π, the primes and the Lambert W function

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Inspired

One day I came accross this formula of Ramanujan (notebooks)

$$\zeta(3) = \frac{7\pi^3}{180} - 2\sum_{n=1}^{\infty} \frac{1}{n^3(e^{2\pi n} - 1)}$$

I tried to understand this formula,
for doing it I made a series of experiments
with one of my favorite program : lindep or PSLQ,
that is Integer Relation algorithm. Lindep is part of *PariGP*
and now PSLQ is no longer a cryptic FORTRAN animal but part of Maple.
I made an interface within maple to write a fortran source from
1 inquiry, compile it on the host computer, run it and come back with the answer. I made one for Mathematica and Maple too.

This lead to more findings

$$\pi = 72\sum_{n=1}^{\infty}\frac{1}{n(e^{\pi n}-1)} - 96\sum_{n=1}^{\infty}\frac{1}{n(e^{2\pi n}-1)} + 24\sum_{n=1}^{\infty}\frac{1}{n(e^{4\pi n}-1)}$$

$$\frac{1}{\pi} = 8\sum_{n=1}^{\infty}\frac{n}{e^{\pi n}-1} - 40\sum_{n=1}^{\infty}\frac{n}{e^{2\pi n}-1} + 32\sum_{n=1}^{\infty}\frac{n}{e^{4\pi n}-1}$$

$$\zeta(3) = 28\sum_{n=1}^{\infty}\frac{1}{n^3(e^{\pi n}-1)} - 37\sum_{n=1}^{\infty}\frac{1}{n^3(e^{2\pi n}-1)} + 7\sum_{n=1}^{\infty}\frac{1}{n^3(e^{4\pi n}-1)}$$

$$\zeta(5) = 24\sum_{n=1}^{\infty}\frac{1}{n^5(e^{\pi n}-1)} - \frac{259}{10}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{2\pi n}-1)} - \frac{1}{10}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{4\pi n}-1)}$$

You see the pattern here ?, $e^{\pi n}$ $e^{2\pi n}$ $e^{4\pi n}$

And some exotic ones like

$e^{\pi} - \pi = 19.99909997919\ldots$ in 1987 with my hp 15c and is still a mystery.

$$\sum_{n=1}^{\infty} \frac{n^3}{e^{2\pi n/7} - 1} = 10.000000000000000190161767888663\ldots$$

$$\sum_{n=1}^{\infty} \frac{n^3}{e^{2\pi n/13} - 1} \cong 119.0000000000000000000000000000000959374585\ldots$$

Getting back to Ramanujan again, an identity with 1.

$$24 \sum_{n=1}^{\infty} \frac{n^{13}}{e^{2\pi n} - 1} = 1$$

In fact, there are more like that

More formulas

$$24\sum_{n=1}^{\infty}\frac{n^{673}}{e^{2\pi n}-1} = \text{a 1077 digit prime}$$

And

$$240\sum_{n=1}^{\infty}\frac{n^{22807}}{e^{2\pi n}-1} = \text{a 71399 digit prime}$$

Following the lead we get :

$$691 = 16\sum_{n=1}^{\infty}\frac{n^{11}}{e^{\pi n}-1} - 2^{16}\sum_{n=1}^{\infty}\frac{n^{11}}{e^{4\pi n}-1}$$

Can we get other primes like that ? ... all the primes ?

But why 691 ?

$$691 = 16\sum_{n=1}^{\infty}\frac{n^{11}}{e^{\pi n}-1} - 2^{16}\sum_{n=1}^{\infty}\frac{n^{11}}{e^{4\pi n}-1}$$

In fact, it comes from this identity with Eisenstein series (Jean-Pierre Serre, cours d'arithmétique, p 157.)

$$Eis_6 = 1 + \frac{65520}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^n$$

Hum, $2^{16} - 16 = 65520$, and 691 is the numerator of B_{12}, a Bernoulli number.

Depending of the person, you could prefer the first version or the 2nd.
Here Eis_6 is the Eisenstein series (not Euler numbers)

Anyhow, we have this approximation of 691…

$$691 \approx \frac{2^4 11!}{\pi^{12}}$$

Well, yes there are others

$$61 \approx \frac{2^8 6!}{\pi^7}$$

And 61 is the 3rd Euler number.

$$E_{510} \approx \frac{2^{512} 510!}{\pi^{511}}$$

E_{510} is a 1062 digits prime,

These numbers come from the expansion of the Dirichlet beta series like

$$\beta(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)^s}$$

Where are the Euler numbers coming from ?

$$\frac{1}{\cos(x)} = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^4}{6!} + \cdots$$

$$\frac{1}{\cosh(x)} = 1 - \frac{x^2}{2!} + \frac{5x^4}{4!} - \frac{61x^4}{6!} + \cdots$$

More generally, can we find all trigonometric expressions that would lead to approximations of primes with π^k ?

Could it be possible to get all the primes with these intriguing expressions with π^k ?

For example, with another expression (trig) we have

$$7 \approx \frac{2^9}{\pi^3\sqrt{(4+2\sqrt{2})}}$$

If we look at A006873 (Number of alternating 4-signed permutations)

1, 1, 7, 47, 497, 6241, 95767, 1704527, 34741217, ...

If the sequence contains a prime then if we have the asymptotic expansion of a(n) it leads to one more approximation of that prime using π^k

In this case , it is

$$a(n) \approx \frac{n!\,8^n}{n\pi^n} \cdot \frac{1}{\sqrt{(4+2\sqrt{2})}}$$

The E.G.F. of A006873 is $\frac{\sin(x)+\cos(3x)}{\cos(4x)}$

But, how the expression of $a(n)$ is found ?

$$a(n) \approx \frac{n!\,8^n}{n\pi^n} \cdot \frac{1}{\sqrt{(4+2\sqrt{2})}}$$

1) **expand** $\frac{\sin(x)+c\quad(3x)}{\cos(4x)}$ into a series
2) Collect coefficients of the e.g.f (with n!)
3) Compute the ratio of $a(n+1)/a(n)$
4) Compute first differences
5) Identify the constant : 2.546479089470325372302…
6) The constant is $\frac{8}{\pi}$
7) Retro engineer the expression to a(n).
8) $\sqrt{(4+2\sqrt{2})}$ is found in $\frac{\Gamma(n)8^n}{\pi^n} = \sqrt{(4+2\sqrt{2})}$

Building one by one each prime from this idea.

$$11 \approx \frac{768}{\pi^4} \qquad 3 \approx \frac{64\sqrt{2}}{\pi^3}$$

$$11 \approx \frac{1944\sqrt{3}}{\pi^5} \qquad 1 \approx \frac{3 \cdot 13!}{2\pi^{14}}$$

$$31 \approx \frac{4 \cdot 10!}{5\pi^{10}} \qquad 17 \approx \frac{4 \cdot 8!}{\pi^8}$$

$$5 \approx \frac{2^6 \cdot 4!}{\pi^5}$$

Not much of a pattern found here

We have some primes with Euler numbers via the Beta Dirichlet series, some Bernoulli numbers…

Can’t we just generate primes with these informations ? Not exactly.

Can’t we just generate some primes with <any> formula ?

What are the known formulas anyway ? Which is the most efficient ?

Who	Year	Comment	Efficiency	How many primes
Eratosthenes	-276 to -194	Sieve	Practical	Computable infinity
Mersenne	1536	Primes of the form 2^p-1	Practical, exact	51
Fermat	1640	Fermat's little theorem	Weak Probable primes	Computable infinity
Euler	1772	Second degree polynomial	Practical	40
Mills and Wright	1947 and 1951	Double exponential	Practical	Less than 5 known exactly
Wilson	Circa 1780	Uses p!	Theoretical	Very few primes
Jones, Sato, Wada, Wiens	1976	25th degree polynomial with 26 variables	Theoretical	Very few primes
John H. Conway	1987	FRACTRAN	Theoretical	Very few primes
Dress, Landreau	2010	6th degree polynomial	Practical	58
Benoit Perichon and al.	2010	26 primes in arithmetic progression	Practical	26
Tomas Oliveira e Silva et al.	2019	Sieve optimized, fastest known prime gererating program	Practical	Computable infinity
Fridman et al.	2019	Prime generating constant	Practical, limited to precision	Computable infinity
A064648	2019	Engel expansion of 0.705230171…	Practical, limited to precision	Computable infinity
Simon Plouffe	2019	Efficient Mills-Wright-like formula	Practical, limited to precision	Computable infinity

The 6’th degree polynomial took months to find

Prime numbers are hard to generate.
The formula of Mills is a good example

if A = 1.3063778838630806904686144926…
then $\lfloor A^{3^n} \rfloor$ is always prime.

a(n) = 2, 11, 1361, 2521008887,
1602223620400981813183 1320183,
4113101149215104800030529 5
3791595317048613962353975 9
9331359499948827704040748 32568499, …
a(21) is 1.214 billion digits long

the triple size at each iteration

Here is the algorithm of Mills seen in the eye of reverse-engineering.

