A set of formulas for primes

by
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December 31, 2018

Abstract

In 1947, W. H. Mills published a paper describing a formula that gives primes : if A = 1.3063778838630806904686144926… then $\lfloor A^{3^n} \rfloor$ is always prime, here $\lfloor x \rfloor$ is the integral part of x. Later in 1951, E. M. Wright published another formula, if $g_0 = \alpha = 1.9287800…$ and $g_{n+1} = 2^{g_n}$ then

$$\lfloor g_n \rfloor = \left\lfloor 2^{\cdots 2^{2^{\alpha}}} \right\rfloor \text{ is always prime.}$$

The primes are uniquely determined by α , the prime sequence is 3, 13, 16381, …
The growth rate of these functions is very high since the fourth term of Wright formula is a 4932 digit prime and the 8'th prime of Mills formula is a 762 digit prime.
A new set of formulas is presented here, giving an arbitrary number of primes minimizing the growth rate. The first one is : if $a_0 = 43.8046877158…$ and $a_{n+1} = (a_n^{\frac{5}{4}})^n$, then if $S(n)$ is the rounded values of a_n, $S(n) = 113, 367, 102217, 1827697, 67201679, 6084503671, ….$ Other exponents can also give primes like 11/10, or 101/100. If a_0 is well chosen then it is conjectured that any exponent > 1 can also give an arbitrary series of primes. When the exponent is 3/2 it is conjectured that all the primes are within a series of trees. The method for obtaining the formulas is explained. All results are empirical.

Résumé

En 1947, W. H. Mills publiait un article montrant une formule qui peut donner un nombre arbitraire de nombres premiers. Si A = 1.3063778838630806904686144926… alors $\lfloor A^{3^n} \rfloor$ donne une suite arbitraire de nombres tous premiers. , ici $\lfloor x \rfloor$ est le plancher de x. Plus tard en 1951, E. M. Wright en proposait une autre, si $g_0 = \alpha = 1.9287800…$ et $g_{n+1} = 2^{g_n}$ alors

$$\lfloor g_n \rfloor = \left\lfloor 2^{\cdots 2^{2^{\alpha}}} \right\rfloor \text{ est toujours premier.}$$

Les premiers consécutifs sont uniquement représentés par α. La suite de premiers est 3, 13, 16381,…
Le taux de croissance de ces 2 fonctions est assez élevé puisque le 4ème terme de la suite de Wright a 4932 chiffres décimaux. La croissance de celle de Mills est moins élevée, le 8ème terme a quand même une taille de 762 chiffres. Une série de formules est présentée ici qui minimise le taux de croissance et qui possède les mêmes propriétés de fournir une suite de premiers de longueur

arbitraire. Si $a_0 = 43.8046877158 \ldots$ et $a_{n+1} = (a_n^{\frac{5}{4}})^n$ alors la suite $S(n) = \{ a_n \}$: l'arrondi de a_n, est une suite de premiers de longueur arbitraire. Ici l'exposant ${}^{5}/_{4}$ peut être abaissé à ${}^{11}/_{10}$, ou même ${}^{101}/_{100}$. Si a_0 est bien choisi il est conjecturé que l'exposant peut être aussi près de 1 que l'on veut. Lorsque l'exposant est $3/2$ il est conjecturé que tous les nombres premiers peuvent être générés par une série d'arbres.

Introduction

The first type of prime formula to consider is for example, given a_0 a real constant > 0 and $a_{n+1} = a(n)10$, if $a_0 = 7.3327334517988679\ldots$ then the sequence $73, 733, 7333, 73327, 733273, \ldots$ is a sequence of primes but fails for obvious reasons after a few terms. If the base is changed to any other fixed size base, taking into account that the average gap between primes is increasing then eventually the process fails to give any more primes.

If we choose a function that grows faster like n^n, we get better results. The best start constant found is $c = 0.265588372943143390897129453665466129 4389\ldots$ giving 19 primes.
But fails at 23 (beginning at n = 3). Here $a_n = \lfloor cn^n \rfloor$.

