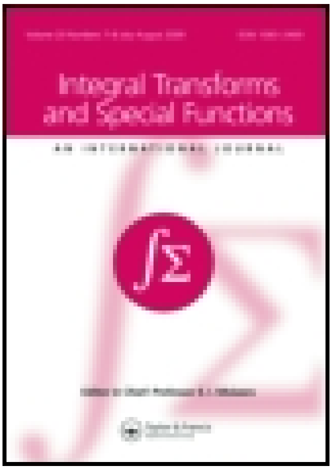

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Several BBP-type formulas for π

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Several BBP-type formulas for π

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In terms of the integral method, we establish two Bailey-Borwein-Plouffe-type (BBP-type) formulas for π with a free parameter. Some BBP-type formulas for π without free parameters are also derived in the same way.

Keywords: integral method; golden ratio; BBP-type formulas for π

2010 Mathematics Subject Classification: Primary: 65B10; Secondary: 40A15

1. Introduction

In 1996, Bailey and Borwein [1] discovered, with the aid of PSLQ algorithm where PS refers to partial sums of squares and LQ is a lower trapezoidal orthogonal decomposition, the surprising Bailey-Borwein-Plouffe-type (BBP-type) formula for π:

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k}\left\{\frac{4}{8k+1}-\frac{2}{8k+4}-\frac{1}{8k+5}-\frac{1}{8+6}\right\}$$

which can be used to generate the nth digit of base-16 for π without computing any prior digits. Subsequently, Adamchik and Wagon [2] found the extension of it:

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k}\left\{\frac{4+8x}{8k+1}-\frac{8x}{8k+2}-\frac{4x}{8k+3}-\frac{2+8x}{8k+4}-\frac{1+2x}{8k+5}-\frac{1+2x}{8k+6}+\frac{x}{8k+7}\right\}.$$

In 2008, Chan [3] gave several interesting π-formulas related to the golden ratio $\phi := (\sqrt{5}+1)/2$. Two of them are displayed as follows:

$$\frac{2\pi\phi^2}{5\sqrt{5-2\sqrt{5}}} = \sum_{k=0}^{\infty} \frac{1}{\phi^{5k}}\left\{\frac{\phi^3}{5k+1}+\frac{\phi}{5k+2}-\frac{1}{5k+3}-\frac{1}{5k+4}\right\},$$

$$\frac{16\pi\phi^2}{5\sqrt{5-2\sqrt{5}}} = \sum_{k=0}^{\infty} \frac{1}{(2\phi)^{5k}}\left\{\frac{8\phi^3}{5k+1}+\frac{4\phi}{5k+2}-\frac{2}{5k+3}-\frac{1}{5k+4}\right\}.$$

Further π-formulas of BBP-type can be seen in the papers.[4–12]

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Inspired by these works, we shall establish several BBP-type formulas for π in accordance with the integral method. Three of them are laid out as follows:

$$\begin{aligned}8\pi &= \sum_{k=0}^{\infty}\left(-\frac{1}{64}\right)^k\left\{\frac{32x}{12k+1}+\frac{32(1-x)}{12k+2}+\frac{8(3-4x)}{12k+3}-\frac{8x}{12k+5}\right.\\&\quad\left.+\frac{4(1-4x)}{12k+6}-\frac{4x}{12k+7}+\frac{3-4x}{12k+9}+\frac{2(1-x)}{12k+10}+\frac{x}{12k+11}\right\},\\\frac{\pi}{3\sqrt{3}} &= \sum_{k=0}^{\infty}\frac{1}{\phi^{6k}}\left\{\frac{\phi^{-1}}{6k+1}-\frac{\phi^{-5}}{6k+5}\right\},\\\frac{4\pi}{3\sqrt{3}} &= \sum_{k=0}^{\infty}\frac{1}{\phi^{12k}}\left\{\frac{9-3\sqrt{5}}{6k+1}+\frac{7-3\sqrt{5}}{6k+2}-\frac{47-21\sqrt{5}}{6k+4}-\frac{369-165\sqrt{5}}{6k+5}\right\}.\end{aligned}$$

The structure of the paper is arranged as follows. We shall deduce two BBP-type formulas for π with a free parameter in Section 2. Some BBP-type formulas for π without free parameters are also offered in Section 3.

