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TRANSCENDENCY OF VARIANTS OF MILLS' CONSTANT

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ABSTRACT. Let $\lfloor x \rfloor$ denote the integer part of x. For every sequence $(C_k)_{k\geq 1}$ of positive integers, we define $\xi(C_k)$ as the smallest real number ξ such that $\lfloor \xi^{C_k} \rfloor$ is a prime number for every positive integer k. The number $\xi(3^k)$ is called Mills' constant. Recently, the author showed that $\xi(3^k)$ is irrational, but the transcendency of this number is still open. In this paper, we give sufficient conditions on $(C_k)_{k\geq 1}$ to determine the transcendency of $\xi(C_k)$. For example, we obtain the following three results:
(A) $\xi(\lfloor b^k \rfloor)$ is irrational for every real number $b \geq 1+\sqrt{2}$;
(B) $\xi((1+\sqrt{2})^k + (1-\sqrt{2})^k)$ is transcendental;
(C) $\xi(r3^k - 1)$ is transcendental for every integer $r \geq 4.003 \times 10^{14}$.
Furthermore, assuming the Riemann hypothesis, $\xi(3^k - 1)$ is transcendental. However, we still do not know whether Mills' constant is transcendental.

1. INTRODUCTION

Let $\mathbb{N}$ be the set of all positive integers. For all real numbers x, we define $\lfloor x \rfloor$ as the greatest integer less than or equal to x. In 1947, Mills [Mil47] constructed a real number $A > 1$ such that

$$\lfloor A^{3^k} \rfloor \text{ is a prime number for all } k \in \mathbb{N}. \tag{1.1}$$

After that, mathematicians started to study variants and generalisations of Mills' result. For example, Wright [Wri51] found a real number $\mu > 1$ such that all of

$$\lfloor 2^{\mu} \rfloor, \quad \lfloor 2^{2^{\mu}} \rfloor, \quad \lfloor 2^{2^{2^{\mu}}} \rfloor, \quad \cdots$$

are prime numbers. We refer the readers to [Dud69] which is a good survey for quickly reviewing early works on prime-representing functions.

The smallest real number A satisfying (1.1) exists, and it is called Mills' constant (see [Fin03]). Let us prove the existence in Lemma 4.1. Assuming the Riemann hypothesis (for short RH), Caldwell and Cheng [CC05] gave a decimal approximation to Mills' constant as follows:

$$1.30637\,78838\,63080\,69046\,86144\,92602\cdots.$$

Proving the irrationality of this number turned out to elude some previous efforts by Alkauskas and Dubickas [AD04] and by the author and Takeda [ST22]. However, the author [Sai25b, Theorem 1.2] recently resolved that Mills' constant is irrational. Furthermore, the author [Sai25b, Proposition 5.2] proved that Mills' constant is transcendental if there exists a real number $\theta \in (0, 1/2)$ such that for every sufficiently large real number x, we find a prime number p satisfying $x \leq p \leq x + x^{\theta}$. Under the RH, Carneiro, Milinovich,

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and Soundararajan [CMS19, Theorem 1.5] showed that for all real numbers $x \geq 4$, there exists a prime number p such that

$$p \in \left[x, x + \frac{22}{25}\sqrt{x}\log x\right].$$

Therefore, even if the RH is true, we still do not know whether Mills' constant is transcendental. Alternatively, we succeed in proving the transcendency of a shifted analogue of Mills' constant as follows.

Theorem 1.1. *There exists a real number $A > 1$ such that $\lfloor A^{3^k-1} \rfloor$ is always a prime number for every $k \in \mathbb{N}$. The smallest real number of such A exists, say ξ. If the Riemann hypothesis is true, then ξ is transcendental.*

This theorem is just one of the representative examples in this paper. To state more general results, we now prepare several notations. Let $(C_k)_{k\geq 1}$ be a positive real sequence. We always write $c_1 = C_1$ and $c_{k+1} = C_{k+1}/C_k$ for all $k \in \mathbb{N}$. Therefore, we have

$$C_k = c_1 c_2 \cdots c_k$$

for all $k \in \mathbb{N}$. We define

$$\mathcal{W}(C_k) = \mathcal{W}((C_k)_{k\geq 1}) = \{A > 1 \colon \lfloor A^{C_k} \rfloor \text{ is a prime number for all } k \in \mathbb{N}\}.$$

For simplicity, we usually use the notation $\mathcal{W}(C_k)$ instead of $\mathcal{W}((C_k)_{k\geq 1})$, and so we remark that $\mathcal{W}(C_k)$ is independent of k. For instance, if $(C_k)_{k\geq 1} = (3^k)_{k\geq 1}$, then

$$\mathcal{W}(3^k) = \{A > 1 \colon \lfloor A^{3^k} \rfloor \text{ is a prime number for every } k \in \mathbb{N}\}.$$

By [Sai25b, Lemma 3.1], if $\mathcal{W}(C_k)$ is non-empty, then the smallest element of $\mathcal{W}(C_k)$ exists, say $\xi(C_k)$.

Essentially by Matomäki [Mat10, Theorem 3][1], we have sufficient conditions that $\mathcal{W}(C_k)$ is non-empty (*i.e.* $\xi(C_k)$ exists) as follows.

Theorem 1.2. *Let $(C_k)_{k\geq 1}$ be a positive real sequence satisfying*

(1) $c_1 > 0$;
(2) $c_{k+1} \geq 2$ *for all* $k \in \mathbb{N}$.

Then, $\mathcal{W}(C_k)$ is non-empty and $\xi(C_k)$ exists.

Is $\xi(C_k)$ irrational? If the answer is "YES", is it transcendental? A Pisot number plays a key role in answering the question. We say that a real algebraic integer $\beta > 1$ is a *Pisot number* if all conjugates of β over $\mathbb{Q}$ lie in the open unit disk of $\mathbb{C}$ except for itself. If the degree of β is 2 or 3, then β is called *quadratic* or *cubic*, respectively. For example, the golden ratio is a quadratic Pisot number. The unique real root of $X^3 - X - 1$ is a cubic Pisot number. In [Sai25b, Theorem 1.5], the author proved the following result.

Theorem 1.3. *Let $(C_k)_{k\geq 1}$ be a positive real sequence satisfying* (1), (2) *in Theorem 1.2, and $c_k \in \mathbb{N}$ for all $k \in \mathbb{N}$. Then, the smallest real number of $\mathcal{W}(C_k)$ exists, say ξ. If $\limsup_{k\to\infty} c_{k+1} \geq 4$, then ξ is transcendental. If $\limsup_{k\to\infty} c_{k+1} = 3$, then either ξ is transcendental, or there exists $m \in \mathbb{N}$ such that ξ^{C_m} is a cubic Pisot number.*

[1]Actually, Matomäki showed that $\mathcal{W}(C_k)$ is uncountable, nowhere dense, and has Lebesgue measure 0 if a sequence $(c_k)_{k\geq 1}$ of real numbers satisfies $c_1 \geq 2$ and (2). There is no essential difference between the initial conditions $c_1 > 0$ and $c_1 \geq 2$ (see [ST22, Remark 3.4]).

This theorem implies that $\xi(3^k)$ is irrational, that is, Mills' constant is irrational. In addition, $\xi(c^k)$ is transcendental for all integers $c \geq 4$. However, we still cannot determine whether $\xi(3^k)$ is transcendental or whether $\xi(2^k)$ is irrational. We summarise the arithmetic properties of $\xi(C_k)$ in Table 1, where $(C_k)_{k\geq 1}$ satisfies the conditions in Theorem 1.3.

$\limsup_{k\to\infty} c_{k+1}$	2	3	4	5	$\cdots$
Irrationality	?	✓	✓	✓	$\cdots$
Transcendency	?	?	✓	✓	$\cdots$

TABLE 1. Arithmetic properties of $\xi(C_k)$

We may find briers for the irrationality and the transcendency from Table 1. In this paper, we aim to extend the condition $c_k \in \mathbb{N}$ to $c_k \in \mathbb{Q} \cap (0, \infty)$, and we find a specific sequence $(C_k)_{k\geq 1}$ braking these briers.

Theorem 1.4. *The following properties are true*:

(A) $\xi(\lfloor b^k \rfloor)$ *is irrational for every fixed real number* $b \geq 1 + \sqrt{2}$;
(B) $\xi((1+\sqrt{2})^k + (1-\sqrt{2})^k)$ *is transcendetal*;
(C) $\xi(r3^k - 1)$ *is transcendental for every fixed integer* $r \geq 4.003 \times 10^{14}$.

Furthermore, we assume the Riemann hypothesis. Then, (C) *is true for every integer* $r \geq 1$.

It is clear that Theorem 1.4 implies Theorem 1.1. We obtain more general results than Theorem 1.4, which we will discuss in the next section.

Remark 1.5. Alkauskas and Dubickas [AD04] constructed a specific transcendental number in $\mathcal{W}(C_k)$ if a sequence $(c_k)_{k\geq 1}$ of rational numbers satisfies $c_1 = 1$, $c_{k+1} > 2.1053$ $(k = 1, 2, 3, \ldots)$, $\limsup_{k\to\infty} c_{k+1} = \infty$, and $C_k = c_1 \cdots c_k \in \mathbb{N}$. Thus, they have already discussed the case $c_k \in \mathbb{Q}$.

Notation 1.6. Let $\mathcal{P}$ be the set of all prime numbers. For all $x \in \mathbb{R}$, $\{x\}$ denotes the fractional part of x *i.e.* $\{x\} = x - \lfloor x \rfloor$. For all finite sets A, $\#A$ denotes the number of elements in A.

2. Types A,B,C and Proof of Theorem 1.4

In this section, we exhibit three types of results which correspond to (A), (B), (C) in Theorem 1.4, say Type A, Type B, and Type C. These names are symbolic and therefore have no deeper meaning.

2.1. **Type A.**

Theorem 2.1. *Suppose that* $(c_k)_{k\geq 1}$ *is a real sequence satisfying*

(A1) $c_1 \geq 1$;
(A2) $c_{k+1} \geq 2$ *for all* $k \in \mathbb{N}$;
(A3) $\limsup_{k\to\infty} c_{k+1} > 40/19$;
(A4) $C_k = c_1 \cdots c_k \in \mathbb{N}$ *for all* $k \in \mathbb{N}$.

Then, $\xi = \xi(C_k)$ exists. Furthermore, either ξ is transcendental, or there exists $m \in \mathbb{N}$ such that ξ^{C_m} is a Pisot number of degree ℓ satisfying

$$1 \le \ell \le 1 + \left(\frac{19}{40} \limsup_{k\to\infty} c_{k+1} - 1 \right)^{-1}. \tag{2.1}$$

In particular, ξ is irrational.

We will prove Theorem 2.1 in Section 7. A feature of Type A is that it does not assume algebraic conditions on $(C_k)_{k\ge1}$ such as divisibility and congruence conditions. This theorem implies the irrationality of $\xi(\lfloor \alpha b^k \rfloor)$, although the algebraic properties of $\lfloor \alpha b^k \rfloor$ are highly unclear for given specific real numbers b and α.

Corollary 2.2. *Let b and α be real numbers with $b > 40/19 = 2.10526\cdots$ and $\alpha \ge \max(\frac{1}{b(b-2)}, \frac{1}{b})$. Then, $\xi = \xi(\lfloor \alpha b^k \rfloor)$ exists. Furthermore, either ξ is transcendental, or there exists $k \in \mathbb{N}$ such that $\xi^{\lfloor \alpha b^k \rfloor}$ is a Pisot number of degree ℓ satisfying*

$$1 \le \ell \le 1 + \left(\frac{19}{40} b - 1 \right)^{-1}$$

In particular, ξ is irrational.

