$\frac{4}{11}\log 2$ IS LIKELY IRRATIONAL

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Abstract. In an effort to modify Apéry's proofs of the irrationality of $\log 2$, $\zeta(2)$, and $\zeta(3)$ to include other, perhaps less well known constants, the author has identified a certain number $R = .25205\ldots$ as a likely candidate for such an irrationality proof. The actual proof, unfinished at the time of a discovery of a different nature, depends on estimating the power to which a prime p divides each rational convergent a_n/b_n, where R is defined by $a_n/b_n \to R$ as $n \to \infty$. A conjecture is salvaged that gives an explicit class of Apéry-type numbers.

1. Introduction

Recall that we may prove that a real number α is irrational by exhibiting a sequence $\{a_n/b_n\}$ of rational numbers (with $b_n \to \infty$) that converges to α with the property that there exist $\delta > 0$ and $C > 0$ such that for all n

$$0 < \left|\alpha - \frac{a_n}{b_n}\right| < \frac{C}{b_n^{1+\delta}}. \tag{1}$$

If $\alpha = c/d$ were *rational*, then we would have

$$\left|\alpha - \frac{a_n}{b_n}\right| = \left|\frac{c}{d} - \frac{a_n}{b_n}\right| = \left|\frac{b_n c - a_n d}{b_n d}\right| \geq \frac{1}{b_n d}$$

since $b_n c - a_n d$ is a nonzero integer. Comparing this with our criterion (1), we obtain

$$\frac{1}{b_n d} \leq \left|\alpha - \frac{a_n}{b_n}\right| < \frac{C}{b_n^{1+\delta}}$$

or $b_n^\delta < C \cdot d$, which is a contradiction because $\{b_n^\delta\}$ is unbounded.

Apéry [1] found such a sequence of rationals for each of the constants $\log 2$, $\zeta(2)$, and $\zeta(3)$. One can view these sequences as arising from recurrence relations satisfied by the summands of

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k},$$
$$\sum_{k=0}^{n} \binom{n}{k}^2\binom{n+k}{k}, \text{ and}$$
$$\sum_{k=0}^{n} \binom{n}{k}^2\binom{n+k}{k}^2$$

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which, under Apéry's method, result in sequences of rational numbers that converge to $\frac{1}{2}\log 2$, $\frac{1}{5}\zeta(2)$, and $\frac{1}{6}\zeta(3)$ respectively, as can be verified numerically with the function `Roger` in Zeilberger's Maple package `AperyRecurrence` [6].

2. Generalization

In search of other numbers than can be proven irrational by Apéry's method, I have used `AperyRecurrence` to study the more general sum

$$\sum_{k=0}^{n} \binom{n}{k}^{p_1} \binom{rn+sk}{k}^{p_2} (t-1)^k \tag{2}$$

with positive integral parameters p_1, p_2, r, s, t. (Later we will see the advantage of using $(t-1)^k$ rather than t^k.)

One can find (again using `Roger`) estimates of δ for various values of these parameters. If $\delta > 0$ for a given summand, then there is a possibility of finding an Apéry-style irrationality proof.

It appears that for p_1 and p_2 greater than 2, there are no good candidates. Moreover, even modifying r, s, and t for the cases $p_1 = p_2 + 1 = 2$ and $p_1 = p_2 = 2$ seems not to give candidates either; that is, $\zeta(2)$ and $\zeta(3)$ are just special cases. Therefore we restrict ourselves to the case $p_1 = p_2 = 1$. (However, it might still be of some interest to find alternative expressions for these numbers, as is done in section 3 for the case $s = p_1 = p_2 = 1$. For example, how are they related to $\zeta(2)$ and $\zeta(3)$?)

For $t \geq 2$, the sum

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (t-1)^k$$

gives a sequence of rational numbers converging to $\frac{1}{2}\log\frac{t}{t-1}$, which is irrational for every t.

A more interesting sum is

$$\sum_{k=0}^{n} \binom{n}{k} \binom{2n+k}{k} (t-1)^k. \tag{3}$$

We first consider the case $t = 2$. Executing Zeilberger's algorithm [5] with the command

```
zeil(binomial(n,k)*binomial(2*n+k,k),k,n,N)
```

reveals that the summand $F(n,k) = \binom{n}{k}\binom{2n+k}{k}$ satisfies the recurrence

$$p_0(n)F(n,k) + p_1(n)F(n+1,k) + p_2(n)F(n+2,k) = G(n,k+1) - G(n,k), \tag{4}$$

where

$$\begin{aligned}
p_0(n) &= -2(17n+28)(2n+1)(n+1),\\
p_1(n) &= 1207n^3 + 4402n^2 + 5021n + 1730,\\
p_2(n) &= -4(17n+11)(2n+3)(n+2),\\
G(n,k) &= R(n,k)\,F(n,k),
\end{aligned}$$

and $R(n,k)$ is a rational function in n and k. Define two sequences $\{a_n\}$, $\{b_n\}$ by the recurrences

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} = 0, \tag{5}$$

$$p_0(n)b_n + p_1(n)b_{n+1} + p_2(n)b_{n+2} = 0 \tag{6}$$

with initial conditions $a_0 = 0$, $a_1 = 1$, $b_0 = 1$, $b_1 = 4$. These conditions ensure that $b_n \in \mathbb{Z}$ for all n; indeed, summing (4) over all integers k gives

$$b_n = \sum_{k=0}^{n} \binom{n}{k}\binom{2n+k}{k}.$$

In general, a_n is a non-integral rational number. The sequence $\{a_n/b_n\}$ begins

$$0, \frac{1}{4}, \frac{865}{3432}, \frac{12643}{50160}, \frac{13619843}{54035520}, \frac{323746091}{1284433920}, \frac{115021083581}{456335953920}, \frac{2224431220019}{8825233697280}, \ldots.$$

The experimental δ for this sequence (given by `Roger`) remains positive for at least several thousand terms, suggesting that the real number

$$R = \lim_{n\to\infty} \frac{a_n}{b_n} = .252053520203616476\ldots$$

may be proven irrational by Apéry's method.

