Indian Journal of Natural Sciences *www.tnsroindia.org.in ©IJONS*

Vol.15 / Issue 87 / Dec / 2024 *International Bimonthly (Print) – Open Access* *ISSN: 0976 – 0997*

RESEARCH ARTICLE

On Some Properties of Summability in Arithmetic - Geometric Series

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Received: 20 July 2024 Revised: 12 Oct 2024 Accepted: 30 Nov 2024

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ABSTRACT

The concept of summability theory has been dealt with several forms in recent years. In this paper, we present a perspective in generalizing Ramanujan summation method by considering Geometric and Arithmetic-Geometric Progressions.Geometric verifications are also included. The applications of geometric and arithmetic – geometric series on Pascal's triangle and other applications in science are also described here.

keywords: Geometric Series, Arithmetic – Geometric series, Ramanujan summation, Pascal's triangle

INTRODUCTION

The concept of summability theory has been prevailing for several years as of now and has been a much studied topic. This idea has paved way for developing new branch of mathematical analysis called "Summability Theory". The purpose of this paper is to present an approach to Ramanujan summation methods for geometric and arithmetic geometric series.

Definition

The Ramanujan summation is defined as $R.S(\sum_{n=1}^{\infty} a_n) = \int_{-1}^{0} s_n \, dn$ (1)
where n is a positive integer and s_n is sum upto first n terms of the divergent series $\sum_{n=1}^{\infty} a_n$ of real numbers.

Theorem 1
The Ramanujan summation of the geometric series $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$
is given by

$$R.S\left(\sum_{n=1}^{\infty} a_n\right) = \begin{cases} \frac{a}{r-1}\left(\frac{1}{logr} - \frac{1}{rlogr} - 1\right) & if \quad r > 1 \\ \frac{a}{1-r}\left(1 - \frac{1}{logr} + \frac{1}{rlogr}\right) & if \;\; 0 < r < 1 \\ -\frac{1}{2}a & if \quad r = 1 \end{cases} \qquad (2)$$

where a is the initial term and r is the common ratio between the terms.

Proof:
For the geometric series , the sum to n terms is given by

$$S_n = \begin{cases} \frac{a(r^n-1)}{r-1} & if\ r > 1 \\ \frac{a(1-r^n)}{1-r} & if\ 0 < r < 1 \\ na & if\ r = 1 \end{cases} \qquad (3)$$

For the case$r > 1$
Substituting (3) in (1) we get,

$$R.S\left(\sum_{n=1}^{\infty} a_n\right) = \int_{-1}^{0} \frac{a(r^n-1)}{r-1} dn = \frac{a}{r-1}\left(\frac{r^n}{logr} - n\right)_{-1}^{0} = \frac{a}{r-1}\left(\frac{1}{logr} - \frac{1}{rlogr} - 1\right)$$

For the case $0 < r < 1$
Substituting (3) in (1) we get,

$$R.S\left(\sum_{n=1}^{\infty} a_n\right) = \int_{-1}^{0} \frac{a(1-r^n)}{1-r}\, dn = \frac{a}{1-r}\left(n - \frac{r^n}{logr}\right)_{-1}^{0} = \frac{a}{1-r}\left(1 - \frac{1}{logr} + \frac{1}{rlogr}\right)$$

For the case $r = 1$,
Substituting (3) in (1) we get,

$$R.S\left(\sum_{n=1}^{\infty} a_n\right) = \int_{-1}^{0} (na)dn = a(\left(\frac{n^2}{2}\right)_{-1}^{0} = -\frac{1}{2}a$$

Geometric Meaning
This can be verified geometrically by taking $a = 1$ and scaling r for few values as follows
We observe that from the shaded portion of Figure 1(a) to Figure 1(b) is that the region representing the area of $S_n$between x- axis and the interval [-1,0] lies below the x- axis and it decreases as the value of common ratio increases.