2. Two BBP-type formulas for π with a free parameter

THEOREM 1 *For a complex number x, there holds the BBP-type formula for π :*

$$\begin{aligned}8\pi &= \sum_{k=0}^{\infty}\left(-\frac{1}{64}\right)^k\left\{\frac{32x}{12k+1}+\frac{32(1-x)}{12k+2}+\frac{8(3-4x)}{12k+3}-\frac{8x}{12k+5}\right.\\&\quad\left.+\frac{4(1-4x)}{12k+6}-\frac{4x}{12k+7}+\frac{3-4x}{12k+9}+\frac{2(1-x)}{12k+10}+\frac{x}{12k+11}\right\}.\end{aligned}$$

Proof Letting $f(x,z)$ stand for the expression

$$\begin{aligned}f(x,z) &= 32x+32(1-x)z+8(3-4x)z^2-8xz^4+4(1-4x)z^5-4xz^6\\&\quad+(3-4x)z^8+2(1-x)z^9+xz^{10},\end{aligned}$$

we have the relation

$$\begin{aligned}\int_0^1\frac{f(x,z)}{1+z^{12}/64}\,\mathrm{d}z &= \int_0^1\sum_{k=0}^{\infty}\left(-\frac{z^{12}}{64}\right)^k f(x,z)\,\mathrm{d}z\\&= \sum_{k=0}^{\infty}\int_0^1\left(-\frac{z^{12}}{64}\right)^k f(x,z)\,\mathrm{d}z\\&= \sum_{k=0}^{\infty}\left(-\frac{1}{64}\right)^k\left\{\frac{32x}{12k+1}+\frac{32(1-x)}{12k+2}+\frac{8(3-4x)}{12k+3}-\frac{8x}{12k+5}\right.\\&\quad\left.+\frac{4(1-4x)}{12k+6}-\frac{4x}{12k+7}+\frac{3-4x}{12k+9}+\frac{2(1-x)}{12k+10}+\frac{x}{12k+11}\right\}.\end{aligned}\tag{2.1}$$

Setting $g(z) = (2+2z+z^2)(4+4z+2z^2+2z^3+z^4)$, we can proceed as follows:

$$\begin{aligned}\int_0^1 \frac{f(x,z)}{1+z^{12}/64}\,\mathrm{d}z &= 64\int_0^1 \frac{\{4x+4(1-3x)z+(12x-5)z^2+2(1-3x)z^3+xz^4\}g(z)}{(2-2z+z^2)(4-4z+2z^2-2z^3+z^4)g(z)}\,\mathrm{d}z \\ &= 64\int_0^1 \frac{4x+4(1-3x)z+(12x-5)z^2+2(1-3x)z^3+xz^4}{(2-2z+z^2)(4-4z+2z^2-2z^3+z^4)}\,\mathrm{d}z \\ &= 32\int_0^1 \left\{\frac{1}{2-2z+z^2}+\frac{(2x-1)(2-4z+z^2)}{4-4z+2z^2-2z^3+z^4}\right\}\mathrm{d}z \\ &= 32\left\{\arctan(z-1)+(2x-1)\arctan\frac{z^2-z}{z-2}\right\}_0^1 \\ &= 8\pi. \end{aligned} \tag{2.2}$$

Then, Theorem 1 is obtained by combining (2.1) with (2.2). ∎

Corollary 2 ($x=0$ in Theorem 1)

$$\pi = \sum_{k=0}^{\infty}\left(-\frac{1}{64}\right)^k\left\{\frac{4}{12k+2}+\frac{3}{12k+3}+\frac{1/2}{12k+6}+\frac{3/8}{12k+9}+\frac{1/4}{12k+10}\right\}.$$

Corollary 3 ($x=1$ in Theorem 1)

$$\pi = \sum_{k=0}^{\infty}\left(-\frac{1}{64}\right)^k\left\{\frac{4}{12k+1}-\frac{1}{12k+3}-\frac{1}{12k+5}-\frac{3/2}{12k+6}-\frac{1/2}{12k+7} -\frac{1/8}{12k+9}+\frac{1/8}{12k+11}\right\}.$$