Proof. Let $C_k = \lfloor \alpha b^k \rfloor$ for every $k \in \mathbb{N}$. We recall that $c_1 = C_1$, and $c_{k+1} = C_{k+1}/C_k$ for every $k \in \mathbb{N}$. Then, it is clear that $(c_k)_{k\ge1}$ satisfies (A3) and (A4) in Theorem 2.1 since $\lim_{k\to\infty} c_{k+1} = b > 40/19$ and $C_k = \lfloor \alpha b^k \rfloor \in \mathbb{N}$ for all $k \in \mathbb{N}$. Furthermore, by $\alpha \ge \max(\frac{1}{b(b-2)}, \frac{1}{b})$, we have

$$c_1 = C_1 = \lfloor \alpha b \rfloor \ge \lfloor (1/b) \cdot b \rfloor = 1,$$
$$c_{k+1} = \frac{C_{k+1}}{C_k} = \frac{\lfloor \alpha b^{k+1} \rfloor}{\lfloor \alpha b^k \rfloor} \ge \frac{\alpha b^{k+1} - 1}{\alpha b^k} \ge b - \frac{1}{\alpha b} \ge b - \frac{1}{\frac{b}{b(b-2)}} = 2$$

for all $k \in \mathbb{N}$. Thus, $(c_k)_{k\ge1}$ satisfies (A1) and (A2). Therefore, Theorem 2.1 implies Corollary 2.2. □

Proof of (A) *in Theorem 1.4.* For all real numbers $b > 2$, we observe that

$$1 \ge \frac{1}{b(b-2)} \iff b^2 - 2b - 1 \ge 0 \iff b \ge 1 + \sqrt{2}.$$

Applying Corollary 2.2 with $\alpha = 1$ and $b \ge 1 + \sqrt{2}$, we obtain (A). □

2.2. **Type B.**

Theorem 2.3. *Suppose that $(c_k)_{k\ge1}$ is a real sequence satisfying*

(B1) $c_1 \ge 1$;
(B2) $c_{k+1} \ge 2$ *for all* $k \in \mathbb{N}$;
(B3) $\limsup_{k\to\infty} c_{k+1} > 40/19$;
(B4) $C_k = c_1 \cdots c_k \in \mathbb{N}$ *for all* $k \in \mathbb{N}$.

Let ϵ be a sufficiently small positive real number such that $I := \{k \in \mathbb{N} \colon c_{k+1} \ge 40/19 + \epsilon\}$ is infinite. We suppose that

(B5) *for all $m \in \mathbb{N}$ there exists $k \in I$ with $k > m$ such that $C_m \mid C_k$.*

Then, $\xi = \xi(C_k)$ exists. Furthermore, either ξ is transcendental, or there exists $g \in \mathbb{N}$ such that ξ^g is a Pisot number of degree ℓ with

$$3 \leq \ell \leq 1 + \left(\frac{19}{40} \limsup_{k\to\infty} c_{k+1} - 1\right)^{-1},$$

and for every sufficiently large $k \in I$, we have $g \mid C_k$. In addition, we suppose that

(B6) *for all $m \in \mathbb{N}$ and $L \in \mathbb{N}$ there exists $k \in I$ with $k > m$ such that*

$$C_k \equiv C_m \bmod LC_m.$$

Then, ξ is transcendental.

Remark 2.4. This theorem implies Theorem 1.3. Therefore, Theorem 2.3 is an extension of Theorem 1.3 that is the previous main result in [Sai25b].

We will prove Theorem 2.3 in Section 8. We note that (B1) to (B4) are coincedent with (A1) to (A4). Thus, by adding the algebraic conditions (B5) and (B6), we can find stronger arithmetic properties of $\xi(C_k)$. A certain class of linear recurrence sequences satisfies (B1) to (B6).

Corollary 2.5. *Let d be an integer greater than or equal to 2. Let $a_1, \dots, a_{d-1}$ be integers, and let $a_0 \in \{-1, 1\}$. Let $(R_k)_{k\geq 1}$ be a sequence of positive integers satisfying*

(B1') $R_1 \geq 1$;
(B2') $R_{k+1}/R_k \geq 2$ *for all* $k \in \mathbb{N}$;
(B3') $\liminf_{k\to\infty} R_{k+1}/R_k > 40/19$;
(B4') $R_{k+d} = a_{d-1}R_{k+d-1} + a_{d-2}R_{k+d-2} + \cdots + a_1R_{k+1} + a_0R_k$ *for all* $k \in \mathbb{N}$.

Then, the number $\xi(R_k)$ exists, and it is transcendental.

We will prove Corollary 2.5 in Section 8.

Proof of (B) *in Theorem 1.4.* Let $R_k = (1+\sqrt{2})^k + (1-\sqrt{2})^k$ for all $k \in \mathbb{N}$. Then, by basic calculation, we have $R_1 = 2$, $R_2 = 4$, and

$$R_{k+2} = 2R_{k+1} + R_k$$

for all $k \in \mathbb{N}$. Therefore, $(R_k)_{k\geq 1}$ satisfies (B1') to (B4'). By applying Corollary 2.5, $\xi(R_k) = \xi((1+\sqrt{2})^k + (1-\sqrt{2})^k)$ is transcendental. □

2.3. **Type C.** For all integers $a_1, a_2, \dots, a_k$, we define $\gcd(a_1, \dots a_k)$ as the gratest positive integers d such that $d \mid a_1$, $d \mid a_2$, … , $d \mid a_k$. For all infinite sequences $(a_k)_{k\geq 1}$ of integers, $\gcd(a_1, a_2, \dots, a_k)$ is decreasing for $k = 1, 2, \dots$, and hence we define

$$\gcd(a_1, a_2, \dots) = \lim_{k\to\infty} \gcd(a_1, a_2, \dots, a_k),$$

which is well-defined. We also see that $\gcd(a_m, a_{m+1}, \dots)$ is increasing for $m = 1, 2, \dots$, and so we define the *asymptotic greatest common divisor of* $(a_k)_{k\geq 1}$ by

$$\lim_{m\to\infty} \gcd(a_m, a_{m+1}, \dots) \in \mathbb{N} \cup \{\infty\}$$

and it is denoted by $\operatorname{agcd}((a_k)_{k\geq 1}) = \operatorname{agcd}(a_k)$.

Theorem 2.6. *Let c be a real number grater than or equal to 3. Assume that there exists $x_0 \geq 1$ such that for all real numbers $x \geq x_0$,*

$$\mathcal{P} \cap [x, x + cx^{1-1/c}] \neq \emptyset. \tag{2.2}$$

Suppose that $(c_k)_{k\geq 1}$ is a sequence of real numbers satisfying

(C1) $c_1 \geq 1$;
(C2) $c_{k+1} \geq c$ *for all* $k \in \mathbb{N}$;
(C3) $C_k = c_1 \cdots c_k \in \mathbb{N}$ *for all* $k \in \mathbb{N}$;
(C4) *for all* $m \in \mathbb{N}$ *there exists* $k \in \mathbb{N}$ *such that* $C_m \mid C_k$;
(C5) $\mathrm{agcd}(C_k) < \infty$.

Then, $\xi = \xi(C_k)$ *exists. Furthermore, either* ξ *is transcendental or there exists* $g \in \mathbb{N}$ *such that* $g \mid \mathrm{agcd}(C_k)$ *and* ξ^g *is a cubic Pisot number less than or equal to*

$$(2x_0^{1/c_2} + 1)^{g/c_1}.$$

In the case $c = 3$, by applying [MTY24, Theorem 1.1] and [CH23, Section 4], Mossinghoff, Trudgian, and Yang [MTY24, Section 2] showed that (2.2) is true for all

$$x \geq \exp(3\exp(32.76)),$$

which implies the following corollary.

Corollary 2.7. *The number* $\xi(r3^k - 1)$ *exsits and it is transcedental for all integers* $r \geq 400296054181892 \approx 4.003 \times 10^{14}$.

It is natural to expect that (2.2) with $c = 3$ is true for all $x \geq 1$. As a partial result, Cully-Hugill [CH23] proved that (2.2) with $c = 155$ is true for all $x \geq 1$. If the expectation were true, then we could replace 4.003×10^{14} with 1.

Corollary 2.8. *Assume that* (2.2) *with* $c = 3$ *is true for all real numbers* $x \geq 1$. *Then, for every fixed integer* $r \geq 1$, *the number* $\xi(r3^k - 1)$ *exists and it is transcendental.*

We will prove Corollaries 2.7 and 2.8 in Section 9.

Proof of (C) *in Theorem 1.4.* By Corollary 2.8, it suffices to show that the RH implies that $x_0 = 1$. As we mentioned in Section 1, assuming the RH, [CMS19, Theorem 1.5] implies that for all real numbers $x \geq 4$, there exists a prime number p such that

$$p \in \left[x, x + \frac{22}{25}\sqrt{x}\log x\right].$$

Therefore, by simple calculation, we obtain (2.2) for all real numbers $x \geq 1$. □

3. A general result

In this section, we provide a key proposition which implies Theorems 2.1, 2.3, and 2.6. Before claiming, let $\mathrm{Tr}(\beta)$ be the trace of β over $\mathbb{Q}$ for every algebraic number β, that is, $\mathrm{Tr}(\beta) = \beta_1 + \cdots + \beta_\ell$, where $\beta_1, \ldots, \beta_\ell$ denote all the conjugates of β over $\mathbb{Q}$.

We consider a positive real number θ satisfying the following condition:

(†) There are $d > 0$ and $x_0 > 0$ such that for every $x \geq x_0$, we have

$$\#(\mathcal{P} \cap [x, x + x^\theta]) \geq \frac{dx^\theta}{\log x}.$$

Theorem 3.1. *The number* 21/40 *satisfies* (†).

Proof. It follows from [BHP01] by Baker, Harman, and Pintz. □

Proposition 3.2. *Let* θ *be a real number in* $[1/2, 1)$ *satisfying* (†). *Suppose that* $(c_k)_{k\geq 1}$ *is a sequence of real numbers satisfying*

(G1) $c_1 \geq 1$;
(G2) $c_{k+1} \geq 2$ *for all* $k \in \mathbb{N}$;

(G3) $\limsup_{k\to\infty} c_{k+1} > 1/(1-\theta)$;
(G4) $C_k = c_1 \cdots c_k \in \mathbb{N}$ *for every sufficiently large* $k \in \mathbb{N}$.

Then, $\xi = \xi(C_k)$ *exists. We further suppose that for every* $m \in \mathbb{N}$*, we have*

$$\xi^{C_m} \notin \mathbb{N}, \tag{3.1}$$

and we suppose that ξ *is algebraic. Let* ϵ *be a fixed small real number with*

$$0 < \epsilon < \limsup_{k\to\infty} c_{k+1} - 1/(1-\theta). \tag{3.2}$$

Let $I = \{k \in \mathbb{N} \colon c_{k+1} \geq 1/(1-\theta) + \epsilon\}$*. Then, the following properties are true*:

(i) *We have*
$$\{\xi^{C_k}\} \leq \frac{2}{c_{k+1}\lfloor \xi^{C_k} \rfloor^{(1-\theta)c_{k+1}-1}}$$
for every sufficiently large $k \in I$;
(ii) *There exists* $g \in \mathbb{N}$ *such that* $\xi^g = \beta$ *is a Pisot number of degree* ℓ *with*
$$2 \leq \ell \leq 1 + \left((1-\theta)\limsup_{k\to\infty} c_{k+1} - 1\right)^{-1}$$
and we have $g \mid C_k$ *for every sufficiently large* $k \in I$;
(iii) *If* $\ell = 2$*, then* $\xi^g = \beta = (1+\sqrt{5})/2$ *and* C_k/g *is an odd prime number for every sufficiently large* $k \in I$;
(iv) *We have* $\mathrm{Tr}(\beta^{C_k/g}) = \lfloor \xi^{C_k} \rfloor$ *for every sufficiently large* $k \in I$.

The rest of the paper is organised as follows. In Section 4, we quickly review the proof of Theorem 1.2. In Section 5, we prepare auxiliary results for proving Proposition 3.2. In Section 6, we complete the proof of Proposition 3.2. In Sections 7, 8, and 9, we prove Theorems 2.1, 2.3, and 2.6, respectively.

