ABSTRACT

AN EXPOSITION OF THE RIEMANN ZETA FUNCTION

by

John Molokach

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Chair: Dr. Guglielmo Fucci

Major Department: Mathematics

This thesis is an exposition of the Riemann zeta function. Included are techniques of analytic continuation and relationships to special functions. Some generalizations of the Riemann zeta function are outlined, as well as the calculation of zeta constants and the development of some identities. Additionally, one of the great unsolved problems of mathematics, the Riemann hypothesis, is discussed.

AN EXPOSITION OF THE RIEMANN ZETA FUNCTION

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John Molokach

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John Molokach

APPROVED BY:

DIRECTOR OF THESIS:

Dr. Guglielmo Fucci

COMMITTEE MEMBER:

Dr. Imre Patyi

COMMITTEE MEMBER:

Dr. David Pravica

COMMITTEE MEMBER:

Dr. Chris Jantzen

CHAIR OF THE DEPARTMENT
OF MATHEMATICS:

Dr. Johannes H. Hattingh

DEAN OF THE
GRADUATE SCHOOL:

Dr. Paul Gemperline

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TABLE OF CONTENTS

Chapter 1: Introduction

The Riemann zeta function is one of the most important special functions in mathematics. Its applications encompass many areas of study, including number theory and physics. Before commenting on its historical development, we begin by outlining a few examples where the Riemann zeta function applies specifically to these two areas.

In number theory, for example, the distribution of primes is studied using the Riemann zeta function. The relation between the Riemann zeta function and the distribution of prime numbers are explained later. In physics, the Riemann zeta function and its generalizations are used in quantum field theory and string theory. For instance, zeta function regularization is used as one possible means of regularization of divergent series and divergent integrals in quantum field theory (see e.g. [65]). The zeta function is also useful for the analysis of dynamical systems [38].

What has now come to be known as the Riemann zeta function has its roots traced to the study of the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}\,, \tag{1.1}$$

which was first shown to be a divergent series in 1360 by Nicole Oresme [37]. The next piece of historical evidence of the (mathematical) study of the harmonic series comes from Pietro Mengoli, who published a proof of its divergence in 1650 [41].

Mengoli's proof is outlined below. To prove it however, we first need a lemma.

Lemma 1.1. *For $x > 1$, we have*

$$\frac{1}{x-1} + \frac{1}{x} + \frac{1}{x+1} > \frac{3}{x}.$$

Proof.

$$\begin{aligned}\frac{1}{x-1}+\frac{1}{x}+\frac{1}{x+1} &= \frac{x(x+1)+(x-1)(x+1)+x(x-1)}{(x-1)x(x+1)} \\ &= \frac{x^2+x+x^2-1+x^2-x}{(x-1)x(x+1)} \\ &= \frac{3x^2-1}{x(x^2-1)} \\ &= \frac{3}{x}\cdot\frac{x^2-1/3}{x^2-1} \\ &> \frac{3}{x},\end{aligned}$$

since $\frac{x^2-1/3}{x^2-1} > 1$ for $x > 1$. ∎

Theorem 1.2. *The series*

$$\sum_{n=1}^{\infty}\frac{1}{n}$$

diverges.

Proof. Assume $\sum_{n=1}^{\infty}\frac{1}{n}$ converges to S. Then

$$\begin{aligned}S &= 1+1/2+1/3+1/4+\cdots \\ &= 1+(1/2+1/3+1/4)+(1/5+1/6+1/7)+\cdots\end{aligned}$$

From the lemma, then

$$S > 1+(3/3)+(3/6)+(3/9)+\cdots = 1+S,$$

which is impossible for any finite S. Since we arrived at a contradiction, we can conclude that $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent. ∎

During the Baroque period, the harmonic series became popular with architects to establish floor plans and elevations (frontal views of the building) [33]. In fact, the term *harmonic* comes from a musical term where the wavelength of a vibrating string produces different pitches, creating "harmonies." These harmonies were accented in music and architecture beginning with the Gothic period in the late 12th century, when architectural drawings emphasized harmonic ratios of width to height in building elevations. The idea was to mimic harmonies in music with notable features in the building design. In these architectural designs of the period, the width/height ratio tended to converge to the harmonic sequence in order to match harmonic tones in music, notably that from Notre Dame [35] (see figure 1.1). In music, the different "harmonics" come from string lengths $1, 1/2, 1/3, \ldots$ (see figure 1.2).

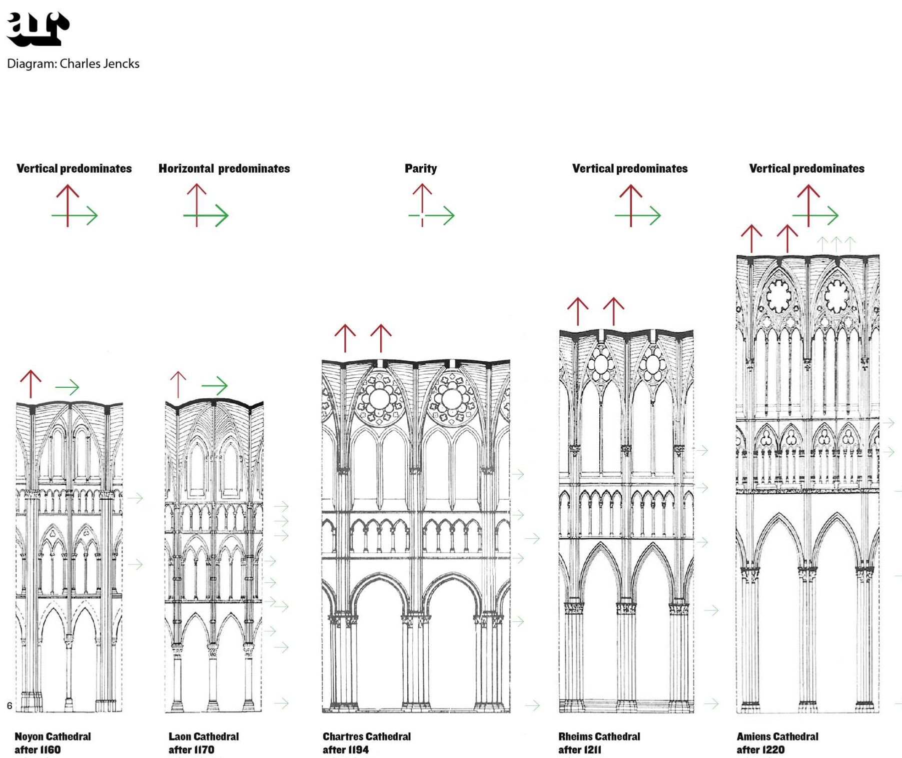

Figure 1.1: Elevations from Gothic architecture in harmonic progression

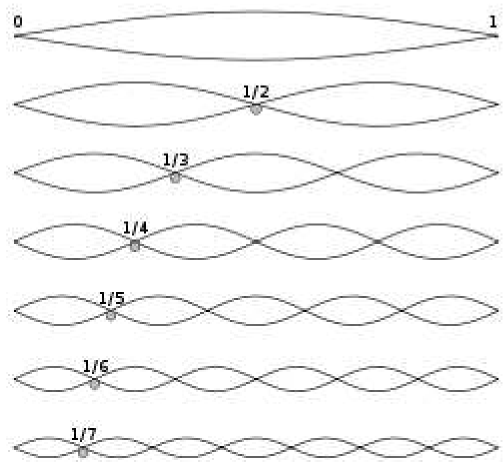

Figure 1.2: Harmonics on a string

The generalization of the harmonic series, known as the $p-$*series*, is

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \tag{1.2}$$

with $p \in \mathbb{R}$. From the integral test, this series can be shown to converge for $p > 1$. The case with $p = 2$ was known by Mengoli to converge, but the exact sum eluded him. He posed the problem of finding the sum in 1644, and after attempts by many mathematicians including John Wallis and the Bernoulli brothers Johann and Jakob, the problem was finally solved in 1735 by Leonard Euler. The problem became known as the "Basel Problem" because the Bernoulli brothers and Euler were from Basel, Switzerland [24].

Further sums for the $p-$series were calculated by Euler. In his book *Introduction to the Analysis of the Infinite*, Euler calculated exact sums for even p from 2 to 26 [27]. Additionally, Euler also connected the series (1.2) to the distribution of primes through the following relation,

$$\zeta(\sigma) = \sum_{n=1}^{\infty} \frac{1}{n^\sigma} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-\sigma}}\,, \tag{1.3}$$

valid for real $\sigma > 1$.

Euler's proof of this formula uses an iterative process to "sieve" prime numbers. The method is attributed to the Greek mathematician Eratosthenes of Cyrene. The sieve process can be described as follows: First, list the natural numbers from 2 to n. Then, starting with 2, mark out all multiples of 2 that follow 2. Then, from the list that remains, mark out all multiples of 3 that follow 3. Continuing this process, one obtains all the primes less than or equal to n. Now we present Euler's proof of (1.3).

Take $\zeta(\sigma) = 1 + 2^{-\sigma} + 3^{-\sigma} + 4^{-\sigma} + \cdots$. Then $2^{-\sigma}\zeta(\sigma) = 2^{-\sigma} + 4^{-\sigma} + 6^{-\sigma} + \cdots$ and subtracting the second equation from the first gives

$$(1 - 2^{-\sigma})\zeta(\sigma) = 1 + 3^{-\sigma} + 5^{-\sigma} + 7^{-\sigma} + \cdots\,. \tag{1.4}$$

Then repeating the process for $3^{-\sigma}$ gives

$$3^{-\sigma}(1 - 2^{-\sigma})\zeta(\sigma) = 3^{-\sigma} + 9^{-\sigma} + 15^{-\sigma} + \cdots\,, \tag{1.5}$$

and subtracting the last two equations gives

$$(1 - 3^{-\sigma})(1 - 2^{-\sigma})\zeta(\sigma) = 1 + 5^{-\sigma} + 7^{-\sigma} + 11^{-\sigma} + \cdots\,. \tag{1.6}$$

Continuing recursively for p prime, we have

$$\lim_{p\to\infty} \prod_{p \text{ prime}} (1 - p^{-\sigma})\zeta(\sigma) = 1, \tag{1.7}$$

for $\sigma > 1$, where the RHS is $\sum_{n=1}^{\infty} n^{-\sigma} - \sum_{n=2}^{\infty} n^{-\sigma}$. Division then gives (1.3), which completes the proof.

Euler used the product formula (1.3) to prove the following fundamental theorem in number theory.

Theorem 1.3. *There are infinitely many primes.*

Proof. Euler proved the infinitude of primes by evaluating his product formula (1.3) at $\sigma = 1$. The left hand side of the formula is a divergent series. Assuming the number of primes is finite (say, N of them), the RHS then becomes a finite product - a contradiction:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n} = \prod_{p \text{ prime}} \frac{1}{1-p^{-1}} &= \frac{1}{(1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{5})\cdots(1-\frac{1}{p_N})} \\ &= \frac{1}{(1/2)(2/3)(4/5)\cdots(p_N-1)/p_N} \\ &= \left(\frac{2}{1}\right)\left(\frac{3}{2}\right)\left(\frac{5}{4}\right)\cdots\left(\frac{p_N}{p_N-1}\right), \end{aligned} \tag{1.8}$$

and since the RHS must diverge along with the left hand side, Euler argued that the numerator of this fraction must be infinite. Therefore, there are infinitely many primes. □

Bernhard Riemann generalized the series (1.2) to be defined for complex values of s, as

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}\,, \tag{1.9}$$

which is valid for $\Re(s) > 1$. Riemann analytically continued this function to be defined for all values $s \in \mathbb{C}$, $s \neq 1$, in his famous paper *On the Number of Prime Numbers less than a Given Quantity*, published in November 1859 [49]. In the following pages, we look closely at this zeta function as Riemann, and later mathematicians, studied it. We give a definition of the function and proceed to expose the details of its analytic continuation. We then outline some methods of calculation of values of the Riemann zeta function, give a summary of some generalizations, list some associated identities, and discuss the famous *Riemann Hypothesis*.

Before concluding this section, we would like to remind the reader of some important formulas that are used throughout this work.

The *gamma function* $\Gamma(s)$ is defined as

$$\Gamma(s) = \int_0^\infty \frac{t^{s-1}}{e^t} dt, \tag{1.10}$$

which is a representation valid for $s \in \mathbb{C}$, with $\Re(s) > 0$. This function can also be analytically continued to all $\{s : s \in \mathbb{C} \setminus Z^-\}$. It can be easily proved that for $n \in \mathbb{N}$,

$$\Gamma(n) = (n-1)!\,, \tag{1.11}$$

which means that $\Gamma(s)$ provides a generalization of the factorial. Later in this work, we see a close interplay between the gamma function and zeta function.

The *reflection formula* for the gamma function can be proved by using the integral representation (1.10) and reads

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}, \tag{1.12}$$

for $s \notin \mathbb{Z}$. The proof can be found in ([11] §7H, 11A).

Through infinite product expansion techniques (see [11] §31), equation (1.12) can be manipulated to produce *Euler's product for the sine function*, which is

$$\sin \pi z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right), \tag{1.13}$$

which also converges for all complex z on a compact set.

Chapter 2: Methods of analytic continuation

In this chapter, we outline three methods for the analytic continuation of the Riemann zeta function.

2.1 Method 1: Hermite method

The following is known as the Hermite method of analytic continuation after the French mathematician Charles Hermite [43], and a portion of the argument given here is referenced in [22]. We begin by using equation (1.10) and make a change of variables $t = nu$, with $n \in \mathbb{N}^+$, so that

$$\begin{aligned} \Gamma(s) = \int_0^\infty t^{s-1} e^{-t}\, dt &= \int_0^\infty (nu)^{s-1} e^{-nu} n\, du \\ &= n^s \int_0^\infty u^{s-1} e^{-nu}\, du. \end{aligned} \tag{2.1}$$

From the above expression, we obtain

$$\frac{\Gamma(s)}{n^s} = \int_0^\infty t^{s-1} e^{-nt}\, dt, \tag{2.2}$$

and hence

$$\zeta(s)\Gamma(s) = \sum_{n=1}^\infty \frac{\Gamma(s)}{n^s} = \sum_{n=1}^\infty \int_0^\infty t^{s-1} e^{-nt}\, dt. \tag{2.3}$$

Because of uniform convergence, we can justify interchanging the sum and integral, giving

$$\zeta(s)\Gamma(s) = \int_0^\infty t^{s-1} \left(\sum_{n=1}^\infty e^{-nt} \right) dt. \tag{2.4}$$

For $t > 0$, we have $0 < e^{-t} < 1$ and therefore $\sum_{n=1}^{\infty} e^{-nt}$ is a convergent geometric series with

$$\sum_{n=1}^{\infty} e^{-nt} = \frac{e^{-t}}{1-e^{-t}} = \frac{1}{e^t - 1}. \tag{2.5}$$

By substituting (2.5) in (2.4), we obtain

$$\zeta(s)\Gamma(s) = \int_0^{\infty} \frac{t^{s-1}}{e^t - 1}\, dt, \tag{2.6}$$

for $\Re(s) > 1$. The integral representation (2.6) can be used to extend the domain of $\zeta(s)$ from $\Re(s) > 1$ to a larger region.

The first step in extending this domain is to rewrite the function $1/(e^t - 1)$ in the integrand of (2.6) in terms of its Laurent expansion:

$$\frac{1}{e^t - 1} = \frac{1}{t} - \frac{1}{2} + \mathcal{O}(t). \tag{2.7}$$

Due to the expansion in (2.7), $1/(e^t - 1) - 1/t$ stays bounded on the interval $[0, 1]$, and hence, we can write for $\Re(s) > 1$,

$$\begin{aligned}
\int_0^{\infty} \frac{t^{s-1}}{e^t - 1}\, dt &= \int_0^1 t^{s-1}\left(\frac{1}{e^t - 1} - \frac{1}{t}\right) dt + \int_0^1 t^{s-1}\left(\frac{1}{t}\right) dt + \int_1^{\infty} \frac{t^{s-1}}{e^t - 1}\, dt \\
&= \int_0^1 t^{s-1}\left(\frac{1}{e^t - 1} - \frac{1}{t}\right) dt + \int_0^1 t^{s-2}\, dt + \int_1^{\infty} \frac{t^{s-1}}{e^t - 1}\, dt \\
&= \int_0^1 t^{s-1}\left(\frac{1}{e^t - 1} - \frac{1}{t}\right) dt + \frac{1}{s-1} + \int_1^{\infty} \frac{t^{s-1}}{e^t - 1}\, dt.
\end{aligned} \tag{2.8}$$

Since the two integrals on the right hand side of (2.8) are analytic functions of s for $\Re(s) > 0$, we can conclude that the left hand side of (2.8) is meromorphic in the half-plane $\Re(s) > 0$ with a simple pole at $s = 1$ of residue 1. According to (2.6), the representation (2.8) coincides with the product $\zeta(s)\Gamma(s)$ when $\Re(s) > 1$, so we can write $\zeta(s)$ for points $s \neq 1$ with $0 < \Re(s) < 1$ as

$$\zeta(s) = \frac{1}{\Gamma(s)} \left[\int_0^1 t^{s-1} \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) dt + \frac{1}{s-1} + \int_1^\infty \frac{t^{s-1}}{e^t - 1}\, dt \right]. \tag{2.9}$$

Since $1/\Gamma(s)$ is an entire function, the relation (2.9) shows that $\zeta(s)$ is meromorphic in the half-plane $\Re(s) > 0$ having a simple pole at $s = 1$ with

$$\mathrm{Res}_{s=1}\zeta(s) = \frac{1}{\Gamma(1)} = 1. \tag{2.10}$$

This implies that

$$\lim_{s \to 1} (s-1)\zeta(s) = 1. \tag{2.11}$$

To simplify equation (2.9), we use the fact that for all points s with $\Re(s) < 1$ we have

$$\frac{1}{s-1} = -\int_1^\infty t^{s-2}\, dt, \tag{2.12}$$

and therefore we can rewrite (2.9) as

$$\begin{aligned}
\zeta(s) &= \frac{1}{\Gamma(s)} \left[\int_0^1 t^{s-1} \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) dt + \frac{1}{s-1} + \int_1^\infty \frac{t^{s-1}}{e^t - 1}\, dt \right] \\
&= \frac{1}{\Gamma(s)} \left[\int_0^1 t^{s-1} \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) dt - \int_1^\infty t^{s-2}\, dt + \int_1^\infty \frac{t^{s-1}}{e^t - 1}\, dt \right] \\
&= \frac{1}{\Gamma(s)} \left[\int_0^1 t^{s-1} \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) dt + \int_1^\infty t^{s-1} \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) dt \right] \\
&= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) dt.
\end{aligned} \tag{2.13}$$

We can now extend the domain of $\zeta(s)$ even further to the left of $\Re(s) = 0$, by repeating this procedure with the next term of the Laurent series in (2.7). To do this, we take the Laurent expansion for $1/(e^t - 1)$ in (2.7) and subtract the first *two* terms.