From the figure for $r = 1$, length equals to 1 and breadth equals to 1 and the area is $-\frac{1}{2}$

Corollary 1
The Ramanujan summation of the series $2^0 + 2^1 + 2^2 + \cdots + 2^n + \cdots$ is
$R.S(\sum_{n=1}^{\infty} a_n) = \frac{1}{2\ln 2} - 1 = -0.27865$(4)

Proof:
$2^0 + 2^1 + 2^2 + \cdots + 2^n + \cdots$ is a geometric series with $a = 1$ and $r = 2$
Substituting $a = 1$ and $r = 2$ in (3)
$S_n = \frac{2^n-1}{2-1} = 2^n - 1$ (5)
Substituting (5) in (1)

$$R.S\left(\sum_{n=1}^{\infty} a_n\right) = \int_{-1}^{0} (2^n - 1)dn = \left[\frac{2^n}{\ln 2} - n\right]_{-1}^{0} = \frac{1}{2\ln 2} - 1$$

$$= \frac{1}{2(0.693147)} - 1 = -0.27865$$

Geometric meaning

This can be verified geometrically as follows

We observe that from the shaded portion of Figure 2 is that the region representing the area of $S_n$between x- axis and the interval [-1,0] lies below the X- axis found to be -0.278652 which equals to $\frac{1}{2\ln 2} - 1$

Theorem 2

The Ramanujan summation of the arithmetic -geometric series

$a + (a+d)r + (a+2d)r^2 + (a+3d)r^3 + \cdots + (a+(n-1)d)r^{n-1} + \cdots$ (6)

is given by

$$R.S(\textstyle\sum_{n=1}^{\infty} a_n) = \begin{cases} \frac{a(1-r)+dr}{(1-r)^2} + \frac{(a-d)}{r\ln r} - \frac{d}{r(\ln r)^2} & if\ r > 0\ and\ r \neq 1 \\ -\frac{a}{2} + \frac{5d}{12} & if\ r = 1 \end{cases} \quad (7)$$

where a is the initial term ,r is the common ratio and d is the common difference between the terms.

Proof:

The sum to n terms of the arithmetic - geometric series is given by

$$s_n = a + (a+d)r + (a+2d)r^2 + (a+3d)r^3 + \cdots + (a+(n-1)d)r^{n-1} \quad (8)$$

$$= a(1 + r + r^2 + \cdots + r^{n-1}) + dr(1 + 2r + 3r^2 + \cdots + (n-1)r^{n-2})$$

$if\ r > 0\ and\ r \neq 1$,then

$$s_n = a\left(\frac{1-r^n}{1-r}\right) + dr(1 + 2r + 3r^2 + \cdots + (n-1)r^{n-2}) \quad (9)$$

Also$1 + 2r + 3r^2 + \cdots + (n-1)r^{n-2} = \frac{d}{dr}(r + r^2 + \cdots + r^{n-1}) =$

$$= \frac{d}{dr}\left(r\left(\frac{1-r^{n-1}}{1-r}\right)\right) = \frac{(n-1)r^n - nr^{n-1} + 1}{(1-r)^2} \quad (10)$$

Substituting (10) in (9) we get,

$$s_n = a\left(\frac{1-r^n}{1-r}\right) + dr\left(\frac{(n-1)r^n - nr^{n-1} + 1}{(1-r)^2}\right)$$

$$= \frac{a(1-r)+dr}{(1-r)^2} - \frac{ar^n}{1-r} + \frac{dr}{(1-r)^2}\left(r^{n-1}\big((n-1)r - n\big)\right) \quad (11)$$

Substituting (11) in (1) we get,

$$R.S\left(\sum_{n=1}^{\infty} a_n\right) = \int_{-1}^{0} s_n dn = \int_{-1}^{0}\left(\frac{a(1-r)+dr}{(1-r)^2} - \frac{ar^n}{1-r} + \frac{dr}{(1-r)^2}\left(r^{n-1}\big((n-1)r-n\big)\right)\right) dn$$

$$= \left[\left(\frac{a(1-r)+dr}{(1-r)^2}\right)n - \frac{a}{1-r}\left(\frac{r^n}{\ln r}\right) + \frac{d(r-1)}{(1-r)^2}\left(n\frac{r^n}{\ln r} - 1\left(\frac{r^n}{(\ln r)^2}\right)\right) - \frac{dr}{(1-r)^2}\frac{r^n}{\ln r}\right]_{-1}^{0}$$

$$= \frac{a(1-r)+dr}{(1-r)^2} + \frac{a}{r\ln r} + \frac{d}{r-1}\left(\frac{1}{r\ln r} - \frac{1}{(\ln r)^2} + \frac{1}{r(\log r)^2}\right) - \frac{d}{(r-1)\ln r}$$

$$= \frac{a(1-r)+dr}{(1-r)^2} + \frac{a}{r\ln r} - \frac{d}{r\ln r} - \frac{d}{r(\ln r)^2} \quad (12)$$