4. Proof of Theorem 1.2

Lemma 4.1 ([ST22, Lemma 4.1] and [Sai25b, Lemma 3.1])**.** *Let* $(C_k)_{k\geq 1}$ *be a sequence of positive real numbers. If* $\mathcal{W}(C_k)$ *is non-empty, then the smallest element of* $\mathcal{W}(C_k)$ *exists.*

Proof. Since $\mathcal{W}(C_k)$ is non-empty and lower bounded, its infimum exists, say ζ. Let k be a fixed positive integer. By the definition of the infimum, we find a sequence $A_1 > A_2 > \cdots$ of $\mathcal{W}(C_k)$ satisfying $\lim_{j\to\infty} A_j = \zeta$. By the right-hand side continuity of $\lfloor \cdot \rfloor$, we obtain

$$\lfloor \zeta^{C_k} \rfloor = \lfloor \lim_{j\to\infty} A_j^{C_k} \rfloor = \lfloor A_j^{C_k} \rfloor \in \mathcal{P}.$$

Therefore, ζ is the smallest element of $\mathcal{W}(C_k)$. □

Lemma 4.2. *Let* $(e_k)_{k\geq 2}$ *be a real sequence with* $e_k \geq 2$ *for all* $k \geq 2$*. Let* ϵ_0 *be a sufficiently small positive real number. Then, there exist real numbers* $X_1 = X_1(\epsilon_0) > 0$ *and* $d_2 > 0$ *such that for all real numbers* X *and* η *satisfying* $X \geq X_1$ *and* $\eta \in [1/2, 1-\epsilon_0]$*, if we have*

$$\#([X, X+X^\eta] \cap \mathcal{P}) \geq \frac{d_2 X^\eta}{\log X},$$

then we can find a sequence $(q_k)_{k\geq 1}$ *of prime numbers such that* $X \leq q_1 \leq X + X^\eta$ *and for all* $k \in \mathbb{N}$*, we have*

$$q_k^{e_{k+1}} \leq q_{k+1} < (q_k+1)^{e_{k+1}} - 1.$$

Proof. This lemma is deduced from [ST22, Lemma 3.2] with $X \coloneqq X$ and $Y \coloneqq X + X^{\eta}$. We note that it essentially follows from Matomäki's results [Mat10, Lemma 9] and [Mat07, Lemma 1.2]. □

Proof of Theorem 1.2. Let ϵ_0 be a positive real number, and let $\eta \in [1/2, 1-\epsilon_0]$. Let $X_1 = X_1(\epsilon)$ be as in Lemma 4.2. Let Y_0 be a sufficiently large real number with $Y_0 \geq X_1$, and we take $Y \geq Y_0$. By the prime number theorem, we have

$$\#([Y, 2Y) \cap \mathcal{P}) \geq \frac{Y}{2\log Y}$$

since $Y \geq Y_0$ and Y_0 is sufficiently large. By decomposing $[Y, 2Y)$ into at least $2Y^{1-\eta}$ intervals with length $Y^{\eta}/2$, there exist an interval $[X, X + X^{\eta}] \subseteq [Y, 2Y)$ and a constant $d_3 > 0$ such that

$$\#([X, X + X^{\eta}] \cap \mathcal{P}) \geq \frac{d_3 X^{\eta}}{\log X},$$

Since $X \geq Y \geq Y_0 \geq X_1$, by applying Lemma 4.2 with $(e_k)_{k\geq 2} \coloneqq (c_k)_{k\geq 2}$, there exists a sequence $(p_k)_{k\geq 1}$ of prime numbers such that for every $k \in \mathbb{N}$, we have

$$p_k^{c_{k+1}} \leq p_{k+1} < (p_k+1)^{c_{k+1}} - 1. \tag{4.1}$$

Therefore, since $C_k = c_1 \cdots c_k$, for every $k \in \mathbb{N}$, by (4.1) we have

$$p_k^{1/C_k} \leq p_{k+1}^{1/C_{k+1}} < (p_{k+1}+1)^{1/C_{k+1}} < (p_k+1)^{1/C_k}.$$

By substituting $k = 1, 2, \ldots$, we deduce that

$$p_1^{1/C_1} \leq p_2^{1/C_2} \leq p_3^{1/C_3} \leq \cdots < (p_3+1)^{1/C_3} < (p_2+1)^{1/C_2} < (p_1+1)^{1/C_1}. \tag{4.2}$$

Thus, both $\lim_{k\to\infty} p_k^{1/C_k}$ and $\lim_{k\to\infty}(p_k+1)^{1/C_k}$ exist, say A and A', respectively. Then, we have $A \leq A'$ by (4.2). Furthermore, by (4.2), we conclude that

$$p_k \leq A_1^{C_k} \leq A_2^{C_k} < p_k + 1,$$

which implies that $p_k = \lfloor A^{C_k} \rfloor$ for every $k \in \mathbb{N}$. Therefore, we obtain $A \in \mathcal{W}(C_k)$. Since $\mathcal{W}(C_k)$ is non-empty, the number $\xi(C_k)$ exists by Lemma 4.1. □

5. Auxiliary results

5.1. **A key dichotomy.** For all real numbers x, we define $\|x\|$ as the length between x and the nearest integer of x.

Lemma 5.1. *For all real numbers $x \geq 1$ and $c \geq 1$, if*

$$\|x\| > 1/(c\lfloor x \rfloor^{c-1}),$$

then we have $\lfloor x \rfloor^c \leq \lfloor x^c \rfloor < (\lfloor x \rfloor + 1)^c - 1$.

Proof. We observe that

$$\begin{aligned} \lfloor x \rfloor^c > \lfloor x^c \rfloor &\implies \lfloor x \rfloor^c > x^c - \{x^c\} \\ &\implies 1 \geq x^c - \lfloor x \rfloor^c. \end{aligned}$$

By the mean value theorem, there exists a real number $y \in [\lfloor x \rfloor, x]$ such that

$$x^c - \lfloor x \rfloor^c = c(x - \lfloor x \rfloor) y^{c-1} \geq c\{x\}\lfloor x \rfloor^{c-1} \geq c\|x\|\lfloor x \rfloor^{c-1},$$

and hence

$$\lfloor x \rfloor^c > \lfloor x^c \rfloor \implies \|x\| \leq 1/(c\lfloor x \rfloor^{c-1}). \tag{5.1}$$

In addition, we have

$$\begin{aligned}\lfloor x^c \rfloor \geq (\lfloor x \rfloor + 1)^c - 1 &\implies 1 \geq (\lfloor x \rfloor + 1)^c - \lfloor x^c \rfloor \\ &\implies 1 \geq (\lfloor x \rfloor + 1)^c - x^c.\end{aligned}$$

By applying the mean value theorem again, there exists a real number $z \in [x, \lfloor x \rfloor + 1]$ such that

$$(\lfloor x \rfloor + 1)^c - x^c = c(\lfloor x \rfloor + 1 - x)z^{c-1} \geq c(1 - \{x\})x^{c-1} \geq c\|x\|\lfloor x \rfloor^{c-1}$$

and hence

$$\lfloor x^c \rfloor \geq (\lfloor x \rfloor + 1)^c - 1 \implies \|x\| \leq 1/(c\lfloor x \rfloor^{c-1}). \tag{5.2}$$

Combining (5.1) and (5.2), we conclude Lemma 5.1. □

Let θ be a real number in $[1/2, 1)$ satisfying (†). Throughtout this subsection, let $(C_k)_{k\geq 1}$ be a sequence of real numbers satisfying (G1), (G2), and (G3). Theorem 1.2 implies that $\mathcal{W}(C_k)$ is non-empty and $\xi(C_k)$ exists. Let $\xi = \xi(C_k)$, and let $p_k = \lfloor \xi^{C_k} \rfloor$ for every $k \in \mathbb{N}$.

Lemma 5.2. *Suppose that for every $k \in \mathbb{N}$ we have* (3.1)*. Let ϵ and I be as in Proposition 3.2. Then, the following* (I) *or* (II) *is true*:

(I) *for infinitely many integers k, we have*

$$\|\xi^{C_k}\| \leq \frac{1}{c_{k+1}\lfloor \xi^{C_k} \rfloor^{c_{k+1}-1}},$$

(II) *there exists $k_0 > 0$ such that for every $k \in I \cap [k_0, \infty)$, we have*

$$p_k^{c_{k+1}} \leq p_{k+1} \leq p_k^{c_{k+1}} + p_k^{\theta c_{k+1}}.$$

Proof. Assume that both (I) and (II) are false. Since (I) is false, there exists a positive integer k_1 such that for every $k \geq k_1$ we have

$$\|\xi^{C_k}\| > \frac{1}{c_{k+1}\lfloor \xi^{C_k} \rfloor^{c_{k+1}-1}}.$$

Then, by Lemma 5.1 with $x \coloneqq \xi^{C_k}$ and $c \coloneqq c_{k+1}$, for all $k \geq k_1$ we obtain

$$\lfloor \xi^{C_k} \rfloor^{c_{k+1}} \leq \lfloor \xi^{C_{k+1}} \rfloor < (\lfloor \xi^{C_k} \rfloor + 1)^{c_{k+1}} - 1,$$

which implies that for all $k \geq k_1$

$$p_k^{c_{k+1}} \leq p_{k+1} < (p_k + 1)^{c_{k+1}} - 1. \tag{5.3}$$

Since (II) is false, there are infinitely many integers $k \in I$ such that

$$p_{k+1} \notin [p_k^{c_{k+1}}, p_k^{c_{k+1}} + p_k^{\theta c_{k+1}}]. \tag{5.4}$$

Then, we set $\delta = \min\{\xi^{C_k} - p_k \mid k = 1, 2, \ldots, k_1\}$. By (3.1), for every integer $k \leq k_1$, we have $p_k = \lfloor \xi^{C_k} \rfloor < \xi^{C_k}$, and hence $\delta > 0$ holds.

Let $\epsilon_0 = 1 - \theta$, and let $X_1 = X_1(\epsilon_0)$ be as in Lemma 4.2. Take a sufficiently large integer $m \in I \cap [k_1, \infty)$ satisfying (5.4), and

$$p_m^{c_{m+1}} \geq X_1, \tag{5.5}$$

$$\delta \geq (2^{-1/C_m}\xi)^{C_{k_1} - C_m}. \tag{5.6}$$

We note that $I = \{k \in \mathbb{N} \colon c_{k+1} \geq 1/(1-\theta) + \epsilon\}$ is an infinite set by (G3) and (3.2). Then, by (5.3) and (5.4), we have

$$p_{m+1} > p_m^{c_{m+1}} + p_m^{\theta c_{m+1}}. \tag{5.7}$$

Let $q_m = p_m$. Since θ satisfies (†) and m is sufficiently large, there exists an absolute constant $d > 0$ such that

$$\#([q_m^{c_{m+1}}, q_m^{c_{m+1}} + q_m^{\theta c_{m+1}}] \cap \mathcal{P}) \geq \frac{d q_m^{\theta c_{m+1}}}{\log q_m^{c_{m+1}}}.$$

Therefore, by (5.5) and Lemma 4.2 with $(e_2, e_3, \ldots) \coloneqq (c_{m+2}, c_{m+3}, \ldots)$, $\epsilon_0 \coloneqq 1 - \theta$, $X \coloneqq q_m^{c_{m+1}}$, and $\eta \coloneqq \theta \in [1/2, 1 - \epsilon_0]$, there exists a sequence $(q_k)_{k \geq m}$ of prime numbers such that

$$q_m^{c_{m+1}} \leq q_{m+1} \leq q_m^{c_{m+1}} + q_m^{\theta c_{m+1}}, \tag{5.8}$$

$$q_k^{c_{k+1}} \leq q_{k+1} < (q_k + 1)^{c_{k+1}} - 1 \tag{5.9}$$

for every integer $k \geq m + 1$. We recall that $c_{m+1} \geq 1/(1 - \theta) + \epsilon$ by the choice of m, and hence (5.8) implies that (5.9) also holds for $k = m$.