7
67
829
12391
218723
4455833
102894377
2655883729
75775462379
2368012611049
80440106764817
2951219812933057
116299525867995629
4899240744635092571
219705395187452015923
10449948501874965563651
525445257345556693801913
27848959374722952425334841
1551723179991864497606172809

Again, for the same reasons mentioned earlier, the process fails to go further, no better example was found. The method used is a homemade Monte-Carlo method that uses Simulated Annealing (principe du recuit simulé in French).

If a function grows too slowly, eventually the average gap between primes increases and the process ceases to give any more primes. The next step was to consider formulas like Mills or Wright. The question was then : is there a way to get a useful formula that grows just enough to produce primes ?

If we consider the recurrence $a_{n+1} = {a_n}^2 - a_n + 1$ that arises in the context of Sylvester sequence. The Sylvester sequence is A000058 of the OEIS catalogue and begins with $a_0 = 2$, like this : (2, 3, 7, 43, 1807, 3263443, 10650056950807,), that sequence has the property that

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \frac{1}{1807} + \frac{1}{3263443} + \cdots$$

The natural extension that comes next is: can we choose a_0 so that a_n will always produce primes ? The answer is yes, when $a(0) = 1.6181418093242092 \ldots$ and by using the $\lfloor x \rfloor$ function we get,

$$a(n) = 2, 3, 7, 43, 1811, 3277913, 10744710357637, \ldots$$

The sequence and formula are interesting for one reason the growth rate is quite smaller than the one of Mills and Wright.

A Formula for primes

What if we choose the exponent to be as small as possible? The problem with that last one is that it is stilll growing too fast, $\mathrm{a}(14) = 9.838 \ldots \times 10^{1667}$. The size of primes doubles in length at each step.

The simplest found was a_0= 43.80468771580293481… and using the { x } , rounding to the nearest integer, we get

$$a_{n+1} = {a_n}^{\frac{5}{4}}$$

Now, what if we carefully choose a_0 so that the exponent is smaller, would it work ? Let's try with 11/10 and start with a larger number.

$$a_{n+1} = {a_n}^{\frac{11}{10}}$$

For example when $S_0 = 100000000000000000000000000000049.31221074776345 \ldots$ and the exponent beeing 11/10 then we get the primes :

100000000000000000000000000000049
15848931924611134852021013733923675 3
524807460249772597364312157022725894401
3908408957924020300919472370957356345933709
709904615855287249312898251180594220053400958 13

343811184035069918804446105763101544331290090895233 3
489724690004200094265557071425023036671550364178496540501
...

If we want a smaller starting value then a_0 has to be bigger, I could get a series of primes when $a_0 = 10^{64} + 57 + \varepsilon$, where $0 < \varepsilon < 0.5$ chosen at random. In this case the exponent is

$$a_{n+1} = a_n^{\frac{21}{20}}$$

If we choose $a_0 = 10^{600} + 543 + \varepsilon$ then we get our formula to be.

