Identities inspired from the Ramanujan Notebooks

Second series

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Abstract

A series of formula are presented that are all inspired from the Ramanujan Notebooks [6]. One of them appears in the notebooks II

$$\zeta(3) = \frac{7\pi^3}{180} - 2\sum_{n=1}^{\infty}\frac{1}{n^3(e^{2\pi n}-1)}$$

That formula inspired others that appeared in 1998, 2006 and 2009 on the author's website and later in literature [1][2][3]. New formulas for π and the Catalan constant are presented and a surprising series of approximations. A new set of identities is given for Eisenstein series.

Une série de formules utilisant l'exponentielle est présentée, ces résultats reprennent ceux apparaissant en 1998, 2006 et 2009 sur [1][2][3]. Elles sont toutes inspirées des Notebooks de Ramanujan tels que

$$\zeta(3) = \frac{7\pi^3}{180} - 2\sum_{n=1}^{\infty}\frac{1}{n^3(e^{2\pi n}-1)}$$

Une nouvelle série pour π et la constante de Catalan sont présentés ainsi qu'une série d'approximations surprenantes. Une série d'identités nouvelles sont présentées concernant les séries d'Eisenstein.

1. Introduction

By taking back the series found in 2006, I extended the search to more general expressions with exponents 1,2 and 4 for the exponential term and found ;

The same pattern is present for powers of π and $\zeta(n)$

1.1 $$\pi = 72\sum_{n=1}^{\infty}\frac{1}{n(e^{\pi n}-1)} - 96\sum_{n=1}^{\infty}\frac{1}{n(e^{2\pi n}-1)} + 24\sum_{n=1}^{\infty}\frac{1}{n(e^{4\pi n}-1)}$$

1.2 $$\frac{1}{\pi} = 8\sum_{n=1}^{\infty}\frac{n}{e^{\pi n}-1} - 40\sum_{n=1}^{\infty}\frac{n}{e^{2\pi n}-1} + 32\sum_{n=1}^{\infty}\frac{n}{e^{4\pi n}-1}$$

1.3 $$\pi^3 = 720\sum_{n=1}^{\infty}\frac{1}{n^3(e^{\pi n}-1)} - 900\sum_{n=1}^{\infty}\frac{1}{n^3(e^{2\pi n}-1)} + 180\sum_{n=1}^{\infty}\frac{1}{n^3(e^{4\pi n}-1)}$$

1.4 $$\zeta(3) = 28\sum_{n=1}^{\infty}\frac{1}{n^3(e^{\pi n}-1)} - 37\sum_{n=1}^{\infty}\frac{1}{n^3(e^{2\pi n}-1)} + 7\sum_{n=1}^{\infty}\frac{1}{n^3(e^{4\pi n}-1)}$$

1.5 $$\pi^5 = 7056\sum_{n=1}^{\infty}\frac{1}{n^5(e^{\pi n}-1)} - 6993\sum_{n=1}^{\infty}\frac{1}{n^5(e^{2\pi n}-1)} + 63\sum_{n=1}^{\infty}\frac{1}{n^5(e^{4\pi n}-1)}$$

1.6 $$\zeta(5) = 24\sum_{n=1}^{\infty}\frac{1}{n^5(e^{\pi n}-1)} - \frac{259}{10}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{2\pi n}-1)} - \frac{1}{10}\sum_{n=1}^{\infty}\frac{1}{n^5(e^{4\pi n}-1)}$$

1.7 $$\pi^7 = \frac{907200}{13}\sum_{n=1}^{\infty}\frac{1}{n^7(e^{\pi n}-1)} - 70875\sum_{n=1}^{\infty}\frac{1}{n^7(e^{2\pi n}-1)} + \frac{14175}{13}\sum_{n=1}^{\infty}\frac{1}{n^7(e^{4\pi n}-1)}$$