Let $q_k = p_k$ for every $1 \leq k \leq m - 1$. Then (5.3) implies that (5.9) is true for every integer $k \geq k_1$. Therefore, in a similar manner with (4.2), we obtain

$$q_{k_1}^{1/C_{k_1}} \leq q_{k_1+1}^{1/C_{k_1+1}} \leq \cdots < (q_{k_1+1} + 1)^{1/C_{k_1+1}} < (q_{k_1} + 1)^{1/C_{k_1}},$$

which implies that $\zeta = \lim_{k \to \infty} q_k^{1/C_k}$ exist. Furthermore, similary with the proof of Theorem 1.2, we obtain

$$\lfloor \zeta^{C_k} \rfloor = q_k \in \mathcal{P} \tag{5.10}$$

for every integer $k \geq k_1$. We note that

$$\zeta < \xi \tag{5.11}$$

holds since $\lfloor \zeta^{C_{m+1}} \rfloor = q_{m+1} < p_{m+1} = \lfloor \xi^{C_{m+1}} \rfloor$ by (5.7) and (5.8).

Let us show that (5.10) is true for every $k \in \mathbb{N}$. Since $\lfloor \zeta^{C_m} \rfloor = q_m = p_m = \lfloor \xi^{C_m} \rfloor$, we have

$$1 \geq |\xi^{C_m} - \zeta^{C_m}| = |\xi^{C_k C_m / C_k} - \zeta^{C_k C_m / C_k}| = (\xi^{C_k})^{C_m/C_k} - (\zeta^{C_k})^{C_m/C_k} \tag{5.12}$$

for every $1 \leq k \leq k_1$, where the last equation follows from (5.11). The mean value theorem yields that

$$(\xi^{C_k})^{C_m/C_k} - (\zeta^{C_k})^{C_m/C_k} \geq (\xi^{C_k} - \zeta^{C_k}) \zeta^{C_m - C_k},$$

and hence

$$\xi^{C_k} - \zeta^{C_k} = |\xi^{C_k} - \zeta^{C_k}| \leq \zeta^{C_k - C_m} \leq \zeta^{C_{k_1} - C_m} \tag{5.13}$$

for every $1 \leq k \leq k_1$. Since (5.12) leads to $\zeta^{C_m} \geq \xi^{C_m} - 1 \geq \xi^{C_m}/2$, we have $\zeta \geq 2^{-1/C_m} \xi$. Therefore, by (5.13), we obtain

$$\xi^{C_k} - (2^{-1/C_m} \xi)^{C_{k_1} - C_m} \leq \xi^{C_k} - \zeta^{C_{k_1} - C_m} \leq \zeta^{C_k} \tag{5.14}$$

By the definition of δ, for every $1 \leq k \leq k_1$, we have

$$\begin{aligned} q_k = p_k \leq \xi^{C_k} - \delta_k &\overset{(5.6)}{\leq} \xi^{C_k} - (2^{-1/C_m} \xi)^{C_{k_1} - C_m} \\ &\overset{(5.14)}{\leq} \zeta^{C_k} < \xi^{C_k} < \lfloor \xi^{C_k} \rfloor + 1 = p_k + 1 = q_k + 1, \end{aligned}$$

and hence (5.10) is true for all $k \in \mathbb{N}$. We conclude that $\zeta \in \mathcal{W}(C_k)$, a contradiction to the minimality of ξ. □

Lemma 5.3. *Suppose that for every $k \in \mathbb{N}$ we have* (3.1)*. Let ϵ and I be as in Proposition 3.2. If* (I) *is false, then for every sufficiently large $k \in I$, we have*

$$\{\xi^{C_k}\} \leq \frac{2}{c_{k+1}\lfloor \xi^{C_k} \rfloor^{(1-\theta)c_{k+1}-1}}. \tag{5.15}$$

Proof. Assume that (I) is false. By Lemma 5.2, there exists $k_0 > 0$ such that for every $k \in I \cap [k_0, \infty)$, we obtain (II). Then, for every sufficiently large $k \in I$, we have

$$p_k = \lfloor \xi^{C_k} \rfloor \leq \xi^{C_k}, \tag{5.16}$$

Furthermore, by (II), for every sufficiently large $k \in I$, we have

$$\xi^{C_{k+1}} \leq p_{k+1} + 1 \leq p_k^{c_{k+1}} + p_k^{\theta c_{k+1}} + 1,$$

which implies that

$$\begin{aligned} \xi^{C_k} &\leq (p_k^{c_{k+1}} + p_k^{\theta c_{k+1}} + 1)^{1/c_{k+1}} \\ &\leq p_k(1 + 2p_k^{(\theta-1)c_{k+1}})^{1/c_{k+1}} \\ &\leq p_k + \frac{2}{c_{k+1}p_k^{(1-\theta)c_{k+1}-1}}. \end{aligned} \tag{5.17}$$

Therefore, by (5.16) and (5.17), we conclude (5.15). □

5.2. **Results from Diophantine approximation.**

Theorem 5.4. *Let $\alpha > 1$ be an algebraic number, and q be a positive integer. Suppose that $0 < s_1 < s_2 < \cdots$ is a sequence of positive integers. Then, either for some $m \geq 0$ the number α^{s_m} is a Pisot number or for each $\epsilon > 0$ there exists a positive integer $k_0 = k_0(\epsilon)$ such that*

$$\|q\alpha^{s_k}\| > e^{-\epsilon s_k}$$

for every $k \geq k_0$.

Proof. See [Dub22, Lemma 6] which is given by Dubickas. This theorem essentially follows from the result [CZ04] by Corvaja and Zannier. □

By Lemmas 5.2 and 5.3, both cases (I) and (II) imply that

$$\limsup_{k\to\infty} \|\xi^{C_k}\|^{1/C_k} < 1.$$

By combining this inequality and Theorem 5.4 with $q = 1$ and $s_k = C_k$, either ξ is transcendental or there exists $m \in \mathbb{N}$ such that ξ^{C_m} is a Pisot number.

In the latter case, we apply Baker's theorem. We say that the height of an algebraic number is the maximum of the absolute values of the relatively prime integer coefficients in its minimal polynomial.

Theorem 5.5 (Baker's theorem [Bak75, Theorem 3.1])**.** *Let $\alpha_1, \ldots, \alpha_n$ be non-zero algebraic numbers with degrees at most d and heights at most A. Further, let $\beta_0, \beta_1, \ldots, \beta_n$ be algebraic numbers with degrees at most d and heights at most $B(\geq 2)$. Let*

$$\Lambda = \beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_n \log \alpha_n.$$

Then either $\Lambda = 0$ or $|\Lambda| > B^{-C}$, where C is an effectively computable number depending only on n, d, A and the original determinations of the logarithms.

5.3. **Lemmas for Pisot numbers.**

Lemma 5.6. *Let β_1 be a Pisot number of degree 2, and let β_2 be the conjugate of β_1 over $\mathbb{Q}$. If $\beta_1+\beta_2=1$, then we have $\beta_1=\frac{1+\sqrt{5}}{2}$ and $\beta_2=\frac{1-\sqrt{5}}{2}$.*

Proof. Let $N=-\beta_1\beta_2\in\mathbb{Z}\setminus\{0\}$. Then, we have $(x-\beta_1)(x-\beta_2)=x^2-x-N$. Therefore, by solving $x^2-x-N=0$, we obtain

$$\beta_1=\frac{1+\sqrt{1+4N}}{2},\quad \beta_2=\frac{1-\sqrt{1+4N}}{2}.$$

By $|\beta_2|<1$, we conclude that $N=1$. □

Lemma 5.7 ([Dub02, Lemma 3])**.** *Let $\ell\in\mathbb{N}$, and let β be a Pisot number of degree ℓ. For all $m\in\mathbb{N}$, β^m is a Pisot number of degree ℓ.*

Proof. Let $\beta_1=\beta,\beta_2,\dots,\beta_\ell$ be all the conjugates of β. Let

$$Q(X)=\prod_{j=1}^{\ell}(X-\beta_j^m)\in\mathbb{Z}[X], \tag{5.18}$$

and let $P\in\mathbb{Z}[X]$ be the minimum polynomial of β^m. Since β^m is an algebraic integer, we may assume that P is a monic polynomial. It is clear that $\deg(\beta^m)\le\ell$, and so we now suppose that $\deg(\beta^m)<\ell$. By this supposition, there exists a non-constant polynomial $R\in\mathbb{Z}[X]$ such that $Q(X)=P(X)R(X)$. By (5.18), $R(X)=\prod_{j\in S}(X-\beta_j^m)$ for some non-empty set $S\subset\{2,3,\dots,\ell\}$. We note that $R(0)$ is an integer, but

$$0<\prod_{j\in S}|\beta_j^m|<1$$

since β is a Pisot number, a contradiction. □

Lemma 5.8. *Let β be a Pisot number of degree ℓ, and let $\beta_1=\beta,\beta_2,\dots,\beta_\ell$ be the conjugates of β over $\mathbb{Q}$. Then, there exists $s_0=s_0(\beta)>0$ such that for all integers $s\ge s_0$, if $\{\beta^s\}<1/2$, then $\mathrm{Tr}(\beta^s)=\beta_1^s+\cdots+\beta_\ell^s=\lfloor\beta^s\rfloor$.*

Proof. It is clear that $\mathrm{Tr}(\beta^s)=\beta_1^s+\cdots+\beta_\ell^s$ from Lemma 5.7. Since β is a Pisot number, we have $|\beta_j|<1$ for all $j=2,3,\dots,\ell$. Therefore, there exists $s_0=s_0(\beta)>0$ such that for all integers $s\ge s_0$ we have $|\beta_2^s+\cdots+\beta_\ell^s|<1/4$. We now take an arbitrary integer s with $s\ge s_0$. Suppose that $\{\beta^s\}<1/2$.

Since $\beta_1^s+\cdots+\beta_\ell^s$ is a non-zero integer from the theory of fields and

$$\beta^s=\beta_1^s=\beta_1^s+\cdots+\beta_\ell^s-(\beta_2^s+\cdots+\beta_\ell^s),$$

we see that

$$\{\beta^s\}=\begin{cases}-(\beta_2^s+\cdots+\beta_\ell^s) & \text{if } \beta_2^s+\cdots+\beta_\ell^s\le 0,\\ 1-(\beta_2^s+\cdots+\beta_\ell^s) & \text{otherwise.}\end{cases}$$

The latter case naver happens since $\{\beta^s\}<1/2$ and $|\beta_2^s+\cdots+\beta_\ell^s|<1/4$, and hence $\{\beta^s\}=-(\beta_2^s+\cdots+\beta_\ell^s)$. Therefore, we conclude that $\lfloor\beta^s\rfloor=\beta^s-\{\beta^s\}=\beta_1^s+\cdots+\beta_\ell^s$. □

Lemma 5.9 ([BD08, Lemma 8])**.** *Let α be a real number greater than 1. Let n and m be positive integers. If α^n and α^m are Pisot numbers, then $\alpha^{\gcd(n,m)}$ is a Pisot number.*

Proof. We may assume that $\gcd(n,m)=1$ by replacing $n/\gcd(n,m)$ and $m/\gcd(n,m)$ with n and m, respectively. Since α^n is an algebraic integer, α is so.