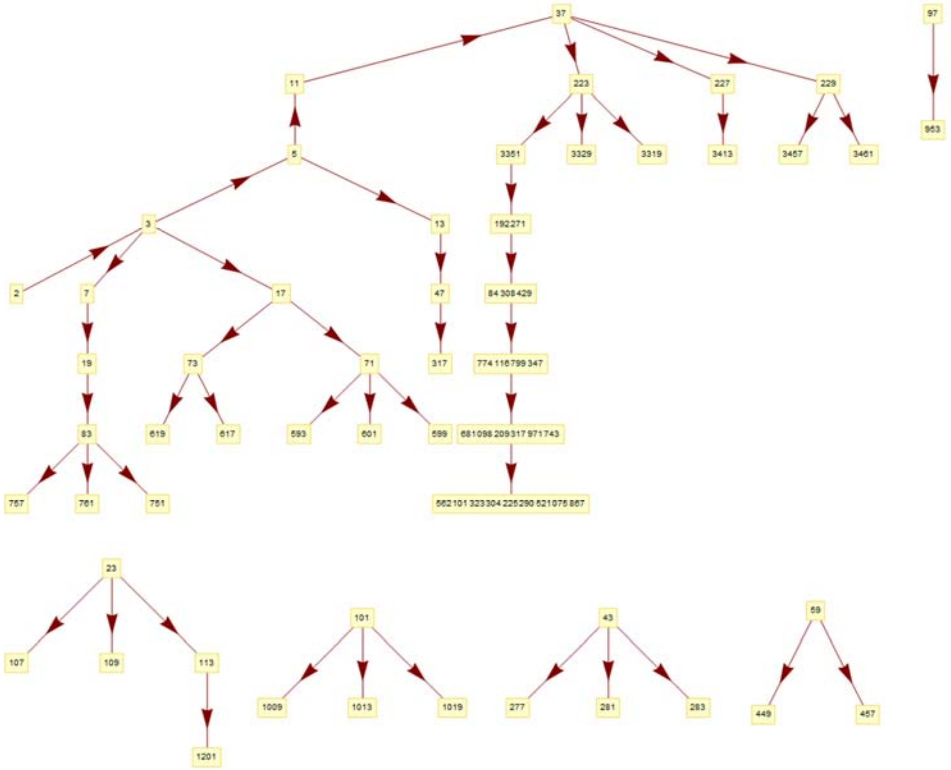
$$a_{n+1} = a_n^{\frac{101}{100}}$$

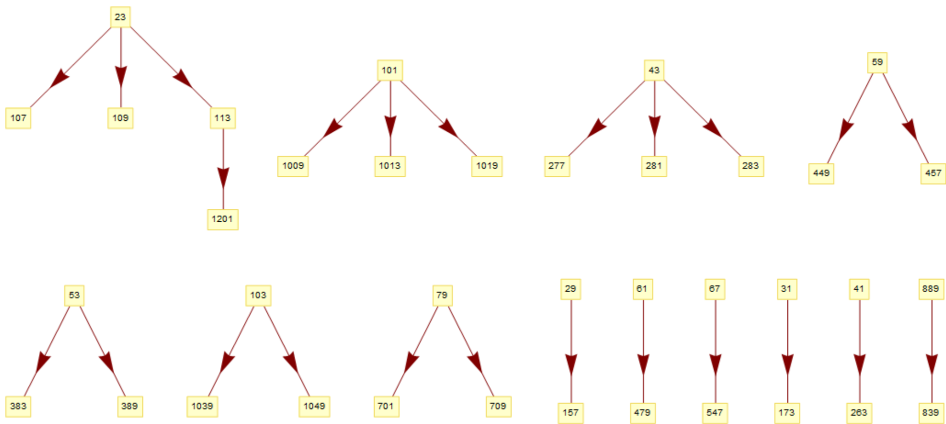
If a(0) = 2.03823915478206876746349086260954825144862477844317361… and the exponent 3/2 then the sequence of primes is :

2
3
5
11
37
223
3331
192271
84308429
774116799347
681098209317971743
562101323304225290104514179
1332667822014585978282511662572214575900 9
153844816227160786960183458743194850623898276519342599327448 9

The natural question that comes next is : Can we generate all the primes with one single exponent ? Here is the tree graph of primes with the exponent 3/2.

Prime trees with the exponent 3/2





Description of the algorithm and method

There are 3 steps

1) First we choose a starting value and exponent (preferably a rational fraction for technical reasons).
2) Use Monte-Carlo method with the Simulated Annealing, in plain english we keep only the values that show primes and ignore the rest. Once we have a series of 4-5 primes we are ready for the next step.
3) We use a formula for forward calculation and backward. The forward calculation is

   Forward : Next smallest prime to { $a(n)^e$ }.

   It is easy to find a probable prime up to thousands of digits. Maple has a limit of about 10000 digits on a Intel core i7 6700K, if I use PFGW I can get a probable prime of 1000000 digits in a matter of minutes.

   Backward : (to check if the formula works)

   Previous prime = solve for x in $x^e - S(n+1)$. Where $S(n+1)$ is the next prime candidate. This is where e needs to be in rational form in order to solve easily in floating point to high precision using Newton-like methods.

