On Multiplicative Sidon Sets

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Abstract

Fix integers $b > a \geq 1$ with $g := \gcd(a, b)$. A set $S \subseteq \mathbb{N}$ is $\{a, b\}$-*multiplicative* if $ax \neq by$ for all $x, y \in S$. For all n, we determine an $\{a, b\}$-multiplicative set with maximum cardinality in $[n]$, and conclude that the maximum density of an $\{a, b\}$-multiplicative set is $\frac{b}{b+g}$.

Erdős [2, 3, 4] defined a set $S \subseteq \mathbb{N}$ to be *multiplicative Sidon*[1] if $ab = cd$ implies $\{a, b\} = \{c, d\}$ for all $a, b, c, d \in S$; see [8–10]. In a similar direction, Wang [13] defined a set $S \subseteq \mathbb{N}$ to be *double-free* if $x \neq 2y$ for all $x, y \in S$, and proved that the maximum density of a double-free set is $\frac{2}{3}$; see [1] for related results. Here the *density* of $S \subseteq \mathbb{N}$ is

$$\lim_{n \to \infty} \frac{|S \cap [n]|}{n} \quad .$$

Pór and Wood [7] generalised the notion of double-free sets as follows. For $k \in \mathbb{N}$, a set $S \subseteq \mathbb{N}$ is k-*multiplicative* (*Sidon*) if $ax = by$ implies $a = b$ and $x = y$ for all $a, b \in [k]$ and $x, y \in S$. Pór and Wood [7] proved that the maximum density of a k-multiplicative set is $\Theta(\frac{1}{\log k})$.

Here we study the following alternative generalisation of double-free sets. For distinct $a, b \in \mathbb{N}$, a set $S \subseteq \mathbb{N}$ is $\{a, b\}$-*multiplicative* if $ax \neq by$ for all $x, y \in S$. Our main result is to determine the maximum density of an $\{a, b\}$-multiplicative set. Assume that $a < b$ throughout.

Say $x \in \mathbb{N}$ is an *i-th subpower* of b if $x = b^i y$ for some $y \not\equiv 0 \pmod{b}$. If x is an i-th subpower of b for some even/odd i then x is an *even/odd* subpower of b.

First suppose that $\gcd(a, b) = 1$. Let T be the set of even subpowers of b. We now prove that T is an $\{a, b\}$-multiplicative set with maximum density. In fact, for all $[n]$,

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[1] Additive Sidon sets have been more widely studied; see the classical papers [5, 11, 12] and the recent survey by O'Bryant [6]. Let $\mathbb{N} := \{1, 2, \dots\}$ and $[n] := \{1, 2, \dots, n\}$.

we prove that $T_n := T \cap [n]$ has maximum cardinality out of all $\{a, b\}$-multiplicative sets contained in $[n]$.

The key to our proof is to model the problem using a directed graph. Let G be the directed graph with $V(G) := [n]$ where $xy \in E(G)$ whenever $bx = ay$ (implying $x < y$). Thus $S \subseteq [n]$ is $\{a, b\}$-multiplicative if and only if S is an independent set in G. If xyz is a directed path in G, then $x = \frac{a}{b}y$ and $z = \frac{b}{a}y$. Thus each vertex y has indegree and outdegree at most 1. Since $xy \in E(G)$ implies $x < y$, G contains no directed cycles. Thus G is a collection of disjoint directed paths. Hence a maximum independent set in G is obtained by taking all the vertices at even distance from the source vertices[2], where a vertex y is a source (indegree 0) if and only if $\frac{a}{b}y$ is not an integer; that is, if $y \not\equiv 0 \pmod{b}$.

We now prove that the vertices at distance d from a source vertex are precisely the d-th subpowers of b. We proceed by induction on $d \geq 0$. Each vertex y of G has an incoming edge if and only if $\frac{a}{b}y \in \mathbb{N}$, which occurs if and only if $y \equiv 0 \pmod{b}$ since $\gcd(a, b) = 1$. Thus the source vertices of G are precisely the 0-th subpowers of b. This proves the $d = 0$ case of the induction hypothesis. Now consider a vertex y at distance d from a source vertex. Thus $y = \frac{b}{a}x$ for some vertex x at distance $d - 1$ from a source vertex. By induction, x is a $(d-1)$-th subpower of b. That is, $x = b^{d-1}z$ for some $z \not\equiv 0 \pmod{b}$. Thus $y = b^d \frac{z}{a}$, which, since $\gcd(a, b) = 1$, implies that $\frac{z}{a}$ is an integer. Hence $\frac{z}{a} \not\equiv 0 \pmod{b}$ and y is a d-th subpower of b, as claimed.