Suppose that α is not a Pisot number. Then, we may take a conjugate β of α satisfying $|\beta|>1$ and $\beta\neq\alpha$. Thus, we have $\beta^n=\alpha^n$ since β^n is also a conjugate of α^n and α^n is a Pisot number. Therefore, $\beta=\alpha e^{2\pi ik/n}$ for some $k\in\{1,2,\dots,n-1\}$. Similarly, we have $\beta^m=\alpha^m$, and hence

$$\alpha^m e^{2\pi ikm/n}=\beta^m=\alpha^m.$$

This implies $\exp(2\pi ikm/n)=1$, a contradiction since $\gcd(n,m)=1$. □

Lemma 5.10. *Let β be a Pisot number of degree $\ell\geq 2$ with conjugates $\beta_1=\beta,\beta_2,\dots,\beta_\ell$. For every k with $\beta_k\notin\mathbb{R}$, if $\beta_k=|\beta_k|e^{2\pi i\phi}$ for some $\phi\in\mathbb{R}$, then ϕ is irrataional.*

Proof. Fix any k with $\beta_k\notin\mathbb{R}$, and let $\beta_k=|\beta_k|e^{2\pi i\phi}$ for some $\phi\in\mathbb{R}$. Then, $\phi\notin\mathbb{Z}$. Then, there exists $k'\in[1,\ell]$ with $k\neq k'$ such that $\beta_{k'}=\overline{\beta_k}=|\beta_k|e^{-2\pi i\phi}$. Suppose that ϕ is rational and let $\phi=a/b$ for some $a\in\mathbb{Z}$ and $b\in\mathbb{N}$. Then, by Lemma 5.7, $\beta_1^b,\dots,\beta_\ell^b$ are pairwise distinct. However, we have

$$\beta_k^b=|\beta_k|^b e^{2\pi ia}=|\beta_k|^b=|\beta_k|^b e^{-2\pi ia}=\beta_{k'}^b,$$

a contradiction. □

Remark 5.11. This lemma follows from Mignotte's result [Mig84], which states that for $N_1,\dots,N_\ell\in\mathbb{Z}$ if $\beta_1^{N_1}\cdots\beta_\ell^{N_\ell}=1$ holds, then $N_1=\cdots=N_\ell=0$.

Lemma 5.12 ([Smy75])**.** *Let β be a Pisot number of degree $\ell\geq 2$ with conjugates $\beta_1=\beta,\beta_2,\dots,\beta_\ell$. For $i\neq j$, if $|\beta_i|=|\beta_j|$, then $\beta_i=\overline{\beta_j}$.*

Proof. Let

$$f(X)=\prod_{1\leq i\leq j\leq\ell}(X-\beta_i\beta_j). \tag{5.19}$$

Then, it is clear that $f(X)\in\mathbb{Z}[X]$ since $f(X)$ is symmetric and β is an algebraic integer. Assume that $|\beta_i|=|\beta_j|$ but $\beta_i\neq\overline{\beta_j}$ for some $i\neq j$. Then, both $(X-\beta_i\overline{\beta_i})$ and $(X-\overline{\beta_j}\beta_j)$ appear in the product (5.19). Therefore, $(X-\phi)^2$ devides $f(X)$, where $\phi:=\beta_i\overline{\beta_i}=\overline{\beta_j}\beta_j$. Since ϕ is an algebraic integer, the product of all the conjugates of ϕ is a non-zero integer. In addition, $|\phi|=|\beta_i|^2<1$ since β is a Pisot number. Therefore, we find one of the conjugates ϕ' of ϕ such that $|\phi'|>1$. Then, $(X-\phi')^2$ devides $f(X)$. However, a root of $f(X)$ with absolute value strictly greater than one forms as $\beta\beta_j$ for some $1\leq j\leq\ell$. All of such roots are distinct, a contradiction. □

Lemma 5.13 ([Dub22, Lemma 8])**.** *Let β be a Pisot number of degree $\ell\geq 2$ with conjugates $\beta_1=\beta,\beta_2,\dots,\beta_\ell$ labelled so that $\beta_1>1>|\beta_2|\geq\cdots\geq|\beta_\ell|$. Then, there is a real number $\lambda>0$ depending only on β such that*

$$|\beta_2^n+\cdots+\beta_\ell^n|\geq|\beta_2|^n n^{-\lambda}$$

for each sufficiently large integer n.

Proof. We may assume that $\ell\geq 3$. If $|\beta_2|>|\beta_3|$, then the lemma is trivial since

$$|\beta_2^n+\cdots+\beta_\ell^n|\geq|\beta_2|^n-\ell|\beta_3|^n\geq|\beta_2|^n/2$$

for every sufficiently large n. If $|\beta_2|=|\beta_3|$, then Lemma 5.12 yields that $\beta_3=\overline{\beta_2}$ and $|\beta_2|>|\beta_4|$. Especially, β_2 is not real. Then, we obtain

$$|\beta_2^n+\cdots+\beta_\ell^n|\geq|\beta_2|^n\Big(|1+(\beta_3/\beta_2)^n|-\ell(|\beta_4|/|\beta_2|)^n\Big).$$

We now define $\log z = \log r + 2\pi i\psi$ if $z = re^{2\pi i\psi}$ $(r \geq 0,\ \psi \in (0,1))$. Let $\beta_2 = |\beta_2|e^{2\pi i\phi}$ for some $\phi \in (0,1)$. For every $n \in \mathbb{N}$, we have

$$|1 + (\beta_3/\beta_2)^n| = |1 + e^{4\pi in\phi}| = 2|\cos(2\pi n\phi)| = 2|\sin(\pi/2 - 2\pi n\phi)| \geq 4\,\|1/2 - 2n\phi\|\,.$$

For every $n \in \mathbb{N}$, we take $k = k(n) \in \mathbb{Z}$ such that

$$\|1/2 - 2n\phi\| = |1/2 + k - 2n\phi|.$$

By Lemma 5.10, $2\pi i\phi$ and πi are linearly independent over $\mathbb{Q}$. Since $2\pi i\phi = \log(\beta_2/|\beta_2|)$ and $\log(-1) = \pi i$, Theorem 5.5 implies that there are constants $C_1, C_2 > 0$ such that for every n, we obtain

$$2\pi|1/2 + k - 2n\phi| = |(k+1)\log(-1) - 2n\log(\beta_2/|\beta_2|)| \geq C_1 n^{-C_2}.$$

Therefore, there exists $\lambda > 0$ such that for every sufficiently large n, we have

$$|\beta_2^n + \cdots + \beta_\ell^n| \geq |\beta_2|^n n^{-\lambda}. \qquad \square$$

In [Dub22], Dubickas proved the transcendency of certain constants deduced from some recurrent sequences by applying [Dub22, Lemma 6] and [Dub22, Lemma 8] (Theorem 5.4 and Lemma 5.13). This is an improvement on the result regarding irrationality by Wagner and Ziegler [WZ21]. A basic idea of the proof comes from these works and [BD08] by Bugeaud and Dubickas which is a study on the set of limit points of $\|\alpha^n\|^{1/n}$ $(n = 1, 2, 3, \ldots)$.

6. Proof of Proposition 3.2

Lemma 6.1. *Let $\alpha > 1$ be an algebraic number, and let $0 < s_1 < s_2 < \cdots$ be a sequence of positive real numbers satisfying that $s_k \in \mathbb{N}$ for sufficiently large k. Let $(t_k)_{k\geq 1}$ be a sequence of positive real numbers with $\liminf_{k\to\infty} t_k > 0$. Suppose that there exists a constant $K > 0$ such that for every sufficiently large $k \in \mathbb{N}$*

$$\|\alpha^{s_k}\| \leq K\alpha^{-s_k t_k}. \tag{6.1}$$

Then, there exists $g \in \mathbb{N}$ such that α^g is a Pisot number of degree ℓ with

$$1 \leq \ell \leq 1 + \left(\limsup_{k\to\infty} t_k\right)^{-1}$$

and for every sufficiently large k, we have $g \mid s_k$. Furthermore, if we have

$$\ell = 1 + \left(\limsup_{k\to\infty} t_k\right)^{-1},$$

then α^g is a unit Pisot number.

Proof. Let $\overline{t} = \limsup_{k\to\infty} t_k$ and $\underline{t} = \liminf_{k\to\infty} t_k$. By (6.1) and Theorem 5.4 with $\epsilon := (\underline{t}/2)\log\alpha$ and $q := 1$, there exists a positive integer m such that α^{s_m} is a Pisot number. Let g be the smallest positive integer such that α^g is a Pisot number.

Let us show that $g \mid s_k$ for every sufficiently large k. If not, there are infinitely many positive integers $n_1 < n_2 < n_3 < \cdots$ such that $g \nmid s_{n_k}$. Then, by (6.1) and Theorem 5.4 with $s_k := s_{n_k}$, $\epsilon := (\underline{t}/2)\log\alpha$ and $q := 1$, there exists $r \in \mathbb{N}$ such that $\alpha^{s_{n_r}}$ is a Pisot number. By Lemma 5.9 and the choice of g, we obtain $g \mid s_{n_r}$, a contradiction.

Let $\beta = \alpha^g$, and let $\beta_1, \beta_2, \dots, \beta_\ell$ be the conjugates over $\mathbb{Q}$ of β labeled so that $\beta = \beta_1 > 1 > |\beta_2| \geq \cdots \geq |\beta_\ell|$. Therefore, (6.1) and Lemma 5.13 with $n \coloneqq s_k/g \in \mathbb{N}$ imply that for every sufficiently large $k \in \mathbb{N}$ we have

$$\begin{aligned} K\alpha^{-s_k t_k} &\geq \|\alpha^{s_k}\| = \|\beta^{s_k/g}\| \\ &= \|\beta_2^{s_k/g} + \cdots + \beta_\ell^{s_k/g}\| \geq |\beta_2|^{s_k/g}(s_k/g)^{-\lambda} \end{aligned}$$

for some constant $\lambda = \lambda(\beta) > 0$. Thus, by taking the logarithm on both sides and taking $\limsup_{k\to\infty}$, we obtain

$$\beta_1^{-\overline{t}/g} = \alpha^{-\overline{t}} \geq |\beta_2|^{1/g},$$

which yields that $\beta_1^{-\overline{t}} \geq |\beta_2|$. Since $\beta_1 \cdots \beta_\ell \in \mathbb{Z} \setminus \{0\}$, we obtain

$$1 \leq |\beta_1 \cdots \beta_\ell| \leq \beta_1^{-(\ell-1)\overline{t}+1},$$

and hence $1 \leq \ell \leq 1+1/\overline{t}$. Furthermore, if $\ell = 1+1/\overline{t}$ is true, then $1 \leq |\beta_1 \cdots \beta_\ell| \leq \beta_1^0 = 1$, which means that β is a Pisot unit. □

Lemma 6.2. *Let $(c_k)_{k\geq 1}$ be a sequence of real numbers satisfying* (G1) *to* (G4). *We further suppose that for every $k \in \mathbb{N}$ we have* (3.1). *If* (I) *in Lemma 5.2 is true, then $\xi(C_k)$ is transcendental.*

Proof. Let $\xi = \xi(C_k)$, and we assume that ξ is algebraic. We furthermore assume that (I) is true, and so

$$J \coloneqq \left\{ n \in \mathbb{N} \colon \|\xi^{C_n}\| \leq \frac{1}{c_{n+1}\lfloor \xi^{C_n} \rfloor^{c_{n+1}-1}} \right\}$$

is infinite. Let $J = \{n_1, n_2, \dots\}$, where $n_1 < n_2 < \cdots$. By Lemma 6.1 with $\alpha \coloneqq \xi$, $s_k \coloneqq C_{n_k}$, and $t_k \coloneqq c_{n_k+1} - 1$, there exists $g \in \mathbb{N}$ such that $\xi^g = \beta$ is a Pisot number of degree ℓ with

$$1 \leq \ell \leq 1 + \left(\limsup_{k\to\infty} c_{n_k+1} - 1 \right)^{-1} \tag{6.2}$$

and $g \mid C_n$ for every sufficiently large $n \in J$. By (G2) and (6.2), we obtain $\ell \in \{1, 2\}$. Suppose that $\ell = 1$, which means that β is an integer. In this case, for sufficiently large $n \in J$, we see that $\beta^{C_n/g}$ is a composite number, but

$$\beta^{C_n/g} = \xi^{C_n} = \lfloor \xi^{C_n} \rfloor$$

is a prime number, a contradiction.