Begin with p=2

a) New prime = Next Prime(p^3)
b) Go to a).

You get the sequence, 2, 11, 1361, 2521008887, 16022236204009818131831320183, ...

The formula of Wright is even worse

E. M. Wright formula (1951)

if $g_0 = \alpha = 1.9287800\ldots$ and $g_{n+1} = 2^{g_n}$ then

$\lfloor g_n \rfloor = \left\lfloor 2^{\cdots 2^{2^{\alpha}}} \right\rfloor$ is always prime.

a(n) = 3, 13, 16381,
The fourth term is 4932 digits long.
No one was able to compute the 5th term of this sequence.
In both cases, it is a good idea but not practical at all.

But if we used Sylvester's sequence rather

A000058 in the OEIS catalogue is
2, 3, 7, 43, 1807, 3263443, 10650056950807,

has the property that

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \frac{1}{1807} + \dots$$

Called the Pierce expansion or egyptian fractions expansion of 0.99999999999…

The sequence is given by the recurrence

$$S_{n+1} = S_n^2 - S_n + 1$$

So starting at 2 we get 3, then 7, 43, ...

But what if we start differently by having

$$S_0 = 1.6181418093242092\ldots$$

Will produce 2, 3, 7, 43, 1811, 3277913, ...
all primes. Nice, but it grows too fast.
The length doubles at each step.

The number

$$S_0 = 1.6181418093242092\ldots$$

Was found using simulated annealing + Monte-Carlo

Simulated annealing is what we call
« le recuit simulé »

Simulated annealing + Monte-Carlo

1) First we choose a starting value and exponent (preferably a rational fraction for technical reasons).

2) Use Monte-Carlo method with the Simulated Annealing, in plain english we keep only the values that show primes and ignore the rest. Once we have a series of 4-5 primes we are ready for the next step.

3) We use a formula for forward calculation and backward.

One example

Hypothesis : there exist a infinite sequence of primes generated by {c· n^n}, c real and { } is the nearest integer.

Yes, if $c = 0.2655883729431433908...$ then the sequence (n $\geq$ 3).

7 , 67, 829, 12391, 218723, 4455833, 102894377, ...

But fails after 19 terms at n=22. The sequence is finite.

We go back to Mills model

What if we use a smaller exponent and test if it works ?

When $a_0 = 43.8046877158029348 1\ldots$ then if $a_{n+1} = a_n^{\frac{5}{4}}$, and use { } to isolate primes.
This is now sequence : A323176
113, 367, 1607, 10177, 102217, 1827697, 67201679,

Now, if you want an even smaller exponent choosing carefully a_0 would it work too ?

Let's try : $a_{n+1} = a_n^{\frac{11}{10}}$

Then if $a_0 =$ 100000000000000000000000000000049.3122107477645

We have this prime sequence :

100000000000000000000000000000049

158489319246111348520210137339236753

5248074602497725973643121570227258944401

3908408957924020300919472370957356345933709

70990461585528724931289825118059422005340095813

...

Continuing according to this idea…

With : $a_{n+1} = a_n^{\frac{101}{100}}$

Then if $a_0 = 10^{500} + 961.49937633785074906096890050\ldots$

I could compute 100 terms of this sequence : a(100) is a 1340 digits prime (only).

http://plouffe.fr/Record%20100%20primes%20sequence.txt

I use a formula for forward calculation and backward calculation.

Forward calculation
Next smallest prime to $\{ a(n)^e \}$

Backward calculation (to check)
Previous prime = solve for x in $x^e - S(n+1)$.

This is the simulated annealing
High speed guessing with a filter

Guess first value (real value)

- 0) Apply $f(S_0)$
- 1) Is $f(S_0)$ prime ?
- 2) If yes, keep the prime in list, if not go to 0
- with new startng value S_0.

- The machine :
- core i7 at 4.4 Ghz with a 220 TB jbod
- 283 billion f.p.o. per second

Finally what could be the sequence with the smallest initial value, like 2 ?

Let's try : $a(n) = \{2^{d^n}\}$
Where d = 1.3007687041481769105525256...

(sequence A306317)

2, 3, 5, 7, 13, 29, 79, 293, 1619, 14947, 269237, 11570443, 1540936027, ...

Can we go backward too ?
Like from any specific prime number using this algorithm ?

Let's say from $10^{100} + 267$ *to* 2 ?

Yes, if the exponent α is inverted,

$$When\ \alpha = 0.38562256415290 \ldots$$

Then we have the sequence :

$10^{100} + 267$, 742123524365563, 542489, 163, 7, 2.

Here a(0) = 2.132221996628413452 and the exponent

$1/\alpha = 2.593209249040428616 7308\ldots$

In 1902, a certain M. Cipolla published a formula for the n'th prime number.

$$p_n = n(\ln(n) + \ln(\ln(n)) - 1 + o(n)$$

On the other hand the formula for the number of primes less or equal to n is

$$\pi(n) = \frac{n}{\ln(n)} \quad (n \to \infty).$$

One formula being the functional inverse of each other.

Actually, no.
Very recently[4,5], a number of people began to realize that these inverses are not as they appear.

$$\text{If } \pi(n) \approx \frac{n}{\ln(n)} = \text{y}$$

then the inverse is -y W(- 1/y) (or for n) to simplify the notation.

This means that $p_n \approx -nW_{-1}\,(-\frac{1}{n})$.
Knowing that the value of $W(-1/n)$ has to be with W_{-1} and not $W\left(0, -\frac{1}{n}\right)$.

Now there is a big question about p_n, $\pi(n)$ and the precision.

As we know the P.N.T. is a major item.

But, in term of precision : it is very rough.

It is true yes, but when $n \rightarrow \infty$. The same with p_n.

We go back to the classic equations.

$$\pi(n) \approx \frac{n}{\ln(n)}$$

Is the classic equation, we change it for,
(see Dusart thesis [2010] for details)

$$\pi(n) \approx \frac{n}{\ln(n) - 1}$$

If the compute the inverse

$$\frac{n}{\ln(n) - 1} = y$$

solve(%,n); gives (y is renamed).

$$p_n \approx -nW_{-1}\left(-\frac{e}{n}\right)$$

With the nth prime we have the formula :

$$p_n \approx n\log(n)$$

Then

$$\pi(n) \approx \frac{n}{W_0(n)}$$

Now we will look at what the error looks like for

$$p_n \approx -nW_{-1}\left(-\frac{e}{n}\right)$$

Then, one by one we eliminate different hypothesis about the difference between our calculated p_n and the real value.

- The size of what is left is comparable to one of the <straight lines> of W_k(n).
- At first sight the value of the difference is a straight line (correlation is 0.9999). It is not.
- What is significant is the magnitude of the difference only.
- The W_k(n) and logarithmic fit are indistinguishable.