$If\ r = 1$, (6) becomes

$$s_n = a + (a+d) + (a+2d) + (a+3d) + \cdots + (a+(n-1)d) \quad (13)$$

Substituting (13) in (1) we get,

$R.S(\sum_{n=1}^{\infty} a_n) = \int_{-1}^{0} s_n dn = \int_{-1}^{0} \left(\frac{n}{2}(2a+(n-1)d)\right) dn$
$= \left[a\frac{n^2}{2} + \frac{d}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\right]_{-1}^{0} = -\frac{a}{2} + \frac{5d}{12}$ ■

Geometric meaning
This can be verified geometrically by taking $a = 1$ and $d = 1$ scaling r for few values as follows.
We observe that from the shaded portion of Figure 3(a) to Figure 3(d) is that the region representing the area of $S_n$between x- axis and the interval [-1,0] lies below the x- axis.

Corallary 1:
If $r = e$, the Ramanujan summation of the arithmetic -geometric series becomes

$$R.S\left(\sum_{n=1}^{\infty} a_n\right) = \frac{1}{e(e-1)^2}\left((a-2d) + e(d(4-e)-a)\right)$$

Also if $a = 1, d = 1$ then,$R.S(\sum_{n=1}^{\infty} a_n) = -0.0291$approximately

Proof
Substituting $r = e$ in (7)

$$R.S\left(\sum_{n=1}^{\infty} a_n\right) = \frac{a(1-e)+de}{(1-e)^2} + \frac{a-2d}{e}$$
$$= \frac{1}{e(e-1)^2}(a - 2d - de^2 - ae + 4ed)$$
$$= \frac{1}{e(e-1)^2}\left((a-2d) + e(d(4-e)-a)\right)$$

If $a = 1, d = 1$ then

$$R.S(\sum_{n=1}^{\infty} a_n) = \frac{1}{e(e-1)^2}(-e^2 + 3e - 1) = -\frac{0.23421}{8.02568} = -0.0291 \text{ approximately}$$ ■

Applications in the Pascals' triangle
We could see Ramanujan like summation can be applied for diagonal elements of Pascal's triangle. which is a triangular array of the binomial coefficients that has wide applications in various mathematical fields like probability theory, combinatorics and algebra.
The first eight rows of the Pascal's triangle is as follows

Theorem 3
The Ramanujan summation of the sum of the diagonal elements along the first slant diagonal of Pascals' triangle is $R.S(\sum_{n=1}^{\infty} a_n) = -\frac{1}{2}$ (14)

Proof:
The sum of the diagonal elements along the first diagonal is given by $1 + 1 + 1 + \cdots + n + \cdots$
which is a geometric progression series
Substituting $a = 1$ and $r = 1$ in (5) we get $S_n = n$ (15)
Substituting (15) in (1) we get

$$R.S\left(\sum_{n=1}^{\infty} a_n\right) = \int_{-1}^{0} n\, dn = -\frac{1}{2}$$

■

Geometric meaning
This can be verified geometrically as follows

We observe that from the shaded portion of Figure 5 is that the region representing the area of $S_n$betweenx- axis and the interval [-1,0] lies below the x- axis found to be -0.5 which equals to $-\frac{1}{2}$

Theorem 4
The Ramanujan summation of the sum of the diagonal elements along the second slant diagonal of Pascal's triangle is $R.S(\sum_{n=1}^{\infty} a_n) = -\frac{1}{12}$(16)

Proof:
The sum of the diagonal elements along the second diagonal is given by
$1 + 2 + 3 + 4 + \cdots + n + \cdots$which is an arithmetic - series where $a = 1$ and $d = 1$
The sum to n terms is given by $s_n = \frac{n}{2}(2a + (n-1)d) = \frac{n}{2}\left(2 + (n-1)\right) = \frac{n}{2}(n+1)$(17)Substituting (17) in (1) we get

$$R.S\left(\sum_{n=1}^{\infty} a_n\right) = \int_{-1}^{0} (\frac{n}{2}(n+1)dn = -\frac{1}{12}$$

Geometric meaning
This can be verified geometrically as follows
We observe that from the shaded portion of Figure 6is that the region representing the area of $S_n$between x-axis and the interval [-1,0] lies below the x axis found to be −0.0833333333333.which is approximately equals to $-\frac{1}{12}$
This result is similar to Ramanujan work on $1 + 2 + 3 + \cdots = -\frac{1}{12}$

Theorem 5
The Ramanujan summation ofthe sum of the diagonal elements along the third slant diagonal of Pascal'triangle is $R.S(\sum_{n=1}^{\infty} a_n) = -\frac{1}{24}$ (18)