Suppose that $\ell = 2$. Then, by (G2) and (6.2), we have

$$\ell = 1 + \left(\limsup_{k\to\infty} c_{n_k+1} - 1 \right)^{-1}.$$

By Lemma 6.1 and $\ell = 2$, β is a quadratic Pisot unit, and hence $\|\beta^n\| = \beta^{-n}$ for all $n \in \mathbb{N}$. Therefore, by (G2), for every sufficiently large $n \in J$, we obtain

$$\xi^{-C_n} = \beta^{-C_n/g} = \|\beta^{C_n/g}\| = \|\xi^{C_n}\| \leq \frac{1}{c_{n+1}\lfloor \xi^{C_n} \rfloor^{c_{n+1}-1}} \leq \frac{1}{2\lfloor \xi^{C_n} \rfloor},$$

a contradiction. □

Proof of Proposition 3.2. Let θ be a real number in $[1/2, 1)$ satisfying (†). Let $(c_k)_{k\geq 1}$ be a sequence of real numbers satisfying (G1) to (G4). By (G1) and (G2), Theorem 1.2 yields that $\mathcal{W}(C_k)$ is non-empty and $\xi = \xi(C_k)$ exists.

We suppose that for every $m \in \mathbb{N}$, we have (3.1), and we further suppose that ξ is algebraic. Let ϵ be a real number with (3.2), and let $I = \{k \in \mathbb{N}\colon c_{k+1} \geq 1/(1-\theta) + \epsilon\}$.

Proof of (i)**:** By Lemma 5.2, we obtain (I) or (II). Furthermore, Lemma 6.2 implies that (I) is false since ξ is algebraic. Therefore, by Lemma 5.3, we have (5.15) for sufficiently large $k \in I$, and hence we obtain (i).

Proof of (ii)**:** Let $\overline{c} = \limsup_{k\to\infty} c_{k+1}$. Let $I = \{n_1, n_2, \ldots\}$, where $0 < n_1 < n_2 < \cdots$. By the definition of I, we have

$$\limsup_{k\to\infty} c_{n_k+1} = \overline{c} \geq 1/(1-\theta) + \epsilon.$$

By $\|x\| \leq \{x\}$ and (i), for sufficiently large $k \in \mathbb{N}$, we have

$$\|\xi^{C_{n_k}}\| \leq \{\xi^{C_{n_k}}\} \leq \frac{2}{c_{n_k+1}\lfloor \xi^{C_{n_k}} \rfloor^{(1-\theta)c_{n_k+1}-1}},$$

and hence by Lemma 6.1 with $\alpha \coloneqq \xi$, $s_k \coloneqq C_{n_k}$, and $t_k \coloneqq (1-\theta)c_{n_k+1} - 1$, there exists $g \in \mathbb{N}$ such that ξ^g is a Pisot number of degree ℓ with

$$1 \leq \ell \leq 1 + \left((1-\theta)\overline{c} - 1\right)^{-1},$$

and for sufficiently large $k \in \mathbb{N}$, we have $g \mid C_{n_k}$.

Let $\beta = \xi^g$. If $\ell = 1$, then β is an integer. Thus, for sufficiently large $k \in I$ we have $\xi^{C_k} = \beta^{C_k/g}$ is an integer, a contradiction to (3.1). Thus, $\ell \neq 1$. Therefore, we obtain (ii).

Proof of (iv)**:** By (i) and (ii), for sufficiently large $k \in I$, we obtain $g \mid C_k$ and

$$\{\beta^{C_k/g}\} = \{\xi^{C_k}\} < 1/2,$$

and hence Lemma 5.8 implies (iv).

Proof of (iii)**:** Suppose that $\ell = 2$. Since $\beta_1\beta_2 \in \mathbb{Z}\setminus\{0\}$ and $\beta_1 \in \mathbb{R}$, we have $\beta_2 \in \mathbb{R}\setminus\{0\}$. By (ii) and (iv), for every sufficiently large $k \in I$, we obtain

$$\beta_2^{C_k/g} = \lfloor \beta_1^{C_k/g} \rfloor - \beta_1^{C_k/g} = -\{\beta_1^{C_k/g}\} < 0,$$

which yields that $\beta_2 < 0$, and C_k/g is odd. Choose a sufficiently large $k \in \mathbb{N}$. Let r and s be positive odd numbers with $C_k/g = rs$. Then, we have

$$\begin{aligned}\lfloor \xi^{C_k} \rfloor = \lfloor \beta_1^{C_k/g} \rfloor &= \beta_1^{C_k/g} + \beta_2^{C_k/g} = \beta_1^{rs} + \beta_2^{rs} \\ &= (\beta_1^r + \beta_2^r)(\beta_1^{r(s-1)} - \beta_1^{r(s-2)}\beta_2^r + \cdots - \beta_1^r\beta_2^{r(s-2)} + \beta_2^{r(s-1)}).\end{aligned}$$

By choosing $r = 1$ and $s = C_k/g$, we have $\beta_1 + \beta_2 = 1$ since $\lfloor \xi^{C_k} \rfloor$ is a prime number and

$$\beta_1^{s-1} - \beta_1^{s-2}\beta_2 + \cdots - \beta_1\beta_2^{s-2} + \beta_2^{s-1}$$

is an integer greater than or equal to $\beta_1^{C_k/g-1} (\geq 2)$. Therefore, Lemma 5.6 implies that $\beta_1 = (1+\sqrt{5})/2$. In addition, if C_k/g is a composite number, then we may choose r and

s as odd numbers ≥ 3. By $\beta_2 = -0.618\cdots$, we observe that

$$\beta_1^r + \beta_2^r \geq \beta_1^3 + \beta_2^3 = 4,$$
$$\beta_1^{r(s-1)} - \beta_1^{r(s-2)}\beta_2^r + \cdots - \beta_1^r\beta_2^{r(s-2)} + \beta_2^{r(s-1)} \geq \beta_1^{r(s-1)} \geq \beta_1^3 \geq 4,$$

and hence $\lfloor \xi^{C_k} \rfloor$ is a composite number, a contradiction. Therefore, we conclude (iii). □

7. Type A: Proof of Theorem 2.1

Suppose that a sequence $(c_k)_{k\geq 1}$ satisfies (A1) to (A4) in Theorem 2.1. Theorem 3.1 yields that $\theta = 21/40$ satisfies (†). Therefore, $(c_k)_{k\geq 1}$ satisfies (G1) to (G4) in Proposition 3.2, and so the constant $\xi = \xi(C_k)$ exists.

If there exists $m \in \mathbb{N}$ such that $\xi^{C_m} \in \mathbb{N}$, then we have (2.1) for $\ell = 1$ since a positive integer is a Pisot number of degree 1. Thus, ξ is an algebraic integer. In particular, if ξ is rational, then ξ must be a rational integer. This is a contradiction to

$$\xi^{C_k} = \lfloor \xi^{C_k} \rfloor \in \mathcal{P}.$$

for every $k \in \mathbb{N}$. Therefore, ξ is irrational.

We may assume that $\xi^{C_m} \notin \mathbb{N}$ for all $m \in \mathbb{N}$. Suppose that ξ is algebraic. Then, by applying (ii) in Proposition 3.2, there exists $g \in \mathbb{N}$ such that $\beta = \xi^g$ is a Pisot number of degree $\ell \geq 2$ with (2.1), and for every sufficiently large $k \in I$, we have $g \mid C_k$. Therefore, by applying Lemma 5.7, there exists $m \in \mathbb{N}$ such that $g \mid C_m$ and $\xi^{C_m} = \xi^{g\cdot C_m/g}$ is a Pisot number of degree ℓ. In particular, ξ is irrational since the degree of ξ^{C_m} is greater than or equal to 2.

8. Type B: Proof of Theorem 2.3 and Corollary 2.5

For all integers a and $q \neq 0$, there uniquly exists a pair (r, s) of integers such that

$$a = rq + s, \quad 0 \leq s < |q|.$$

Then we define $a \bmod q = s$.

Lemma 8.1 ([Dub02, Lemma 2] and [Sai25a, Lemma 2.1])**.** *Let $(b_k)_{k\geq 1}$ be a sequence of integers satisfying that there exist integers $a_{d-1}, \ldots, a_0$ such that for every $k \in \mathbb{N}$ we have*

$$b_{k+d} = a_{d-1}b_{k+d-1} + a_{d-2}b_{k+d-2} + \cdots + a_0 b_k \tag{8.1}$$

Let q be an integer. If a_0 and q are coprime, then $(b_k \bmod q)_{k\geq 1}$ is purely periodic i.e. there exists $L \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ we have $b_k \equiv b_{k+L} \bmod q$.

Proof. If $q = 1$, then it is trivial. Thus, we assume $q \neq 1$. Since

$$(b_{k+d-1} \bmod q, b_{k+d-2} \bmod q, \ldots, b_k \bmod q) \in \{0, 1, \ldots, q-1\}^{d+1}$$

for all $k = 1, 2, \ldots$, there are integers r and s with $r > s$ such that

$$\begin{aligned}&(b_{r+d-1} \bmod q, b_{r+d-2} \bmod q, \ldots, b_r \bmod q)\\ &= (b_{s+d-1} \bmod q, b_{s+d-2} \bmod q, \ldots, b_s \bmod q).\end{aligned}$$

Let $L = r - s$. Then, by (8.1) with $k = r$ and $k = s$, we obtain $b_{r+d} \equiv b_{s+d} \bmod q$. Again applying (8.1) with $k = r+1$ and $k = s+1$, we have $b_{r+d+1} \equiv b_{s+d+1} \bmod q$. By iterating this argument, we have $b_k \equiv b_{k+L} \bmod q$ for all $k \geq s$. In addition, since a_0 and q are coprime, there exists $u \in \{1, 2, \ldots, q-1\}$ such that $a_0 u \equiv 1 \bmod q$. By (8.1), we obtain

$$b_k \equiv ub_{k+d} - ua_{d-1}b_{k+d-1} - ua_{d-2}b_{k+d-2} - \cdots - ua_1 b_{k+1} \mod q \tag{8.2}$$

for all $k \in \mathbb{N}$. Since $b_k \equiv b_{k+L} \bmod q$ for all $s \le k \le s+d$, (8.2) with $k = s-1$ implies that $b_{s-1} = b_{s+L-1}$. By iterating this argument, we conclude that $b_k = b_{k+L}$ for all $k \in \mathbb{N}$. □

Proof of Theorem 2.3. Suppose that a sequence $(c_k)_{k\ge1}$ satisfies (B1) to (B5) in Theorem 2.3. By choosing $\theta = 21/40$, $(c_k)_{k\ge1}$ satisfies (G1) to (G4) in Proposition 3.2 and the constant $\xi = \xi(C_k)$ exists.

Suppose that $\xi^{C_m} \in \mathbb{N}$ for some $m \in \mathbb{N}$. Then by (B5), there exists $k \in I$ with $k > m$ such that $C_m \mid C_k$, and hence

$$\xi^{C_k} = (\xi^{C_m})^{C_k/C_m}$$

is a composite number, but $\xi^{C_k} = \lfloor \xi^{C_k} \rfloor$ is a prime number, a contradiction. Therefore, we have $\xi^{C_m} \notin \mathbb{N}$ for every $m \in \mathbb{N}$.

Assume that ξ is algebraic. Then, we obtain (i), (ii), (iii), and (iv) in Proposition 3.2. Especially, by (ii), there exists $g \in \mathbb{N}$ such that $\xi^g = \beta$ is a Pisot number of degree ℓ with

$$2 \le \ell \le 1 + \left(\frac{19}{40} \limsup_{k\to\infty} c_{k+1} - 1 \right)^{-1},$$

and for sufficiently large $k \in I$, we have $g \mid C_k$.