So, by adding and subtracting we obtain

$$\begin{aligned}\zeta(s)\Gamma(s) &= \int_0^1 t^{s-1}\left(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\right)dt - \int_0^1 t^{s-1}\left(\frac{1}{2}\right)dt + \int_1^\infty t^{s-1}\left(\frac{1}{e^t-1}-\frac{1}{t}\right)dt \\ &= \int_0^1 t^{s-1}\left(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\right)dt - \frac{1}{2s} + \int_1^\infty t^{s-1}\left(\frac{1}{e^t-1}-\frac{1}{t}\right)dt. \quad (2.14)\end{aligned}$$

Both integrals on the right hand side of (2.14) yield analytic functions for $\Re(s) > -1$. From (2.14), we can write $\zeta(s)$ for $\Re(s) > -1$, as

$$\zeta(s) = \frac{1}{\Gamma(s)}\left[\int_0^1 t^{s-1}\left(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\right)dt - \frac{1}{2s} + \int_1^\infty t^{s-1}\left(\frac{1}{e^t-1}-\frac{1}{t}\right)dt\right]. \quad (2.15)$$

This is now meromorphic for $\Re(s) > -1$ with a simple pole of residue 1 at $s = 1$. Moreover, since $1/(s\,\Gamma(s)) = 1/\Gamma(s+1)$, $\zeta(s)$ is analytic at $s = 0$. For points s with $-1 < \Re(s) < 0$ we have

$$\int_1^\infty t^{s-1}\,dt = -\frac{1}{s}. \quad (2.16)$$

So (2.15) and (2.16) give

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\left(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}\right)dt, \quad (2.17)$$

for values of s in the strip $-1 < \Re(s) < 0$.

We now proceed with the explicit evaluation of the integral in (2.17). To do so, we first rewrite the expression appearing in parentheses in the integrand by using Euler's product formula for the sine function (1.13). Taking the logarithm of both sides of this formula gives

$$\ln(\sin \pi z) = \ln z + \sum_{n=1}^{\infty} \ln\left(1 - \frac{z^2}{n^2}\right), \quad (2.18)$$

which is valid for all z in a compact region of the cut complex plane. By differentiating

(2.18) we obtain

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{-2z/n^2}{1 - \frac{z^2}{n^2}} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}, \tag{2.19}$$

which is valid for all $z \notin \mathbb{Z}$. We now make the change of variables $z = it/(2\pi)$, multiply (2.19) by $i/(2\pi)$ and use the fact that $\coth(x) = i\cot(ix)$, to obtain

$$\frac{i}{2} \cot\left(\frac{it}{2}\right) = \frac{1}{2} \coth\left(\frac{t}{2}\right) = \frac{1}{t} + \sum_{n=1}^{\infty} \frac{2t}{t^2 + 4n^2\pi^2}. \tag{2.20}$$

Since $\frac{1}{2}\coth\left(\frac{t}{2}\right) = \frac{1}{e^t - 1} + \frac{1}{2}$ we can rewrite (2.20) as

$$\frac{1}{e^t - 1} = \frac{-1}{2} + \frac{\coth(t/2)}{2} = \frac{1}{t} - \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2t}{t^2 + 4n^2\pi^2}. \tag{2.21}$$

The use of (2.21) in (2.17) allows us to write (for $-1 < \Re(s) < 1$)

$$\begin{aligned} \zeta(s)\Gamma(s) &= 2\int_0^{\infty} t^{s-1} \sum_{N=1}^{\infty} \frac{t}{t^2 + (2N\pi)^2}\, dt \\ &= 2\sum_{N=1}^{\infty} \int_0^{\infty} \frac{t^s}{t^2 + (2N\pi)^2}\, dt, \end{aligned} \tag{2.22}$$

where the sum and integral can be interchanged due to uniform convergence.

Our next task is to compute the integral that appears in (2.22). To do this, let $-1 < \Re(s) < 0$ and N be a fixed real number. Then we define

$$I_N(s) = \int_0^{\infty} \frac{t^s}{t^2 + (2N\pi)^2}\, dt. \tag{2.23}$$

In order to compute $I_N(s)$, we consider the following integral

$$I_{N,C}(s) = \int_C \frac{z^s}{z^2 + (2N\pi)^2}\, dz, \tag{2.24}$$

where C is the "keyhole contour" shown in figure (2.1).

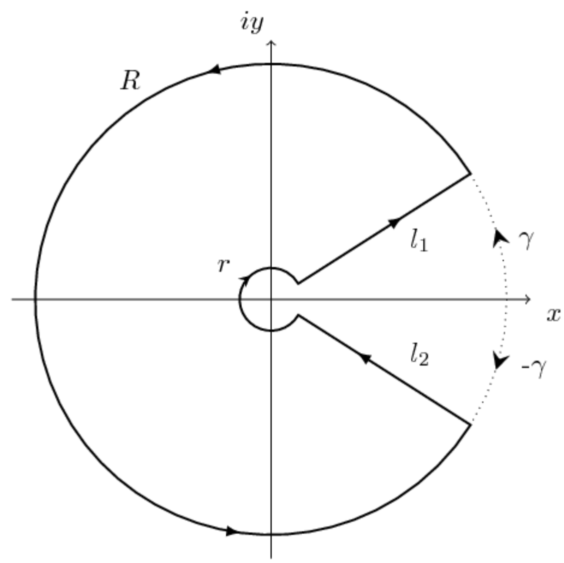

Figure 2.1: Keyhole Contour

We parameterize C by writing $C = C_r + l_1 + C_R + l_2$ as follows:

$$C_r = re^{-i\theta}, \;\; \gamma \le \theta \le 2\pi - \gamma, \tag{2.25}$$

$$l_1 = te^{i\gamma}, \;\; r \le t \le R, \tag{2.26}$$

$$C_R = Re^{i\theta}, \;\; \gamma \le \theta \le 2\pi - \gamma, \tag{2.27}$$

$$l_2 = te^{i(2\pi-\gamma)}, \;\; R \le t \le r. \tag{2.28}$$

Then by denoting

$$f(z) = \frac{z^s}{z^2 + (2N\pi)^2}, \tag{2.29}$$

we can rewrite $I_{N,C}(s)$ as a sum

$$I_{N,C}(s) = \int_{C_r} f(z)\,dz + \int_{l_1} f(z)\,dz + \int_{C_R} f(z)\,dz + \int_{l_2} f(z)\,dz. \tag{2.30}$$

Now we focus on each integral on the right hand side of (2.30) separately. For the first integral, we have

$$\int_{C_r} f(z)dz = \int_{\gamma}^{2\pi-\gamma} \frac{(re^{-i\theta})^s}{r^2e^{-2i\theta} + (2N\pi)^2}\,(-ir)e^{i\theta}\,d\theta = -ir^{s+1}\int_{\gamma}^{2\pi-\gamma} \frac{(e^{i\theta})^{1-s}}{r^2e^{-2i\theta} + (2N\pi)^2}\,d\theta. \tag{2.31}$$

This integral converges to 0 as $r \to 0$, and $\gamma \to 0$. For the second integral, we get

$$\int_{l_1} f(z)\,dz = \int_r^R \frac{(te^{i\gamma})^s}{t^2e^{2i\gamma} + (2N\pi)^2}\,e^{i\gamma}\,dt = \int_r^R \frac{t^s(e^{i\gamma})^{s+1}}{t^2e^{2i\gamma} + (2N\pi)^2}\,dt. \tag{2.32}$$

In the limits $r \to 0, R \to \infty$, and $\gamma \to 0$, we have

$$\int_0^\infty \frac{t^s}{t^2 + (2N\pi)^2}\,dt = I_N(s). \tag{2.33}$$

For the third integral, we obtain instead,

$$\begin{aligned}\int_{C_R} f(z)\,dz &= \int_{\gamma}^{2\pi-\gamma} \frac{(Re^{i\theta})^s}{R^2e^{2i\theta} + (2N\pi)^2}\,iRe^{i\theta}\,d\theta \\ &= i\int_{\gamma}^{2\pi-\gamma} \frac{(Re^{i\theta})^{s+1}}{R^2e^{2i\theta} + (2N\pi)^2}\,d\theta \\ &= iR^{s-1}\int_{\gamma}^{2\pi-\gamma} \frac{(e^{i\theta})^{s+1}}{e^{2i\theta} + \left(\frac{2N\pi}{R}\right)^2}\,d\theta. \end{aligned} \tag{2.34}$$

For $-1 < \Re(s) < 0$, we have that (2.34) converges to 0 as $R \to \infty$, and $\gamma \to 0$. For the fourth integral, we get

$$\begin{aligned}\int_{l_2} f(z)dz &= \int_R^r \frac{(te^{i(2\pi-\gamma)})^s}{t^2 e^{2i(2\pi-\gamma)} + (2N\pi)^2}\, e^{i(2\pi-\gamma)} dt \\ &= -\int_r^R \frac{(te^{i(2\pi-\gamma)})^s}{t^2 e^{2i(2\pi-\gamma)} + (2N\pi)^2}\, e^{i(2\pi-\gamma)} dt \\ &= -\int_r^R \frac{t^s (e^{i(2\pi-\gamma)})^{s+1}}{t^2 e^{2i(2\pi-\gamma)} + (2N\pi)^2}\, dt. \end{aligned} \tag{2.35}$$

Taking the limits $r \to 0, R \to \infty$, and $\gamma \to 0$, we have

$$-e^{2\pi i s} \int_0^\infty \frac{t^s}{t^2 + (2N\pi)^2}\, dt = -e^{2\pi i s} I_N(s). \tag{2.36}$$

By combining the results in (2.31), (2.33), (2.34), and (2.36), we finally obtain

$$I_{N,C}(s) = (1 - e^{2i\pi s})\ I_N(s) = -2ie^{i\pi s} \frac{e^{i\pi s} - e^{-i\pi s}}{2i}\ I_N(s) = -2ie^{i\pi s} \sin(\pi s)\ I_N(s). \tag{2.37}$$

On the other hand, $I_{N,C}(s)$ in (2.24) can be computed by using the residue theorem. The integrand $f(z)$ has (simple) poles at $z = \pm 2N\pi i$, with residues

$$\mathrm{Res}_{z=2N\pi i} f(z) = \left.\frac{z^s}{z + 2N\pi i}\right|_{z=2N\pi i} = \frac{(2\pi N i)^{s-1}}{2} \tag{2.38}$$

$$\mathrm{Res}_{z=-2N\pi i} f(z) = \left.\frac{z^s}{z - 2N\pi i}\right|_{z=-2N\pi i} = \frac{(-2\pi N i)^{s-1}}{2}, \tag{2.39}$$

and therefore, we have

$$\begin{aligned} I_{N,C}(s) &= 2\pi i \left(\frac{(2\pi N i)^{s-1}}{2} + \frac{(-2\pi N i)^{s-1}}{2} \right) = 2\pi i \frac{(2\pi N)^{s-1}}{2} \left(i^{s-1} + (-i)^{s-1} \right) \\ &= 2\pi i \frac{(2\pi N)^{s-1}}{2} \left(\frac{(e^{i\pi/2})^s}{i} - \frac{(e^{3i\pi/2})^s}{i} \right) = -2i\pi (2\pi N)^{s-1} e^{i\pi s} \frac{(e^{i\pi/2})^s - (e^{-i\pi/2})^s}{2i} \\ &= -2i\pi (2\pi N)^{s-1} e^{i\pi s} \sin\left(\frac{\pi s}{2}\right). \end{aligned} \tag{2.40}$$

By using (2.40) in the relation (2.37), we obtain

$$-2ie^{i\pi s}\sin \pi s\; I_N(s) = -2i\pi(2\pi N)^{s-1}e^{i\pi s}\sin\frac{\pi s}{2}, \tag{2.41}$$

which gives

$$I_N(s) = \pi(2\pi N)^{s-1}\frac{\sin\frac{\pi s}{2}}{\sin \pi s}. \tag{2.42}$$

At this point, we use the results (2.42) and (2.22) to obtain

$$\begin{aligned}\zeta(s)\Gamma(s) &= 2\sum_{N=1}^{\infty}\pi(2\pi N)^{s-1}\frac{\sin\frac{\pi s}{2}}{\sin \pi s}\\ &= 2(2\pi)^{s-1}\frac{\pi}{\sin(\pi s)}\left(\sum_{N=1}^{\infty}\frac{1}{N^{1-s}}\right)\sin\left(\frac{\pi s}{2}\right)\\ &= 2(2\pi)^{s-1}\frac{\pi}{\sin(\pi s)}\zeta(1-s)\sin\left(\frac{\pi s}{2}\right).\end{aligned} \tag{2.43}$$

We divide both sides of (2.43) by $\Gamma(s)$ and use equation (1.12) to finally get

$$\zeta(s) = 2(2\pi)^{s-1}\Gamma(1-s)\zeta(1-s)\sin\left(\frac{\pi s}{2}\right), \tag{2.44}$$

which is *Riemann's functional equation* for the zeta function.

The right hand side of (2.44) is well defined not only for $-1 < \Re(s) < 0$, but also for the larger region $\Re(s) < 0$. So we can use this equation to define $\zeta(s)$ for all $s \neq 1$.

2.2 Method 2: Euler transform

In order to outline this method, we first need to introduce the alternating zeta function, otherwise known as the Dirichlet eta function. Let $\sigma > 0$ be a real number.

Then

$$\eta(\sigma) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{\sigma}} = 1 - 2^{-\sigma} + 3^{-\sigma} - 4^{-\sigma} + \cdots , \tag{2.45}$$

where the series converges due to Leibniz's alternating series test. The reason for considering $\eta(\sigma)$ lies in its relation with the Riemann zeta function. In fact,

$$\begin{aligned} \zeta(\sigma) - \eta(\sigma) &= \sum_{n=1}^{\infty} n^{-\sigma} - \sum_{n=1}^{\infty} (-1)^{n-1} n^{-\sigma} \\ &= \sum_{n=1}^{\infty} 2(2n)^{-\sigma} \\ &= 2 \cdot 2^{-\sigma} \sum_{n=1}^{\infty} n^{-\sigma} \\ &= 2^{1-\sigma} \zeta(\sigma). \end{aligned} \tag{2.46}$$

So we have for $\sigma > 0, \sigma \neq 1$,

$$\zeta(\sigma) = \frac{1}{(1 - 2^{1-\sigma})} \, \eta(\sigma). \tag{2.47}$$

The relation (2.47) implies that the analytic continuation of $\eta(\sigma)$ provides the analytic continuation of $\zeta(\sigma)$.

In what follows, we need some definitions.

Definition 2.1. Let $\{a_n\}$ be an increasing sequence of real numbers. The forward difference operator, Δ is the operation on $\{a_n\}$ defined as $\Delta\{a_n\} \equiv \{a_{n+1} - a_n\}$.

Definition 2.2. Let $\{a_n\}$ be an increasing sequence of real numbers. The higher order difference operator, Δ^k is the operation on $\{a_n\}$ defined as $\Delta^k\{a_n\} \equiv \{\Delta^{k-1}\{a_{n+1}\} - \Delta^{k-1}\{a_n\}\}$. As an example, we consider

$$\Delta^2 a_n = \Delta a_{n+1} - \Delta a_n$$

$$= (a_{n+2} - a_{n+1}) - (a_{n+1} - a_n)$$
$$= a_{n+2} - 2a_{n+1} + a_n, \tag{2.48}$$

and

$$\begin{aligned}\Delta^3 a_n &= \Delta^2 a_{n+1} - \Delta^2 a_n \\ &= (a_{n+3} - 2a_{n+2} + a_{n+1}) - (a_{n+2} - 2a_{n+1} + a_n) \\ &= a_{n+3} - 3a_{n+2} + 3a_{n+1} - a_n. \end{aligned} \tag{2.49}$$

In general,

$$\Delta^k a_n = \sum_{j=0}^{k} (-1)^j \binom{k}{j} a_{n+k-j}. \tag{2.50}$$

The following method of analytic continuation of the Riemann zeta function has been outlined in [53] and [54]. First, we consider the Euler transform of an alternating series. Let S be a convergent alternating series:

$$S = \sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots . \tag{2.51}$$

Then, we can write S also as

$$S = \frac{1}{2}a_1 + \frac{1}{2}[(a_1 - a_2) - (a_2 - a_3) + \cdots]. \tag{2.52}$$

Continuing this process on the terms in brackets gives

$$S = \frac{1}{2}a_1 + \frac{1}{4}(a_1 - a_2) + \frac{1}{4}\left[(a_1 - 2a_2 + a_3) - (a_2 - 2a_3 + a_4) + \cdots\right], \tag{2.53}$$

and in general

$$S = \sum_{j=0}^{k-1} \frac{\Delta^j a_1}{2^{j+1}} + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\Delta^k a_n}{2^k}, \tag{2.54}$$

where $\Delta^0 a_n = a_n$ and

$$\Delta^k a_n = \Delta^{k-1} a_n - \Delta^{k-1} a_{n+1} = \sum_{m=0}^{k} (-1)^m \binom{k}{m} a_{n+m}, \tag{2.55}$$

for $k \geq 1$. Since

$$\frac{1}{2^k} \sum_{n=1}^{\infty} (-1)^{n-1} \Delta^k a_n < \frac{S}{2^k} \to 0 \text{ as } k \to \infty,$$

then the quantity S can be expressed as

$$S = \sum_{j=0}^{\infty} \frac{\Delta^j a_1}{2^{j+1}}, \tag{2.56}$$

which is the Euler transform of (2.51).

Now working from equations (2.47) and (2.51), we can use this Euler transform to analytically continue $\zeta(\sigma)$ to $\zeta(s)$ for complex $s = \sigma + it, s \neq 1$ as follows:

First, we apply the Euler transform to $\eta(\sigma)$:

$$\begin{aligned} \eta(\sigma) &= \sum_{j=0}^{\infty} \frac{(-1)^j}{(j+1)^\sigma} = \sum_{j=0}^{\infty} \frac{\Delta^j 1^{-\sigma}}{2^{j+1}} \\ &= \sum_{j=0}^{\infty} \frac{1 - \binom{j}{1} 2^{-\sigma} + \binom{j}{2} 3^{-\sigma} - \cdots + (-1)^j \binom{j}{j} (j+1)^{-\sigma}}{2^{j+1}}, \end{aligned} \tag{2.57}$$

where $\Delta 1^{-\sigma} = 1 - 2^{-\sigma}$, and (2.50) is used to write $\Delta^j 1^{-\sigma} = 1 - \binom{j}{1} 2^{-\sigma} + \binom{j}{2} 3^{-\sigma} - \cdots + (-1)^j \binom{j}{j} (j+1)^{-\sigma}$. So for $\sigma > 0, \sigma \neq 1$ we have

$$\zeta(\sigma) = \frac{1}{(1 - 2^{1-\sigma})} \sum_{j=0}^{\infty} \frac{\Delta^j 1^{-\sigma}}{2^{j+1}}. \tag{2.58}$$

Next, we invoke a lemma.