Corollary 4 ($x=\frac{3}{4}$ in Theorem 1)

$$\pi = \sum_{k=0}^{\infty}\left(-\frac{1}{64}\right)^k\left\{\frac{3}{12k+1}+\frac{1}{12k+2}-\frac{3/4}{12k+5}-\frac{1}{12k+6}-\frac{3/8}{12k+7} +\frac{1/16}{12k+10}+\frac{3/32}{12k+11}\right\}.$$

Theorem 5 *For a complex number x, there holds the BBP-type formula for π :*

$$\frac{2(1+x)}{3\sqrt{3}}\pi = \sum_{k=0}^{\infty}\frac{1}{\phi^{12k}}\left\{\frac{(\sqrt{5}-1)x}{12k+1}+\frac{9-3\sqrt{5}}{12k+2}+\frac{7-3\sqrt{5}}{12k+4}-\frac{(5\sqrt{5}-11)x}{12k+5} +\frac{(13\sqrt{5}-29)x}{12k+7}-\frac{47-21\sqrt{5}}{12k+8}-\frac{369-165\sqrt{5}}{12k+10}-\frac{(89\sqrt{5}-199)x}{12k+11}\right\}.$$

Proof Letting $u(x,z)$ stand for the expression

$$u(x,z) = (\sqrt{5}-1)x + (9-3\sqrt{5})z + (7-3\sqrt{5})z^3 - (5\sqrt{5}-11)xz^4 + (13\sqrt{5}-29)xz^6 - (47-21\sqrt{5})z^7 - (369-165\sqrt{5})z^9 - (89\sqrt{5}-199)xz^{10},$$

it is not difficult to show

$$\begin{aligned}\int_0^1 \frac{u(x,z)}{1-(z/\phi)^{12}}\,dz &= \int_0^1 \sum_{k=0}^{\infty}\left(\frac{z}{\phi}\right)^{12k} u(x,z)\,dz\\ &= \sum_{k=0}^{\infty}\int_0^1 \left(\frac{z}{\phi}\right)^{12k} u(x,z)\,dz\\ &= \sum_{k=0}^{\infty}\frac{1}{\phi^{12k}}\left\{\frac{(\sqrt{5}-1)x}{12k+1}+\frac{9-3\sqrt{5}}{12k+2}+\frac{7-3\sqrt{5}}{12k+4}-\frac{(5\sqrt{5}-11)x}{12k+5}\right.\\ &\quad\left.+\frac{(13\sqrt{5}-29)x}{12k+7}-\frac{47-21\sqrt{5}}{12k+8}-\frac{369-165\sqrt{5}}{12k+10}-\frac{(89\sqrt{5}-199)x}{12k+11}\right\}.\end{aligned} \tag{2.3}$$

Setting $v(z)=1-z^4$, the integral can be calculated as follows:

$$\begin{aligned}\int_0^1 \frac{u(x,z)}{1-(z/\phi)^{12}}\,dz &= 2\int_0^{1/\phi}\frac{(x+3z+z^3+3z^5+xz^6)v(z)}{(1+z^4+z^8)v(z)}\,dz\\ &= 2\int_0^{1/\phi}\frac{x+3z+z^3+3z^5+xz^6}{1+z^4+z^8}\,dz\\ &= 2x\int_0^{1/\phi}\frac{1+z^6}{1+z^4+z^8}\,dz + 2\int_0^{1/\phi}\frac{3z+z^3+3z^5}{1+z^4+z^8}\,dz\\ &= 2x\int_0^{1/\phi}\frac{1+z^2}{1+z^2+z^4}\,dz + \int_0^{1/\phi^2}\frac{3+z+3z^2}{1+z^2+z^4}\,dz\\ &= x\int_0^{1/\phi}\left\{\frac{1}{1+z+z^2}+\frac{1}{1-z+z^2}\right\}dz\\ &\quad+\int_0^{1/\phi^2}\left\{\frac{1}{1+z+z^2}+\frac{2}{1-z+z^2}\right\}dz\\ &= \frac{2x}{\sqrt{3}}\left\{\arctan\frac{2z+1}{\sqrt{3}}+\arctan\frac{2z-1}{\sqrt{3}}\right\}_0^{1/\phi}\\ &\quad+\frac{2}{\sqrt{3}}\left\{\arctan\frac{2z+1}{\sqrt{3}}+2\arctan\frac{2z-1}{\sqrt{3}}\right\}_0^{1/\phi^2}\\ &= \frac{2(1+x)}{3\sqrt{3}}\pi.\end{aligned} \tag{2.4}$$