Conclusions

There are no proofs of all this, just empirical results. In practical terms, we have now a way to generate an arbitrary series of primes with (so far) a minimal growth function. The formulas are much smaller in growth rate than of the 2 historical results of Mills and Wright. Perhaps there is even a simpler formulation, I did not find anything simpler. In the appendix, the 50'th term of the sequence beginning with $10^{500} + 961$ is given, breaking the record of known series of primes in either a polynomial (46 values) or primes in arithmetic progression (26 values).

Appendix

Value of a(0) for $a_{n+1} = a_n^{\frac{5}{4}}$.

a(0) = (2600 digits)

```
43.8046877158029348185966456256908949508103708713749518407406132875267041950609\
3664963337107816445739567943558075746653930014636424735710790652072233219699229\
3264945523871320676837539601640703027906901627234335816037601910027530786667742\
4699462324971177019386074328699571966485110591669754207733416768610281388237076\
1457089916472910767210053536764714815683021515348888782413463002196590898912153\
7807818768276618328571083354683602327177557419282082356357338008605137724420625\
6355959550708768409691378766548276776271408551872012425050013678129199364572781\
6813484896708864364878618010738012262326340551265300995594932556249327764325520\
8194746104359540189525699560278047888104384271808675933509955483655110570975535\
0785549077202528823776864085546060978256419523714505379764311017583547117340087\
4119475081308042239895607717353189481000696610499835674474037156885696125148581\
8456576179531740469553899959017477684394053845672138911212798487820145291551046\
7629440955925010855475725385626417446232721714485692213455756668913904742940022\
8509137974545537496866153147014899471237034869279337802008797927308327868037503\
8887734734110132599896243006668693634844327103981195681572642575883497205246107\
7249503005611993311614482718587460518670384095172358532168924929832717916742877\
5928083798545071506803190331910539166466084176778289275156307759698474412455449\
4235840260513685727576427182775664320370267582830712839903568563176995646262746\
1416682265277556094918638843788662398254929224611054007569004788919208610982286\
8199056832247214421586610021379123061307988471051671497283110156867428960680180\
5503966626145521465670913389257049550857812576171202023957487357339788493712676\
9426225000014887911004760565168355053802557466312278070529726060791106644597456\
7661803493305130350891168486525370221416972540649352435163491961811066911684870\
8294575729521396968670925609170964827383150968820007346616462008754667125912162\
9887393234505470764134731474624437303908697918037642878970154570903124414003971\
0357272367808690366664091433772132766429665349666355316481741675272445299039148\
2286893771949904585620265483705408090265640475448008548569022696419278537015273\
9052515666046613248284007441714998112109106552831729128554638499405603790836674\
4055864672832545083146225507711246656708938743897521843934561767824939322527151\
9446656406377385343422986304586480251980132774829846894886302934727512320855666\
7311057826570702055978747565264786935758838660918835790956716726895968781893980\
6197577089835533428250070076463861672738236689824429341266143008249630480529249\
5142244721354743123104963974936146293364440867656174513881761811216195861355409\
7226590982832455160532348892270853032526217953271203753936241844699444780852653\
926712756105661
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50'th prime in the formula $a_{n+1} = a_n^{\frac{101}{100}}$ and $a_0 = 10^{500} + 961 + \epsilon, 0 < \epsilon < 0.5$

S(50)= (807 digits).

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129729528971426122166658259081315435974871367309456840812055525509563976052536464197\
821936120784492089449745630948278142648656401758919926499683620493424145145363861773\
044716845814540511418289754542689191694327904116242782241131052138054549585683795895\
226460529926493834263717492409387560259409231253958370245042303023794648019244182073\
576593618946511947995963350548413770285593359081097306798650486731513585054871329096\
194202981055877907668708729761964242992640744211230936407662435884639367683685800000\
716124853576007781499789743771269181463159253173337794440878414346193538514506034277\
502087533266305538298562224619861085522581430515597209416207494298867400378422593043\
260350351208262898632520628116793338057678207643439460644660886621181985756002255888\
259043523402372168932260997906477619348535003398763
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