Lemma 2.3. *Fix $k \geq 0$ and let $(s)_k$ be the Pochhammer symbol that denotes the product $s(s+1)\cdots(s+k-1)$ with $s_0 = 1, s_1 = s$. Then*

$$\Delta^k n^{-s} = (s)_k \int_0^1 \cdots \int_0^1 \left(n + \sum_{i=1}^{k} x_i\right)^{-(s+k)} dx_1 \cdots dx_k, \text{ for } k = 1, 2, \ldots. \tag{2.59}$$

Proof. We prove equation (2.59) by induction. The base case is when $k = 1$.

$$s\int_0^1 (n+x_1)^{-(s+1)}dx_1 = s\left[\frac{(n+x_1)^{-s}}{-s}\right]_0^1 = -\left[(n+1)^{-s} - n^{-s}\right] = n^{-s} - (n+1)^{-s} = \Delta n^{-s}, \tag{2.60}$$

valid for $s > 1$. Assuming (2.59) is true for k iterated integrals, we introduce x_{k+1}, and compute (2.59) with $k+1$ iterations as follows,

$$(s)_{k+1} \int_0^1 \cdots \int_0^1 \left(n + \sum_{i=1}^{k+1} x_i\right)^{-(s+k+1)} dx_1 \cdots dx_{k+1}. \tag{2.61}$$

Since (2.61) is absolutely integrable, we use Fubini's Theorem, and rewrite it as

$$(s)_k \int_0^1 \cdots \int_0^1 (s+k) \int_0^1 \left(n + \sum_{i=1}^{k+1} x_i\right)^{-(s+k+1)} dx_{k+1}\, dx_1 \cdots dx_k. \tag{2.62}$$

In (2.62), the innermost integral can be computed, which gives

$$\begin{aligned} &(s+k) \int_0^1 \left(n + \sum_{i=1}^{k+1} x_i\right)^{-(s+k+1)} dx_{k+1} \\ &= (s+k) \left[\frac{(n + \sum_{i=1}^{k+1} x_i)^{-(s+k)}}{-(s+k)}\right]_0^1 \end{aligned}$$

$$= (s+k)\left[\frac{(n+1+\sum_{i=1}^{k} x_i)^{-(s+k)} - (n+\sum_{i=1}^{k} x_i)^{-(s+k)}}{-(s+k)}\right]$$
$$= \left(n+\sum_{i=1}^{k} x_i\right)^{-(s+k)} - \left(n+1+\sum_{i=1}^{k} x_i\right)^{-(s+k)}. \tag{2.63}$$

By substituting this result into (2.62), we have that

$$(s)_{k+1}\int_0^1 \cdots \int_0^1 \left(n+\sum_{i=1}^{k+1} x_i\right)^{-(s+k+1)} dx_1 \cdots dx_{k+1}$$
$$= (s)_k \int_0^1 \cdots \int_0^1 \left(n+\sum_{i=1}^{k} x_i\right)^{-(s+k)} dx_1 \cdots dx_k$$
$$- (s)_k \int_0^1 \cdots \int_0^1 \left(n+1+\sum_{i=1}^{k} x_i\right)^{-(s+k)} dx_1 \cdots dx_k$$
$$= \Delta^k n^{-s} - \Delta^k (n+1)^{-s}$$
$$= \Delta^{k+1} n^{-s}, \tag{2.64}$$

which completes the proof. □

Now we return to our earlier result (2.58) to state the following theorem.

Theorem 2.4. *The analytic continuation of $\zeta(s)$ for all complex $s \neq 1$ is given by*

$$\zeta(s) = \frac{1}{(1-2^{1-s})}\sum_{n=0}^{\infty}\frac{\Delta^n 1^{-s}}{2^{n+1}} = \frac{1}{(1-2^{1-s})}\sum_{n=0}^{\infty}\frac{1}{2^{n+1}}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(k+1)^{-s}. \tag{2.65}$$

Proof. From lemma 2.3, it follows that

$$|\Delta^k n^{-s}| \leq |(s)_k|/n^{\sigma+k} \text{ whenever } \sigma + k \geq 0, \tag{2.66}$$

for $k = 0, 1, 2, \ldots$, where $(s)_0 = 1$. Now let A be a compact set in the half plane $\sigma > 1-k$, and let M_k denote the maximum of $|(s)_k|/n^{\sigma+k}$ on A. Due to the *Weierstrass-M test*, (2.66) implies that $\sum M_n$ has the property

$$\sum M_n \geq \left| \sum_{n=1}^{\infty} (-1)^{n+1} \Delta^k n^{-s} \right|, \tag{2.67}$$

on A. By the triangle inequality, the Euler transform of $\sum M_n$ then dominates the Euler transform of (2.67), which, since $\Delta^j \Delta^k = \Delta^{j+k}$, is

$$\sum_{j=0}^{\infty} \frac{\Delta^j \Delta^k 1^{-s}}{2^{j+1}} = \sum_{j=k}^{\infty} \frac{\Delta^j 1^{-s}}{2^{j+1-k}}. \tag{2.68}$$

This means that (2.68) converges absolutely and uniformly on A to an entire function. Multiplying (2.68) by $1/2^k$ and adding $\sum_{j=0}^{k-1} \Delta^j 1^{-s}/2^{j+1}$ then produces the series in (2.57), which, since k is arbitrary, also converges absolutely and uniformly on A to an entire function. Because the series in (2.45) has zeroes at the (simple) poles of $(1-2^{1-s})^{-1}$ except at $s = 1$ (see [54]), the theorem is proved. $\square$

2.3 Method 3: Another contour

Our third method of analytic continuation is described in [43]. In this method, we consider the function

$$\frac{\pi \cot(\pi z)}{z^s}, \tag{2.69}$$

in the domain $G_N : |z| \leq N + 1/2,\ x = \Re(z) \geq a$ $(0 < a < 1)$, as shown in Figure 2.2.

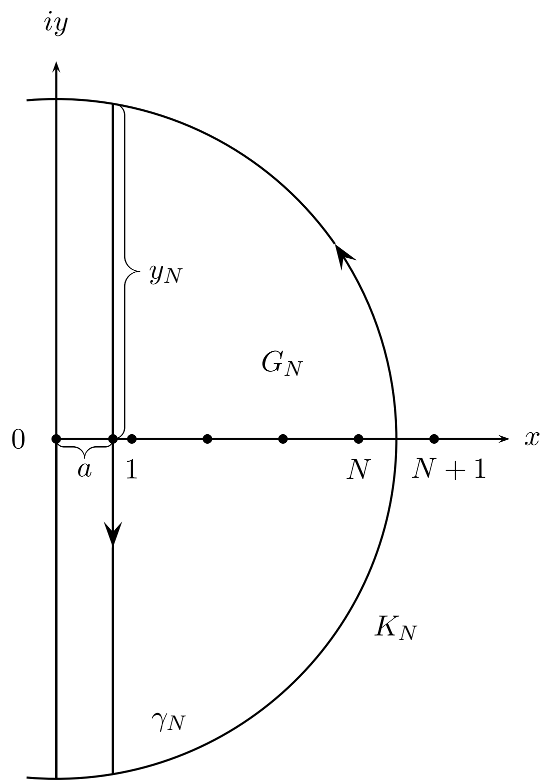

Figure 2.2: G_N

Let $\gamma_N = \partial G_N$ be the positively oriented boundary of G_N and choose a branch such that the function (2.69) is real on the positive real axis. Also in G_N, the poles of (2.69) are at the points $1, 2, \ldots, N$, with $N \in \mathbb{N}$. We apply the residue theorem to (2.69) and use the fact that the residue at $z = n$ is $1/n^s$ to claim

$$\frac{1}{2i} \int_{\gamma_N} \frac{\cot(\pi z)}{z^s}\, dz = \sum_{n=1}^{N} \frac{1}{n^s}, \tag{2.70}$$

with γ_N traversed in the counterclockwise direction. To do so, we need the following result.

Lemma 2.5. *Let Ω be a region of $\mathbb{C}$ enclosed by a simple curve C. Let $F(z)$ be a function with simple zeroes and no poles in the interior of Ω. Then for any analytic function $f(z)$ in Ω we have*

$$\sum_{k=1}^{N} f(z_k) = \frac{1}{2\pi i} \int_C f(z) \left(\frac{d}{dz} \ln F(z) \right) dz. \tag{2.71}$$

Proof. From Cauchy's integral formula, given an analytic function f defined in Ω, we have

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz, \tag{2.72}$$

where $z_0 \in \Omega$. By using (2.72), we can write the sum $f(z_1) + f(z_2) + \cdots + f(z_N)$ as

$$\sum_{k=1}^{N} f(z_k) = \frac{1}{2\pi i} \int_C f(z) \sum_{k=1}^{N} \frac{1}{z - z_k}\, dz, \tag{2.73}$$

where the points $\{z_1, z_2, \ldots, z_N\}$ belong to Ω. We now assume that the values $\{z_1, z_2, \ldots, z_N\}$ are simple zeroes of the function $F(z)$. Under this assumption, $F(z)$ can be written as

$$F(z) = A(z) \prod_{k=1}^{N} (z - z_k), \tag{2.74}$$

where $A(z) \neq 0$ is an analytic function in G_N. By taking the logarithm of $F(z)$, we have

$$\ln F(z) = \ln A(z) + \sum_{k=1}^{N} \ln(z - z_k), \tag{2.75}$$

Upon differentiation of (2.75), we obtain

$$\frac{d}{dz} \ln F(z) = \frac{A'(z)}{A(z)} + \sum_{k=1}^{N} \frac{1}{z - z_k}, \tag{2.76}$$

which is analytic in G_N. By substituting (2.76) into (2.73), we obtain

$$\sum_{k=1}^{N} f(z_k) = \frac{1}{2\pi i}\int_C f(z)\left(\frac{d}{dz}\ln F(z) - \frac{A'(z)}{A(z)}\right) dz. \tag{2.77}$$

Since $\dfrac{A'(z)}{A(z)}$ is analytic in G_N, we can conclude from the Cauchy Integral Theorem that

$$\int_C \frac{A'(z)}{A(z)}\, dz = 0, \tag{2.78}$$

and therefore the claim follows. $\square$

We would like to point out that this lemma can be generalized to $F(z)$ having zeroes of multiplicity higher than one.

By applying the result of the previous lemma to $f(z) = z^{-s}$, we obtain

$$\sum_{k=1}^{N} z_k^{-s} = \frac{1}{2\pi i}\int_C z^{-s}\left(\frac{d}{dz}\ln F(z)\right) dz. \tag{2.79}$$

If we set $F(z) = \sin(\pi z)$, which has simple zeroes for $z \in \mathbb{Z}$, and choose the contour $C = \gamma_N$, we obtain the representation (2.70), where γ_N encloses the values $\{1, 2, \ldots, N\}$. The integral in (2.70) can be computed by writing it as a sum:

$$\frac{1}{2i}\int_{\gamma_N} \frac{\cot(\pi z)}{z^s}\, dz = \frac{1}{2i}\int_{K_N} \frac{\cot(\pi z)}{z^s}\, dz + \frac{1}{2i}\int_{a+iy_N}^{a-iy_N} \frac{\cot(\pi z)}{z^s}\, dz, \tag{2.80}$$

where $a - iy_N$ and $a + iy_N$ are the points of intersection of the circle $|z| = N + 1/2$ and the line $x = a$, K_N is the circular arc contained in γ_N, and the last integral in (2.80) is taken along the straight line $x = a$.

Because $|\cot(\pi z)|$ is bounded on every circle $|z| = N+1/2$ by a fixed finite number

M (see e.g. [43] §10.7), we obtain for the first integral

$$\left| \frac{1}{2i} \int_{K_N} \frac{\cot(\pi z)}{z^s} \, dz \right| \leq \frac{\pi}{2} \cdot \frac{M}{(N+1/2)^{s-1}}, \tag{2.81}$$

for $s > 1$, which tends to zero as $N \to \infty$.

Next, we write the second integral as a sum:

$$\frac{1}{2i} \int_{a+iy_N}^{a-iy_N} \frac{\cot(\pi z)}{z^s} \, dz = -\frac{1}{2i} \int_{a}^{a+iy_N} \frac{\cot(\pi z)}{z^s} \, dz + \frac{1}{2i} \int_{a}^{a-iy_N} \frac{\cot(\pi z)}{z^s} \, dz. \tag{2.82}$$

The integrals on the right side of (2.82) can be rewritten in a different form thanks to the following relations.

$$\frac{\cot(\pi z)}{2i} = -\frac{1}{2} - \frac{1}{e^{-2\pi i z} - 1}, \tag{2.83}$$

for $\Im(z) > 0$ and

$$\frac{\cot(\pi z)}{2i} = \frac{1}{2} + \frac{1}{e^{2\pi i z} - 1}, \tag{2.84}$$

for $\Im(z) < 0$. The formula (2.83) allows us to express the first integral on the right hand side of (2.82) as

$$\begin{aligned} -\frac{1}{2i} \int_{a}^{a+iy_N} \frac{\pi \cot(\pi z)}{z^s} \, dz &= \int_{a}^{a+iy_N} \left(\frac{z^{-s}}{2} + \frac{z^{-s}}{e^{-2\pi i z} - 1} \right) dz \\ &= \frac{1}{2} \left[\frac{(a+iy_N)^{1-s}}{1-s} - \frac{a^{1-s}}{1-s} \right] + \int_{a}^{a+iy_N} \left(\frac{z^{-s}}{e^{-2\pi i z} - 1} \right) dz. \end{aligned} \tag{2.85}$$

In the limit $y_N \to \infty$, we have that $\dfrac{(a+iy_N)^{1-s}}{1-s} \to 0$ for $s > 1$, which implies that

$$-\frac{1}{2i} \int_{a}^{a+iy_N} \frac{\pi \cot(\pi z)}{z^s} \, dz = \frac{1}{2} \left(\frac{a^{1-s}}{s-1} \right) + \int_{a}^{a+iy_N} \left(\frac{z^{-s}}{e^{-2\pi i z} - 1} \right) dz + \mathcal{O}\left(\frac{1}{N} \right). \tag{2.86}$$

Taking the result (2.84) and proceeding in a likewise manner, we can write the second integral on the right hand side of (2.82) as

$$\frac{1}{2i}\int_a^{a-iy_N} \frac{\pi\cot(\pi z)}{z^s}\,dz = \frac{1}{2}\left(\frac{a^{1-s}}{s-1}\right) + \int_a^{a-iy_N}\left(\frac{z^{-s}}{e^{2\pi iz}-1}\right)dz + \mathcal{O}\left(\frac{1}{N}\right). \quad (2.87)$$

We can now use the results (2.87), (2.86), (2.82) and (2.80) to rewrite (2.70) as

$$\sum_{n=1}^{N}\frac{1}{n^s} = \frac{a^{1-s}}{s-1} + \int_a^{a+iy_N}\left(\frac{z^{-s}}{e^{-2\pi iz}-1}\right)dz + \int_a^{a-iy_N}\left(\frac{z^{-s}}{e^{2\pi iz}-1}\right)dz + \mathcal{O}\left(\frac{1}{N}\right). \quad (2.88)$$

If we take the limit as $N \to \infty$, then both of the integrals on the right hand side are convergent. In fact, for $y > 0$ we have

$$\int_a^{a+iy_N}\left|\frac{z^{-s}}{e^{-2\pi iz}-1}\,dz\right| \le \int_a^{a+iy_N}\frac{|z|^{-s}}{e^{2\pi y}-1}\,|dz|, \quad (2.89)$$

and for $y < 0$,

$$\int_a^{a-iy_N}\left|\frac{z^{-s}}{e^{2\pi iz}-1}\,dz\right| \le \int_a^{a-iy_N}\frac{|z|^{-s}}{e^{-2\pi y}-1}\,|dz|. \quad (2.90)$$

Additionally, the integrals on the right hand side of (2.89) and (2.90) are convergent for all s. Now by taking the limit as $N \to \infty$ in equation (2.88), we obtain

$$\zeta(s) = \frac{a^{1-s}}{s-1} + \left[\int_a^{a+i\infty}\left(\frac{z^{-s}}{e^{-2\pi iz}-1}\right)dz + \int_a^{a-i\infty}\left(\frac{z^{-s}}{e^{2\pi iz}-1}\right)dz\right], \quad (2.91)$$

which is valid for $\Re(s) > 1$. At this point, it is important to mention that although (2.91) has been derived under the assumption $\Re(s) > 1$, it is valid for all $s \neq 1$. In fact, the first term on the right hand side of (2.91) is a meromorphic function with a simple pole at $s = 1$. Moreover, since

1. $\dfrac{z^{-s}}{e^{-2\pi iz}-1}$ is continuous for $z = a + it, t \in [0, \infty)$ and $\dfrac{z^{-s}}{e^{2\pi iz}-1}$ is continuous

for $z = a + it, t \in (-\infty, 0)$,

2. the partial derivative $\dfrac{\partial}{\partial z}\left(\dfrac{z^{-s}}{e^{-2\pi iz}-1}\right)$ is continuous for $z = a + it, t \in [0, \infty)$, and the partial derivative $\dfrac{\partial}{\partial z}\left(\dfrac{z^{-s}}{e^{2\pi iz}-1}\right)$ is continuous for $z = a + it, t \in (-\infty, 0]$, and,

3. $\int_a^{a+i\infty}\left(\frac{z^{-s}}{e^{-2\pi iz}-1}\right)dz$ is uniformly convergent on $z = a + it, t \in [0, \infty)$ and $\int_a^{a-i\infty}\left(\frac{z^{-s}}{e^{2\pi iz}-1}\right)dz$ is uniformly convergent on $z = a + it, t \in (-\infty, 0]$,

theorem 15.2 of [43], ensures that both integrals on the right hand side of (2.91) are analytic functions of s. This implies that (2.91) represents a meromorphic function of s with a simple pole at $s = 1$.