Then, we get Theorem 5 by substituting (2.3) into (2.4). ■

COROLLARY 6 ($x \to \infty$ in Theorem 5)

$$\frac{\pi}{3\sqrt{3}} = \sum_{k=0}^{\infty} \frac{1}{\phi^{6k}} \left\{ \frac{\phi^{-1}}{6k+1} - \frac{\phi^{-5}}{6k+5} \right\}.$$

COROLLARY 7 ($x = 0$ in Theorem 5)

$$\frac{4\pi}{3\sqrt{3}} = \sum_{k=0}^{\infty} \frac{1}{\phi^{12k}} \left\{ \frac{9-3\sqrt{5}}{6k+1} + \frac{7-3\sqrt{5}}{6k+2} - \frac{47-21\sqrt{5}}{6k+4} - \frac{369-165\sqrt{5}}{6k+5} \right\}.$$

3. Some BBP-type formulas for π without free parameters

PROPOSITION 8 (A BBP-type formula for π)

$$\frac{\sqrt{7}+1}{6\sqrt{3}}\pi = \sum_{k=0}^{\infty} \frac{(-1)^{\binom{k}{2}}}{2k+1} \left(\frac{3}{4+\sqrt{7}} \right)^k.$$

Proof Firstly, we have the relation

$$\begin{aligned}
\int_0^1 \frac{9+3(4-\sqrt{7})z^2}{9+(23-8\sqrt{7})z^4}\,\mathrm{dz} &= \int_0^1 \frac{1+((4-\sqrt{7})/3)z^2}{1+((23-8\sqrt{7})/9)z^4}\,\mathrm{dz} \\
&= \int_0^1 \sum_{k=0}^{\infty} \left(\frac{8\sqrt{7}-23}{9} z^4 \right)^k \left\{ 1 + \frac{4-\sqrt{7}}{3} z^2 \right\} \mathrm{dz} \\
&= \sum_{k=0}^{\infty} \int_0^1 \left(\frac{8\sqrt{7}-23}{9} z^4 \right)^k \left\{ 1 + \frac{4-\sqrt{7}}{3} z^2 \right\} \mathrm{dz} \\
&= \sum_{k=0}^{\infty} \left(\frac{8\sqrt{7}-23}{9} \right)^k \left\{ \frac{1}{4k+1} + \frac{(4-\sqrt{7})/3}{4k+3} \right\} \\
&= \sum_{k=0}^{\infty} (-1)^k \left(\frac{3}{4+\sqrt{7}} \right)^{2k} \left\{ \frac{1}{4k+1} + \frac{3/(4+\sqrt{7})}{4k+3} \right\} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^{\binom{k}{2}}}{2k+1} \left(\frac{3}{4+\sqrt{7}} \right)^k. \qquad (3.1)
\end{aligned}$$

Secondly, we can proceed as follows:

$$\begin{aligned}
&\int_0^1 \frac{9+3(4-\sqrt{7})z^2}{9+(23-8\sqrt{7})z^4}\,\mathrm{dz} \\
&= \frac{3}{2} \int_0^1 \left\{ \frac{1}{3+(\sqrt{21}-\sqrt{3})z+(4-\sqrt{7})z^2} + \frac{1}{3-(\sqrt{21}-\sqrt{3})z+(4-\sqrt{7})z^2} \right\} \mathrm{dz} \\
&= \frac{\sqrt{3}}{\sqrt{7}-1} \left\{ \arctan \frac{\sqrt{7}+\sqrt{3}-1}{\sqrt{3}} + \arctan \frac{\sqrt{7}-\sqrt{3}-1}{\sqrt{3}} \right\}
\end{aligned}$$

$$= \frac{\sqrt{3}}{\sqrt{7}-1}\arctan\sqrt{3}$$
$$= \frac{\sqrt{7}+1}{6\sqrt{3}}\pi. \tag{3.2}$$