The number R did not appear in Plouffe's Inverter [3] as a widely known constant (and the advanced search was inoperable at the time), so it seemed that R was in fact a new candidate for irrationality. What remains, then, is to *prove* that $\{a_n/b_n\}$ satisfies the irrationality criterion (1) for some $\delta > 0$. To do this we must estimate $b_n d_n$, where d_n is the denominator of a_n. The leading terms of (6) give the estimate

$$\left(-68 + 1207N - 136N^2\right) b_n \approx 0,$$

where N is the shift operator in n: $Nb_n = b_{n+1}$. Thus $b_n = O(\alpha^n)$, where $\alpha = \frac{71+17\sqrt{17}}{16}$ is a zero of the above quadratic polynomial.

A conjectural upper bound for d_n is $11 \cdot 2^{n-2}\, l_{2n}$, where $l_k = \operatorname{lcm}(1,2,\ldots,k)$. However, this is too crude, as it results in an asymptotic of $\frac{11}{4}(2e^2)^n$ with $2e^2 > \alpha$. One possible approach to a refinement is the determination of the power $\operatorname{ord}_p(d_n)$ to which each prime p divides d_n. For example, for each of the primes $p = 3$, 97, 337 we have $\operatorname{ord}_p(d_n) = \operatorname{ord}_p(l_{2n}) = \lfloor \log_p(2n) \rfloor$; in these cases, l_{2n} is the best we can do. For most primes, however, we can do substantially better. The identity

$$\operatorname{ord}_7(a_n) = \lfloor \log_7(2n-1) \rfloor + \left\lfloor \log_7 \tfrac{n}{3} \right\rfloor - \left\lfloor \log_7 \tfrac{2n-1}{5} \right\rfloor$$

for $p = 7$ holds for the first fifteen thousand n. Additionally, $p = 199$ seems to be similar to 7 in this regard. In general, $\operatorname{ord}_p(d_n)$ varies much more frequently than in these special cases; however, it is still conceivable that explicit bounds exist. Thus it would seem that R is likely irrational.

3. Closed forms

But, alas, an all-too-late consultation with the now-defunct Inverse Symbolic Calculator [2] reveals that R is just $\frac{4}{11}\log 2$, the irrationality of which does not require extensive analysis. And once this is known it is not difficult to guess that

$$\frac{2t}{6t+5} \log \tfrac{t}{t-1}$$

is a general expression for the real number arising as the limit of a_n/b_n, where $a_0 = 0$, $a_1 = 1$, $b_0 = F(0,0)$, $b_1 = F(1,0) + F(1,1)$, $F(n,k) = \binom{n}{k}\binom{2n+k}{k}(t-1)^k$ is the summand of (3), and a_n, b_n satisfy the recurrence given for $F(n,k)$ by Zeilberger's algorithm.

In general, the sum (2) seems to satisfy a recurrence of order 2 whenever $s = 1$. One may repeat the same procedure for $r = 3$, $s = 1$ to find the expression

$$\frac{6t^2}{24t^2 - 6t - 1} \log \tfrac{t}{t-1}$$

for the limit of a_n/b_n. For $r = 4$, $s = 1$ we obtain

$$\frac{12t^3}{60t^3 - 18t^2 - 4t - 1} \log \tfrac{t}{t-1}.$$

With many more cases, it becomes clear that the general form (when $s = 1$) is

$$\frac{r!\, t^{r-1}}{(r+1)!\, t^{r-1} - \cdots - (r-2)!} \log \tfrac{t}{t-1}.$$

(In finding the rational coefficients of $\log \tfrac{t}{t-1}$, *Mathematica*'s `Rationalize` function outperforms Maple's `convert(f, rational)` in correctness.) One divides the denominator by $r!\, t^{r-1}$ and interpolates a rational function for each coefficient to experimentally determine the general expression

$$\left((r+1) - \sum_{k=1}^{r-1} \frac{r-k}{k(k+1)t^k}\right)^{-1} \log \tfrac{t}{t-1}. \tag{7}$$

References

[1] Roger Apéry, "Irrationalité de $\zeta(2)$ et $\zeta(3)$", *Astérisque* **61** (1979), 11–13.

[2] Centre for Experimental and Constructive Mathematics, *Inverse Symbolic Calculator*, `http://oldweb.cecm.sfu.ca/projects/ISC/ISCmain.html`.

[3] Simon Plouffe, *Plouffe's Inverter*, `http://pi.lacim.uqam.ca/eng/`.

[4] Alfred van der Poorten, "A proof that Euler missed... Apéry's proof of the irrationality of $\zeta(3)$", *Mathematical Intelligencer* **1** (1979), 195–203.

[5] Doron Zeilberger, "A fast algorithm for proving terminating hypergeometric identities", *Discrete Mathematics* **80** (1990), 207–211.

[6] Doron Zeilberger, "Computerized deconstruction", *Advances in Applied Mathematics* **30** (2003), 633–654.