Now let us consider values of s in the interval $-1 < s < 0$. In the limit as $a \to 0$, the first term of (2.91) converges to zero. The first integral on the right hand side,

$$\int_a^{a+i\infty}\left(\frac{z^{-s}}{e^{-2\pi iz}-1}\right)dz, \tag{2.92}$$

converges uniformly for $0 \leq a \leq t$, with $t > 0$. In fact, by using the following estimate for the integrand in (2.92)

$$\frac{z^{-s}}{e^{-2\pi iz}-1} = \frac{i}{2\pi}z^{-s-1}\left(1 + \mathcal{O}(z)\right), \tag{2.93}$$

we can conclude that there exists a disk $|z| \leq 2t$ in which $|1 + \mathcal{O}(z)| < 2\pi$, so that

$$\left|\frac{z^{-s}}{e^{-2\pi iz}-1}\right| \leq |z|^{-s-1} \leq |y|^{-s-1}. \tag{2.94}$$

Since the integral

$$\int_0^1 y^{-s-1}\,dy,$$

converges for $\Re(s) < 0$, the integral (2.92) converges uniformly for $0 \leq a \leq t$. It follows from the uniform convergence of the integral that

$$\lim_{a \to 0} \int_{a}^{a+i\infty} \left(\frac{z^{-s}}{e^{-2\pi i z} - 1} \right) dz = i \int_{0}^{\infty} \left(\frac{|y|^{-s} e^{i\pi s/2}}{e^{2\pi y} - 1} \right) dy. \tag{2.95}$$

It can be shown in the same way that

$$\lim_{a \to 0} \int_{a}^{a+i\infty} \left(\frac{z^{-s}}{e^{2\pi i z} - 1} \right) dz = i \int_{0}^{-\infty} \left(\frac{|y|^{-s} e^{i\pi s/2}}{e^{-2\pi y} - 1} \right) dy. \tag{2.96}$$

Thus, as $a \to 0$, equation (2.91) becomes

$$\zeta(s) = i e^{-i\pi s/2} \int_{0}^{\infty} \left(\frac{y^{-s}}{e^{2\pi y} - 1} \right) dy + i e^{i\pi s/2} \int_{0}^{-\infty} \left(\frac{|y|^{-s}}{e^{-2\pi y} - 1} \right) dy. \tag{2.97}$$

In the second integral, we introduce the new variable $y = -u$ and combine the two integrals from (2.97) to obtain the formula

$$\begin{aligned} \zeta(s) &= i(e^{i\pi s/2} - e^{-i\pi s/2}) \int_{0}^{\infty} \left(\frac{y^{-s}}{e^{2\pi y} - 1} \right) dy \\ &= 2 \sin\left(\frac{\pi s}{2} \right) \int_{0}^{\infty} \left(\frac{y^{-s}}{e^{2\pi y} - 1} \right) dy, \end{aligned} \tag{2.98}$$

valid for $\Re(s) < 0$. If we then substitute $2\pi y = x$, we get

$$\zeta(s) = 2(2\pi)^{s-1} \sin\left(\frac{\pi s}{2} \right) \int_{0}^{\infty} \left(\frac{x^{-s}}{e^{x} - 1} \right) dx. \tag{2.99}$$

This representation has been derived for $-1 < s < 0$. However, since both sides are analytic functions in the half-plane $\Re(s) < 0$, the formula is valid in this whole half-plane.

We now relate (2.99) to the formula (2.6), already derived in section 2.1 for $\Re(s) >$

1. Let s lie in the half-plane $\Re(s) < 0$. Then

$$\Gamma(1-s)\zeta(1-s) = \int_0^\infty \frac{x^{-s}}{e^x - 1}\, dx; \tag{2.100}$$

therefore, we can use (2.100) in (2.99) to once again arrive at

$$\zeta(s) = 2(2\pi)^{s-1}\Gamma(1-s)\zeta(1-s)\sin\left(\frac{\pi s}{2}\right),$$

which is Riemann's functional equation (2.44). Since both sides are analytic functions for $s \in \mathbb{C}$, the functional equation holds for *all values of s.*

Chapter 3: Calculation of zeta constants

3.1 The Basel problem and $\zeta(2n)$

Here we revisit the Basel problem we first introduced in chapter 1 and recall that Euler was the first to present a solution. We present two of Euler's solutions [52] here.

Euler's solutions

Euler's first solution was found by writing $\sin x$ as a product of normalized linear factors, giving us

$$\sin x = x\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\cdots, \tag{3.1}$$

which is the expansion of Euler's product formula for $\sin(x)$ in (1.13), valid for all x.

Expanding the product in (3.1), we get

$$\sin x = x - \left(\sum_{n=1}^{\infty} \frac{1}{(n\pi)^2}\right) x^3 + \mathcal{O}(x^5). \tag{3.2}$$

Here, to calculate $\zeta(2)$, we take the *third* derivative of (3.2) and evaluate at $x = 0$ to get

$$-1 = -6\left(\sum_{n=1}^{\infty} \frac{1}{(n\pi)^2}\right). \tag{3.3}$$

Multiplying (3.3) by $-\pi^2/6$ then gives

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \tag{3.4}$$

Euler produced another solution 6 years later in 1741 which is based on calculating the integral

$$\int_0^1 \frac{\arcsin x}{\sqrt{1-x^2}}\,dx, \tag{3.5}$$

in two ways. On one hand, the integral can be calculated using the change of variables $u = \arcsin x$ so that (3.5) becomes

$$\int_0^{\pi/2} u\,du = \frac{\pi^2}{8}. \tag{3.6}$$

On the other hand, we can use the binomial expansion

$$(1+y)^{-t} = \frac{1}{\Gamma(t)}\sum_{n=0}^{\infty}(-1)^n\frac{\Gamma(t+n)}{n!}y^n,$$

with $t = 1/2$ and $y = -x^2$ to write

$$\frac{1}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty}\frac{\Gamma(n+1/2)}{\sqrt{\pi}n!}x^{2n} = \sum_{n=0}^{\infty}\frac{(2n)!}{2^{2n}n!n!}x^{2n} = \sum_{n=0}^{\infty}\binom{2n}{n}\left(\frac{x}{2}\right)^{2n}, \tag{3.7}$$

valid for $|x| < 1$. The result (3.7) has been obtained by using

$$\Gamma(n+1/2) = \frac{\sqrt{\pi}(2n)!}{2^{2n}n!},$$

found in [59]. Integrating the right hand side of (3.7) term by term, gives the Taylor expansion about $x = 0$ for $\arcsin x$,

$$\arcsin x = \int \frac{1}{\sqrt{1-x^2}}\,dx = \sum_{n=0}^{\infty}\binom{2n}{n}\frac{x^{2n+1}}{(2n+1)2^{2n}}, \tag{3.8}$$

valid once again for $|x| < 1$. Euler then multiplies (3.8) by

$$\frac{1}{\sqrt{1-x^2}}$$

and integrates from $x = 0$ to $x = 1$ to write (3.5) as

$$\int_0^1 \frac{\arcsin x}{\sqrt{1-x^2}}\,dx = \sum_{n=0}^{\infty}\left[\binom{2n}{n}\frac{1}{(2n+1)2^{2n}}\int_0^1 \frac{x^{2n+1}}{\sqrt{1-x^2}}\,dx\right]. \tag{3.9}$$

To calculate the right hand side of (3.9), we need the following lemma.

Lemma 3.1. *Let $\alpha \in \mathbb{N}^+$. Then*

$$I_\alpha(x) = \int \frac{x^\alpha}{\sqrt{1-x^2}}\,dx = \frac{\alpha-1}{\alpha}\int \frac{x^{\alpha-2}}{\sqrt{1-x^2}}\,dx - \frac{x^{\alpha-1}}{\alpha}\sqrt{1-x^2}. \tag{3.10}$$

Proof. By using integration by parts, we obtain

$$\begin{aligned}
I_\alpha(x) &= -x^{\alpha-1}\sqrt{1-x^2} + (\alpha-1)\int x^{\alpha-2}\sqrt{1-x^2}\,dx \\
&= -x^{\alpha-1}\sqrt{1-x^2} + (\alpha-1)\int \frac{x^{\alpha-2}(1-x^2)}{\sqrt{1-x^2}}\,dx \\
&= -x^{\alpha-1}\sqrt{1-x^2} + (\alpha-1)\int \frac{x^{\alpha-2}}{\sqrt{1-x^2}}\,dx - (\alpha-1)\int \frac{x^\alpha}{\sqrt{1-x^2}}\,dx \\
&= -x^{\alpha-1}\sqrt{1-x^2} + (\alpha-1)\int \frac{x^{\alpha-2}}{\sqrt{1-x^2}}\,dx - (\alpha-1)I_\alpha(x). \qquad (3.11)
\end{aligned}$$

Then

$$\alpha I_\alpha(x) = -x^{\alpha-1}\sqrt{1-x^2} + (\alpha-1)\int \frac{x^{\alpha-2}}{\sqrt{1-x^2}}\,dx, \tag{3.12}$$

and, hence,

$$I_\alpha(x) = \frac{\alpha-1}{\alpha}\int \frac{x^{\alpha-2}}{\sqrt{1-x^2}}\,dx - \frac{x^{\alpha-1}}{\alpha}\sqrt{1-x^2}. \tag{3.13}$$

□

By using (3.13), we get

$$\int_0^1 \frac{x^{2n+1}}{\sqrt{1-x^2}}\,dx = I_{2n+1}(1) - I_{2n+1}(0) = \frac{2n}{2n+1}\int_0^1 \frac{x^{2n-1}}{\sqrt{1-x^2}}\,dx. \tag{3.14}$$

The last relation implies that for $n \in \mathbb{N}^+$, $I_{2n+1}(x) = \frac{2n}{2n+1} I_{2n-1}(x)$, with $I_1(x) = 1$, and therefore,

$$\begin{aligned}
\int_0^1 \frac{x^{2n+1}}{\sqrt{1-x^2}}\,dx &= \begin{cases} 1, & n = 0 \\ \displaystyle\prod_{k=1}^{n} \frac{2k}{2k+1}, & n \geq 1 \end{cases} \\
&= \frac{2\cdot 4\cdot 6\cdots 2n}{3\cdot 5\cdot 7\cdots (2n+1)} \\
&= \frac{(2\cdot 4\cdot 6\cdots 2n)^2}{(2n+1)!} \\
&= \frac{(2^n n!)^2}{(2n+1)(2n)!} \\
&= \frac{2^{2n}}{\binom{2n}{n}(2n+1)}.
\end{aligned} \tag{3.15}$$

By substituting (3.15) in (3.9), we obtain

$$\int_0^1 \frac{\arcsin x}{\sqrt{1-x^2}}\,dx = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}, \tag{3.16}$$

which is equivalent to the solution to the Basel problem, since

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{3}{4}\zeta(2). \tag{3.17}$$

From the results (3.16) and (3.17), once more we conclude that

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Additional proofs for the result of the Basel problem can be found, for instance, in [10] and [20].

3.2 Using the reflection formula

As a result of the analytic continuation, we found that $\zeta(s)$ satisfies the reflection formula:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi}{2}s\right)\Gamma(1-s)\zeta(1-s).$$

Two immediate results coming from this formula are

- Using $s = 1/2$ we obtain $\Gamma(1/2) = \sqrt{\pi}$.
- $\zeta(-2n) = 0$ for $n \in \mathbb{N}^+$.

However, the above reflection formula can be used to obtain additional interesting results.

Calculating $\zeta(0)$

The value $\zeta(0)$ can be calculated using the following method. First, we multiply (2.44) by $(1-s)$ and take the limit as $s \to 1$ to obtain

$$\lim_{s\to1}(1-s)\zeta(s) = \lim_{s\to1} 2(2\pi)^{s-1}(1-s)\Gamma(1-s)\zeta(1-s)\sin\left(\frac{\pi s}{2}\right). \tag{3.18}$$

By using (2.11), and the fact that $(1-s)\Gamma(1-s) = \Gamma(2-s)$, we get

$$-1 = \lim_{s\to1} 2(2\pi)^{s-1}\Gamma(2-s)\zeta(1-s)\sin\left(\frac{\pi s}{2}\right) = 2\zeta(0), \tag{3.19}$$

which implies that

$$\zeta(0) = -\frac{1}{2}. \tag{3.20}$$

Calculating $\zeta(2n)$

In order to compute the values $\zeta(2n)$, for $n \in \mathbb{N}^+$, we use equation (2.44) and the double angle sine formula to obtain

$$\zeta(2n) = \frac{(2\pi)^{2n}\zeta(1-2n)}{2\Gamma(2n)\cos(n\pi)} = \frac{(-1)^n(2\pi)^{2n}}{2(2n-1)!}\zeta(1-2n). \tag{3.21}$$

To simplify this equation, we introduce the *Bernoulli numbers*.

Definition 3.2. The Bernoulli numbers B_n, with $n \in \mathbb{N}_0$, are a sequence of numbers given by

$$B_n = \lim_{x\to 0} \frac{d^n}{dx}\left(\frac{x}{e^x-1}\right), \tag{3.22}$$

or equivalently

$$\frac{x}{e^x-1} = \sum_{k=0}^{\infty} \frac{B_k}{k!}x^k. \tag{3.23}$$

The following lemma provides a relation between the Bernoulli numbers and the Riemann zeta function.

Lemma 3.3. *For* $n \in \mathbb{N}$,

$$B_n = (-1)^{n+1}n\zeta(1-n). \tag{3.24}$$

Proof. We start the proof by writing (2.6) as

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^1 \frac{t}{e^t-1}t^{s-2}\,dt + \frac{1}{\Gamma(s)}\int_1^{+\infty} \frac{t^{s-1}}{e^t-1}\,dt, \tag{3.25}$$

valid for $\Re(s) > 1$. The second integral on the right hand side of (3.25) converges for all $s \in \mathbb{C}$, and therefore defines an entire function of s that we denote by $F(s)$. By using (3.23) in (3.25), we get

$$\zeta(s) = \frac{1}{\Gamma(s)} \sum_{k=0}^{\infty} \frac{B_k}{k!} \int_0^1 t^{k+s-2}\, dt + \frac{F(s)}{\Gamma(s)}. \tag{3.26}$$

Upon calculating the integral on the right hand side of (3.26) and setting $s = 1-n+\varepsilon$, with $\varepsilon > 0$, we obtain

$$\zeta(1-n+\varepsilon) = \frac{1}{\Gamma(1-n+\varepsilon)} \left[\sum_{k=0}^{\infty} \frac{B_k}{k!(k-n+\varepsilon)} \right] + \frac{F(1-n+\varepsilon)}{\Gamma(1-n+\varepsilon)}. \tag{3.27}$$

The use of (1.12) allows us to write

$$\frac{1}{\Gamma(1-n+\varepsilon)} = \frac{\Gamma(n-\varepsilon)\sin[\pi(n-\varepsilon)]}{\pi}, \tag{3.28}$$

and by expanding the angle difference in $\sin[\pi(n-\varepsilon)]$, we get

$$\begin{aligned} \frac{1}{\Gamma(1-n+\varepsilon)} &= \frac{\Gamma(n-\varepsilon)\left[\sin(\pi n)\cos(\pi\varepsilon) - \cos(\pi n)\sin(\pi\varepsilon)\right]}{\pi} \\ &= \frac{\Gamma(n-\varepsilon)}{\pi}\left[(-1)^{n+1}\sin(\pi\varepsilon)\right] = \frac{\Gamma(n-\varepsilon)}{\pi}(-1)^{n+1}\left[\pi\varepsilon + \mathcal{O}(\varepsilon^3)\right] \\ &= \Gamma(n-\varepsilon)(-1)^{n+1}\left[\varepsilon + \mathcal{O}(\varepsilon^3)\right]. \end{aligned} \tag{3.29}$$

We now substitute the result (3.29) into (3.27) to obtain

$$\begin{aligned} \zeta(1-n+\varepsilon) =& \Gamma(n-\varepsilon)(-1)^{n+1}\left[\varepsilon + \mathcal{O}(\varepsilon^3)\right] \left[\sum_{k=0}^{\infty} \frac{B_k}{k!(k-n+\varepsilon)} \right] \\ &+ \Gamma(n-\varepsilon)(-1)^{n+1}\left[\varepsilon + \mathcal{O}(\varepsilon^3)\right] F(1-n+\varepsilon). \end{aligned} \tag{3.30}$$

When taking the limit as $\varepsilon \to 0$, the expansion of the first term of the right hand side of (3.30) is zero except when $k = n$, while the second term is zero for all n. Therefore, we obtain

$$\zeta(1-n) = \varepsilon\Gamma(n)(-1)^{n+1}\frac{B_n}{n!\varepsilon} = (-1)^{n+1}\frac{B_n}{n}, \tag{3.31}$$

from which the claim (3.24) follows. $\square$

We can use (3.24) to rewrite (3.21) as

$$\zeta(2n) = \frac{(-1)^n(2\pi)^{2n}}{2(2n-1)!}\zeta(1-2n) = \frac{(-1)^{n+1}(2\pi)^{2n}}{2(2n)!}B_{2n}. \tag{3.32}$$

3.3 Apéry's Constant and $\zeta(2n+1)$

Roger Apéry proved that $\zeta(3)$ is irrational in 1979 [2]. For this reason, $\zeta(3)$ is known as *Apéry's Constant*. It is not known if any other values $\zeta(2n+1)$, with $n = \{2, 3, \ldots\}$ are irrational. In fact, proving whether or not $\zeta(2n+1)$ can be written in terms of a finite number of irrational numbers is still an open problem. To obtain a formula for $\zeta(2n+1)$, we evaluate (2.44) for $s = 2x+1$. With this definition of s, we obtain

$$\zeta(2x+1) = 2(2\pi)^{2x}\Gamma(-2x)\zeta(-2x)\sin\left[\frac{\pi(2x+1)}{2}\right], \tag{3.33}$$

which is valid for $x \neq 0, x \in \mathbb{C}$. We now use (1.12) along with the double angle sine formula to get

$$\zeta(2x+1) = \frac{\pi(2\pi)^{2x}\zeta(-2x)}{\Gamma(2x+1)\cos\left[\frac{\pi(2x+1)}{2}\right]} = \frac{-\pi(2\pi)^{2x}\zeta(-2x)}{\Gamma(2x+1)\sin(\pi x)}. \tag{3.34}$$

At this point, since

$$\lim_{x\to n}\frac{\zeta(-2x)}{\sin(x\pi)}=\frac{\zeta'(-2n)\cdot(-2)}{\pi\cos(n\pi)}=\frac{2}{\pi}(-1)^{n+1}\zeta'(-2n), \tag{3.35}$$

and

$$\lim_{x\to n}\Gamma(2x+1)=(2n)!, \tag{3.36}$$

we obtain

$$\zeta(2n+1)=\lim_{x\to n}\zeta(2x+1)=\frac{2(2\pi)^{2n}(-1)^n\zeta'(-2n)}{(2n)!}. \tag{3.37}$$

Currently, it is not known if the constants $\zeta'(-2n)$ can be written in terms of a finite number of irrational numbers. However, infinite series representations exist for $\zeta(3)$, which include

$$\zeta(3)=\frac{5}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3\binom{2k}{k}}, \tag{3.38}$$

used by Apéry to prove the irrationality of $\zeta(3)$ in [2];

$$\zeta(3)=\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n}{n^2}, \tag{3.39}$$

where $H_n=\sum_{k=1}^{n}\frac{1}{k}$ is called the n-th *harmonic number*, found in [58];

$$\zeta(3)=\frac{7\pi^3}{180}-2\sum_{n=1}^{\infty}\frac{1}{n^3(e^{2n\pi}-1)}, \tag{3.40}$$

found in [48]; and

$$\zeta(3)=\frac{\pi^3}{28}+\frac{16}{7}\sum_{n=1}^{\infty}\frac{1}{n^3(e^{n\pi}+1)}-\frac{2}{7}\sum_{n=1}^{\infty}\frac{1}{n^3(e^{2n\pi}+1)}, \tag{3.41}$$

attributed to Simon Plouffe in [55]. We now prove the following formula for $\zeta(3)$.