Suppose that $\ell = 2$. By (iii), we have $\beta^g = (1+\sqrt{5})/2$ and C_k/g is an odd prime number for sufficiently large k. Then, we take a sufficiently large $m \in I$ with $g \mid C_m$. By (B5), there exists $k \in I$ with $k > m$ such that $C_m \mid C_k$. Therefore, C_m/g is a divisor of C_k/g, a contradiction.

In addition, we now suppose (B6) and let us show that ξ is transcendental. Let $\beta_1, \beta_2, \dots, \beta_\ell$ be all the conjugates over $\mathbb{Q}$ of β labeled so that

$$\xi^g = \beta = \beta_1 > 1 > |\beta_2| \ge \cdots \ge |\beta_\ell|.$$

Take a sufficiently large integer $m \in I$ such that $g \mid C_m$. Let $\alpha_1 = \beta_1^{C_m/g}, \dots, \alpha_\ell = \beta_\ell^{C_m/g}$. By Lemma 5.7, α_1 is a Pisot number of degree ℓ. For every $n \in \mathbb{N}$, we define

$$S(n) = \mathrm{Tr}(\alpha^n) = \alpha_1^n + \cdots + \alpha_\ell^n.$$

Since α_1 is an algebraic integer of degree ℓ, there exist integers $e_0, e_1, \dots, e_\ell$ such that for all $j \in \{1, 2, \dots, \ell\}$, we have

$$\alpha_j^\ell = e_{\ell-1}\alpha_j^{\ell-1} + e_{\ell-2}\alpha_j^{\ell-2} \cdots + e_0,$$

where $e_0 = \pm\alpha_1 \cdots \alpha_\ell$. Thus, for all non-negative integers n and for all $j \in \{1, 2, \dots, \ell\}$

$$\alpha_j^{n+\ell} = e_{\ell-1}\alpha_j^{n+\ell-1} + e_{\ell-2}\alpha_j^{n+\ell-2} \cdots + e_0\alpha_j^n.$$

By taking summation over $j \in \{1, 2, \dots, \ell\}$ on the both sides, we obtain

$$S(n+\ell) = e_{\ell-1}S(n+\ell-1) + e_{\ell-2}S(n+\ell-2) + \cdots + e_0S(n).$$

Since β is a Pisot number and m is sufficiently large, we have

$$S(1) = \beta_1^{C_m/g} + \cdots + \beta_\ell^{C_m/g} > |\beta_1^{C_m/g} \cdots \beta_\ell^{C_m/g}| = |\alpha_1 \cdots \alpha_\ell| = |e_0|.$$

We note that $S(1)$ is a prime number since $S(1) = \mathrm{Tr}(\beta^{C_m/g}) = \lfloor \xi^{C_m} \rfloor$ by (iv), and hence $S(1)$ and e_0 are coprime. Therefore, Lemma 8.1 with $(b_k)_{k\ge1} \coloneqq (S(n))_{n\ge1}$ and $q \coloneqq S(1)$ implies that $(S(n) \bmod S(1))_{n\ge1}$ has period L for some $L \in \mathbb{N}$. By (B6), there exists $k \in I$ with $k > m$ such that $C_k \equiv C_m \bmod LC_m$. Therefore, there exists $n \in \mathbb{N}$ such that

$$C_k = LC_m n + C_m = (Ln+1)C_m,$$

and hence

$$\lfloor \xi^{C_k} \rfloor = \mathrm{Tr}(\beta^{C_k/g}) = \mathrm{Tr}(\alpha^{C_k/C_m}) = S(C_k/C_m) = S(Ln+1) \equiv S(1) \equiv 0 \bmod S(1).$$

This is a contradiction since $\lfloor \xi^{C_k} \rfloor > \lfloor \xi^{C_m} \rfloor = S(1)$ and $\lfloor \xi^{C_k} \rfloor$ is a prime number. □

Proof of Corollary 2.5. Suppose that $(R_k)_{k\geq 1}$ is a sequence of positive integers satisfying (B1') to (B4'). Let $c_1 = R_1$ and $c_{k+1} = R_{k+1}/R_k$ for all $k \in \mathbb{N}$. Then, it is clear that $(c_k)_{k\geq 1}$ satisfies (B1) to (B4) in Theorem 2.3. We note that (B5) follows from (B6). Therefore, we shall show (B6).

Let L and m be arbitrary positive integers. By (B4'), we have

$$R_{k+d} = a_{d-1}R_{k+d-1} + a_{d-2}R_{k+d-2} + \cdots + a_1 R_{k+1} + a_0 R_k$$

for all $k \in \mathbb{N}$, where $a_0 = \pm 1$. Therefore, since a_0 and LR_m are coprime, Lemma 8.1 with $(b_k)_{k\geq 1} \coloneqq (R_k)_{k\geq 1}$ and $q \coloneqq LR_m$ yields that $(R_k \bmod LR_m)_{k\geq 1}$ is purely periodic with period U for some $U \in \mathbb{N}$, and hence choosing $k = m + U$,

$$R_k = R_{m+U} \equiv R_m \bmod LR_m.$$

Thus, (B6) is true. Therefore, Theorem 2.3 implies that $\xi(R_k)$ exists and it is transcendental. □

9. Type C: Proof of Theorem 2.6 and Corollaries 2.7 and 2.8

Lemma 9.1. *Let c be a real number greater than or equal to 2. Suppose that there exists $x_0 \geq 1$ such that for all real numbers $x \geq x_0$ we have (2.2). Let $(c_k)_{k\geq 1}$ be a sequence of real numbers satisfies*

- $c_1 > 0$;
- $c_{k+1} \geq c$ *for all* $k \in \mathbb{N}$.

Then, for all prime numbers $p > x_0^{1/c_2}$, there exists $A \in \mathcal{W}(C_k)$ such that $p = \lfloor A^{C_1} \rfloor$.

Proof. Let p_1 be an arbitrary prime number with $p_1 > x_0^{1/c_2}$. Then, by applying (2.2) with $x = p_1^{c_2} \geq x_0$, there exists $p_2 \in \mathcal{P}$ such that

$$p_1^{c_2} \leq p_2 < p_1^{c_2} + c p_1^{c_2(1-1/c)} \leq p_1^{c_2} + c_2 p_1^{c_2-1}. \tag{9.1}$$

Let $f(x) = x^{c_2}$. Then by the Taylor expansion of $f(x)$ around $x = p_1$, there exists $\eta \in (p_1, p_1+1)$ such that $f(p_1+1) = f(p_1) + f'(p_1) + f''(\eta)/2$, and hence

$$(p_1+1)^{c_2} = p_1^{c_2} + c_2 p_1^{c_2-1} + c_2(c_2-1)\eta^{c_2-2} > p_1^{c_2} + c_2 p_1^{c_2-1} + 1. \tag{9.2}$$

Combining (9.1) and (9.2), we have $p_1^{c_2} \leq p_2 < (p_1+1)^{c_2} - 1$. Let m be an integer greater than or equal to 2. By iterating this argument, we construct a sequence of prime numbers $(p_k)_{k\geq 1}$ such that

$$p_k^{c_{k+1}} \leq p_{k+1} < (p_k+1)^{c_{k+1}} - 1.$$

Similarly with the proof of Theorem 1.2, the limit $\lim_{k\to\infty} p_k^{1/C_k}$ exists, say A. In addition, we have $A \in \mathcal{W}(C_k)$ and $\lfloor A^{C_1} \rfloor = p_1$. □

Proof of Theorem 2.6. Suppose that a sequence $(c_k)_{k\geq 1}$ satisfies (C1) to (C5). Then, $(c_k)_{k\geq 1}$ satisfies (B1) to (B5). Thus, by Theorem 2.3, the number $\xi = \xi(C_k)$ exists, and either ξ is transcendental or there exists g such that $\xi^g = \beta$ is a Pisot number of degree ℓ with

$$3 \leq \ell \leq 1 + \left(\frac{19}{40} \limsup_{k\to\infty} c_{k+1} - 1 \right)^{-1} \leq 1 + \left(\frac{57}{40} - 1 \right)^{-1} = 1 + \frac{40}{17} < 4$$

and for every sufficiently large $k \in I$ we have $g \mid C_k$. Therefore, $\ell = 3$. We note that (C3) leads to $I = \{k \in \mathbb{N} \colon c_{k+1} \geq 40/19 + \epsilon\} = \mathbb{N}$, and hence $g \mid \operatorname{agcd}(C_k)$ since $g \mid C_k$ for every sufficiently large $k \in \mathbb{N}$.

By Bertrand's postulate (or Bertrand-Chebyshev theorem), we find a prime number $p_1 \in [x_0^{1/c_2}, 2x_0^{1/c_2}]$. Therefore, by applying Lemma 9.1, there exists $A \in \mathcal{W}(C_k)$ such that $\lfloor A^{C_1} \rfloor = p_1$. Since ξ is the smallest real number of $\mathcal{W}(C_k)$, we obtain

$$\xi \leq A \leq (p_1 + 1)^{1/C_1} \leq (2x_0^{1/c_2} + 1)^{1/c_1}.$$

Therefore, $\xi^g = \beta$ is a cubic Pisot number less than or equal to

$$(2x_0^{1/c_2} + 1)^{g/c_1}. \qquad \square$$

Lemma 9.2. *There are exactly* 20 *cubic Pisot numbers less than or equal to* 3. *Let* $\alpha_1, \ldots, \alpha_{20}$ *be such Pisot numbers labeled so that* $1 < \alpha_1 < \cdots < \alpha_{20} < 3$. *Then, we obtain the list of* α_j $(j = 1, 2, \ldots, 20)$ *as Table 2.*

Proof. Let M be a given positive integer. If α is a cubic Pisot number with $1 < \alpha \leq M$, then the minimal polynomial $f(X)$ of α over $\mathbb{Q}$ forms as

$$f(X) = X^3 - a_2X^2 - a_1X - a_0, \quad (a_2, a_1, a_0 \in \mathbb{Z},\ a_0 \neq 0).$$

Let α' and α'' be the conjugates of α. Then, we have $|\alpha'| < 1$ and $|\alpha''| < 1$ since α is a Pisot number. Therefore, the relation between roots and coefficients implies that

$$\begin{aligned} a_2 &= \alpha + \alpha' + \alpha'' \in (-1, M + 2), \\ |a_1| &= |\alpha\alpha' + \alpha'\alpha'' + \alpha\alpha''| < 2M + 1, \\ 0 < |a_0| &= |\alpha\alpha'\alpha''| < M. \end{aligned}$$

By substituting $M = 3$, we have $a_2 \in [0, 4]$, $|a_1| \leq 6$, and $1 \leq |a_0| \leq 2$. Thus, there are $5 \cdot 13 \cdot 4 = 260$ candidates of polynomials whose roots contain a cubic Pisot number. By Akiyama's result [Aki00, Lemma 1], $f(X)$ is the minimal polynomial of some cubic Pisot number if and only if

$$|a_1 - 1| < a_2 + a_0 \quad \text{and} \quad a_0^2 - a_1 < \operatorname{sgn}(a_0)(1 + a_2a_0), \tag{9.3}$$

where $\operatorname{sgn}(x) = 1$ if $x > 0$; $\operatorname{sgn}(x) = -1$ if $x < 0$. Using Mathematica, for all $a_2 \in [0, 4]$, $|a_1| \leq 6$, and $1 \leq |a_0| \leq 2$, we check whether (a_0, a_1, a_2) satisfies (9.3). After that, we examine whether all real roots of $f(X)$ are less than or equal to 3, and obtain Table 2. In the appendix, we give a program using Mathematica to find cubic Pisot numbers up to a given positive real number. $\square$

Proof of Corollary 2.7 . Fix an arbitrary integer $r \geq 1$. Let $C_k = r3^k - 1$ for every $k \in \mathbb{N}$. Then, (C1) and (C3) are clearly valid. Let us verify (C2), (C4), and (C5) in Theorem 2.6.