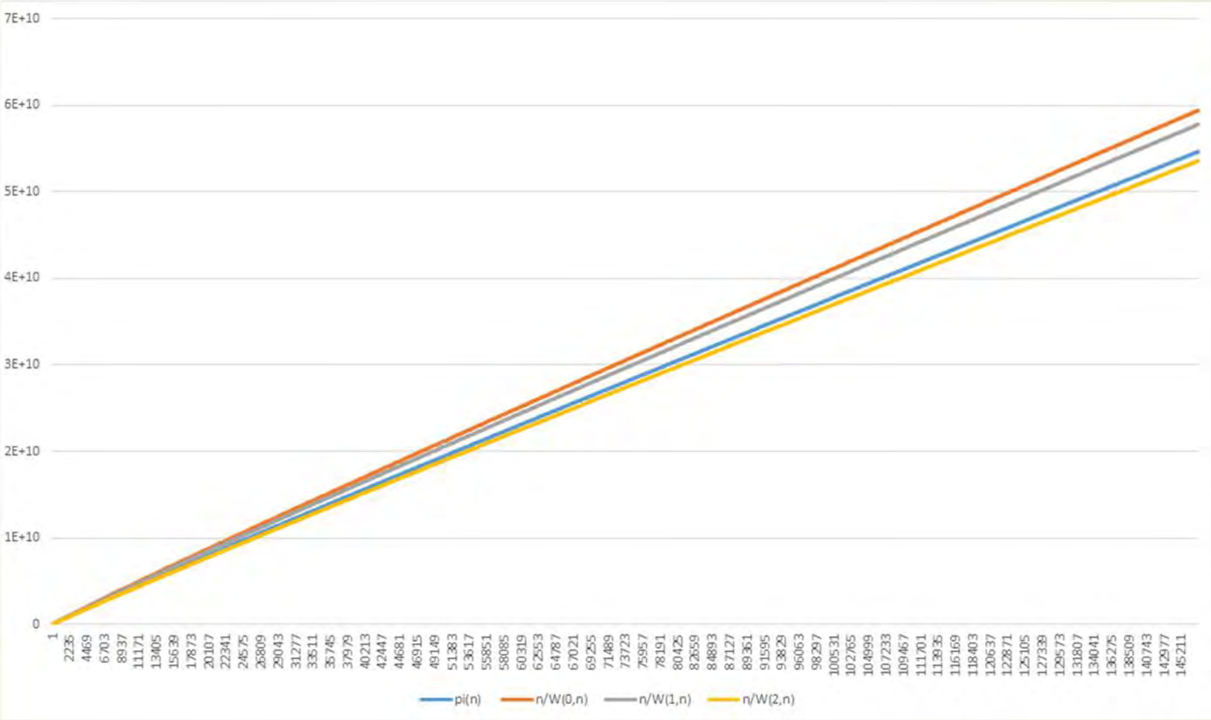
For this comparison we need to consider the extent of the known tables of p_n.

My own table is up to 20000 billions.
The known long range table is only up to 10^{24}.
(powers of 10 only).

The table for $\pi(n)$ is up to 27000 billions and the long range goes up to 10^{27}.

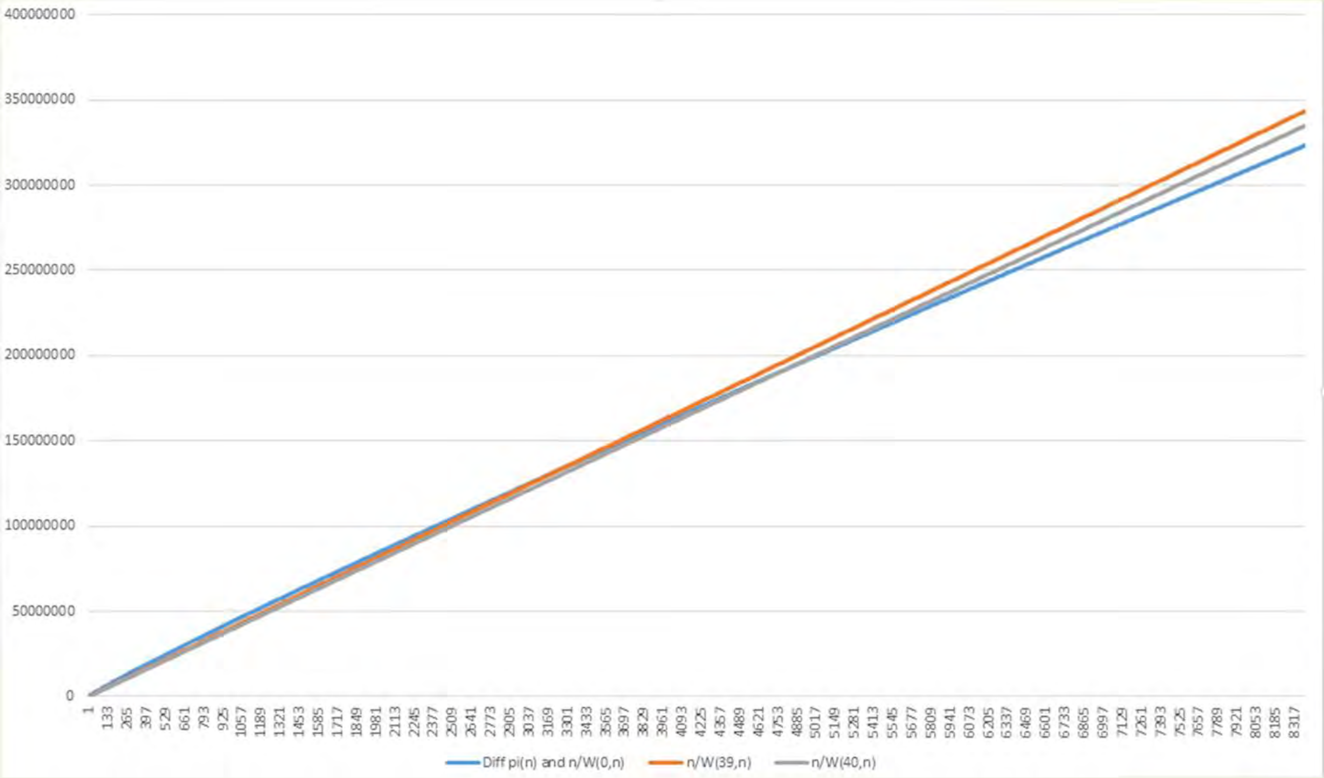
The ponctual very large known primes are useless for this study for a very simple reason :
We do not know the rank of these primes.

$\frac{n}{W_k(n)}$ for large values of n



$\pi(n)$ is the straight line in blue between the values of $\frac{n}{|W_k(n)|}$

Since the values lies in between, it is natural to use differences to realize we still have the <u>same thing</u> but smaller



$$\frac{n}{W_{39}(n)} < \pi(n) - \frac{n}{W_0(n)} < \frac{n}{W_{40}(n)}$$

This is strange.

The known behaviour of p_n and $\pi(n)$ is not exactly predictable, the evaluation of Riemann with $li(x)$ not simple.

The last computations of p_n and $\pi(n)$ were 'difficult'. (Months of computer time).

From numerical evidence then

$$p_n \approx -nW_{-1}\left(-\frac{e}{n}\right) - \frac{n}{W_0(n)}$$

The surprise is that : what is left (again) is something that resembles exactly what we had in the first place (!). The 'curve' is still a 'straight line' but the magnitude is smaller.

The only plausible explanation is that we have here the *matryoshka principle* : russian puppets.

$$p_n \approx n\log(n)$$

$$-nW_{-1}\left(-\frac{e}{n}\right) - \frac{n}{W_0(n)}$$

is more precise
Tested at n=10^{24}

$$p_{10^{24}} \approx 583086425504749834767 17666$$

The real value being
58310039994836584070534263

Now if we continue with this *matryoshka* principle, what is the next term !

For $p_{10^{24}}$ by using a bisection method to find the next term in the form of

$$\frac{n}{W_k(n)}$$

The next terms are 114 and 96606.
$W_k(n)$ can be approximated with formula 4.20 in Corless et al.

$$W_k(n) \approx \log(n) + 2\pi ki + \log(\log(n) + 2\pi ki)$$

$$p_{10^{24}} = -10^{24} W_{-1}\left(-\frac{e}{10^{24}}\right) - \frac{10^{24}}{W_0(10^{24})} + \frac{n}{W_{114}(10^{24})} + \frac{n}{W_{96606}(10^{24})} + \cdots$$

The value is
583100399948247999494935554 compared to
583100399948365840705342632
(12 exact digits).
With 3 terms : 6 exact digits.