Then, Proposition 8 is achieved by combining (3.1) with (3.2). ■

PROPOSITION 9 (A BBP-type formula for π)

$$\frac{\sqrt{3}+\sqrt{5}}{12}\pi = \sum_{k=0}^{\infty}\frac{(-1)^{\binom{k}{2}}}{2k+1}\left(\frac{1}{4+\sqrt{15}}\right)^k.$$

Proof On one hand, it is not difficult to show

$$\begin{aligned}\int_0^1 \frac{1+(4-\sqrt{15})z^2}{1+(31-8\sqrt{15})z^4}\,\mathrm{d}z &= \int_0^1 \sum_{k=0}^{\infty}\{(8\sqrt{15}-31)z^4\}^k\{1+(4-\sqrt{15})z^2\}\,\mathrm{d}z\\
&= \sum_{k=0}^{\infty}\int_0^1 \{(8\sqrt{15}-31)z^4\}^k\{1+(4-\sqrt{15})z^2\}\,\mathrm{d}z\\
&= \sum_{k=0}^{\infty}(8\sqrt{15}-31)^k\left\{\frac{1}{4k+1}+\frac{4-\sqrt{15}}{4k+3}\right\}\\
&= \sum_{k=0}^{\infty}(-1)^k\left(\frac{1}{4+\sqrt{15}}\right)^{2k}\left\{\frac{1}{4k+1}+\frac{1/(4+\sqrt{15})}{4k+3}\right\}\\
&= \sum_{k=0}^{\infty}\frac{(-1)^{\binom{k}{2}}}{2k+1}\left(\frac{1}{4+\sqrt{15}}\right)^k. \end{aligned}\tag{3.3}$$

On the other hand, the integral can be evaluated as follows:

$$\begin{aligned}&\int_0^1 \frac{1+(4-\sqrt{15})z^2}{1+(31-8\sqrt{15})z^4}\,\mathrm{d}z\\
&= \frac{1}{2}\int_0^1\left\{\frac{1}{1+(\sqrt{5}-\sqrt{3})z+(4-\sqrt{15})z^2}+\frac{1}{1-(\sqrt{5}-\sqrt{3})z+(4-\sqrt{15})z^2}\right\}\mathrm{d}z\\
&= \frac{1}{\sqrt{5}-\sqrt{3}}\left\{\arctan(\sqrt{5}-\sqrt{3}+1)+\arctan(\sqrt{5}-\sqrt{3}-1)\right\}\\
&= \frac{1}{\sqrt{5}-\sqrt{3}}\arctan\frac{1}{\sqrt{3}}\\
&= \frac{\sqrt{3}+\sqrt{5}}{12}\pi. \end{aligned}\tag{3.4}$$

Then, we gain Proposition 9 by substituting (3.3) into (3.4). ■

PROPOSITION 10 (A BBP-type formula for π)

$$\frac{\pi}{3\sqrt{3}} = \sum_{k=0}^{\infty}\frac{1}{(649+180\sqrt{13})^k}\left\{\frac{\sqrt{13}-3}{6k+1}-\frac{109\sqrt{13}-393}{6k+5}\right\}.$$

Proof Firstly, we have the relation

$$\begin{aligned}
\int_0^1 \frac{(119+33\sqrt{13})-2z^4}{2-(1298-360\sqrt{13})z^6}\,\mathrm{d}z &= \int_0^1 \frac{(119+33\sqrt{13})/2-z^4}{1-(649-180\sqrt{13})z^6}\,\mathrm{d}z \\
&= \int_0^1 \sum_{k=0}^{\infty}\{(649-180\sqrt{13})z^6\}^k \left\{\frac{119+33\sqrt{13}}{2}-z^4\right\}\mathrm{d}z \\
&= \sum_{k=0}^{\infty}\int_0^1 \{(649-180\sqrt{13})z^6\}^k \left\{\frac{119+33\sqrt{13}}{2}-z^4\right\}\mathrm{d}z \\
&= \sum_{k=0}^{\infty}(649-180\sqrt{13})^k \left\{\frac{(119+33\sqrt{13})/2}{6k+1}-\frac{1}{6k+5}\right\} \\
&= \sum_{k=0}^{\infty}\frac{1}{(649+180\sqrt{13})^k} \left\{\frac{(119+33\sqrt{13})/2}{6k+1}-\frac{1}{6k+5}\right\}.
\end{aligned} \tag{3.5}$$