Theorem 3.4.

$$\zeta(3) = \frac{2\pi^2 \ln 2}{7} - \frac{16}{7} \sum_{n=1}^{\infty} \frac{4^n}{8n^3 \binom{2n}{n}}. \tag{3.42}$$

Proof. We start the proof with a formula proved by Euler in 1772 [26],

$$\frac{7}{16}\zeta(3) = \frac{\pi^2 \ln 2}{8} + \int_0^{\pi/2} t \ln(\sin t)\, dt. \tag{3.43}$$

Setting $x = \sin t$ in (3.43) and solving for $\zeta(3)$, we get

$$\zeta(3) = \frac{2\pi^2 \ln 2}{7} + \frac{16}{7} \int_0^1 \frac{\arcsin x \ln x}{\sqrt{1-x^2}}\, dx. \tag{3.44}$$

Using integration by parts, we rewrite the integral on the right hand side of (3.44) to get

$$\int_0^1 \frac{\arcsin x \ln x}{\sqrt{1-x^2}}\, dx = \lim_{a \to 0} \left[\frac{(\arcsin x)^2 \ln x}{2} \right]_a^1 - \int_0^1 \frac{(\arcsin x)^2}{2x}\, dx. \tag{3.45}$$

Since the first term on the right hand side of (3.45) is zero, we use (3.45) in (3.44) to get

$$\zeta(3) = \frac{2\pi^2 \ln 2}{7} - \frac{16}{7} \int_0^1 \frac{(\arcsin x)^2}{2x}\, dx. \tag{3.46}$$

By using the Taylor expansion about $x = 0$ for $(\arcsin x)^2$ (see e.g., [15])

$$(\arcsin x)^2 = \sum_{n=1}^{\infty} \frac{(2x)^{2n}}{2n^2 \binom{2n}{n}}, \tag{3.47}$$

in (3.46), we obtain

$$\zeta(3) = \frac{2\pi^2 \ln 2}{7} - \frac{16}{7} \int_0^1 \left[\sum_{n=1}^{\infty} \frac{2^{2n-2} x^{2n-1}}{n^2 \binom{2n}{n}} \right] dx = \frac{2\pi^2 \ln 2}{7} - \frac{16}{7} \sum_{n=1}^{\infty} \frac{2^{2n-2}}{2n^3 \binom{2n}{n}}, \tag{3.48}$$

where integral and summation are interchanged due to uniform convergence, giving the desired claim. □

We conclude this section by giving a general formula for $\zeta(2n+1)$, found in [3],

$$\zeta(2n+1) = \left(\frac{\pi}{2}\right)^{2n+1} \lim_{m\to\infty} \frac{1}{m^{2n+1}} \sum_{k=1}^{m} \cot^{2n+1}\left(\frac{k\pi}{2m+1}\right). \tag{3.49}$$

3.4 Rapidly converging series

There are rapidly converging series for the Riemann zeta function, which can be used to calculate its values numerically. In [46], formula (1.9) is rewritten as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{4}{n^{s-2}(4n^2-1)} - \sum_{n=1}^{\infty} \frac{1}{n^s(4n^2-1)}, \tag{3.50}$$

valid for $\Re(s) > 1$, which provides accelerated convergence through recursive computation. For example, Olkkonen and Olkkonen in [46] prove

$$\zeta(3) = 8\ln(2) - 4 - \sum_{n=1}^{\infty} \frac{1}{n^3(4n^2-1)}, \tag{3.51}$$

and note that the series

$$\zeta(3) = \frac{4}{3}\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^3}, \tag{3.52}$$

needs 1600 terms for 9 decimal place accuracy, while (3.51) needs only 120 terms for the same accuracy. Recursive computation occurs for $s > 3$ by using (3.51) in (3.52), which allows us to write

$$\zeta(5) = \sum_{n=1}^{\infty} \frac{4}{n^3(4n^2-1)} - \sum_{n=1}^{\infty} \frac{1}{n^5(4n^2-1)} \tag{3.53}$$

$$= 32\ln(2) - 16 - 4\zeta(3) - \sum_{n=1}^{\infty} \frac{1}{n^5(4n^2-1)}, \tag{3.54}$$

and then continuing this process for greater values of s.

Another rapidly converging series for $n \in \mathbb{N}^+, n > 1, n \equiv 3 \pmod 4$, attributed to Ramanujan in [55] is

$$\zeta(n) = \frac{2^{n-1}\pi^n}{(n+1)!} \sum_{k=0}^{(n+1)/2} (-1)^{k-1} \binom{n+1}{2k} B_{n+1-2k} B_{2k} - 2 \sum_{k=1}^{\infty} \frac{1}{k^n(e^{2\pi k} - 1)}, \tag{3.55}$$

where B_k is the kth Bernoulli number.

Further rapidly converging series and integral representations for the Riemann zeta function can be found in [18] and [56], where in the latter, for example, a formula for $\zeta(2)$ is given as

$$\zeta(2) = \frac{5}{3} \sum_{n=0}^{\infty} \binom{2n}{n} \frac{(-1)^n}{16^n(2n+1)^2}. \tag{3.56}$$

The formula (3.56) gives 9 decimal places of accuracy with only the first 9 terms, while the sum of 1,000,000 terms of the expansion of (3.4) yields only 5 decimal places of accuracy.

Accelerated convergence for series that can calculate values of the Riemann zeta function is also found in what is known as a *BBP-type formula.* The formula known as the BBP formula is named after David Bailey, Jonathan Borwein, and Simon Plouffe [6], with

$$\pi = \sum_{n=0}^{\infty} \frac{1}{16^n} \left(\frac{4}{8n+1} - \frac{2}{8n+4} - \frac{1}{8n+5} - \frac{1}{8n+6} \right). \tag{3.57}$$

A surprising result of the formula is that it can be used to extract the nth digit of π in base 16, without needing to calculate the first $n-1$ digits [6]. Other constants can be written as a summation of this type, including values of the Riemann zeta function, such as

$$\zeta(2) = \frac{3}{16} \sum_{n=0}^{\infty} \frac{1}{64^n} \left(\frac{16}{6n+1} - \frac{24}{6n+2} - \frac{8}{6n+3} - \frac{6}{6n+4} + \frac{1}{6n+5} \right), \tag{3.58}$$

found in [4]. It can be shown that the formula (3.58) achieves 13 decimal places accuracy when summing only the first 6 terms.

3.5 Plots of the Riemann zeta function

In section (3.2), we discovered that $\zeta(-2n) = 0$ for $n \in \mathbb{N}^+$. These are called the *trivial zeros* of the Riemann zeta function, shown in figure (3.1).

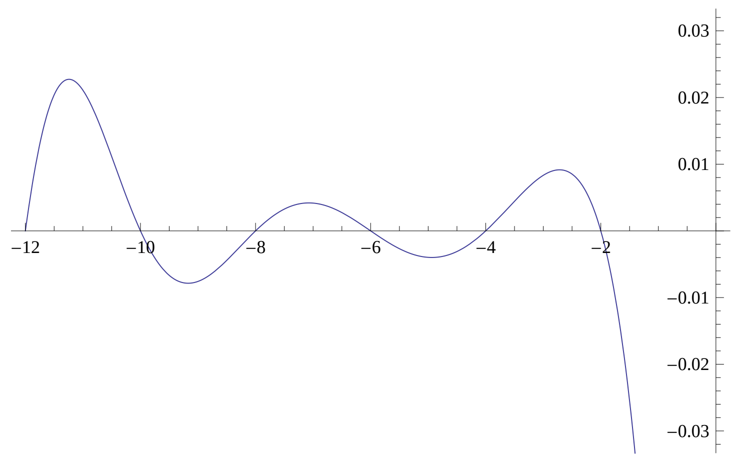

Figure 3.1: Trivial zeros of the Riemann zeta function, $\zeta(x)$ for $x < 0$.

We also mentioned that it is not known if the constants $\zeta'(-2n)$ can be written in terms of a finite number of irrational numbers. It is clear from figure (3.1) that these derivatives alternate sign.

In chapter 2, we showed that the Riemann zeta function is meromorphic with a simple pole at $s = 1$. The pole can be seen in figure (3.2), where it is evident that

$$\lim_{x \to 1^-} \zeta(x) = -\infty, \text{ and } \lim_{x \to 1^+} \zeta(x) = +\infty, \tag{3.59}$$

where $x = \Re(s)$.

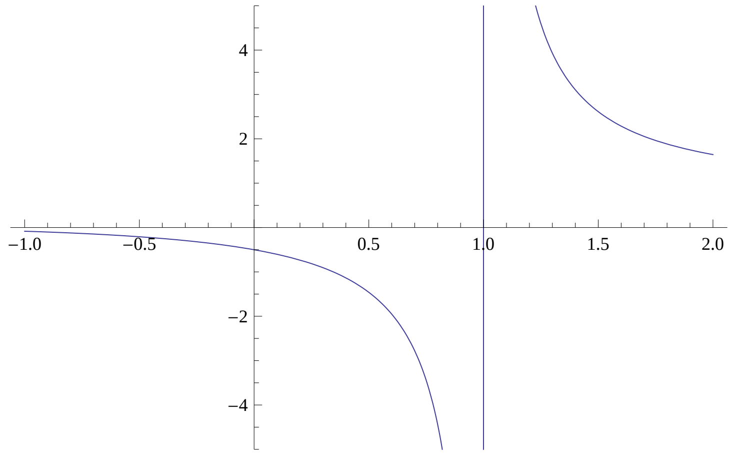

Figure 3.2: A plot $\zeta(x)$ with $x \in \mathbb{R}$, with a simple pole at $x = 1$.

We also showed in section (3.2) that $\zeta(2n) = c_{2n}\pi^{2n}$, with $c_{2n} \in \mathbb{Q}$, and $n \in \mathbb{N}^+$. The values appear on the graph in figure (3.3).

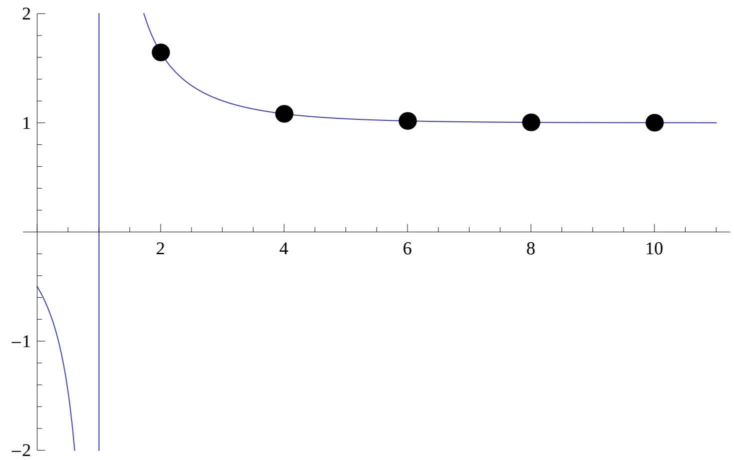

Figure 3.3: A plot of $\zeta(x)$ with $x \in \mathbb{R}$, with $x > 1$ and points at $(2n, \zeta(2n))$.

There are zeros of the Riemann zeta function apart from those shown in figure (3.1) that we discuss further in chapter 6. These are called the *nontrivial zeros* of the Riemann zeta function. It is conjectured that all of these zeros occur on the line $\Re(s) = 1/2$. The first few of the nontrivial zeros can be seen in figure (3.4), while an example of a plot where $\Re(s) \neq 1/2$, in figure (3.5) shows no zeros.

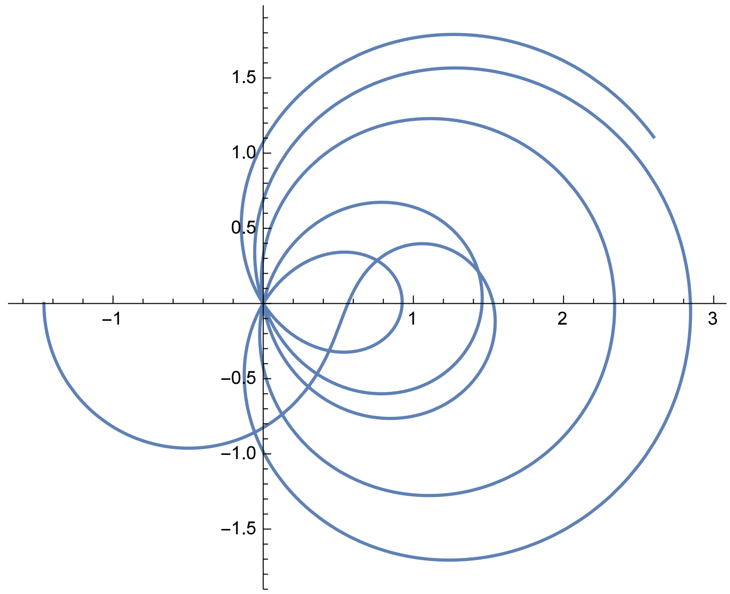

Figure 3.4: Polar graph of $\zeta(1/2 + it)$ with $0 \leq t \leq 35$.

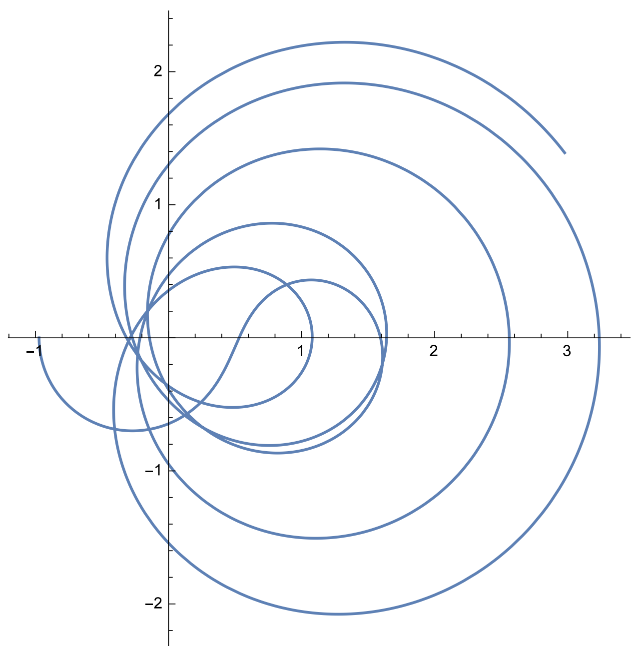

Figure 3.5: Polar graph of $\zeta(1/3 + it)$ with $0 \leq t \leq 35$.

Chapter 4: Generalizations of the Riemann zeta function

4.1 Functions that generalize $\zeta(s)$.

General Dirichlet series

The general Dirichlet series is defined as

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}, \tag{4.1}$$

where $a_n, s \in \mathbb{C}$, and λ_n, is a strictly increasing sequence of real numbers that tends to infinity. The series (4.1) converges for values of s that depend on the coefficients a_n.

Dirichlet series

The general Dirichlet series specializes to what is known as an *"ordinary"* Dirichlet series when $\lambda_n = \ln(n)$. In this case, (4.1) reduces to

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \tag{4.2}$$

(see [47], p. 640), where $F(s)$ is a *Dirichlet generating function* for the coefficients a_n. Once again, the region of convergence of (4.2) depends on the coefficients a_n. In the case $a_n = 1$, the Dirichlet series becomes the Riemann zeta function which, as we have already shown, converges for all $\Re(s) > 1$ and can be analytically continued to all $s \in \mathbb{C} \setminus \{1\}$.

Many number theoretical functions are coefficients for series of Dirichlet generating functions ([47], p. 640), including

$$\frac{\zeta(s-1)}{\zeta(s)} = \sum_{n=1}^{\infty} \phi(n) n^{-s} \quad \text{for} \quad \Re(s) > 2, \tag{4.3}$$

$$(\zeta(s))^k = \sum_{n=1}^{\infty} d_k(n) n^{-s} \quad \text{for} \quad \Re(s) > 1, \tag{4.4}$$

and

$$\zeta'(s) = \sum_{n=2}^{\infty} \ln(n) n^{-s}, \tag{4.5}$$

where $\phi(n)$ is called *Euler's totient function*, which is the number of positive integers $\leq n$ that are relatively prime to n, and $d_k(n)$ is the function that counts the number of ways of expressing n as the product of k factors, with the order of factors taken into account. For example, $d_2(12) = 6$ because 12 can be factored into 2 factors 6 ways, with the ordered pairs being $\{(1,12),(12,1),(2,6),(6,2),(3,4),(4,3)\}$, and $d_3(12) = 18$ because 12 can be factored into 3 factors 18 ways, with the ordered triples being $\{(1,1,12),(1,12,1),(12,1,1),(2,2,3),(2,3,2),(3,2,2),(1,3,4),(1,4,3),(3,1,4),(3,4,1),$ $(4,1,3),(4,3,1),(1,2,6),(1,6,2),(2,1,6),(2,6,1),(6,1,2),(6,2,1)\}$.

Hurwitz zeta function

The Hurwitz zeta function represents a generalization of the Riemann zeta function named after the German mathematician Adolf Hurwitz. It is defined as (see [47], p. 607)

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, \tag{4.6}$$

which is valid for $\Re(s) > 1$, and $a \notin -\mathbb{Z} \cup \{0\}$. The Hurwitz zeta function can be analytically continued to all $s \in \mathbb{C} \setminus \{1\}$, and becomes the Riemann Zeta function in the case when $a = 1$.