In 1994, B. Salvy published a paper to dig out an algorithm to get dozens of terms in the Cipolla formula :

$$p_n \sim n\left(\ln(n) + \ln(\ln(n)) - 1 + \frac{\ln(\ln(n)) - 2}{\ln(n)} - \frac{\ln(\ln(n))^2 - 6\ln(\ln(n)) + 11}{2\ln(n)^2} + \cdots\right)$$

The formula is quite similar to the asymptotic expansion of $W_0(n)$

$$W_0(n) \approx L_1 - L_2 + \frac{L_2}{L_1} + \frac{L_2(-2 + L_2)}{2L_1^2} + \frac{L_2(6 - 9L_2 + 2L_2^2)}{6L_1^3} + \frac{L_2(-12 + 36L_2 - 22L_2^2 + 36L_2^3)}{12L_1^4} + \cdots$$

Here $L_1 = \ln(n)$ *and* $L_2 = \ln(\ln(n))$.

In principle, with enough terms and with n >> 1 it should do the thing.

Not exactly, with 72 terms of Cipolla-Salvy formula we get 12 digits exact too.

There is a limit to it.

The expansion in Lambert functions is much simpler.

For the moment, the only clue I have about the 4th term is that it is proportional to the $[log_2(n) + 1]$, that is the log of n in base 2.

For the info, here is the 10th term of the Cipolla-Salvy expansion.

```
k*ln(k)*(1+(ln(ln(k))-1)/ln(k)+(ln(ln(k))-2)/ln(k)^2+(-1/2*
ln(ln(k))^2+3*ln(ln(k))-11/2)/ln(k)^3+(1/3*ln(ln(k))^3-7/2*
ln(ln(k))^2+14*ln(ln(k))-131/6)/ln(k)^4+(-1/4*ln(ln(k))^4+
23/6*ln(ln(k))^3-49/2*ln(ln(k))^2+159/2*ln(ln(k))-1333/12)/
ln(k)^5+(1/5*ln(ln(k))^5-49/12*ln(ln(k))^4+73/2*ln(ln(k))^3
-367/2*ln(ln(k))^2+3143/6*ln(ln(k))-13589/20)/ln(k)^6+(-1/6
*ln(ln(k))^6+257/60*ln(ln(k))^5-1193/24*ln(ln(k))^4+1027/3*
ln(ln(k))^3-17917/12*ln(ln(k))^2+47053/12*ln(ln(k))-193223/
40)/ln(k)^7+(1/7*ln(ln(k))^7-89/20*ln(ln(k))^6+959/15*ln(ln
(k))^5-13517/24*ln(ln(k))^4+6657/2*ln(ln(k))^3-39769/3*ln(
ln(k))^2+493568/15*ln(ln(k))-32832199/840)/ln(k)^8+(-1/8*ln
(ln(k))^8+643/140*ln(ln(k))^7-14227/180*ln(ln(k))^6+34097/
40*ln(ln(k))^5-76657/12*ln(ln(k))^4+616679/18*ln(ln(k))^3-\
642111/5*ln(ln(k))^2+36780743/120*ln(ln(k))-893591051/2520)
/ln(k)^9+(1/9*ln(ln(k))^9-1321/280*ln(ln(k))^8+119603/1260*
ln(ln(k))^7-218809/180*ln(ln(k))^6+1328803/120*ln(ln(k))^5-\
2696687/36*ln(ln(k))^4+33904723/90*ln(ln(k))^3-40633409/30*
ln(ln(k))^2+7921124011/2520*ln(ln(k))-2995314311/840)/ln(k)
^10)
```

But , let's go back to $\pi(n)$, we had

$$\pi(n) \approx \frac{n}{W_0(n)}$$

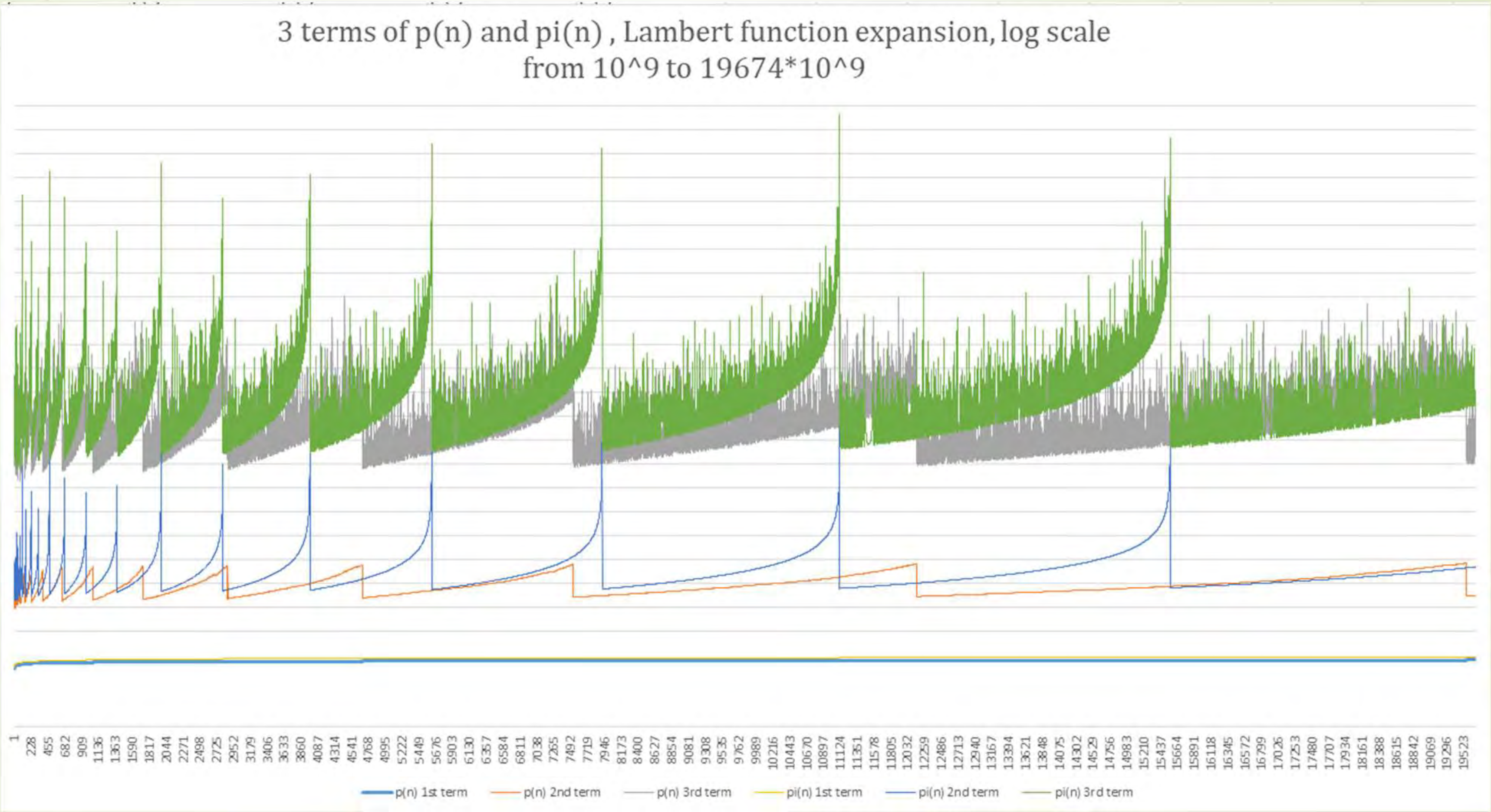
We apply the same scheme let's say for 10^7 for $\pi(n)$ and p_n .

$$\frac{p_{10^7}}{10^7} = -W_{-1}\left(\frac{-e}{10^7}\right) - \frac{1}{W_0(10^7)} + \frac{1}{W_{22}(10^7)} + \frac{1}{W_{763}(10^7)} + \frac{1}{W_{5323546}(10^7)} - \cdots$$

$$\frac{\pi(10^7)}{10^7} = \frac{1}{W_0(10^7)} - \frac{1}{W_{22}(10^7)} + \frac{1}{W_{640}(10^7)} + \frac{1}{W_{2174463}(10^7)} - \cdots$$

It is similar, how similar is it ?

Graph of the coefficients of $\pi(n)$ and p_n in the Lambert expansion, every billion from 10^9 to $19674 \cdot 10^9$, log scale.