Secondly, we can proceed as follows:

$$\begin{aligned}
&\int_0^1 \frac{(119+33\sqrt{13})-2z^4}{2-(1298-360\sqrt{13})z^6}\,\mathrm{d}z \\
&= \frac{119+33\sqrt{13}}{2}\int_0^1 \frac{2+(11-3\sqrt{13})z^2}{2+(11-3\sqrt{13})z^2+(119-33\sqrt{13})z^4}\,\mathrm{d}z \\
&= \frac{119+33\sqrt{13}}{2}\int_0^1 \frac{1}{2+(\sqrt{13}-3)z+(11-3\sqrt{13})z^2}\,\mathrm{d}z \\
&\quad + \frac{119+33\sqrt{13}}{2}\int_0^1 \frac{1}{2-(\sqrt{13}-3)z+(11-3\sqrt{13})z^2}\,\mathrm{d}z \\
&= \frac{393+109\sqrt{13}}{2\sqrt{3}}\left\{\arctan\frac{7+\sqrt{13}}{3\sqrt{3}+\sqrt{39}}+\arctan\frac{1-\sqrt{13}}{3\sqrt{3}+\sqrt{39}}\right\} \\
&= \frac{393+109\sqrt{13}}{2\sqrt{3}}\arctan\frac{1}{\sqrt{3}} \\
&= \frac{393+109\sqrt{13}}{12\sqrt{3}}\pi.
\end{aligned} \tag{3.6}$$

Then, Proposition 10 is obtained by combining (3.5) with (3.6). ■

PROPOSITION 11 (A BBP-type formula for π)

$$\frac{\pi}{19\sqrt{21}-87} = \sum_{k=0}^{\infty}\frac{1}{(55+12\sqrt{21})^k}\left\{\frac{23+5\sqrt{21}}{6k+1}-\frac{2}{6k+5}\right\}.$$

Proof On one hand, it is not difficult to show

$$\begin{aligned}
\int_0^1 \frac{(23+5\sqrt{21})-2z^4}{2-(110-24\sqrt{21})z^6}\,\mathrm{d}z &= \int_0^1 \frac{(23+5\sqrt{21})/2-z^4}{1-(55-12\sqrt{21})z^6}\,\mathrm{d}z \\
&= \int_0^1 \sum_{k=0}^{\infty}\{(55-12\sqrt{21})z^6\}^k \left\{\frac{23+5\sqrt{21}}{2}-z^4\right\}\mathrm{d}z \\
&= \sum_{k=0}^{\infty}\int_0^1 \{(55-12\sqrt{21})z^6\}^k \left\{\frac{23+5\sqrt{21}}{2}-z^4\right\}\mathrm{d}z \\
&= \sum_{k=0}^{\infty}(55-12\sqrt{21})^k \left\{\frac{(23+5\sqrt{21})/2}{6k+1}-\frac{1}{6k+5}\right\} \\
&= \sum_{k=0}^{\infty}\frac{1}{(55+12\sqrt{21})^k} \left\{\frac{(23+5\sqrt{21})/2}{6k+1}-\frac{1}{6k+5}\right\}. \qquad (3.7)
\end{aligned}$$