This proves that the even subpowers of b form a maximum independent set in G. That is, T_n is an $\{a, b\}$-multiplicative set of maximum cardinality in $[n]$. To illustrate this proof, the following table shows two examples of the graph G with $b = 3$. Observe that the i-th row consists of the i-th subpowers of 3 regardless of a.

$a = 1$ and $b = 3$									$a = 2$ and $b = 3$											
1	2	4	5	7	8	10	11	···	1	2	4	5	7	8	10	11	13	14	16	···
↓	↓	↓	↓	↓	↓	↓	↓			↓	↓			↓	↓			↓	↓	
3	6	12	15	21	24	30	33	···		3	6			12	15			21	24	···
↓	↓	↓	↓	↓	↓	↓	↓				↓			↓					↓	
9	18	36	45	63	72	90	99	···			9			18					36	···
↓	↓	↓	↓	↓	↓	↓	↓							↓					↓	
27	48	108	135	189	216	270	297	···						27					48	···
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮												⋮	

[2]Note that this is not necessarily the only maximum independent set—for a path component with odd length, we may take the vertices at odd distance from the source of this path. This observation readily leads to a characterisation of all maximum independent sets in G, and thus of all $\{a, b\}$-multiplicative sets in $[n]$ with maximum cardinality. Details omitted.

We now bound $|T_n|$ from above. Observe that

$$T_n = \left\{ b^{2i}y : 0 \le i \le \tfrac{1}{2}\log_b n,\ 1 \le y \le \frac{n}{b^{2i}},\ y \not\equiv 0 \pmod{b} \right\} \ .$$

Thus

$$\begin{aligned}
|T_n| &\le \sum_{i=0}^{\lfloor (\log_b n)/2 \rfloor} \left\lceil \frac{b-1}{b}\frac{n}{b^{2i}} \right\rceil \\
&\le 1 + \tfrac{1}{2}(\log_b n) + \frac{(b-1)n}{b} \sum_{i \ge 0} \frac{1}{b^{2i}} \\
&\le 1 + \tfrac{1}{2}(\log_b n) + \frac{(b-1)n}{b} \frac{b^2}{b^2-1} \\
&= 1 + \tfrac{1}{2}(\log_b n) + \frac{bn}{b+1} \ .
\end{aligned}$$

We now bound $|T_n|$ from below. Observe that

$$T_n = [n] \setminus \left\{ b^{2i+1}y : 0 \le i \le \tfrac{1}{2}((\log_b n) - 1),\ 1 \le y \le \frac{n}{b^{2i+1}},\ y \not\equiv 0 \pmod{b} \right\} \ .$$

Thus

$$\begin{aligned}
|T_n| &\ge n - \sum_{i=0}^{\lfloor ((\log_b n)-1)/2 \rfloor} \left\lceil \frac{b-1}{b}\frac{n}{b^{2i+1}} \right\rceil \\
&\ge n - \tfrac{1}{2}((\log_b n) + 1) - \frac{(b-1)n}{b^2} \sum_{i \ge 0} \frac{1}{b^{2i}} \\
&\ge n - \tfrac{1}{2}((\log_b n) + 1) - \frac{(b-1)n}{b^2} \frac{b^2}{b^2-1} \\
&= n - \tfrac{1}{2}((\log_b n) + 1) - \frac{n}{b+1} \\
&= \frac{bn}{b+1} - \tfrac{1}{2}((\log_b n) + 1) \ .
\end{aligned}$$

These upper and lower bounds on $|T_n|$ imply that

$$|T_n| = \frac{bn}{b+1} + \Theta(\log_b n) \ .$$

Hence the density of T is $\frac{b}{b+1}$, and because T_n is optimal for each n, no $\{a,b\}$-multiplicative set has density greater than $\frac{b}{b+1}$.