This is where we can apply the *duck principle.*

If an animal has a beak like a duck , feathers like a duck, the color of a duck, quacks like a duck and has 2 feet like a duck then it's a duck.

With this Lambert function expansion the 2 quantities $\pi(n)$ and p_n are the same (except for the first term of p_n).

Recently, a certain André LeClair and Guillerme França (2014) had a formula for the nth zero of Riemann's Zeta function.

It follows the same idea. If $N(n)$ is the number of non-trivial zeros (considering only the imaginary part) then

$$N(n) \approx \frac{n}{2\pi}\log\left(\frac{n}{2\pi}\right) - \frac{n}{2\pi} + \frac{11}{8}$$

By inverting (functionally) the formula we obtain a formula for the nth zero.

$$\sigma(n) \approx \frac{(8n-11)\pi}{4\,W\left(\frac{8n-11}{8e}\right)}$$

The formula is spectacular in precision.

$$\sigma(1) \approx -\frac{3}{4}\frac{\pi}{W\left(-\frac{3}{8e}\right)}$$

Is 14.5213469… when the real value is 14.13472514

So precise that they could evaluate precise values of $\sigma(n)$ with $\mathrm{n} = 10^{1000}$ by using an additional Newton-like interpolation.

We have here a quantum leap compared to previous models.

Again, if we go back to the classic known equations.

$$\sigma(n) \approx \frac{2\pi n}{\log(n)}$$

And $N(n)$ (Riemann) is

$$N(n) \approx \frac{n}{2\pi} log(\frac{n}{2\pi e})$$

And now by solving for n in each case we get

$$\frac{N(n)}{2\pi n} \approx \frac{1}{\mathrm{W}\left(\frac{n}{e}\right)} \qquad \frac{\sigma(n)}{2\pi n} \approx -W_{-1}\left(\frac{-2\pi}{n}\right)$$

$$\frac{p_n}{n} \approx -W_{-1}\left(-\frac{e}{n}\right) - \frac{1}{W_0(n)} + \frac{1}{W_{\{2\,\mathrm{lo}\ \ (n)\}+1}(n)}$$

If we collect the 4 formulas we found, dividing by either n or $2\pi n$ we get

$\frac{\pi(n)}{n} \approx \frac{1}{W_0(n)}$	$\frac{p_n}{n} \approx -W_{-1}\left(-\frac{e}{n}\right)$
$\frac{N(n)}{2\pi n} \approx \frac{1}{\mathrm{W}\left(\frac{n}{e}\right)}$	$\frac{\sigma(n)}{2\pi n} \approx -W_{-1}\left(\frac{-2\pi}{n}\right)$

$$\pi(n) \cong n\left(\frac{1}{W_0(n)} - \frac{1}{W_{\mathrm{k}}(n)} \cdots\right) \qquad p_n \cong n\left(-W_{-1}\left(\frac{-e}{n}\right) - \frac{1}{W_0(n)} \cdots\right)$$

$$\sigma_n = 2\pi n\left(-W_{-1}\left(\frac{-2\pi}{n}\right) \cdots\right) \qquad N(n) = 2\pi n\left(\frac{1}{W_0\left(\frac{n}{e}\right)} + \frac{1}{W_1(n)} \cdots\right)$$

From there, 2 possibles directions

If the Euler principle applies then we should have a sum and a product on each side.

Or, the expression with primes needs to be completed with an expression using the zeros of the Zeta function $(1/2 + it)$, then it has to match with the equivalent expression with the primes on the other 2 equations.

I leave this question as an exercice to be completed…

Here is a model we can try for $\pi(n)$ and p_n

$$\frac{\pi(n)}{n} \approx \frac{1}{W_0(n)} - \sum_{k=1}^{\infty} \frac{1}{W_k(n) + e^{kf(n)}}$$

$$p_n \approx -W_{-1}\left(\frac{-e}{n}\right) - \frac{1}{W_0(n)} - \sum_{k=1}^{\infty} \frac{1}{W_k(n) + e^{kg(n)}}$$

Here $f(n)$ and $g(n)$ are found using a logarithmic fit.

For $f(n)$, at $n = 4635000018752$, the formula gives 146388867645773 *exactly.*

For g(n) , the formula is quite similar.

Comparing the 2 :

$$f(n) = 5.14071313388525388606186555508885 + .0480894831299088001055089594166636 \ln(n)$$

$$g(n) = 5.14252590354189118976617708563 62 + .04783919797825572908551130822 9899 \ln(n)$$

Is this just a coincidence ?

Let's try another model ?

$$\frac{\pi(n)}{n} \approx \frac{1}{W_0(n)} - \sum_{k=1}^{\infty} \frac{1}{W_k(n) + k^{f(n)}}$$

$$p_n \approx -W_{-1}\left(\frac{-e}{n}\right) - \frac{1}{W_0(n)} - \sum_{k=1}^{\infty} \frac{1}{W_k(n) + k^{g(n)}}$$

$$g(n) = 4.8493349460 + 0.0557287326 \ln(n)$$
$$f(n) = 4.8604042576 + 0.068197411 \ln(n)$$

I think that there is a possibility that one could find eventually an exact formula for $\pi(n), p_n$, $N(n)$ and $\sigma(n)$.

And now, something completely different.

This is Viète formula (1593).

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}}{2} \cdot \dots$$

As far as mathematic is concerned, nobody knows what is the binary expansion of $\frac{2}{\pi}$. This is just a bunch of zeros and ones at random. Perhaps we will never find the patterns in it.

But here is another approach to the problem.

If we are looking at individual bits of this number we do not see anything.

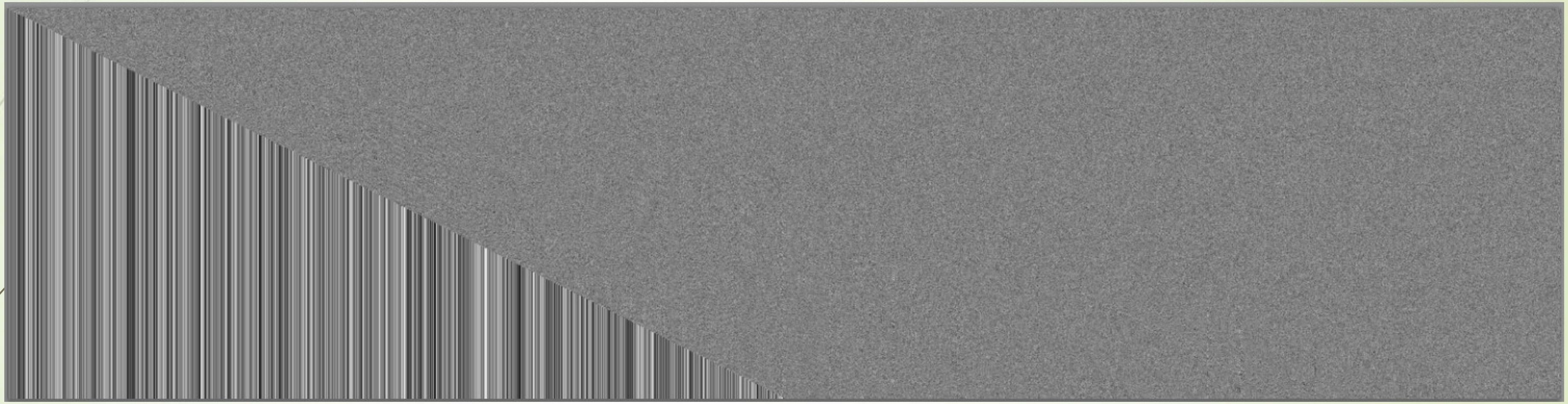
But let's consider this instead.
We align all the partial products in Viète expansion and look at the bits as a whole, everything is computed in binary.

