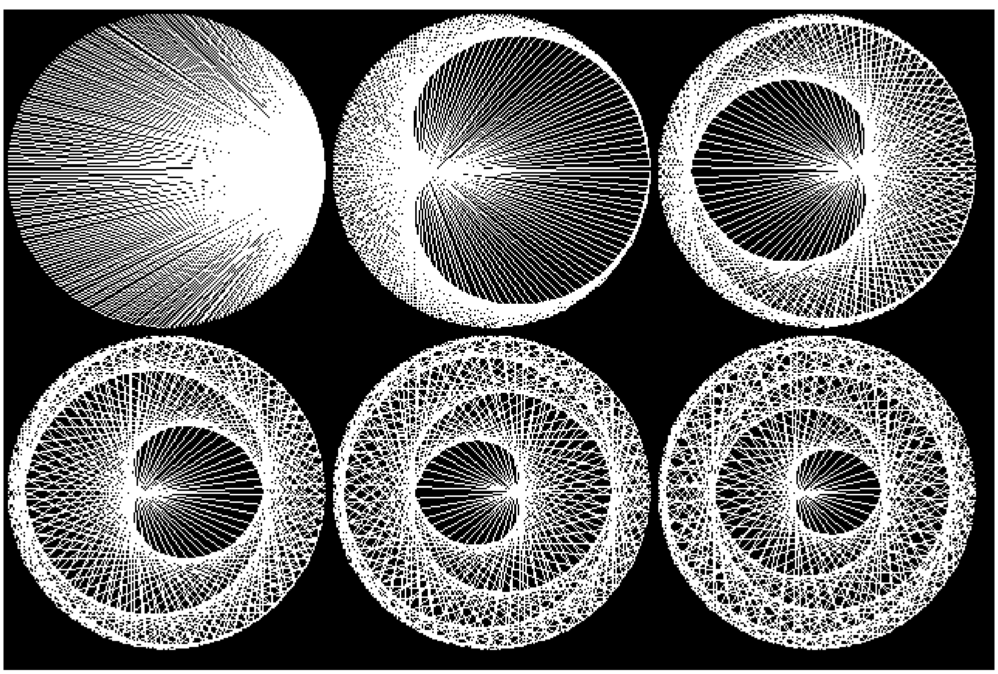
The reflection of light rays in a cup of coffee
or
the curves obtained with b^n mod p

by Simon Plouffe
based on works done in the years 1974-79

Keywords Congruences, light rays, primitive roots, trigonometric sums, hypocycloids, epicycloids, binary expansion, n-ary expansion of 1/p.

Take a circle centered at (0,0), divide it into p parts and take 2^n mod p, if 2 is a primitive root of p then you will have this nice drawing of a cardioïd. That same figure can be obtained by a source at (1,0) that projects p rays at the p equally spaced points on the circumference. If the rays are reflected once then we obtain the curve. You may obtain the same curve by looking at a cup of coffee when you are under the sun during day, a thing that does not happen often in Vancouver (!). The following 6 images are the 5 first reflections of a source of light (the sun at point =infinity) that hits the side of a ideal cup of coffee and rebounds on the side 5 times. The number of rays are 257 in this case.



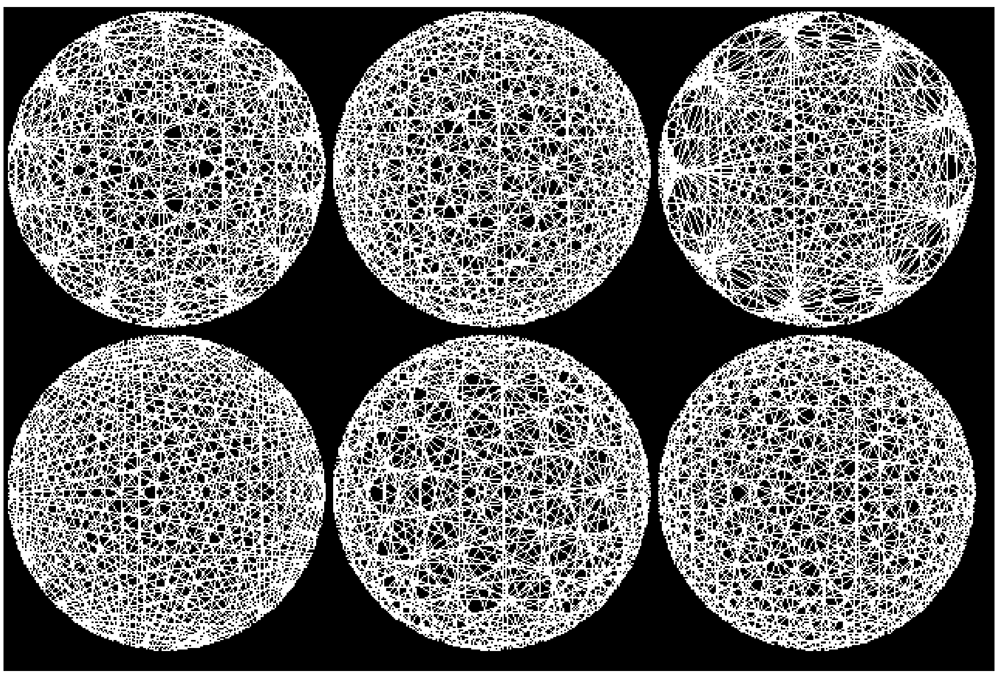
Figures obtained by reflecting a light source 5 times
on the side of a cup of coffee with 257 rays