Proof:
The sum of the diagonal elements along the third diagonal of Pascal's triangle is given by
$1 + 3 + 6 + \cdots n + \cdots$
The sum to n terms of the series is given by$S_n = \binom{n+2}{3} = \frac{n^3+3n^2+2n}{6}$ (19)
Substituting (19) in (1) we get

$$R.S\left(\sum_{n=1}^{\infty} a_n\right) = \int_{-1}^{0} (\frac{n^3 + 3n^2 + 2n}{6}\Bigg) dn = -\frac{1}{24} \quad ■$$

Geometric meaning
This can be verified geometrically as follows.
We observe that from the shaded portion of (Figure 7) is that the region representing the area of $S_n$between x- axis and the interval [-1,0] lies below the x- axis found to be −0.0416666666667 which is approximately equals to $-\frac{1}{24}$

Other Applications in Science
Here are some additional applications of geometric and arithmetic geometric series in various scientific fields. Scientists across various disciplines can leverage geometric and arithmetic geometric series whenever exponential growth, decay, or repeated processes with a constant factor come into play. For instances in physics, in diffraction gratings and in geometric optics, in Chemistry, Chemical Kinetics and serial dilutions, in biology, Signal Transduction, population Genetics and in other sciences, earthquake magnitudes, ecology etc

CONCLUSION

We have discussed the Ramanujan summation for geometic and arithmetic-geometric series and we have applied this to Pascal's triangle and we got nice results. We have used Desmos graphing software tool to create graphs presented in the figures. Further we can construct Ramanujan summation for various other series also using the techniques presented in this paper.

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Vol.15 / Issue 87 / Dec / 2024 *International Bimonthly (Print) – Open Access* *ISSN: 0976 – 0997*

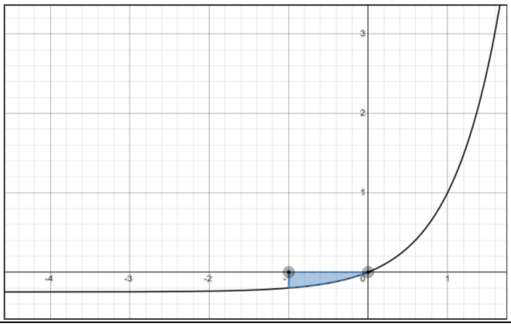

Figure 1(a)
Graph for $\int_{-1}^{0} \frac{a(r^n-1)}{r-1} dn$
showing area of S_n between x- axis and the interval [-1,0] for $r = 5$

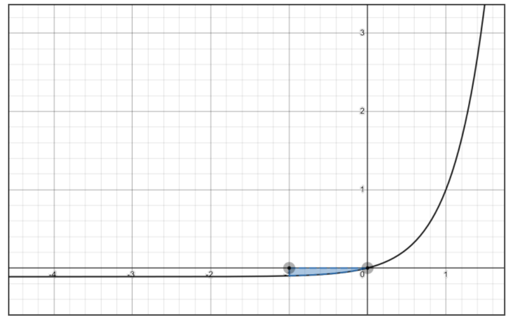

Figure 1(b)
Graph for $\int_{-1}^{0} \frac{a(r^n-1)}{r-1} dn$
showing area of S_n between x- axis and the interval [-1,0] for $r = 10$

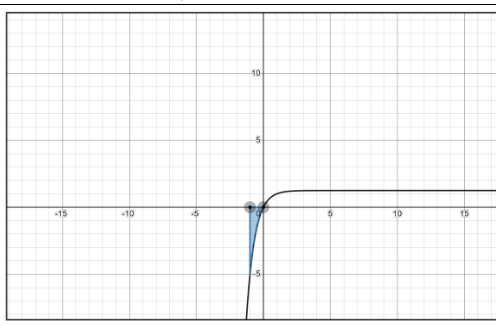

Figure 1(c)
Graph for $\int_{-1}^{0} \frac{a(1-r^n)}{1-r} dn$
showing area of S_n between x- axis and the interval [-1,0] for $r = 0.2$

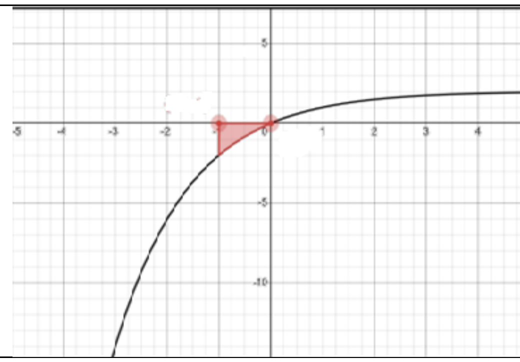

Figure 1(d)
Graph for $\int_{-1}^{0} \frac{a(1-r^n)}{1-r} dn$
showing area of S_n between x- axis and the interval [-1,0] for $r = 0.5$