The first line is $\frac{\sqrt{2}}{2}$

The 2nd line is $\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2}$

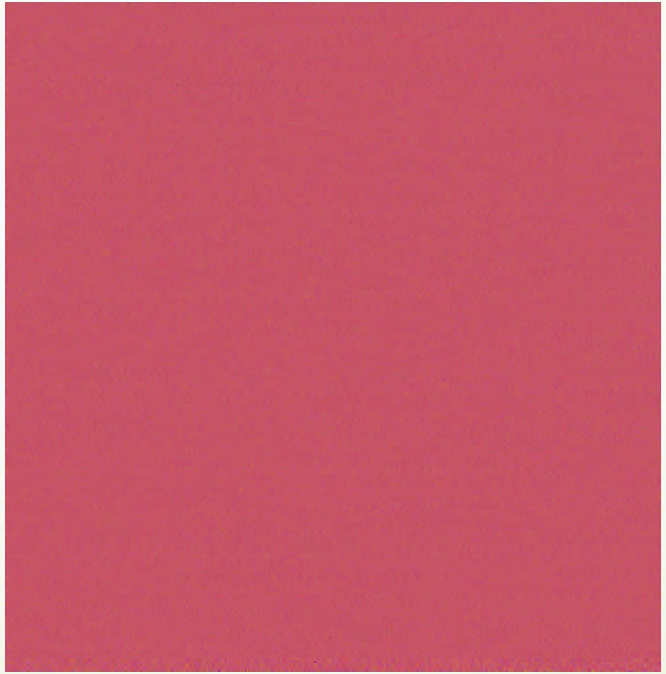
The 3rd line is $\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}$

With 99715 bits x 25000 terms gives this :

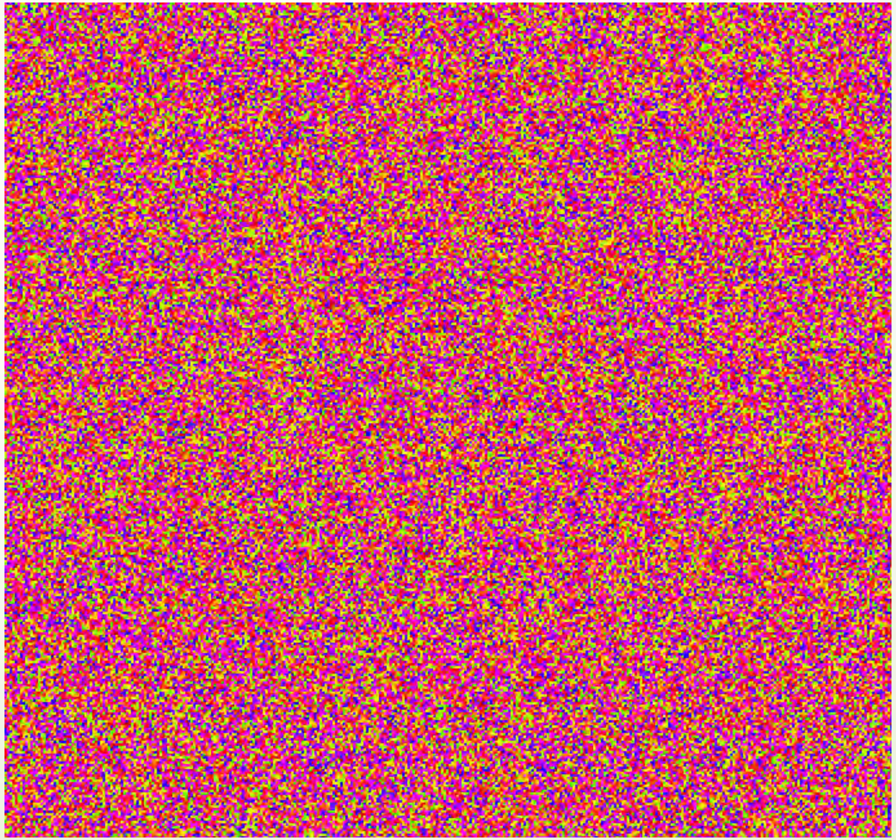


An image of 2.5 billion pixels.
The long vertical lines at the left are the slow convergence of the product to $\frac{2}{\pi}$.
We don't see much of a pattern here.

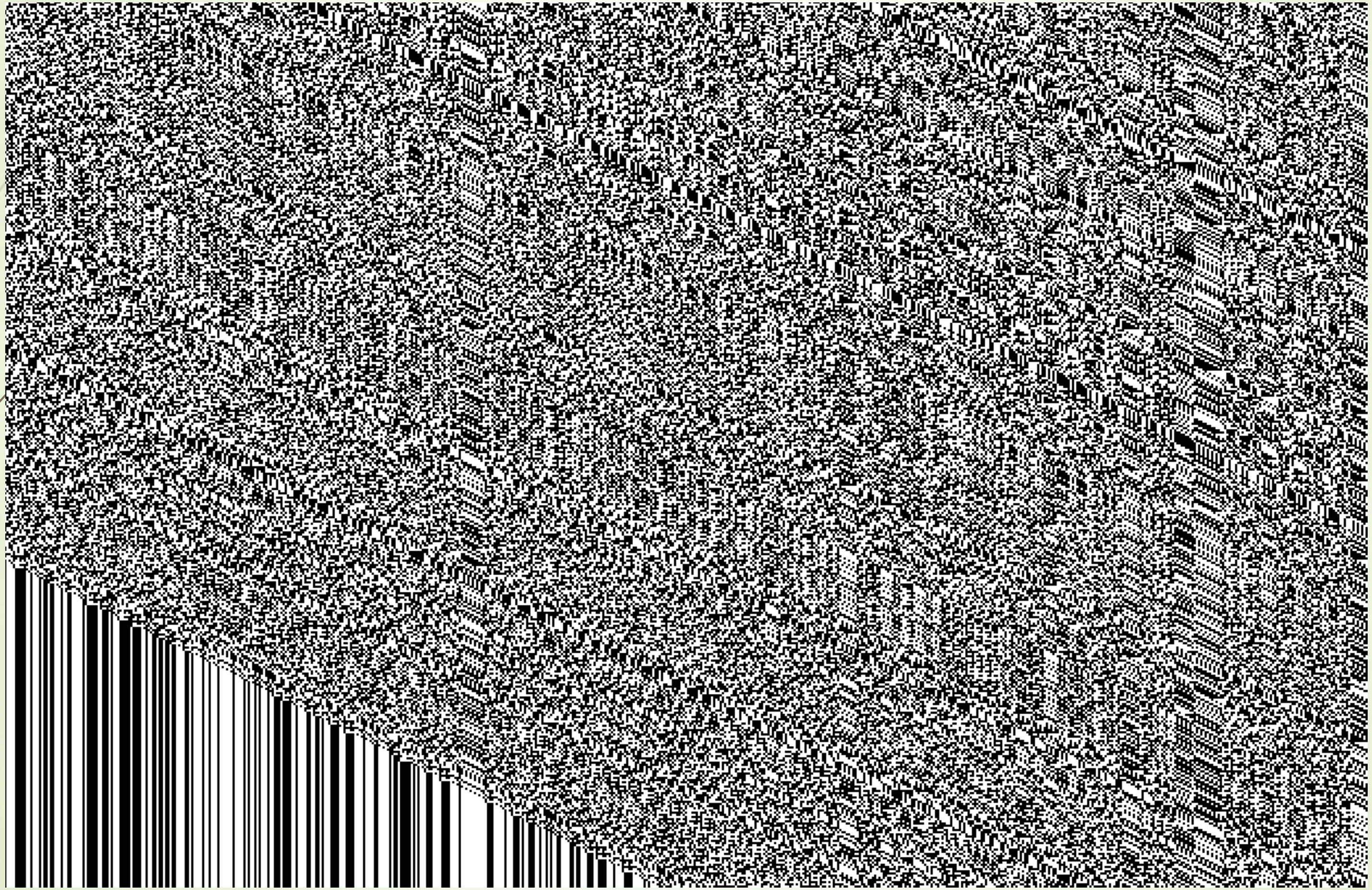
As a comparison, this is an image of the first 100 millions decimal digits of Pi, colorized with 10 colors, blue, green, red, yellow… the image is 10000 x 10000.



If we zoom in : this is pretty much random data.
One experiment was done with the first 1000 billion digits of π : 10000 images of 10000 x 10000 (see web page).



