Plouffe's Constant

N.B. A detailed on-line essay by S. Finch was the starting point for this entry.

Define the function

$$\rho(x) \equiv \begin{cases} 1 & \text{for } x < 0 \\ 0 & \text{for } x \geq 0. \end{cases}$$

Let

$$a_n = \sin(2^n) = \begin{cases} \sin 1 & \text{for } n = 0 \\ 2a_0\sqrt{1-a_0{}^2} & \text{for } n = 1 \\ 2a_{n-1}(1-2a_{n-2}{}^2) & \text{for } n \geq 2, \end{cases}$$

then

$$\sum_{n=0}^{\infty} \frac{\rho(a_n)}{2^{n+1}} = \frac{1}{2\pi}.$$

For

$$b_n = \cos(2^n) = \begin{cases} \cos 1 & \text{for } n = 0 \\ 2b_{n-1}{}^2 - 1 & \text{for } n \geq 1, \end{cases}$$

and

$$\sum_{n=0}^{\infty} \frac{\rho(b_n)}{2^{n+1}} = 0.4756260767\ldots.$$

Letting

$$c_n = \tan(2^n) = \begin{cases} \tan 1 & \text{for } n = 0 \\ \frac{2c_{n-1}}{1-c_{n-1}^2} & \text{for } n \geq 1, \end{cases}$$

then

$$\sum_{n=0}^{\infty} \frac{\rho(c_n)}{2^{n+1}} = \frac{1}{\pi}.$$

Plouffe asked if the above processes could be ``inverted.'' He considered

$$\begin{aligned} \alpha_n &= \sin(2^n \sin^{-1} \tfrac{1}{2}) \\ &= \begin{cases} \frac{1}{2} & \text{for } n = 0 \\ \frac{1}{2}\sqrt{3} & \text{for } n = 1 \\ 2\alpha_{n-1}(1 - 2\alpha_{n-2}^2) & \text{for } n \geq 2, \end{cases} \end{aligned}$$

giving

$$\sum_{n=0}^{\infty} \frac{\rho(\alpha_n)}{2^{n+1}} = \tfrac{1}{12},$$

and

$$\beta_n = \cos(2^n \cos^{-1} \tfrac{1}{2}) = \begin{cases} \frac{1}{2} & \text{for } n = 0 \\ 2\beta_{n-1}^2 - 1 & \text{for } n \geq 1, \end{cases}$$

giving

$$\sum_{n=0}^{\infty} \frac{\rho(\beta_n)}{2^{n+1}} = \tfrac{1}{2},$$

and

$$\gamma_n = \tan(2^n \tan^{-1} \tfrac{1}{2}) = \begin{cases} \frac{1}{2} & \text{for } n = 0 \\ \frac{2\gamma_{n-1}}{1-\gamma_{n-1}^2} & \text{for } n \geq 1, \end{cases}$$

giving

$$\sum_{n=0}^{\infty} \frac{\rho(\alpha_n)}{2^{n+1}} = \frac{1}{\pi}\tan^{-1}(\tfrac{1}{2}).$$

The latter is known as Plouffe's constant.

Borwein and Girgensohn (1995) extended Plouffe's γ_n to *arbitrary* Real *x*, showing that if

$$\xi_n = \tan(2^n \tan^{-1} x) = \begin{cases} x & \text{for } n = 0 \\ \frac{2\xi_{n-1}}{1-\xi_{n-1}^2} & \text{for } n \geq 1 \\ & \quad \text{and } |\xi_{n-1}| \neq 1 \\ -\infty & \text{for } n \geq 1 \\ & \quad \text{and } |\xi_{n-1}| = 1, \end{cases}$$

then

$$\sum_{n=0}^{\infty} \frac{\rho(\xi_n)}{2^{n+1}} = \begin{cases} \frac{\tan^{-1} x}{\pi} & \text{for } x \geq 0 \\ 1 + \frac{\tan^{-1} x}{\pi} & \text{for } x < 0. \end{cases}$$

Borwein and Girgensohn (1995) also give much more general recurrences and formulas.

References

Borwein, J. M. and Girgensohn, R. ``Addition Theorems and Binary Expansions.'' *Canad. J. Math.* **47**, 262-273, 1995.

Finch, S. ``Favorite Mathematical Constants.'' http://www.mathsoft.com/asolve/constant/plff/plff.html

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