For the Hurwitz zeta function, one can find asymptotic expansions for large and small values of a. We present the small-a expansion here since it is quite straightforward to obtain. First, we write (4.6) as

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} = \frac{1}{a^s} + \sum_{n=1}^{\infty} \frac{1}{(n+a)^s}, \tag{4.7}$$

and use (2.2) to write

$$\frac{\Gamma(s)}{(n+a)^s} = \int_0^{\infty} t^{s-1} e^{-(n+a)t}\, dt. \tag{4.8}$$

By using the integral representation (4.8) in (4.7), we get

$$\begin{aligned} \zeta(s,a) &= \frac{1}{a^s} + \frac{1}{\Gamma(s)} \sum_{n=1}^{\infty} \int_0^{\infty} t^{s-1} e^{-(n+a)t}\, dt \\ &= \frac{1}{a^s} + \frac{1}{\Gamma(s)} \sum_{n=1}^{\infty} \int_0^{\infty} t^{s-1} e^{-at} e^{-nt}\, dt. \end{aligned} \tag{4.9}$$

Due to uniform convergence for $\Re(s) > 1$ in (4.9), we interchange summation and integral and use

$$\sum_{n=1}^{\infty} e^{-nt} = \frac{1}{e^t - 1},$$

to obtain

$$\zeta(s,a) = \frac{1}{a^s} + \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \frac{e^{-at}}{e^t - 1}\, dt. \tag{4.10}$$

This, along with the use of the following small-a expansion

$$e^{-at} = \sum_{k=0}^{\infty} (-1)^k \frac{a^k t^k}{k!},$$

allows us to write

$$\zeta(s,a) = \frac{1}{a^s} + \frac{1}{\Gamma(s)} \sum_{k=0}^{\infty} (-1)^k \frac{a^k}{k!} \int_0^{\infty} \frac{t^{s+k-1}}{e^t - 1}\, dt. \tag{4.11}$$

By recalling (2.6), we can write

$$\zeta(s,a) = \frac{1}{a^s} + \frac{1}{\Gamma(s)} \sum_{k=0}^{\infty} (-1)^k \frac{a^k}{k!} \Gamma(s+k)\zeta(s+k), \tag{4.12}$$

which represents the small-a expansion of $\zeta(s,a)$. Large-a expansions can be found in ([47], p. 610).

For $\zeta(s,k+1)$ with $k \in \mathbb{N}_0$, the Hurwitz zeta function represents a shift in the beginning term of the Riemann zeta function. In fact,

$$\zeta(s,k+1) = \sum_{n=1}^{\infty} \frac{1}{(n+k)^s} = \sum_{n=k+1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{k} \frac{1}{n^s} = \zeta(s) - H_k^s, \tag{4.13}$$

where H_k^s, is called the kth *generalized harmonic number* at $s \in \mathbb{C}$.

We can use (4.13) to give a representation of the Hurwitz zeta function $\zeta(s,a)$ in terms of the Riemann zeta function when the second argument a is a *"half-integer,"* namely $a = k + 1/2$ with $k \in \mathbb{N}_0$. To prove this relation, we use the following lemma.

Lemma 4.1. *For $k \in \mathbb{N}_0$, and $s \in \mathbb{C}$,*

$$\zeta(s, k+1/2) = 2^s \zeta(s, 2k) - \zeta(s,k). \tag{4.14}$$

Proof. Using (4.6), we can rewrite the right hand side of (4.14) in series form to get

$$\begin{aligned} 2^s\zeta(s,2k) - \zeta(s,k) &= \sum_{n=0}^{\infty}\left[\frac{2^s}{(n+2k)^s} - \frac{1}{(n+k)^s}\right] \\ &= 2^s\sum_{n=0}^{\infty}\left[\frac{1}{(n+2k)^s} - \frac{1}{(2n+2k)^s}\right]. \end{aligned} \tag{4.15}$$

Next, we note that the expansion of the right hand side of (4.15) is a telescoping sum.

$$\begin{aligned} &2^s\sum_{n=0}^{\infty}\left[\frac{1}{(n+2k)^s} - \frac{1}{(2n+2k)^s}\right] \\ &= 2^s\left[\frac{1}{(2k)^s} - \frac{1}{(2k)^s} + \frac{1}{(2k+1)^s} - \frac{1}{(2k+2)^s} + \frac{1}{(2k+2)^s} - \frac{1}{(2k+4)^s} + \cdots\right] \\ &= \sum_{n=0}^{\infty}\frac{2^s}{(2n+2k+1)^s} = \sum_{n=0}^{\infty}\frac{1}{(n+k+1/2)^s} = \zeta(s,k+1/2). \end{aligned} \tag{4.16}$$

□

We now return to the aforementioned representation of $\zeta(s,k+1/2)$ which is outlined in the following theorem.

Theorem 4.2. *For $k \in \mathbb{N}_0$, and $s \in \mathbb{C}$, one has*

$$\zeta(s,k+1/2) = (2^s-1)\zeta(s) - 2^s\sum_{n=1}^{2k-1}\frac{1}{n^s} + \sum_{n=1}^{k-1}\frac{1}{n^s}. \tag{4.17}$$

Proof. By using (4.13) in (4.14) we have

$$\begin{aligned} \zeta(s,k+1/2) &= 2^s\left[\zeta(s) - \sum_{n=0}^{2k-1}\frac{1}{n^s}\right] - \zeta(s) + \sum_{n=0}^{k-1}\frac{1}{n^s} \\ &= (2^s-1)\zeta(s) - 2^s\sum_{n=1}^{2k-1}\frac{1}{n^s} + \sum_{n=1}^{k-1}\frac{1}{n^s}. \end{aligned} \tag{4.18}$$

$\square$

Other properties and relations involving $\zeta(s,a)$ can be found in ([17], [45], [47], and [62]).

Polylogarithm and Lerch transcendent

The *polylogarithm* function is defined as

$$\mathrm{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}, \tag{4.19}$$

(see [47], p. 611), which is analytic for $|z| < 1$ and can be extended by analytic continuation to $|z| \geq 1$. The term *dilogarithm* is used in the case where $s = 2$. The polylogarithm function becomes the Riemann zeta function when $z = 1$, namely $\mathrm{Li}_s(1) = \zeta(s)$.

The polylogarithm function for $s \in \mathbb{C}$, satisfies the following recursion

Theorem 4.3. *For any $s \in \mathbb{C}$ and $z \in \mathbb{C}$, one has*

$$\frac{\partial}{\partial z}\mathrm{Li}_s(z) = \frac{1}{z}\mathrm{Li}_{s-1}(z). \tag{4.20}$$

Proof.

$$\frac{\partial}{\partial z}\mathrm{Li}_s(z) = \frac{\partial}{\partial z}\sum_{n=1}^{\infty}\frac{z^n}{n^s} = \sum_{n=1}^{\infty}\frac{nz^{n-1}}{n^s} = \frac{1}{z}\sum_{n=1}^{\infty}\frac{z^n}{n^{s-1}} = \frac{1}{z}\mathrm{Li}_{s-1}(z).$$

$\square$

We list a few values of $\mathrm{Li}_s(z)$ with $s \in \mathbb{Z}$ as follows:

$$\mathrm{Li}_1(z) = \sum_{n=1}^{\infty} \frac{z^n}{n} = -\ln(1-z), \tag{4.21}$$

$$\mathrm{Li}_0(z) = \sum_{n=1}^{\infty} z^n = \frac{z}{1-z}, \tag{4.22}$$

$$\mathrm{Li}_{-1}(z) = \sum_{n=1}^{\infty} nz^n = \frac{z}{(1-z)^2}, \tag{4.23}$$

$$\mathrm{Li}_{-2}(z) = \sum_{n=1}^{\infty} n^2 z^n = \frac{z(z+1)}{(1-z)^3}, \mathrm{Li}_{-3}(z) \qquad = \sum_{n=1}^{\infty} n^3 z^n = \frac{z(z^2+4z+1)}{(1-z)^4}. \tag{4.24}$$

The graphs of the five functions above are shown in the plot below.

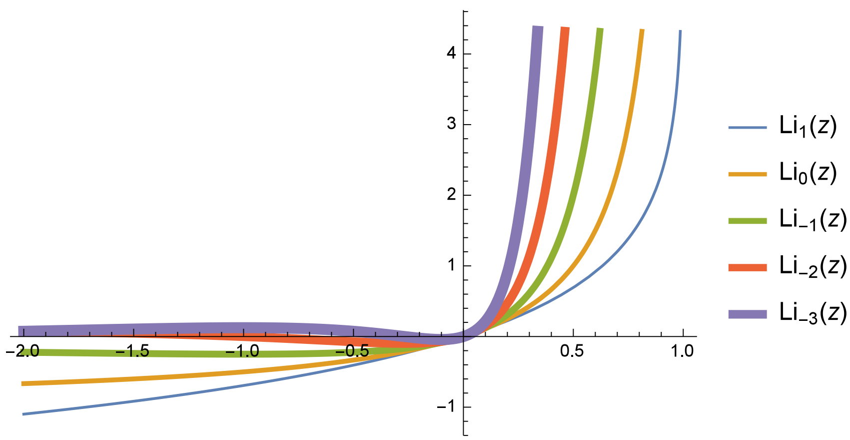

Figure 4.1: Plot of $\mathrm{Li}_s(z)$ with $z \in \mathbb{R}$ and $s = \{-3, -2, -1, 0, 1\}$.

In 1889, Jonquiére related the polylogarithm to the Hurwitz zeta function [36] by the formula

$$\mathrm{Li}_s(e^{2\pi i a}) + e^{\pi i s}\mathrm{Li}_s(-e^{2\pi i a}) = \frac{(2\pi)^s e^{\pi i s/2}}{\Gamma(s)}\zeta(1-s, a), \tag{4.25}$$

valid for $\Re(s) > 0$, $\Im(a) > 0$ or $\Re(s) > 1$, $\Im(a) = 0$. Other properties and relations involving $\mathrm{Li}_s(z)$ can be found in ([36], [47], and [57]).

The *Lerch transcendent* is a generalization of the polylogarithm, defined as

$$\Phi(z,s,a)=\sum_{n=0}^{\infty}\frac{z^n}{(a+n)^s}, \tag{4.26}$$

(see [47], p. 611), valid for $\Re(s)>1, |z|<1$, and $a\notin -\mathbb{Z}\cup\{0\}$. Special cases of the Lerch transcendent include,

$$\zeta(s,a)=\Phi(1,s,a), \tag{4.27}$$

$$\mathrm{Li}_s(z)=z\Phi(z,s,1), \tag{4.28}$$

$$\zeta(s)=\Phi(1,s,1). \tag{4.29}$$

Other properties and relations involving $\Phi(z,s,a)$ can be found in ([36], [45], [47], and [57]).

Multiple zeta function

The multiple zeta function was first considered by Euler [13] and is defined as

$$\zeta(s_1,s_2,\ldots,s_N)=\sum_{n_1>n_2>\cdots>n_N}\prod_{k=1}^{N}\frac{1}{n_k^{s_k}}, \tag{4.30}$$

valid for $\sum_{k=1}^{N}s_k>N$ for all N, where the two-variable case becomes

$$\zeta(a,b)=\sum_{n_1>n_2>0}\frac{1}{n_1^a n_2^b}=\sum_{n_1=1}^{\infty}\frac{1}{n_1^a}\sum_{n_2=1}^{n_1-1}\frac{1}{n_2^b}, \tag{4.31}$$

and obeys the reflection formula [13]

$$\zeta(a)\zeta(b)=\zeta(a,b)+\zeta(b,a)+\zeta(a+b), \tag{4.32}$$

valid for $\Re(a) > 1, \Re(b) > 1$. Other properties and relations involving the multiple zeta function can be found in ([1], [5], and [47]).

Barnes zeta function

The Barnes zeta function, introduced by E.W. Barnes [8] in 1901, is defined by

$$\zeta_N(s, w \mid a_1, \ldots, a_N) = \sum_{n_1 > n_2 > \cdots > n_N \geq 0} \frac{1}{(w + n_1 a_1 + \cdots + n_N a_N)^s}, \tag{4.33}$$

where $\Re(w) > 0$, $\Re(a_k) > 0$, and $\Re(s) > N$. The Barnes zeta function has a meromorphic continuation to all $s \in \mathbb{C}$ where $s \neq \{1, 2, \ldots, N\}$. In the case $w = N = a_1 = 1$, the Barnes zeta function becomes the Riemann zeta function. We refer the reader to [8] for additional information about $\zeta_N(s, w \mid a_1, \ldots, a_N)$.

Epstein zeta function

The Epstein zeta function [9] is named after the German mathematician Paul Epstein, and is defined as

$$\sum (am^2 + bmn + cn^2)^{-s}, \tag{4.34}$$

where the sum is taken over all ordered pairs (m, n) such that $\{m, n \in \mathbb{Z} \mid (m, n) \neq (0, 0)\}$, $\{a, b, c\} \in \mathbb{R}$ and $a > 0$ with the discriminant $b^2 - 4ac < 0$.

Epstein zeta functions are used in the calculations of *lattice sums*, or sums over an array of points. These calculations are discussed in [16], where a class of Epstein zeta functions are used in crystal physics. Other properties and relations involving the Epstein zeta function can be found in ([9] and [17]).

Spectral zeta function

The spectral zeta function [63] is defined as the sum of reciprocals of powers of eigenvalues λ_i of an elliptic, positive, and self-adjoint operator $\mathcal{O}$,

$$\zeta_{\mathcal{O}}(s) = \sum_{\lambda_i} \lambda_i^{-s} = \mathrm{Tr}(\mathcal{O}^{-s}), \tag{4.35}$$

where Tr is the trace. Here the sequence

$$0 \leq \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_k \leq \cdots \text{ with } \lambda_k \to \infty$$

is the complete set of eigenvalues for $\mathcal{O}$, listed in increasing order.

It can be proved [28] that for a second-order elliptic self-adjoint operator acting on functions defined on a compact D-dimension space, we have the asymptotic behavior for the eigenvalues $\lambda_n \sim n^{2/D}$ as $n \to \infty$. This implies that $\zeta_{\mathcal{O}}(s)$ converges for $\Re(s) > D/2$. One example of such an operator is $\mathcal{O} = -\Delta$, where Δ is the *Laplacian* on a compact region of $\mathbb{R}^n$,

$$\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}.$$

The function $\zeta_{\mathcal{O}}(s)$ can be analytically continued to all $\{s \in \mathbb{C} \mid s \neq \frac{D-k}{2}, k = \{0, 1, 2, \ldots, D-1\}\} \cup \{s \in \mathbb{C} \mid s \neq \frac{1-2n}{2}, n \in \mathbb{N}_0\}$ [28].

Nearest integer zeta function

A little known, yet interesting, generalization of the Riemann zeta function is the *"nearest integer zeta function"* which arose from a solution to a problem proposed by Seung-Jin Bang in [7], which reads

> *Let $a(n)$ be the integer closest to $\sqrt[3]{n}$. Evaluate $\sum_{n=1}^{\infty} a(n)^{-4}$.*

In the same publication, a solution is given by Jonathan M. Borwein and Leo C. Hsu, who also considered the general case

$$S_N(s) = \sum_{n=1}^{\infty} \left[(n^{1/N})\right]^{-s}, \tag{4.36}$$

with $N \in \mathbb{N}$, where $[x]$ denotes the nearest integer to x. The relation (4.36) is therefore called the *nearest integer zeta function*.

When $s > 3$, the Riemann zeta function appears in some calculations of (4.36). For instance, one can prove that

$$S_2(s) = 2\zeta(s-1), \tag{4.37}$$

$$S_3(s) = 3\zeta(s-2) + 4^{-s}\zeta(s), \tag{4.38}$$

and

$$S_4(s) = 4\zeta(s-3) + \zeta(s-1). \tag{4.39}$$

It is also interesting to note that when s is restricted to natural numbers, $S_N(n)$ is a polynomial in π, whose coefficients are algebraic numbers when $n - N$ is odd. As an example, the solution to the problem proposed by Bang is

$$S_3(4) = \frac{\pi^2}{2} + \frac{\pi^4}{23040}. \tag{4.40}$$

4.2 η, λ, and β functions

In this section we describe some well known Dirichlet series: the Dirichlet eta function, Dirichlet beta function, and Dirichlet lambda function along with their relations to the Riemann zeta function.

Dirichlet $\eta-$function

The Dirichlet $\eta-$function was first mentioned in section 2.2 and is once again defined as

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s},$$

valid for $\Re(s) > 0$. The function $\eta(s)$ can be extended to all $s \neq 1$ through analytic continuation. Recall from (2.47) that the Dirichlet $\eta-$function is related to the Riemann zeta function through the relation

$$\eta(s) = (1 - 2^{1-s})\zeta(s). \tag{4.41}$$

The Dirichlet $\eta-$function can be written in terms of the Lerch transcendent as

$$\eta(s) = \Phi(-1, s, 1). \tag{4.42}$$

Dirichlet $\lambda-$function

The Dirichlet $\lambda-$function is defined as ([45], p.32)

$$\lambda(s) = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s}, \tag{4.43}$$

valid for $\Re(s) > 1$. The function (4.43) can be extended to all $s \neq 1$ through anayltic continuation. The Dirichlet $\lambda-$function is related to the Riemann zeta function as follows.