If we zoom on Viète image, we see this :

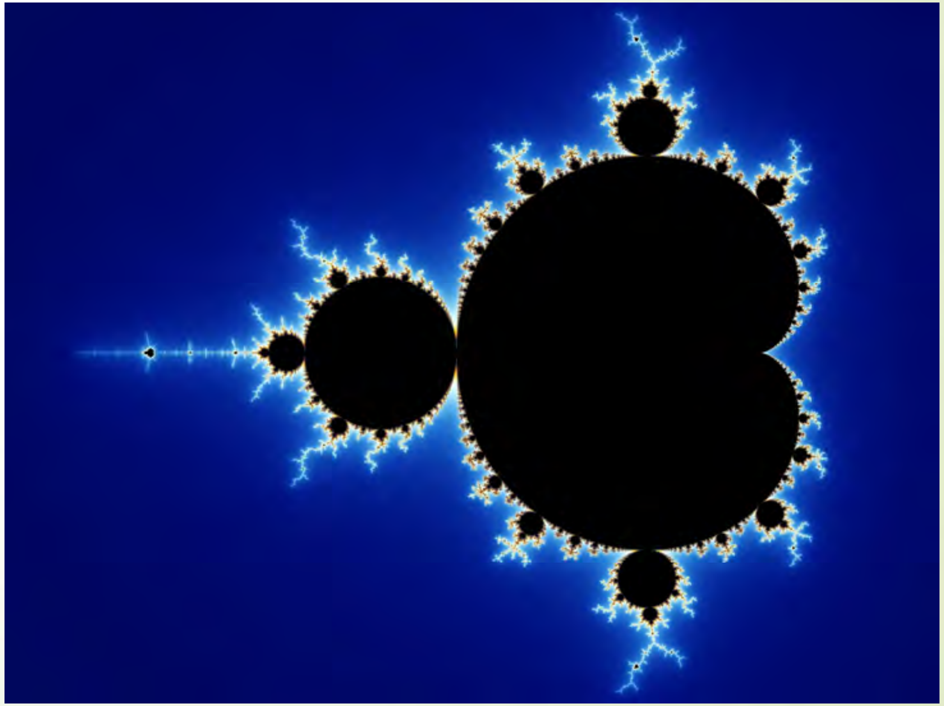


I have no explanation for it.
The successive square roots are producing the effect.
The thing we can say is that : from one term to the other : bits are not random and the pattern is quite persistant.

Are there any other algebraic curiosities like that ?

When experimenting with square roots and square roots of square roots, I wanted to know if a persistant pattern occurs in the Mandelbrot set.

As we know, there are similarities. I wanted to know what if we approach one limit point ? If the limit point is algebraic, are there any patterns in these algebraic numbers ? (in binary perhaps ?).



Well, I have found one formula :

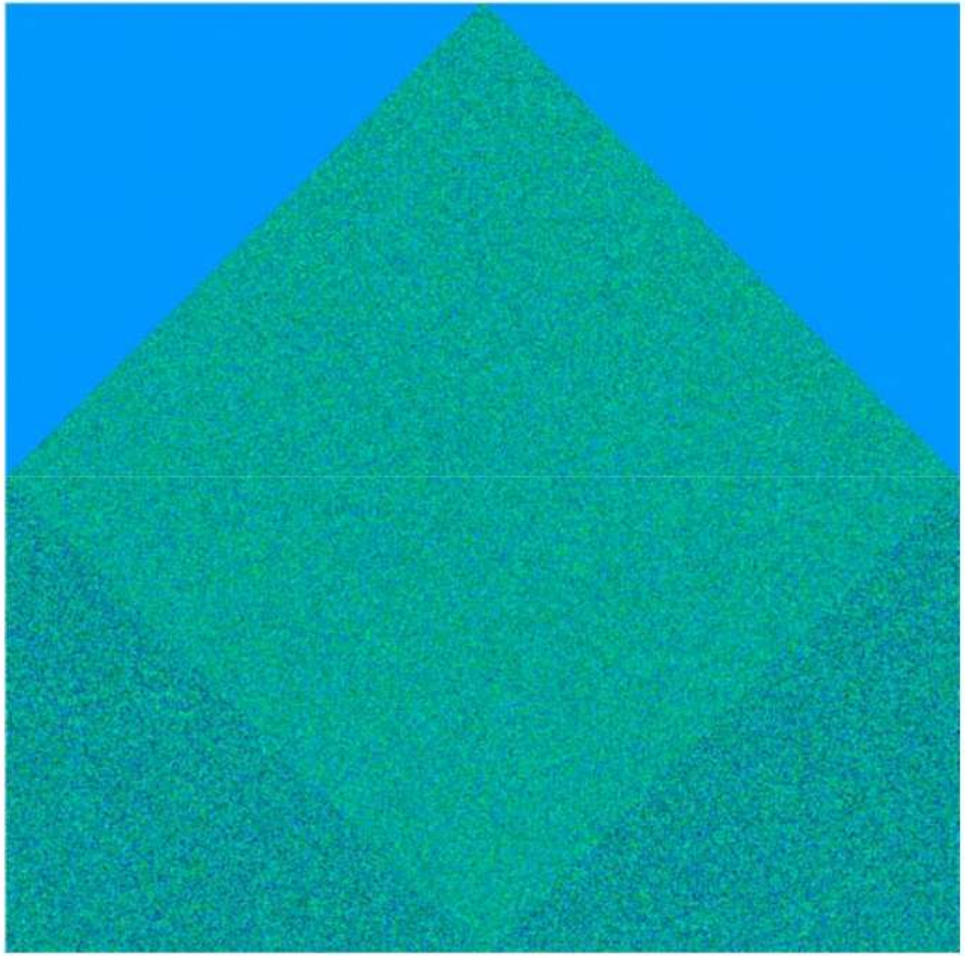
If $f(n) = 1 + \frac{\sqrt[4]{16^{4n}+1}}{16^{n}+1}$ then the binary expansion of f

has a very, very long and persistant pattern.

When $n = 4096$ then at position 1342238724 there are 4118 successive bits of this number that are all ‘0’.

(estimated) at n = 1000000 the persistant pattern goes up to the 8000001500000 4’th position.
At n= 1000000000 the position is at 8.0 x 10^{18}.

The first 270 million bits of $1 + \frac{\sqrt{2 \cdot 4^{8192} + 2\sqrt{16^{8192}+1}}}{2^{16386}}$

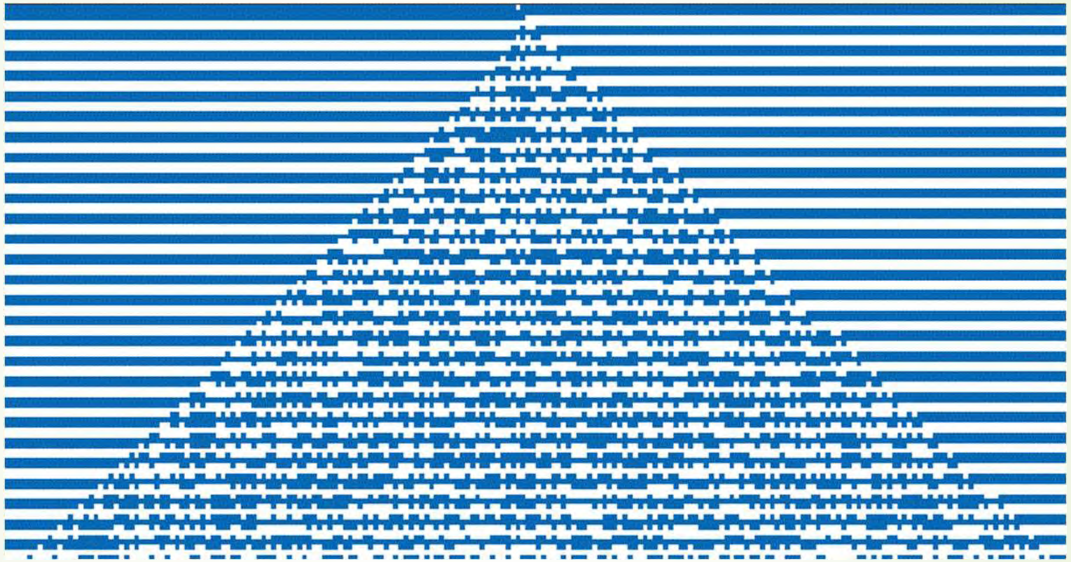


In false color :
The bits were colorized to enhance the contrast.