On the other hand, the integral can be calculated as follows:

$$\begin{aligned}
\int_0^1 \frac{(23+5\sqrt{21})-2z^4}{2-(110-24\sqrt{21})z^6}\,\mathrm{d}z &= \frac{23+5\sqrt{21}}{2}\int_0^1 \frac{2+(5-\sqrt{21})z^2}{2+(5-\sqrt{21})z^2+(23-5\sqrt{21})z^4}\,\mathrm{d}z \\
&= \frac{23+5\sqrt{21}}{2}\int_0^1 \frac{1}{2+(\sqrt{7}-\sqrt{3})z+(5-\sqrt{21})z^2}\,\mathrm{d}z \\
&\quad + \frac{23+5\sqrt{21}}{2}\int_0^1 \frac{1}{2-(\sqrt{7}-\sqrt{3})z+(5-\sqrt{21})z^2}\,\mathrm{d}z \\
&= \frac{29\sqrt{3}+19\sqrt{7}}{2\sqrt{3}}\left\{\arctan\frac{4+\sqrt{3}+\sqrt{7}}{3+\sqrt{21}}+\arctan\frac{4-\sqrt{3}-\sqrt{7}}{3+\sqrt{21}}\right\} \\
&= \frac{29\sqrt{3}+19\sqrt{7}}{2\sqrt{3}}\arctan 1 \\
&= \frac{29\sqrt{3}+19\sqrt{7}}{8\sqrt{3}}\pi. \qquad (3.8)
\end{aligned}$$

Then, we get Proposition 11 by substituting (3.7) into (3.8). ■

PROPOSITION 12 (A BBP-type formula for π)

$$36\sqrt{3}\pi = \sum_{k=0}^{\infty}\frac{1}{(77+20\sqrt{13})^k}\left\{\frac{81\sqrt{13}-81}{6k+1}-\frac{63-9\sqrt{13}}{6k+2}+\frac{31-7\sqrt{13}}{6k+4}-\frac{19\sqrt{13}-61}{6k+5}\right\}.$$

Proof Firstly, we have the relation

$$\begin{aligned}
&\int_0^1 \frac{81(\sqrt{13}-1)-9(7-\sqrt{13})z+(31-7\sqrt{13})z^3-(19\sqrt{13}-61)z^4}{1-z^6/(77+20\sqrt{13})}\,\mathrm{d}z \\
&= \int_0^1 \sum_{k=0}^{\infty}\left(\frac{z^6}{77+20\sqrt{13}}\right)^k \Big\{81\left(\sqrt{13}-1\right)-9\left(7-\sqrt{13}\right)z \\
&\qquad + \left(31-7\sqrt{13}\right)z^3-\left(19\sqrt{13}-61\right)z^4\Big\}\,\mathrm{d}z
\end{aligned}$$

$$= \sum_{k=0}^{\infty} \int_0^1 \left(\frac{z^6}{77 + 20\sqrt{13}} \right)^k \left\{ 81\left(\sqrt{13} - 1\right) - 9\left(7 - \sqrt{13}\right) z \right.$$
$$\left. + \left(31 - 7\sqrt{13}\right) z^3 - \left(19\sqrt{13} - 61\right) z^4 \right\} \,\mathrm{d}z$$
$$= \sum_{k=0}^{\infty} \frac{1}{(77 + 20\sqrt{13})^k} \left\{ \frac{81\sqrt{13} - 81}{6k+1} - \frac{63 - 9\sqrt{13}}{6k+2} + \frac{31 - 7\sqrt{13}}{6k+4} - \frac{19\sqrt{13} - 61}{6k+5} \right\}. \tag{3.9}$$

Secondly, we can proceed as follows:

$$\int_0^1 \frac{81(\sqrt{13} - 1) - 9(7 - \sqrt{13})z + (31 - 7\sqrt{13})z^3 - (19\sqrt{13} - 61)z^4}{1 - z^6/(77 + 20\sqrt{13})} \,\mathrm{d}z$$
$$= 162 \int_0^{2/(1+\sqrt{13})} \frac{3 - z + z^3 - 3z^4}{1 - z^6} \,\mathrm{d}z$$
$$= 162 \int_0^{2/(1+\sqrt{13})} \frac{3 - z + 3z^2}{1 + z^2 + z^4} \,\mathrm{d}z$$
$$= 162 \int_0^{2/(1+\sqrt{13})} \left\{ \frac{2}{1 + z + z^2} + \frac{1}{1 - z + z^2} \right\} \mathrm{d}z$$
$$= \frac{324}{\sqrt{3}} \left\{ 2 \arctan \frac{\sqrt{13} + 2}{3\sqrt{3}} + \arctan \frac{\sqrt{13} - 4}{3\sqrt{3}} - \arctan \frac{1}{\sqrt{3}} \right\}$$
$$= 36\sqrt{3}\pi. \tag{3.10}$$