Theorem 4.4.

$$\lambda(s) = (1 - 2^{-s})\zeta(s). \tag{4.44}$$

Proof. Since $\lambda(s)$ represents the sum of the odd-indexed terms of $\zeta(s)$, we add the even indexed terms to (4.44) to get

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^s}+\sum_{n=1}^{\infty}\frac{1}{(2n)^s}=\zeta(s) \tag{4.45}$$

$$\lambda(s)+2^{-s}\zeta(s)=\zeta(s) \tag{4.46}$$

$$\lambda(s)=(1-2^{-s})\zeta(s),$$

which is the desired claim. □

Dirichlet $\beta-$function

The Dirichlet $\beta-$function is defined as ([45], p.33)

$$\beta(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^s}, \tag{4.47}$$

valid for $\Re(s)>0$. The analytic continuation of $\beta(s)$ to all $s\in\mathbb{C}$ is given by [45]

$$\beta(1-s)=\left(\frac{\pi}{2}\right)^{-s}\sin\left(\frac{\pi}{2}s\right)\Gamma(s)\beta(s). \tag{4.48}$$

The function $\beta(s)$ can be written in terms of the Lerch transcendent as

$$\beta(s)=2^{-s}\Phi(-1,s,1/2). \tag{4.49}$$

The Dirichlet $\beta-$function is also related to the *Euler numbers*, E_n ([47], p. 588), which are defined through the series

$$\frac{1}{\cosh(t)}=\frac{2}{e^t+e^{-t}}=\sum_{n=0}^{\infty}E_n\frac{t^n}{n!}, \tag{4.50}$$

valid for $|t| < \frac{\pi}{2}$. Since $f(t) = \cosh(t)$ is an even function, it follows that $E_{2n+1} = 0$ for $n \in \mathbb{N}$. Using (4.50), we prove that values for $\beta(s)$ at integer values of s ([45], p. 36) are

$$\beta(-2n) = \frac{E_{2n}}{2}, \tag{4.51}$$

$$\beta(-2n-1) = 0, \tag{4.52}$$

and

$$\beta(2n+1) = \frac{(-1)^n E_{2n}}{2(2n)!} \left(\frac{\pi}{2}\right)^{2n+1}. \tag{4.53}$$

First, we prove equation (4.51) using the same method outlined in section 3.2. To begin with, we consider (2.2) and write

$$\frac{(-1)^{n+1}\Gamma(s)}{(2n-1)^s} = (-1)^{n+1} \int_0^\infty t^{s-1} e^{-(2n-1)t}\, dt. \tag{4.54}$$

Now we utilize (4.54) in (4.47) to get

$$\begin{aligned} \Gamma(s) \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^s} &= \sum_{n=1}^{\infty} (-1)^{n+1} \int_0^\infty t^{s-1} e^{-(2n-1)t}\, dt \\ \Gamma(s)\beta(s) &= \int_0^\infty t^{s-1} \left(\sum_{n=1}^{\infty} (-1)^{n+1} e^{(1-2n)t} \right) dt, \end{aligned} \tag{4.55}$$

where we have interchanged the summation and integral due to uniform convergence. In order to calculate the sum on the right hand side of (4.55), we note that $|e^{-2t}| < 1$ for $t > 0$, which then allows us to write

$$\sum_{n=1}^{\infty} (-1)^{n+1} e^{(1-2n)t} = -e^t \sum_{n=1}^{\infty} (-e^{-2t})^n = \frac{e^{-t}}{1+e^{-2t}} = \frac{1}{e^t + e^{-t}}. \tag{4.56}$$

By substituting the result (4.56) into (4.55), we get

$$\Gamma(s)\beta(s) = \int_0^\infty \frac{t^{s-1}}{e^t + e^{-t}}\,dt, \tag{4.57}$$

which can be rewritten as

$$\beta(s) = \frac{1}{2\Gamma(s)} \int_0^1 \frac{2}{e^t + e^{-t}} t^{s-1}\,dt + \frac{1}{\Gamma(s)} \int_1^\infty \frac{t^{s-1}}{e^t + e^{-t}}\,dt. \tag{4.58}$$

Here, we note that the second integral on the right hand side of (4.58) converges for all $s \in \mathbb{C}$, and therefore defines an entire function of s that we denote $G(s)$. By using (4.50) in (4.58), we get

$$\beta(s) = \frac{1}{2\Gamma(s)} \sum_{k=0}^{\infty} \frac{E_k}{k!} \int_0^1 t^{k+s-1}\,dt + \frac{G(s)}{\Gamma(s)}. \tag{4.59}$$

Upon calculating the integral on the right hand side of (4.59) and setting $s = -2n+\varepsilon$, with $\varepsilon > 0$, we obtain

$$\beta(-2n+\varepsilon) = \frac{1}{2\Gamma(-2n+\varepsilon)} \left[\sum_{k=0}^{\infty} \frac{E_k}{k!(k-2n+\varepsilon)} \right] + \frac{G(-2n+\varepsilon)}{\Gamma(-2n+\varepsilon)}. \tag{4.60}$$

The use of (1.12) now allows us to write

$$\frac{1}{2\Gamma(-2n+\varepsilon)} = \frac{\Gamma(1+2n-\varepsilon)\sin[\pi(-2n+\varepsilon)]}{2\pi}, \tag{4.61}$$

and by expanding the angle sum in $\sin[\pi(-2n+\varepsilon)]$, we get

$$\begin{aligned} \frac{1}{2\Gamma(-2n+\varepsilon)} &= \frac{\Gamma(1+2n-\varepsilon)\left[\sin(-2n\pi)\cos(\pi\varepsilon) + \cos(-2n\pi)\sin(\pi\varepsilon)\right]}{2\pi} \\ &= \frac{\Gamma(1+2n-\varepsilon)}{2\pi}\sin(\pi\varepsilon) = \frac{\Gamma(1+2n-\varepsilon)}{2\pi}\left[\pi\varepsilon + \mathcal{O}(\varepsilon^3)\right] \end{aligned}$$

$$= \frac{1}{2}\Gamma(1+2n-\varepsilon)\left[\varepsilon + \mathcal{O}(\varepsilon^3)\right]. \tag{4.62}$$

At this point, we substitute the result (4.62) into (4.60) to obtain

$$\begin{aligned}\beta(-2n+\varepsilon) = \frac{1}{2}\Gamma(1+2n-\varepsilon)(-1)^{n+1}\left[\varepsilon + \mathcal{O}(\varepsilon^3)\right]\left[\sum_{k=0}^{\infty}\frac{E_k}{k!(k-2n+\varepsilon)}\right] \\ + \Gamma(1+2n-\varepsilon)\left[\varepsilon + \mathcal{O}(\varepsilon^3)\right]G(-2n+\varepsilon).\end{aligned} \tag{4.63}$$

When taking the limit as $\varepsilon \to 0$, the expansion of the first term of the right hand side of (4.63) vanishes except when $k = 2n$, while the second term vanishes for all n. Therefore, we obtain

$$\beta(-2n) = \varepsilon\Gamma(1+2n)\frac{E_{2n}}{2(2n)!\varepsilon} = \frac{E_{2n}}{2}, \tag{4.64}$$

from which the claim (4.51) follows.

Additionally, equation (4.52) can be proved by setting $n = k + 1/2$ in (4.51), and by using the fact that $E_{2k+1} = 0$.

Equation (4.53) can be shown to hold by using (4.51) in (4.48), together with (1.12). This allows us to write

$$\begin{aligned}\beta(2n+1) &= \left(\frac{\pi}{2}\right)^{2n}\sin(-n\pi)\Gamma(-2n)\beta(-2n) \\ &= \frac{\left(\frac{\pi}{2}\right)^{2n}\pi\sin(-n\pi)}{\Gamma(2n+1)2\sin(-n\pi)\cos(-n\pi)}\frac{E_{2n}}{2} \\ &= \frac{(-1)^n E_{2n}}{2(2n)!}\left(\frac{\pi}{2}\right)^{2n+1}.\end{aligned}$$

Some beta constants include $\beta(1)$, which is the famous Gregory-Leibniz series,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}. \tag{4.65}$$

The sum in (4.65) is proved by writing

$$\arctan(x) = \int_0^x \frac{1}{1+t^2}\, dt = \int_0^x \sum_{n=0}^{\infty} (-1)^n t^{2n}\, dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \tag{4.66}$$

which is valid for $-1 < x \leq 1$. Substituting $x = 1$ into (4.66) produces (4.65).

A general expression is not known for $\beta(2n)$, but $\beta(2)$ (denoted by G) is *Catalan's constant*, which is defined as

$$G = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \approx 0.915965594. \tag{4.67}$$

We close this chapter by proving a relation between the η and β functions.

Theorem 4.5. *The η and β functions are related to the polylogarithm function by the formula*

$$Li_s(\pm i) = -2^{-s}\eta(s) \pm i\beta(s). \tag{4.68}$$

Proof. In the case where $z = i$, we take $n \in \mathbb{N}$ and write

$$\begin{aligned}
\mathrm{Li}_s(i) = \mathrm{Li}_s(e^{i\pi/2}) &= \sum_{n=1}^{\infty} \frac{e^{i(\pi/2)n}}{n^s} = \sum_{n=1}^{\infty} \frac{e^{i(\pi/2)(2n)}}{(2n)^s} + \sum_{n=1}^{\infty} \frac{e^{i(\pi/2)(2n-1)}}{(2n-1)^s} \\
&= \sum_{n=1}^{\infty} \frac{(e^{i\pi})^n}{(2n)^s} + e^{i\pi/2} \sum_{n=1}^{\infty} \frac{(e^{i\pi})^{(n-1)}}{(2n-1)^s} \\
&= 2^{-s} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s} + i \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^s}
\end{aligned}$$

$$
\begin{aligned}
&= -2^{-s} \sum_{n=1}^{\infty} \frac{(-1)^{(n-1)}}{n^s} + i \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^s} \\
&= -2^{-s} \eta(s) + i\beta(s). \qquad (4.69)
\end{aligned}
$$

The proof for the case $z = -i$ can be completed in the same fashion. $\square$

Chapter 5: Identities involving the zeta function

In this chapter we list some identities involving the zeta function. With one exception, we state these without proof. There are many reference books with multitudes of identities involving the Riemann zeta function (see, e.g., [57]). In what follows, we list a few of these identities that have relations with certain special functions.

An identity with the Möbius function

Definition 5.1. The *Möbius function*, $\mu(n)$, is

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ has one or more repeated integer factors} \\ 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n \text{ is a product of } k \text{ distinct primes.} \end{cases} \tag{5.1}$$

A relation involving this function and the Riemann zeta function, for $\Re(s) > 1$, is

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}, \tag{5.2}$$

(see [47], pp. 639–640, [62], p. 45). It is known that the convergence of the above formula for $\Re(s) > 1/2$ is equivalent to the Riemann hypothesis [60]. The Riemann hypothesis is discussed later in this work.

It is also known [44] that $\dfrac{1}{\zeta(n)}$ for $n \in \mathbb{N}, n \geq 2$, gives the probability that n positive integers chosen at random from a finite set are relatively prime.

Prime counting function

Given $\pi(x)$, the prime counting function, which counts the number of primes less than or equal to $x \in \mathbb{R}$, one can prove that

$$\ln \zeta(s) = s \int_0^\infty \frac{\pi(x)}{x(x^s - 1)}\, dx, \tag{5.3}$$

for $\Re(s) > 1$, (see [62], p. 49). Later in this work, we discuss another relation between $\pi(x)$ and $\zeta(s)$, which concerns the Riemann hypothesis [49].

Rising factorial series

As mentioned previously in lemma 3.3, let $(s)_n$ be the Pochhammer symbol that denotes the product $s(s+1)\cdots(s+n-1)$, with $s_0 = 1, s_1 = s$. Then,

$$\zeta(s) = \frac{s}{s-1} - \sum_{n=1}^{\infty} (\zeta(s+n) - 1)\, \frac{(s)_n}{(n+1)!}, \tag{5.4}$$

valid for all $s \neq 1$ ([57], p. 247).

Hasse's combinatorial series

In 1930, Helmut Hasse [31] proved the following formulas:

$$\zeta(s) = \frac{1}{(1-2^{1-s})} \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (k+1)^{-s},$$

and

$$\zeta(s) = \frac{1}{s-1} \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{(k+1)^{s-1}}, \tag{5.5}$$

both valid for $s \neq 1 + \frac{2\pi i n}{\ln(2)}$ with $n \in \mathbb{Z}$. The first of these two has also been proved in section (2.2).

Constants represented by series of the Riemann zeta function

Here we include a partial list of identities where certain constants can be represented as infinite series involving the Riemann zeta function. For $n \in \mathbb{N}$:

$$1 = \sum_{n=2}^{\infty} (\zeta(n) - 1)\,, \tag{5.6}$$

$$\frac{3}{4} = \sum_{n=1}^{\infty} (\zeta(2n) - 1)\,, \tag{5.7}$$

$$\ln 2 = \sum_{n=1}^{\infty} \frac{\zeta(2n) - 1}{n}, \tag{5.8}$$

found in [57], and

$$\pi = \sum_{n=1}^{\infty} \frac{(3^n - 1)\zeta(n+1)}{4^n}, \tag{5.9}$$

$$\ln \pi = \sum_{n=2}^{\infty} \frac{(2(3/2)^n - 3)(\zeta(n) - 1)}{n}, \tag{5.10}$$

found in [14]. Also, we define the *Euler-Mascheroni constant*, γ, as

Definition 5.2.

$$\gamma = \int_1^{\infty} \left(\frac{1}{x} - \frac{1}{\lfloor x \rfloor} \right) dx, \tag{5.11}$$

where $\lfloor x \rfloor$ is the floor function, which has the following relation with the Riemann zeta function,

$$\sum_{n=2}^{\infty} (-1)^n \frac{\zeta(n)}{n} = \gamma, \tag{5.12}$$

attributed to Euler in 1731 (see [39], p. 532).

An integral representation

Earlier we used an integral representation of $\zeta(s)$ found in equation (2.6) to perform the analytic continuation of the Riemann zeta function. Another integral representation, valid for all $s \neq 1$, is the *Abel-Plana formula* ([47], p.604), which gives the explicit formula

$$\zeta(s) = \frac{2^{s-1}}{s-1} - 2^s \int_0^\infty \frac{\sin(s \arctan t)}{(1+t^2)^{s/2}(e^{\pi t}+1)}\, dt, \tag{5.13}$$

in terms of elementary functions. Further integral representations for $\zeta(s)$ can be found in ([47], §25.5).

Generating functions for $\zeta(2n)$

For $|x| < 1$, we have (see [14], p. 254),

$$-\frac{\pi}{2} x \cot(\pi x) = \sum_{n=0}^{\infty} \zeta(2n) x^{2n}. \tag{5.14}$$

The above formula can be used to also write

$$\frac{\pi}{2} x \tan(\pi x) = \sum_{n=0}^{\infty} \zeta(2n)(4^n - 1) x^{2n}, \tag{5.15}$$

via the identity $2\cot(2\theta) = \cot(\theta) - \tan(\theta)$.

An identity involving the generalized harmonic number

Given H_n^s, the generalized harmonic number with

$$H_n^s = \sum_{k=1}^{n} \frac{1}{k^s},$$

we prove the following relation,

$$\sum_{n=1}^{\infty} \frac{H_n^s}{n^s} = \frac{\zeta(s)^2 + \zeta(2s)}{2}, \tag{5.16}$$

valid for $\Re(s) > 1$.

Proof. First we square H_n^s, writing the expansion in a symmetric $n \times n$ square matrix,

$$\begin{aligned}
&1 + 2^{-s} + 3^{-s} + 4^{-s} + \cdots + n^{-s} \\
&+ 2^{-s} + 4^{-s} + 6^{-s} + 8^{-s} + \cdots + (2n)^{-s} \\
&+ 3^{-s} + 6^{-s} + 9^{-s} + 12^{-s} + \cdots + (3n)^{-s} \\
&+ 4^{-s} + 8^{-s} + 12^{-s} + 16^{-s} + \cdots + (4n)^{-s} \\
&+ \cdots \\
&+ n^{-s} + (2n)^{-s} + (3n)^{-s} + (4n)^{-s} + \cdots + (n^2)^{-s}
\end{aligned} \tag{5.17}$$

and take the limit of the sum as $n \to \infty$. In doing so, we now have that (5.17) is $\zeta(s)^2$, where the main diagonal is $\zeta(2s)$. We call the remaining diagonally-arranged terms of (5.17) *minor diagonals* and note that these have identical pairs above and below the main diagonal. We now want to compute the sum of the minor diagonals.

Now the terms of the infinite expansion (5.17) can be rearranged without changing the sum, since $\zeta(s)$ is absolutely convergent for $s > 1$. Therefore, we can take the value $\zeta(s)^2$ by summing the main diagonal and the minor diagonals. The first minor diagonal above the main diagonal is

$$2^{-s} + 6^{-s} + 12^{-s} + \cdots = 1^{-s}2^{-s} + 2^{-s}3^{-s} + 3^{-s}4^{-s} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^s(n+1)^s}. \tag{5.18}$$

Continuing to calculate sums of further minor diagonals whose first terms are

$(k+1)^{-s}$, we have that the sum of each one is

$$\sum_{k=1}^{\infty} \frac{1}{n^s(n+k)^s}, \tag{5.19}$$

With this is mind, we can write the sum of all the minor diagonals in $\zeta(s)^2$ as

$$2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{n^s(n+k)^s},$$

which then allows us to write

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{n^s(n+k)^s} = \frac{\zeta(s)^2 - \zeta(2s)}{2}. \tag{5.20}$$

Note that the expansion of the series on the left hand side of (5.20) includes the terms in either triangle above or below the main diagonal of (5.17).

We now show that

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{n^s(n+k)^s} = \sum_{n=1}^{\infty} \frac{H_n^s}{n^s} - \zeta(2s). \tag{5.21}$$

Expanding the right hand side of (5.21), we obtain

$$\sum_{n=1}^{\infty} \frac{H_n^s}{n^s} - \zeta(2s) = \left[1 + 2^{-s}(1+2^{-s}) + 3^{-s}(1+2^{-s}+3^{-s}) + 4^{-s}(1+2^{-s}+3^{-s}+4^{-s}) + \cdots\right]$$

$$- \left[1 + 2^{-s}2^{-s} + 3^{-s}3^{-s} + 4^{-s}4^{-s} + \cdots\right]$$

$$= 2^{-s} + 3^{-s}(1+2^{-s}) + 4^{-s}(1+2^{-s}+3^{-s}) + \cdots$$

$$\begin{aligned} &= 2^{-s} \\ &+ 3^{-s} + 6^{-s} \\ &+ 4^{-s} + 8^{-s} + 12^{-s} \\ &+ \cdots, \end{aligned} \tag{5.22}$$

which is the same collection of terms in the triangle above or below the main diagonal of (5.17).

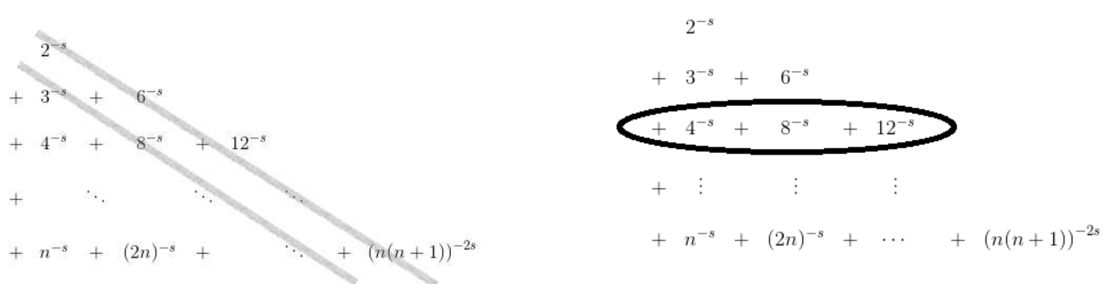

Figure 5.1: The triangular arrangement of terms is $\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{n^s(n+k)^s}$. The diagonal sums are $\frac{\zeta(s)^2-\zeta(2s)}{2}$, and the horizontal sums are $\sum_{n=1}^{\infty}\frac{H_n^s}{n^s} - \zeta(2s)$.