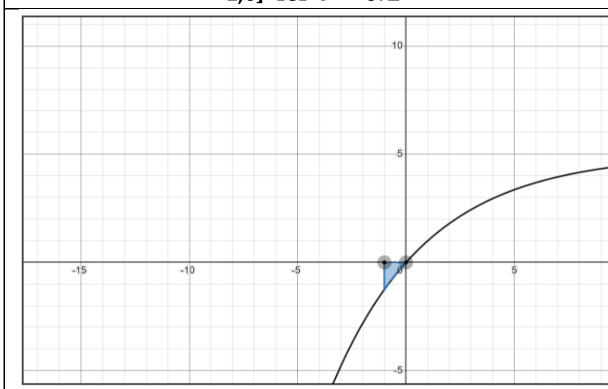

Figure 1(e)
Graph for $\int_{-1}^{0} \frac{a(1-r^n)}{1-r} dn$
showing area of S_n between x- axis and the interval [-1,0] for $r = 0.8$

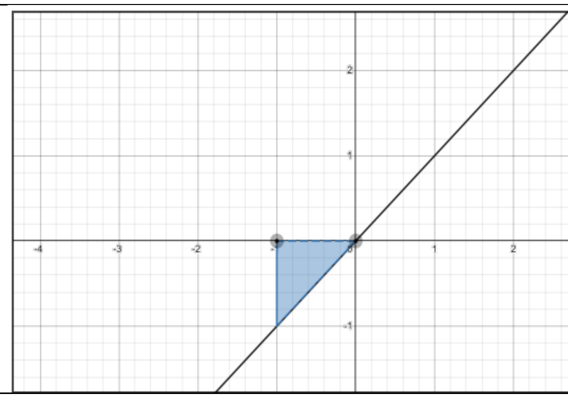

Figure 1(f)
Graph for $\int_{-1}^{0} (na) dn$
showing area of S_n between x- axis and the interval [-1,0] for $r = 1$

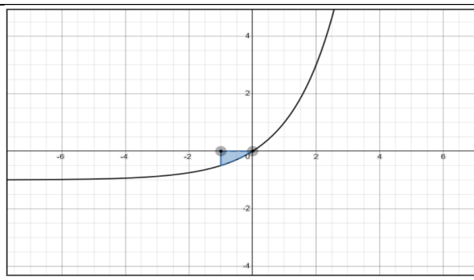

Figure 2
Graph for $\int_{-1}^{0}(2^n-1)dn$
howing area of S_n between x- axis and the interval [-1,0]

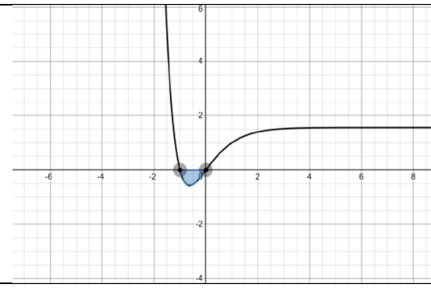

Figure 3(a)
Graph for
$$\int_{-1}^{0}\left(a\left(\frac{1-r^n}{1-r}\right)+dr\left(\frac{(n-1)r^n-nr^{n-1}+1}{(1-r)^2}\right)\right)dn$$
showing area of S_n between x- axis and the interval [-1,0] for $r = 0.2$

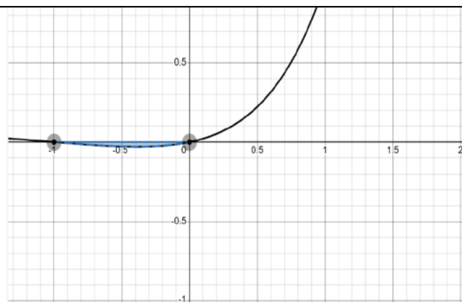

Figure 3(b)
Graph for
$$\int_{-1}^{0}\left(a\left(\frac{1-r^n}{1-r}\right)+dr\left(\frac{(n-1)r^n-nr^{n-1}+1}{(1-r)^2}\right)\right)dn$$
showing area of S_n between x- axis and theinterval [-1,0] for $r = 0.8$