1.8 $$\zeta(7) = \frac{304}{13}\sum_{n=1}^{\infty}\frac{1}{n^7(e^{\pi n}-1)} - \frac{103}{4}\sum_{n=1}^{\infty}\frac{1}{n^7(e^{2\pi n}-1)} + \frac{19}{52}\sum_{n=1}^{\infty}\frac{1}{n^7(e^{4\pi n}-1)}$$

For Catalan constant, I find this new identity :

1.9 $$K = 11\sum_{n=1}^{\infty}\frac{1}{n^2(\cosh(\pi n)-1)} - \frac{71}{2}\sum_{n=1}^{\infty}\frac{1}{n^2(\cosh(2\pi n)-1)} + 11\sum_{n=1}^{\infty}\frac{1}{n^2(\cosh(4\pi n)-1)}$$

For $1/\pi^2$, by varying the function at the numerator, I find this:

1.10 $$\frac{1}{\pi^2} = 4\sum_{n=1}^{\infty}\frac{\sigma_1(n)n}{e^{\pi n}} - 64\sum_{n=1}^{\infty}\frac{\sigma_1(n)n}{e^{2\pi n}} + 64\sum_{n=1}^{\infty}\frac{\sigma_1(n)n}{e^{4\pi n}}$$

Here, $\sigma_1(n)$ is Euler sigma function of order 1, Actually the same coefficients as with cosh (kπn), k=1,2 et 4.

1.11 $$\frac{1}{\pi^2} = 2\sum_{n=1}^{\infty}\frac{n^2}{\cosh(\pi n) - 1} - 32\sum_{n=1}^{\infty}\frac{n^2}{\cosh(2\pi n) - 1} + 32\sum_{n=1}^{\infty}\frac{n^2}{\cosh(4\pi n) - 1}$$

The pattern persist for $1/\pi^3$ but apparently for no other powers of π

1.12 $$\frac{1}{\pi^3} = 4\sum_{n=1}^{\infty}\frac{\sigma_1(n)n^2}{e^{\pi n}} - 128\sum_{n=1}^{\infty}\frac{\sigma_1(n)n^2}{e^{2\pi n}} + 256\sum_{n=1}^{\infty}\frac{\sigma_1(n)n^2}{e^{4\pi n}}$$

2. Experiments with fractional exponent

I was compiling a table of values for the Inverter [9] and found that for some arguments the closeness to rational numbers, these are the 2 examples that are the most striking.

2.1 $$\sum_{n=1}^{\infty}\frac{n^3}{e^{2\pi n/7} - 1} = 10.000000000000000190161767888663\ldots$$

2.2 $$\sum_{n=1}^{\infty}\frac{n^3}{e^{2\pi n/13} - 1} \cong 119.00000000000000000000000000000000959374585\ldots$$

The precision is 15 and 31 decimal digits and for an argument of $2\pi n/163$ the precision is 435 decimal digits. Other series of the form $\sum_{n=1}^{\infty}\frac{n^3}{e^{\frac{2\pi n}{k}}-1}$ are also producing near integers when k is not a multiple of 2,3 and 5. For the exponent one can obtain near integers when the exponent of n is 4m-1, m > 0. This fact is related to properties of Eisenstein series which is ; if $240 \mid k^4 - 1$ then the series is near an integer. But it is not always that it produces approximations since I have this identity for π. We see the pattern [1,2,4] again with the exponent.