We now drop the assumption that $\gcd(a,b) = 1$. Let $g := \gcd(a,b)$. Since $ax = by$ if and only if $\frac{a}{g}x = \frac{b}{g}y$, a set S is $\{a,b\}$-multiplicative if and only if S is $\{\frac{a}{g},\frac{b}{g}\}$-multiplicative. Since $\frac{b/g}{b/g+1} = \frac{b}{b+g}$, we have the following result.

Theorem 1. *Fix integers $b > a \ge 1$. Let $g := \gcd(a,b)$. Then for every integer $n \in \mathbb{N}$, the even subpowers of $\frac{b}{g}$ in $[n]$ are an $\{a,b\}$-multiplicative set in $[n]$ with maximum cardinality. And the even subpowers of $\frac{b}{g}$ are an $\{a,b\}$-multiplicative set with density $\frac{b}{b+g}$, which is maximum.*

Note that if $g = a$ then $b \geq 2g$ and $b + g \leq \frac{3}{2}b$, and if $g < a$ then $a \geq 2g$ and $b + g \leq b + a < \frac{3}{2}b$. In both cases the density bound $\frac{b}{b+g}$ in Theorem 1 is at least $\frac{2}{3}$, which is the bound obtained by Wang [13] for the $a = 1$ and $b = 2$ case.

In conclusion, we propose a further generalisation of double-free sets. Let $A, B \subset \mathbb{N}$. Say $S \subset \mathbb{N}$ is $\{A, B\}$-multiplicative if $ax = by$ implies $\{a, x\} = \{b, y\}$ for all $a \in A$ and $b \in B$ and $x, y \in S$. One case is easily dealt with. For some prime number b, let $A := [b - 1]$ and $B := \{b\}$. Then $\gcd(a, b) = 1$ for all $a \in A$. Thus the even subpowers of b are $\{A, B\}$-multiplicative, and have maximum density.

References

[1] Jean-Paul Allouche, André Arnold, Jean Berstel, Srečko Brlek, William Jockusch, Simon Plouffe, and Bruce E. Sagan. A relative of the Thue-Morse sequence. *Discrete Math.*, 139(1–3):455–461, 1995. doi:10.1016/0012-365X(93)00147-W.

[2] Paul Erdős. On sequences of integers no one of which divides the product of two others and some related problems. *Izvestiya Naustno-Issl. Inst. Mat. i Meh. Tomsk*, 2:74–82, 1938.

[3] Paul Erdős. On some applications of graph theory to number theoretic problems. *Publ. Ramanujan Inst. No.*, 1:131–136, 1968/1969.

[4] Paul Erdős. Some applications of graph theory to number theory. In *The Many Facets of Graph Theory (Proc. Conf., Western Mich. Univ., Kalamazoo, Mich., 1968)*, pp. 77–82. Springer, Berlin, 1969.

[5] Paul Erdős and Pál Turán. On a problem of Sidon in additive number theory, and on some related problems. *J. London Math. Soc.*, 16:212–215, 1941. doi:10.1112/jlms/s1-16.4.212.

[6] Kevin O'Bryant. A complete annotated bibliography of work related to Sidon sequences. *Electron. J. Combin.*, DS11, 2004. http://www.combinatorics.org/Surveys/index.html.

[7] Attila Pór and David R. Wood. Colourings of the Cartesian product of graphs and multiplicative Sidon sets. *Combinatorica*, 29(4):449–466, 2009. doi:10.1007/s00493-009-2257-0.

[8] Imre Z. Ruzsa. Erdős and the integers. *J. Number Theory*, 79(1):115–163, 1999. doi:10.1006/jnth.1999.2395.

[9] Imre Z. Ruzsa. Additive and multiplicative Sidon sets. *Acta Math. Hungar.*, 112(4):345–354, 2006. doi:10.1007/s10474-006-0102-0.

[10] András Sárközy. Unsolved problems in number theory. *Period. Math. Hungar.*, 42(1-2):17–35, 2001. doi:10.1023/A:1015236305093.

[11] Simon Sidon. Ein Satz űber trigonometrische Polynome und seine Anwendung in der Theorie der Fourier-Reihen. *Math. Ann.*, 106:536–539, 1932.

[12] James Singer. A theorem in finite projective geometry and some applications to number theory. *Trans. Amer. Math. Soc.*, 43(3):377–385, 1938. doi:10.2307/1990067.

[13] Edward T. H. Wang. On double-free sets of integers. *Ars Combin.*, 28:97–100, 1989.