by Theodor Schneider (1911–1988, German). This result in the literature is known as Gelfond-Schneider theorem. Their result was extended by Alan Baker in the 1960s in his work on lower bounds for linear forms in any number of logarithms (of algebraic numbers). The transcendental number $2^{\sqrt{2}}$ (first proved in 1930 by Rodion Kuzmin, 1891–1949, Russia) is known as the Gelfond–Schneider constant (or Hilbert number), and the transcendental number $e^\pi = (e^{i\pi})^{-i} = (-1)^{-i}$ is known as Gelfond’s constant. The numbers $e^{-\pi/2} = i^i$ and $e^{\pi\sqrt{n}}$ are also transcendental. $\operatorname{Log} 2$ (base 10) can be shown to be transcendental using the Gelfond–Schneider theorem. However, their theorem does not help to determine whether numbers such as e^e, π^π, or π^e are transcendental, since both the bases and exponents are transcendental numbers. The trigonometric functions $\sin x, \ \cos x, \ \tan x, \ \csc x, \ \sec x, \ \cot x,$ and their hyperbolic counterparts, for any nonzero algebraic number $x,$ expressed in radians are transcendental. It is also not yet known if the numbers $\pi + e, \ \pi - e, \ \pi e$ are rational, algebraic, irrational, or transcendental. However, it is certain that both $\pi + e$ and πe cannot be rational (or algebraic). In fact, if both are rational then $(\pi + e)^2 - 4\pi e$ is rational. But this gives $(\pi - e)^2$, and so $\pi - e$ is algebraic. But then adding and subtracting $\pi - e$ with $\pi + e,$ we find that both π and *e* are algebraic, which contradicts the fact that both are transcendental. For more details of transcendental numbers, see Baker [52] and Siegel [477].

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