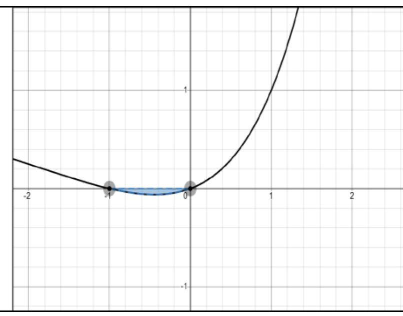

Figure 3(c)
Graph for
$\int_{-1}^{0}\left(a\left(\frac{1-r^n}{1-r}\right)+dr\left(\frac{(n-1)r^n-nr^{n-1}+1}{(1-r)^2}\right)\right)dn$showing area of S_n between x- axis and the interval [-1,0] for $r = 2$

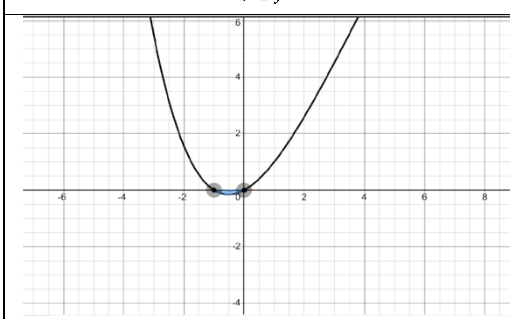

Figure 3(d)
Graph for
$\int_{-1}^{0}\left(a\left(\frac{1-r^n}{1-r}\right)+dr\left(\frac{(n-1)r^n-nr^{n-1}+1}{(1-r)^2}\right)\right)dn$ showing area of S_n between x- axis and the interval [-1,0] for $r = 4$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

Figure 4 Pascal's Triangle

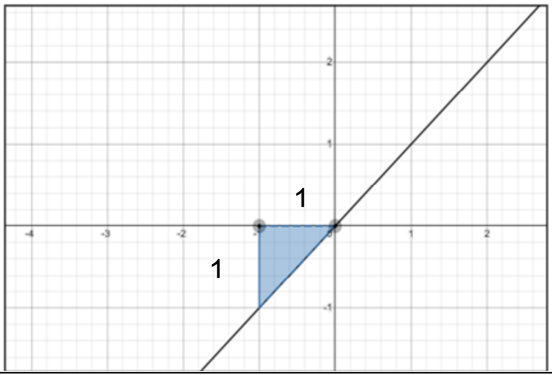

Figure 5
Graph for$\int_{-1}^{0} \mathbf{n}\,\mathbf{dn}$
howing area of $\mathbf{S_n}$*between x- axis and the interval [-1,0]*

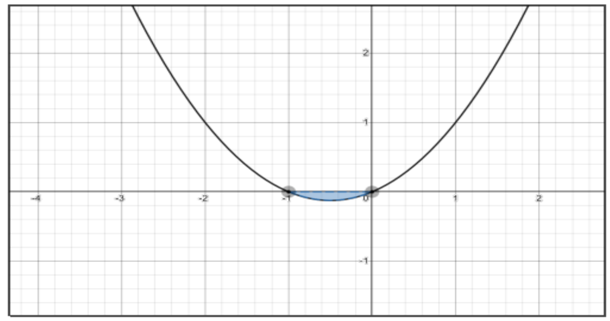

Figure 6
Graph of$\int_{-1}^{0}(\frac{\mathbf{n}}{2}(\mathbf{n}+\mathbf{1})\mathbf{dn}$
showing area of S_n*between x- axis and the interval [-1,0]*

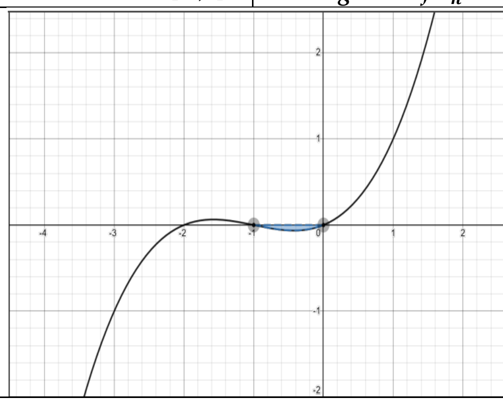

Figure 7
Graph of $\int_{-1}^{0}\left(\frac{\mathbf{n^3+3n^2+2n}}{\mathbf{6}}\right)\mathbf{dn}$
showing area of $S_n$between x- axis and the interval [-1,0]