Remarks :

- Most of these drawing were done by hand first using a ruler and compass, I took 257 because it is easy to construct a polygon with 256 sides (relatively speaking). When apple's became more available, I could at least experiment more...For many values taking 257 instead of 256 do not change the figure that much, if you plot it using a computer screen and all the reflections it does matter.

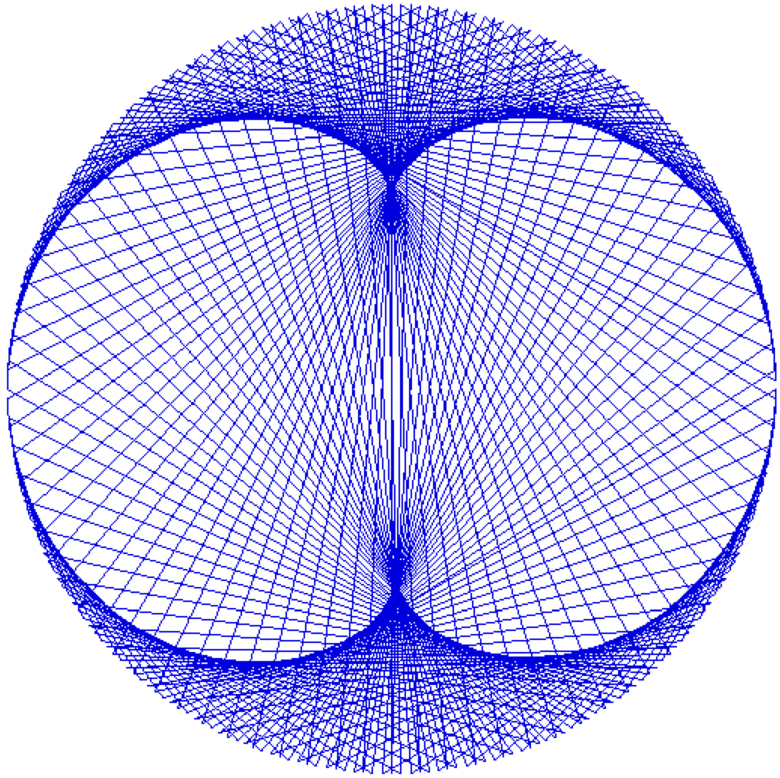
- The second figure is a cardioïd and can also be generated with a circle divided into p (prime) parts and by taking **2^n mod p**, 2 being a **primitive root of p** (so that there are p-1 residues). For doing the figure take the successive residues (mod p) and **join** them with a line.
- By plotting the residues mod p joined by lines it is the same as representing 1/p in base 2 by considering the binary expansion of that number (which has a period of p-1 since 2 is a primitive root). For this we **map** the number x in [0,1] to x -> exp(2*Pi*i*x).
- If we **plot** 1/257 in base 10 or equivalently if we plot the residues of 10^n mod 257 we obtain a strange figure with 9 cusps and many other structures **AND** it is also the 57th reflection of the light hitting the side of a cup of coffee... This fact (as far as I know) is not easily explained. If you ever find an explanation please let me know ! send email to plouffe@math.uqam.ca

  See image #3 of this template.



- The number of different figures obtained with a prime p are (p-1)/2 + 1 , the figures are eventually repeating after a number of reflections.

- The number of principal cusps are b-1 when b is a primitive root of p and b relatively small. It is difficult to come with a general formula. For 3^n mod 257 , see below, we have (as expected), that rule is no longer valid when b gets larger, I have no explanation for the general case. See below the other templates.



With lots of experiments, I came with this formula, it explains many cases like 10^n mod 257 which has 23 secondary cusps, **not all**. For any p and any b there are no (not known to me) other formulas.

The number H of secondary cusps are equal to

$$H = \left[\frac{p}{b}\right] + 1 - \left(b\left(\left[\frac{p}{b}\right] + 1\right) - p\right)$$

for $p \gg b$.

Other templates, see the whole here : bluecircles.html

Reflections 6 to 11
Reflections 12 to 17
Reflections 18 to 23
Reflections 24 to 29
Reflections 30 to 35
Reflections 36 to 41
Reflections 42 to 47
Reflections 48 to 53
Reflections 54 to 59 ...figure of 10^n mod 257
Reflections 60 to 65 ...figure of 5^n mod 257.
Reflections 66 to 71
Reflections 72 to 77
Reflections 78 to 83
Reflections 84 to 89
Reflections 90 to 95
Reflections 96 to 101

 ...figures are repeating from that point after 128 reflections: (p-1)/2. =128, also the figure of 3^n mod 257.