2.3 $$\begin{aligned}\frac{\pi}{10} = -\sum_{n=1}^{\infty}\frac{1}{n(e^{\pi n}-1)} + 4\sum_{n=1}^{\infty}\frac{1}{n(e^{2\pi n}-1)} \\ - 1\sum_{n=1}^{\infty}\frac{1}{n(e^{4\pi n}-1)} + \sum_{n=1}^{\infty}\frac{1}{n(e^{\pi n/5}-1)} - 4\sum_{n=1}^{\infty}\frac{1}{n(e^{2\pi n/5}-1)} \\ + \sum_{n=1}^{\infty}\frac{1}{n(e^{4\pi n/5}-1)}\end{aligned}$$

2.4 $$\frac{7\pi}{120} = -2\sum_{n=1}^{\infty}\frac{1}{n(e^{\pi n}-1)} - \sum_{n=1}^{\infty}\frac{1}{n(e^{\pi n/5}-1)} + 4\sum_{n=1}^{\infty}\frac{1}{n(e^{2\pi n/5}-1)} - 1\sum_{n=1}^{\infty}\frac{1}{n(e^{4\pi n/5}-1)}$$

2.5 $$\begin{aligned}3\log(\varphi) = &-4\sum_{n=1}^{\infty}\frac{1}{n(e^{\pi n}-1)} + 10\sum_{n=1}^{\infty}\frac{1}{n(e^{2\pi n}-1)} \\ &- 4\sum_{n=1}^{\infty}\frac{1}{n(e^{4\pi n}-1)} + 4\sum_{n=1}^{\infty}\frac{1}{n(e^{\frac{\pi n}{5}}-1)} - 10\sum_{n=1}^{\infty}\frac{1}{n(e^{2\pi n/5}-1)} \\ &+ 4\sum_{n=1}^{\infty}\frac{1}{n(e^{4\pi n/5}-1)}\end{aligned}$$

2.6 $$\sum_{n=1}^{\infty}\frac{1}{n(e^{\pi n/5}-1)} - 4\sum_{n=1}^{\infty}\frac{1}{n(e^{2\pi n/5}-1)} + \sum_{n=1}^{\infty}\frac{1}{n(e^{4\pi n/5}-1)} = \frac{\pi}{40} - \frac{3\ln(\pi)}{2} + 2ln\Gamma\left(\frac{1}{4}\right) - \frac{7\ln(2)}{4}$$

2.7 $$\log(\varphi) = \sum_{n=1}^{\infty}\frac{1}{n(e^{\frac{\pi n}{5}}-1)} - 2\sum_{n=1}^{\infty}\frac{1}{n(e^{\frac{2\pi n}{5}}-1)} + \sum_{n=1}^{\infty}\frac{1}{n(e^{\frac{4\pi n}{5}}-1)} + \frac{\pi}{120} - \frac{\ln(2)}{4}$$

where φ is the golden ratio.

Since the generic series for π is with $\sum_{n=1}^{\infty}\frac{1}{n(e^{\pi n}-1)}$ that series is the log of the well known Euler partition function $F(x) = \prod_{n=1}^{\infty}\frac{1}{1-x^n}$ series which means that an identity can be translated into the partition function as well.

2.8 $$\frac{F(e^{\pi/5})^5 F(e^{4\pi/5})^5}{F(e^{2\pi/5})^{20}} = \frac{F(e^{2\pi})^{28} F(e^{4\pi})^{31}}{F(e^{\pi})^7}$$

The exact expression for $\sum_{n=1}^{\infty}\frac{1}{n(e^{\pi n}-1)}$ can be found also by a variety of methods, one of them is to simply try it in the Integer Relations algorithm to find (2.6). For

The exact expression for k=2/7 was found by Bill Gosper using the Computer Algebra system Macsyma and a set of personal routines.

2.9 $$\sum_{n=1}^{\infty}\frac{n^3}{e^{2\pi n/7}-1} = \frac{-1}{240} + \frac{1}{320}\left(301 + 210\sqrt{2}\,7^{1/4} + 120\sqrt{7} + 90\sqrt{2}\,7^{3/4}\right)\frac{\pi^2}{\Gamma(\frac{3}{4})^8}$$

In fact that series is the series of Eisenstein which are:

2.10 $$E_4(q) = 1 + 240\sum_{k=1}^{\infty}\sigma_3(k)q^{2k}$$

2.11 $$E_8(q) = 1 + 480\sum_{k=1}^{\infty}\sigma_7(k)q^{2k}$$

2.12 $$E_{12}(q) = 1 + \frac{65520}{691}\sum_{k=1}^{\infty}\sigma_{11}(k)q^{2k}$$

For $E_4(e^{\pi/10})$, $E_4(e^{\pi/5}), E_4(e^{2\pi/5})$ we can get respectively

2.13 $$\frac{\pi^2}{\Gamma(3/4)^8}\left(\frac{5313}{4} + 630\sqrt{5} + 90\sqrt{360 + 161\sqrt{5}}\right)$$

2.14 $$\frac{\pi^2}{\Gamma(3/4)^8}\left(\frac{(483}{4}90\sqrt{5})\right)$$

2.15 $$\frac{\pi^2}{\Gamma(3/4)^8}\left((5313 + 2520\sqrt{5} - 32\sqrt{\frac{91125}{2} + \frac{326025\sqrt{5}}{16})}\right)$$

Note, $E_4(e^{\pi/10}) = 10000.000000000000000000012378\ldots$

2.16 $$2\sum_{n=1}^{\infty}\frac{n^3}{e^{\pi n/5}-1} - 28\sum_{n=1}^{\infty}\frac{n^3}{e^{2\pi n/5}-1} + 32\sum_{n=1}^{\infty}\frac{n^3}{e^{4\pi n/5}-1} - 257\sum_{n=1}^{\infty}\frac{n^3}{e^{\pi n}-1} + 251\sum_{n=1}^{\infty}\frac{n^3}{e^{2\pi n}-1} = 0$$

2.17 $$8\sum_{n=1}^{\infty}\frac{n^7}{e^{\pi n/5}-1} - 2192\sum_{n=1}^{\infty}\frac{n^7}{e^{2\pi n/5}-1} + 2048\sum_{n=1}^{\infty}\frac{n^7}{e^{4\pi n/5}-1} + 208897\sum_{n=1}^{\infty}\frac{n^7}{e^{\pi n}-1} - 208761\sum_{n=1}^{\infty}\frac{n^7}{e^{2\pi n}-1} = 0$$

The pattern observed earlier [1,2,4] can be translated into Eisenstein series identities that are new (?).

2.18 $$-E_4\left(e^{\pi/10}\right) + 14E_4\left(e^{\pi/5}\right) - 16E_4\left(e^{2\pi/5}\right) + \frac{1288}{11}E_4\left(e^{\pi/2}\right) = 0$$

2.19 $$E_8\left(e^{\pi/10}\right) - 274E_8\left(e^{\pi/5}\right) + 256E_8\left(e^{2\pi/5}\right) + \frac{3133472}{121}E_8\left(e^{\pi/2}\right) = 0$$

2.20 $$-E_{12}\left(e^{\pi/10}\right) + 4034E_{12}\left(e^{\pi/5}\right) - 4096E_{12}\left(e^{2\pi/5}\right) + \frac{7811747968}{2081}E_{12}\left(e^{\pi/2}\right) = 0$$

I could not find any other exact fractional values, the other ones are only translatable into approximations like, here F as in (2.8).

2.21	$F(e^{\pi})^8 \approx \frac{e^{\pi}}{2^4}$, 4 digits
2.22	$F(e^{\pi/2})^{16} \approx \frac{e^{5\pi}}{2^{16}}$, 6 digits

2.23	$F(e^{\pi/8})^{64} \approx \frac{e^{85\pi}}{2^{128}}$, 36 digits
2.24	$F(e^{\pi/32})^{256} \approx \frac{e^{1365\pi}}{2^{768}}$, 173 digits

3. Conclusion

As far as the author knows the formulas with arguments [1/5,2/5,4/5] are new, the formula for Catalan , π and $1/\pi$ as well. No other exact formula was found for the fractional exponent but the search was limited to the Farey set of order 60 only.

References

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