We also obtain (C2) since for every $k \in \mathbb{N}$

$$C_{k+1} = r3^{k+1} - 1 = 3(r3^k - 1) + 2 \geq 3C_k.$$

For (C4), it follows that $\gcd(r, C_m) = 1$ for every $m \in \mathbb{N}$ since $C_m = r3^m - 1$. Thus,

$$3^m \equiv r^{-1} \bmod C_m,$$

where r^{-1} denotes the inverse of r in $\mathbb{Z}/C_m\mathbb{Z}$. Let $\varphi(n)$ be the number of positive integers up to n which is coprime to n. Euler's theorem implies that

$$r3^{m\varphi(C_m)+m} - 1 = r(3^m)^{\varphi(C_m)+1} - 1 \equiv r \cdot r^{-1} - 1 \equiv 0 \bmod C_m.$$

j	α_j	Minimal polynomial of α_j
1	$1.32471795724474602596\cdots$	X^3-X-1
2	$1.46557123187676802665\cdots$	X^3-X^2-1
3	$1.75487766624669276004\cdots$	X^3-2X^2+X-1
4	$1.83928675521416113255\cdots$	X^3-X^2-X-1
5	$2.14789903570478735402\cdots$	X^3-X^2-2X-1
6	$2.20556943040059031170\cdots$	X^3-2X^2-1
7	$2.24697960371746706105\cdots$	X^3-2X^2-X+1
8	$2.26953084208114277085\cdots$	X^3-X^2-2X-2
9	$2.32471795724474602596\cdots$	X^3-3X^2+2X-1
10	$2.35930408597177642073\cdots$	X^3-2X^2-2
11	$2.51154714169453198401\cdots$	X^3-X^2-3X-2
12	$2.52137970680456756960\cdots$	X^3-3X^2+2X-2
13	$2.54681827688408207913\cdots$	X^3-2X^2-X-1
14	$2.65896708191699407934\cdots$	X^3-2X^2-X-2
15	$2.76929235423863141524\cdots$	X^3-3X^2+X-1
16	$2.83117720720833690413\cdots$	X^3-2X^2-2X-1
17	$2.83928675521416113255\cdots$	X^3-4X^2+4X-2
18	$2.87938524157181676810\cdots$	X^3-3X^2+1
19	$2.89328919630449778890\cdots$	X^3-3X^2+X-2
20	$2.91963956583941814511\cdots$	X^3-2X^2-2X-2

TABLE 2. All cubic Pisot numbers ≤ 3

Therefore, we have $C_m \mid C_k$ for $k = m\varphi(C_m)+m$, which is (C4).

For (C5), it suffices to show that

$$\operatorname{agcd}(C_k) = \begin{cases} 2 & \text{if } r \text{ is odd,} \\ 1 & \text{if } r \text{ is even.} \end{cases} \tag{9.4}$$

We now take an arbitrary prime factor p of $C_m = r3^m - 1$. Then, $3^m \equiv r^{-1} \bmod p$ since $\gcd(r,p)=1$. Therefore, we have

$$C_{m+1} = r3^{m+1} - 1 = r(3^m)\cdot 3 - 1 \equiv 3 - 1 \equiv 2 \bmod p. \tag{9.5}$$

If r is even, then we have $\gcd(C_m, C_{m+1}) = 1$ for every $m \in \mathbb{N}$, and hence $\operatorname{agcd}(C_k) = 1$.

Suppose that r is odd. Then, by (9.5), it is clear that the prime factor of $\gcd(C_m, C_{m+1})$ is only 2. Furthermore, if $r \equiv 1 \bmod 4$, then

$$C_{2k+1} = r3^{2k+1} - 1 \equiv (-1)^{2k+1} - 1 \equiv -2 \not\equiv 0 \bmod 4$$

for all $k \in \mathbb{N}$. If $r \equiv 3 \bmod 4$, then

$$C_{2k} = r3^{2k} - 1 \equiv 3(-1)^{2k} - 1 \equiv 2 \not\equiv 0 \bmod 4$$

for all $k \in \mathbb{N}$. Therefore, $\gcd(C_m, C_{m+1}) = 2$ for all $m \in \mathbb{N}$, and hence we have (9.4) and (C5).

By applying Theorem 2.6, $\xi = \xi(r3^k - 1)$ exists, and either ξ is transcendental or there exists $g \in \{1, 2\}$ such that ξ^g is a cubic Pisot number satisfying

$$\xi^g \leq (2x_0^{1/c_2} + 1)^{g/c_1} \leq (2x_0^{1/3} + 1)^{2/(3r-1)}.$$

Let κ be the unique real root of $X^3 - X - 1$, which is approximately equal to $1.324717\cdots$. Then by Siegel's result [Sie44], κ is the smallest Pisot number. Therefore, if we have

$$(2x_0^{1/3} + 1)^{2/(3r-1)} < \kappa \quad \Longleftrightarrow \quad \frac{2\log(2x_0^{1/3} + 1)}{3\log\kappa} + \frac{1}{3} < r,$$

then ξ is transcendental. Choosing $x_0 = \exp(3\exp(32.76))$ by [MTY24, Section 2] and using Mathematica, we obtain

$$\frac{2\log(2x_0^{1/3} + 1)}{3\log\kappa} + \frac{1}{3} = 400296054181891.5\cdots. \qquad \square$$

Proof of Corollary 2.8. Assume that (2.2) is true for $x_0 = 1$. Similarly with the proof of Corollary 2.7, either $\xi = \xi(r3^k - 1)$ is transcendental or there exists $g \in \{1, 2\}$ such that ξ^g is a cubic Pisot number satisfying

$$\beta = \xi^g \leq (2x_0^{1/c_2} + 1)^{g/c_1} = 3^{g/(3r-1)},$$

where $g \in \{1, 2\}$. Assume that ξ is algebraic. Then, in the case $r \geq 3$, we have

$$\beta \leq 3^{2/(3r-1)} \leq 3^{1/4} < 1.320 < 1.3247 < \kappa,$$

a contradiction since κ is the smallest Pisot number. Therefore, $\xi = \xi(r3^k - 1)$ is transcendental for all $r \geq 3$.

In the case $r = 2$, we have $g = 1$ by (9.4). Therefore, we obtain

$$\beta \leq 3^{1/(3\cdot 2-1)} = 3^{1/5} = 1.2457\cdots < 1.3247 < \kappa,$$

a contradiction.

In the case $r = 1$ and $g = 1$, we see that

$$\beta = \xi^g (= \xi) \leq 3^{1/(3-1)} = 3^{1/2} = 1.732\cdots < 1.754 < \alpha_3.$$

This yileds that $\xi \in \{\alpha_1, \alpha_2\}$. By the definition of ξ, both $\lfloor \xi^{3^1-1} \rfloor$ and $\lfloor \xi^{3^2-1} \rfloor$ are prime numbers, but we observe that

$$\lfloor \alpha_1^{3^1-1} \rfloor = 1, \quad \lfloor \alpha_2^{3^2-1} \rfloor = 21,$$

a contradiction.

In the case $r = 1$ and $g = 2$, we see that

$$\beta = \xi^g (= \xi^2) \leq 3^{2/(3-1)} = 3,$$

and hence $\xi \in \{\alpha_1^{1/2}, \alpha_2^{1/2}, \ldots, \alpha_{20}^{1/2}\}$. Since $\lfloor (\alpha_j^{1/2})^{3^1-1} \rfloor = 1$ for all $j \in \{1, 2, 3, 4\}$, we have $\xi \in \{\alpha_5^{1/2}, \alpha_6^{1/2}, \ldots, \alpha_{20}^{1/2}\}$. Let us recall that $\lfloor \xi^{3^k-1} \rfloor$ is a prime number for every $k \in \mathbb{N}$,

but we observe that

$$\lfloor(\alpha_5^{1/2})^{3^2-1}\rfloor = 21 = 3\cdot 7, \qquad \lfloor(\alpha_6^{1/2})^{3^3-1}\rfloor = 29226 = 2\cdot 3\cdot 4871,$$
$$\lfloor(\alpha_7^{1/2})^{3^2-1}\rfloor = 25 = 5^2, \qquad \lfloor(\alpha_8^{1/2})^{3^2-1}\rfloor = 26 = 2\cdot 13,$$
$$\lfloor(\alpha_9^{1/2})^{3^4-1}\rfloor = 2\cdot 3\cdot 43\cdot 1750616861141, \qquad \lfloor(\alpha_{10}^{1/2})^{3^2-1}\rfloor = 30 = 2\cdot 3\cdot 5,$$
$$\lfloor(\alpha_{11}^{1/2})^{3^2-1}\rfloor = 39 = 3\cdot 13, \qquad \lfloor(\alpha_{12}^{1/2})^{3^2-1}\rfloor = 40 = 2^3\cdot 5,$$
$$\lfloor(\alpha_{13}^{1/2})^{3^2-1}\rfloor = 42 = 2\cdot 3\cdot 7, \qquad \lfloor(\alpha_{14}^{1/2})^{3^2-1}\rfloor = 49 = 7^2,$$
$$\lfloor(\alpha_{15}^{1/2})^{3^2-1}\rfloor = 58 = 2\cdot 29, \qquad \lfloor(\alpha_{16}^{1/2})^{3^2-1}\rfloor = 64 = 2^6,$$
$$\lfloor(\alpha_{17}^{1/2})^{3^2-1}\rfloor = 64 = 2^6, \qquad \lfloor(\alpha_{18}^{1/2})^{3^2-1}\rfloor = 68 = 2^2\cdot 17,$$
$$\lfloor(\alpha_{19}^{1/2})^{3^2-1}\rfloor = 70 = 2\cdot 5\cdot 7, \qquad \lfloor(\alpha_{20}^{1/2})^{3^2-1}\rfloor = 72 = 2^3\cdot 3^2.$$

All of them are composite numbers, a contradiction. □

Appendix A. Cubic Pisot numbers up to a given positive integer

In this appendix, we give a program using Mathematica to obtain a list of all cubic Pisot numbers up to a given positive real number M. We apply it to describe Table 2. It is based on the proof of Lemma 9.2.

```
(*Input M as a positive real number*)
M = Input["Enter a positive real number"];

(*Fix a range of coefficients (a0,a1,a2)*)
ceilM = Ceiling[M];
coeffs =
  Flatten[Table[{a0, a1, a2}, {a0, -ceilM + 1,
     ceilM - 1}, {a1, -2*ceilM, 2*ceilM}, {a2, 0, ceilM + 1}], 2];
coeffs = DeleteCases[coeffs, {0, _, _}];

(*Select polynomials f(x)=x^3-a2*x^2-a1*x-a0 of which all real roots are
less than or equal to M, and make a list of {alpha, a0, a1, a2},
where alpha is the maxmum real root of f(x) (it should be a Pisot number)*)
list = {};
max = Length[pisotcoeffs];
For[j = 1, j <= max, j++,
  a0 = pisotcoeffs[[j, 1]];
  a1 = pisotcoeffs[[j, 2]];
  a2 = pisotcoeffs[[j, 3]];
  poly = x^3 - a2*x^2 - a1*x - a0;
  realroots = x /. NSolve[poly == 0, x, Reals, WorkingPrecision -> 50];
If[Select[realroots, # > M &] == {},
list = Join[list, {{Max[realroots], a0, a1, a2}}] ];
];

(*Sort the list*)
list = SortBy[list, First];
```

```
max = Length[list];

(*Print*)
Print[StringForm[
   "The number of cubic Pisot numbers less than or equal to `` is
exactly ``, and the list is as follows:\n
a cubic Pisot number, its minimal polynomial", M, max]];

For[j = 1, j <= max, j++,
 alpha = list[[j, 1]];
 a0 = list[[j, 2]]; a1 = list[[j, 3]]; a2 = list[[j, 4]];
 poly = x^3 - a2*x^2 - a1*x - a0;
 Print[StringForm[" alpha_`` = `` , ``", j, NumberForm[alpha, 25],
   TraditionalForm[poly]]]
 ]
```

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