This is the binary expansion of f(100) (same as above)

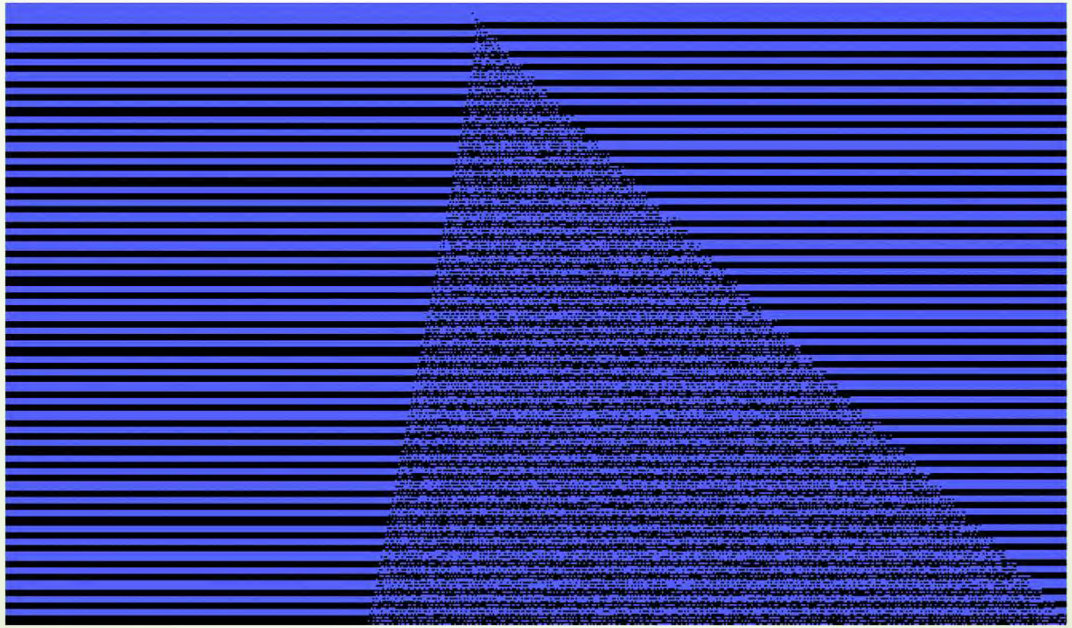
f(100)=1.00000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000010
000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000
000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000
000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000
000000000000000000000000000000000000000000000000000000000000000000000000001111111111111111111111111111111111
111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111
111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111
111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111
111111111111111111111111111111111111111101100000000000000000000000000000000000000000000000000000000000000000
000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000
000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000
000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000
000001010011111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111
111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111
111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111
111111111111111111111111111111111111111111111111111111111111111111111111111111001010011000000000000000000000
000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000
000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000
000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000
000000000000000000000000000000000000000000100101111110111111111111111111111111111111111111111111111111111111
111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111
111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111
111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111111
111111110001101001011110000000000000000000000000000000000000000000000000000000000000000000000000000000000000
000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000
000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000000
000000000000000000000000000000000000000000000000000000000000000000000000000001011010101011111001111111111111
11111111111111111111111111111111111111111111111111111111111111...

When we zoom in :

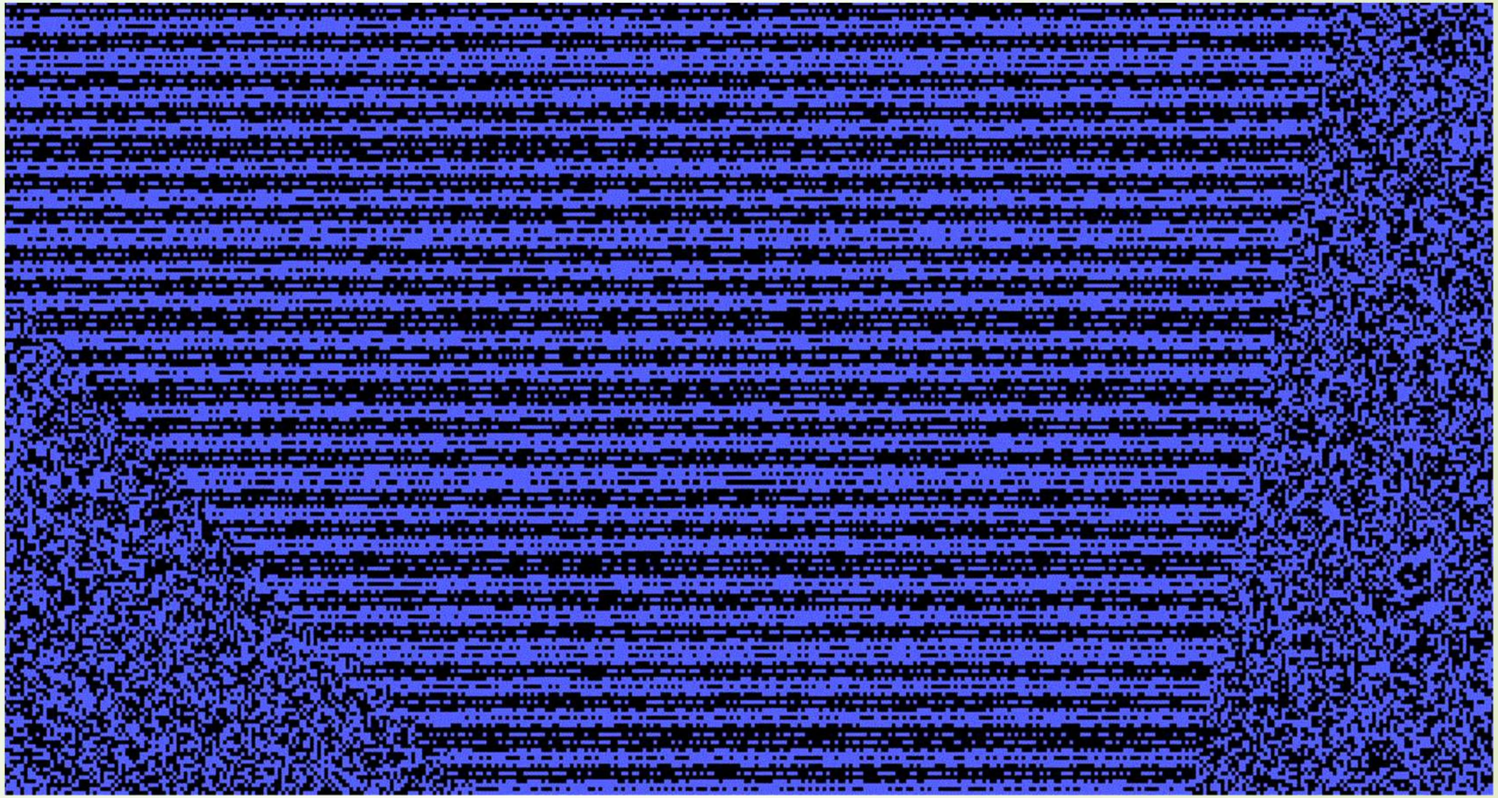


The first 446 million bits of the number

$$1+\frac{\sqrt{2\cdot 32^{16383}+3\sqrt{4^{16383}+1}}}{2\cdot 16^{32766}}$$



At another point (depending on the choice of width)



Zoom at the bit level (from an image of 332 m pixels.)

The data

- In all, 74 TB of mathematical data, mostly numbers.
- 41 TB of primes, from 2 to 80594098476893 (2602 billion entries)
- 5.6 TB of ζ zeros, 103 billion zeros
- OEIS tables (and extended tables)
- The Inverter, 41 digits (small version with 11.3 billion entries), 64 digits, 17.2 billion entries
- Inverter 41 : 1.008 TB, http://plouffe.fr/ip/
- Inverter 64 : 2.15 TB.
- High resolution images : 1206000, 1.773 TB.

Errors found…

- Prime[8200000000] and Prime[93000000000] just hangs in Mathematica.

- The tables of zeros at http://www.lmfdb.org/zeros/zeta/ do contains errors when the decimal expansion finishes by 00. 11 errors where found.

The index and the corresponding zeros do not match.

In both cases, they are working on the problem.

Thank you for your attention

Merci de votre attention

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