Then Proposition 12 is achieved by combining (3.9) with (3.10). ■

Proposition 13 (A BBP-type formula for π)

$$\sqrt{\frac{7 + 5\sqrt{2}}{2}}\pi = \sum_{k=0}^{\infty} \frac{1}{(17 + 12\sqrt{2})^k} \left\{ \frac{4 + 3\sqrt{2}}{8k+1} + \frac{2 - \sqrt{2}}{8k+3} - \frac{3\sqrt{2} - 4}{8k+5} - \frac{2 - \sqrt{2}}{8k+7} \right\}.$$

Proof On one hand, it is not difficult to show

$$\int_0^1 \frac{(4 + 3\sqrt{2}) + (2 - \sqrt{2})z^2 - (3\sqrt{2} - 4)z^4 - (2 - \sqrt{2})z^6}{1 - z^8/(17 + 12\sqrt{2})} \,\mathrm{d}z$$
$$= \int_0^1 \sum_{k=0}^{\infty} \left(\frac{z^8}{17 + 12\sqrt{2}} \right)^k \{(4 + 3\sqrt{2}) + (2 - \sqrt{2})z^2 - (3\sqrt{2} - 4)z^4 - (2 - \sqrt{2})z^6\} \,\mathrm{d}z$$
$$= \sum_{k=0}^{\infty} \int_0^1 \left(\frac{z^8}{17 + 12\sqrt{2}} \right)^k \{(4 + 3\sqrt{2}) + (2 - \sqrt{2})z^2 - (3\sqrt{2} - 4)z^4 - (2 - \sqrt{2})z^6\} \,\mathrm{d}z$$
$$= \sum_{k=0}^{\infty} \frac{1}{(17 + 12\sqrt{2})^k} \left\{ \frac{4 + 3\sqrt{2}}{8k+1} + \frac{2 - \sqrt{2}}{8k+3} - \frac{3\sqrt{2} - 4}{8k+5} - \frac{2 - \sqrt{2}}{8k+7} \right\}. \tag{3.11}$$

On the other hand, the integral can be evaluated as follows:

$$\begin{aligned}
&\int_0^1 \frac{(4+3\sqrt{2})+(2-\sqrt{2})z^2-(3\sqrt{2}-4)z^4-(2-\sqrt{2})z^6}{1-z^8/(17+12\sqrt{2})}\,dz \\
&=\sqrt{1+\sqrt{2}}\int_0^{1/\sqrt{1+\sqrt{2}}} \frac{(1-z^2)\{(4+3\sqrt{2})+4(1+\sqrt{2})z^2+(4+3\sqrt{2})z^4\}}{1-z^8}\,dz \\
&=\sqrt{1+\sqrt{2}}\int_0^{1/\sqrt{1+\sqrt{2}}} \frac{(4+3\sqrt{2})+4(1+\sqrt{2})z^2+(4+3\sqrt{2})z^4}{1+z^2+z^4+z^6}\,dz \\
&=\sqrt{14+10\sqrt{2}}\int_0^{1/\sqrt{1+\sqrt{2}}} \left\{\frac{1}{1+z^2}+\frac{1}{\sqrt{2}(1+\sqrt{2}z+z^2)}+\frac{1}{\sqrt{2}(1-\sqrt{2}z+z^2)}\right\}dz \\
&=\sqrt{14+10\sqrt{2}}\left\{\arctan\frac{1}{\sqrt{1+\sqrt{2}}}+\arctan\frac{\sqrt{1+\sqrt{2}}+\sqrt{2}}{\sqrt{1+\sqrt{2}}}-\arctan\frac{\sqrt{1+\sqrt{2}}-\sqrt{2}}{\sqrt{1+\sqrt{2}}}\right\} \\
&=\sqrt{\frac{7+5\sqrt{2}}{2}}\pi. \qquad (3.12)
\end{aligned}$$

Then, we gain Proposition 13 by substituting (3.11) into (3.12). ■

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