Thus, using (5.20) and (5.21), we obtain

$$\sum_{n=1}^{\infty} \frac{H_n^s}{n^s} - \zeta(2s) = \frac{\zeta(s)^2 - \zeta(2s)}{2}, \tag{5.23}$$

and

$$\sum_{n=1}^{\infty} \frac{H_n^s}{n^s} = \frac{\zeta(s)^2 + \zeta(2s)}{2},$$

which completes the proof. □

Chapter 6: The Riemann hypothesis

> “*If I were to awaken after having slept a thousand years, my first question would be: Has the Riemann Hypothesis been proven?*”
>
> - David Hilbert

It is known from the functional equation (2.44), that $\zeta(-2n) = 0$, for $n \in \mathbb{N}$, and that $\zeta(s) \neq 0$ for $\Re(s) \leq 0$ when $s \neq -2n$. These are called the *trivial zeros* of the Riemann zeta function. Additionally, from Euler's product formula (1.3), we know that $\zeta(s) \neq 0$ for $\Re(s) \geq 1$. The remaining region $0 < \Re(s) < 1$ is called the *critical strip*. It is known that all nontrivial zeros of $\zeta(s)$ lie in the critical strip. A conjecture about the critical strip is the *Riemann hypothesis*, which is the following conjecture.

Conjecture 6.1. *If* $\zeta(s) = 0$ *in the critical strip, then* $\Re(s) = 1/2$.

The conjecture, first made by Riemann in his 1859 paper *On the Number of Prime Numbers less than a Given Quantity* [49], has far reaching implications in number theory, and in particular, in the study of the distribution of prime numbers. The proof of this conjecture is an open problem whose solution would immediately resolve many other conjectures ([60], [64]).

In this chapter we look at the history of the Riemann hypothesis, state a few selected “equivalent statements” of the conjecture, and highlight one that is especially easy to understand. We also discuss some attempts at proving Riemann's conjecture and some functions that are used to study the zeros of $\zeta(s)$ in the critical strip.

6.1 Voronin Universality Theorem

In 1975, Sergei Voronin proved a remarkable property of $\zeta(s)$ in the critical strip [61], which is the following theorem.

Theorem 6.2. *Let $0 < r < 1/4$ and suppose $g(s)$ is a nonvanishing continuous function on the disk $|s| \leq r$ that is analytic on $|s| < r$. Then for any $\varepsilon > 0$, there exists a positive real number τ such that*

$$\max_{|s|\leq r} \left| \zeta\left(s + \frac{3}{4} + i\tau\right) - g(s) \right| < \varepsilon.$$

Additionally, the values τ have "positive lower density," which is illustrated in the following inequality.

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0,T] : \max_{|s|\leq r} \left| \zeta\left(s + \frac{3}{4} + i\tau\right) - g(s) \right| < \varepsilon \right\} > 0.$$

Loosely speaking, the theorem states that any nonvanishing analytic function can be approximated uniformly by certain imaginary shifts of $\zeta(s)$ in the critical strip.

6.2 Statements of the Riemann hypothesis

Riemann's statement

In [49], Riemann states

> *"One now finds indeed approximately this number of real roots within these limits, and it is very probable that all roots are real. Certainly one would wish for a stricter proof here; I have meanwhile temporarily put aside the search for this after some fleeting futile attempts, as it appears unnecessary for the next objective of my investigation."*

Here, Riemann is discussing a variant of the Riemann zeta function, whose roots are real rather than on the critical line $\Re(s) = 1/2$. This variant is known as the Riemann Xi function [40], which we discuss later in this chapter.

Hilbert's statement

In 1900, David Hilbert presented a list of 23 problems at the International Congress of Mathematicians [34]. Proving the Riemann hypothesis is mentioned as part of problem 8 in the list. In Hilbert's words:

> "*...it still remains to prove the correctness of an exceedingly important statement of Riemann, viz., that the zero points of the function* $\zeta(s)$ *defined by the series*
>
> $$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$$
>
> *all have the real part* $1/2$ *, except the well-known negative integral real zeros. As soon as this proof has been successfully established, the next problem would consist in testing more exactly Riemann's infinite series for the number of primes below a given number and, especially, to decide whether the difference between the number of primes below a number* x *and the integral logarithm of* x *does in fact become infinite of an order not greater than* $1/2$ *in* x."

Many of Hilbert's problems have been resolved, at least in part. However, the Riemann hypothesis question has remained unsolved, prompting The Clay Mathematics Institute, 100 years after Hilbert's problems, to list it as one of the *Millennium Problems.*

Millennium Problems - The Clay Mathematics Institute

The proof of the Riemann hypothesis is listed as problem 4 of the seven Millennium Problems. These problems were listed in 2000 by the Clay Mathematics Institute to document some of the most difficult problems facing mathematicians at the end of the second millennium. The official statement of the problem of the Riemann hypothesis is listed in [12], and reads:

Riemann hypothesis. *The nontrivial zeros of* $\zeta(s)$ *have real part equal to* $\frac{1}{2}$.

Bombieri [12] goes on to say,

"*In the opinion of many mathematicians the Riemann hypothesis, and its extension to general classes of* $L-$*functions, is probably today the most important open problem in pure mathematics.*"

Solving any of the Millennium Problems has a prize of one million US dollars. Since the listing of the problems, only one of the seven has been solved.

6.3 Selected equivalent statements of the Riemann hypothesis

The work [17], lists several equivalent statements of the Riemann hypothesis. In fact, it is noted that the statement can be reformulated into many diverse and seemingly unrelated ways. The following is a summary of some of these equivalent statements.

The probability of having even or odd distinct prime factors

One of the equivalent statements of the Riemann hypothesis is eloquently illustrated in [64]. Herbert S. Wilf, upon his commencement of being editor of *The American Mathematical Monthly*, wrote a greeting to the readers of the publication, in which he gave a form of the Riemann hypothesis he claimed should be explanable to a gifted group of high school tenth graders. Here, Wilf defines *squarefree* positive integers as the set of natural numbers, minus any number that is divisible by the square of an integer larger than 1. (It follows that the squarefree positive integers are each a product of distinct primes.) He then divides the squarefree positive integers into two nonintersecting subsets he calls *red numbers* and *blue numbers*, with the red numbers being the squarefree positive integers having an even number of prime factors and the blue numbers being those with an odd number of prime factors. With this in mind, the Riemann Hypothesis is stated as

Conjecture 6.3. *Fix $\varepsilon > 0$. Then there exists N such that for all $n > N$ the number of blue numbers in $[1, n]$ does not differ from the number of red numbers in $[1, n]$ by more than $n^{1/2+\varepsilon}$.*

In simple language, this means that the number of blue numbers and the number of red numbers does not differ by much more than the square root of the total number of numbers in any such collection. The previous conjecture is also stated in [17] in terms of the *Liouville function*, defined as

$$\lambda(n) = (-1)^{\omega(n)}, \tag{6.1}$$

where $\omega(n)$ is the number of, not necessarily distinct, prime factors of n, counted with multiplicity. For example, $\lambda(2) = \lambda(3) = \lambda(5) = \lambda(7) = \lambda(8) = -1$, while $\lambda(1) = \lambda(4) = \lambda(6) = \lambda(9) = \lambda(10) = 1$. With this in mind, the Riemann hypothesis is equivalent to the following statement:

For every fixed $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \lambda(k)}{n^{1/2+\varepsilon}} = 0.$$

This can be translated into the statement that an integer has equal probability of having an odd number or an even number of distinct prime factors. Another way of saying this is that the sequence

$$\{\lambda(k)\}_{k=1}^{\infty} = \{1, -1, -1, 1, -1, 1, -1, -1, 1, 1, -1, -1, -1, 1, 1, 1, -1, -1, -1, -1, 1, \ldots\},$$

has the property that the difference between the number of -1's and 1's is, again, not much larger than the square root of the number of terms.

Other equivalent statements

In ([17], pp. 45-52), many statements are listed as equivalent forms of the Riemann hypothesis. We now list a few of these having relevance to ideas mentioned earlier in this work. First, we list a relation about the Riemann zeta function evaluated at the odd integers $\{3, 5, 7, \ldots\}$.

Theorem 6.4. *The Riemann hypothesis holds if and only if*

$$\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n!\zeta(2n+1)} = \mathcal{O}(x^{-1/4}), \tag{6.2}$$

as $x \to \infty$.

Another equivalence is a remarkable integral formula that connects the zeros of the Riemann zeta function in the critical strip to the Euler-Mascheroni constant, γ.

Theorem 6.5. *The Riemann hypothesis holds if and only if*

$$\int_0^{\infty} \frac{1-12t^2}{(1+4t^2)^3} \int_{1/2}^{\infty} \ln |\zeta(\sigma + it)| \, d\sigma \, dt = \frac{(3-\gamma)\pi}{32}. \tag{6.3}$$

Victor Moll [42] adds that evaluating this integral "*might be hard.*" A third equivalence is given in [51] and concerns the family of curves $|\zeta(\sigma + it)|$.

Theorem 6.6. *The Riemann hypothesis is true if*

$$\frac{\partial |\zeta(s)|}{\partial \sigma} < 0, \tag{6.4}$$

for $t > 2\pi$ *and* $0 \leq \sigma < 1/2$ *with* $s = \sigma + it$.

This statement is equivalent to saying that $|\zeta(s)|$ is monotone decreasing for $t > 2\pi$ and $0 \leq \sigma < 1/2$, and visual evidence of this can be seen in the figures below.

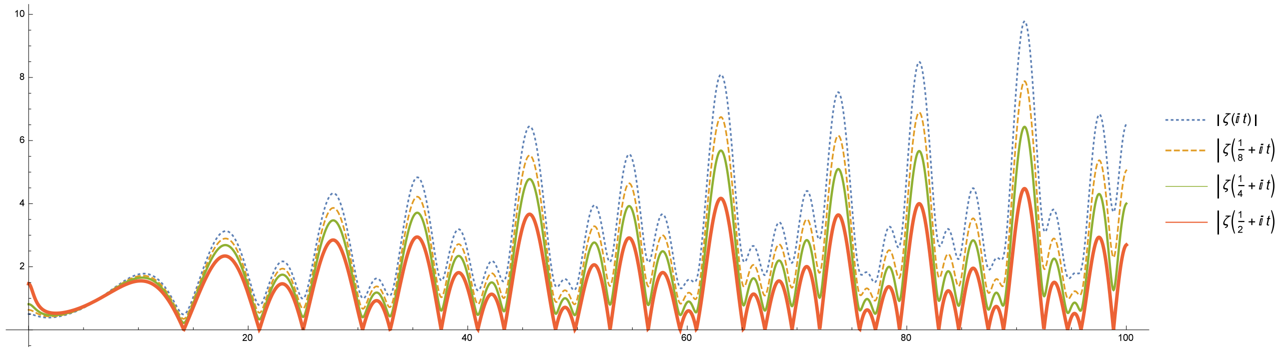

Figure 6.1: Plots of $|\zeta(\sigma + it)|$ with $\sigma = \{0, 1/8, 1/4, 1/2\}$ and $0 \leq t \leq 100$.

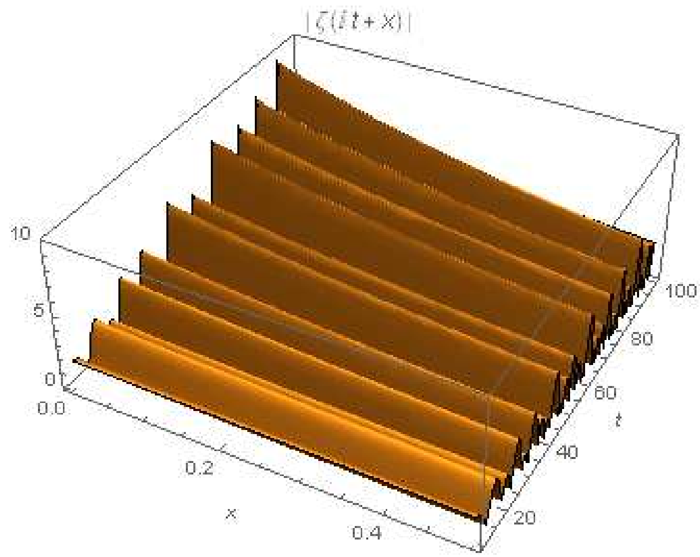

Figure 6.2: $|\zeta(x + it)|$ with $0 \leq x \leq 1/2$ and $10 \leq t \leq 100$.

6.4 Functions used to study the nontrivial zeros of $\zeta(s)$

Riemann-Siegel Function

The Riemann-Siegel function is [17]

$$Z(t) = e^{i\vartheta(t)}\zeta\left(\frac{1}{2} + it\right), \tag{6.5}$$

where $\vartheta(t) = \arg\left(\Gamma\left(\frac{1}{4}+\frac{1}{2}it\right)\right) - \ln\left(\frac{\pi}{2}\right)t$, the argument is chosen so that $Z(t)$ is real valued, and $\arg\left(\Gamma\left(\frac{1}{4}+\frac{1}{2}it\right)\right)$ assumes its principal value ([47], p.606). The function $Z(t)$ has the property of changing sign infinitely many times. Because $|Z(t)| = |\zeta\left(\frac{1}{2}+it\right)|$, it follows that $Z(t)$ is zero whenever $\zeta\left(\frac{1}{2}+it\right)$ is zero. Therefore, zeros of $\zeta\left(\frac{1}{2}+it\right)$ can be identified by recording where $Z(t)$ changes sign.

ξ and Ξ functions

In [49], Riemann produced a modified version of the Riemann zeta function

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s), \tag{6.6}$$

which is valid for all $s \in \mathbb{C}$. The function $\xi(t)$ found in the translation from David Wilkins in [49] was changed in the literature of Edmund Landau [40] to Ξ, with $\Xi(t) = \xi(\frac{1}{2}+it)$, giving

$$\Xi(t) = -\frac{1}{2}\left(t^2+\frac{1}{4}\right)\pi^{it/2-1/4}\Gamma\left(\frac{1}{4}-\frac{1}{2}it\right)\zeta\left(\frac{1}{2}-it\right). \tag{6.7}$$

The right hand side of (6.7) was shown by Riemann [49] to be real valued for $t \in \mathbb{R}$. In fact, by denoting

$$\sum_{n=1}^{\infty} e^{-n^2\pi x} = \psi(x), \tag{6.8}$$

one can prove ([25], p.17)

$$\Xi(t) = \xi\left(1/2+it\right) = 4\int_1^{\infty} \frac{d[x^{3/2}\psi'(x)]}{dx} x^{-1/4}\cos\left(\frac{t}{2}\ln x\right)dx. \tag{6.9}$$

It is also noted in ([25], p.17) that

$$\xi(s) = \sum_{n=0}^{\infty} a_{2n} \left(s - \frac{1}{2}\right)^{2n}, \tag{6.10}$$

where

$$a_{2n} = 4 \int_1^{\infty} \frac{d[x^{3/2}\psi'(x)]}{dx} x^{-1/4} \frac{(\frac{1}{2}\ln x)^{2n}}{(2n)!}\, dx. \tag{6.11}$$

If we let $s = 1/2 + it$ in (6.10), we get

$$\xi(1/2 + it) = \Xi(t) = \sum_{n=0}^{\infty} a_{2n} \left(it\right)^{2n} = -\sum_{n=0}^{\infty} (-1)^n a_{2n} t^{2n}; \tag{6.12}$$

therefore, $\Xi(t)$ is real-valued for $t \in \mathbb{R}$. The Ξ function has the property

$$\Xi(-t) = \Xi(t), \tag{6.13}$$

valid for all $t \in \mathbb{R}$. It is the Ξ function that Riemann was referring to in [49], stating

> "*One now finds indeed approximately this number of real roots within these limits, and it is very probable that all roots are real.*"

This is equivalent to the statement that the complex valued function $\Xi(t + i\sigma)$ has $|\Xi(t + i\sigma)| > 0$ for all $\sigma \neq 0$. Plots of $\Xi(t)$ for $t \in \mathbb{R}$ and $|\Xi(t + i\sigma)|$ for $t, \sigma \in \mathbb{R}$ are included below.

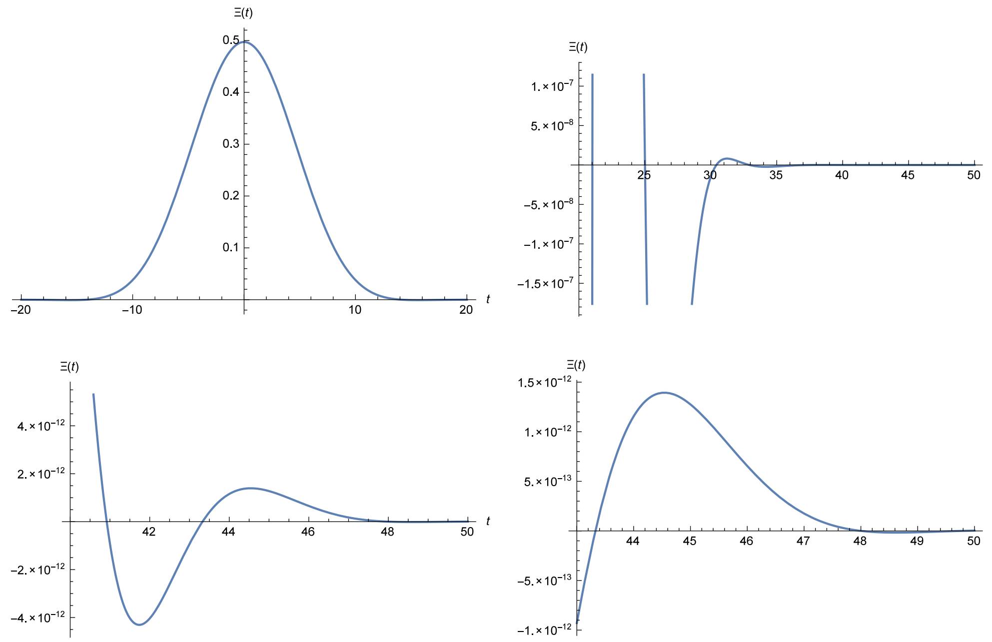

Figure 6.3: Plots of $\Xi(t)$ on varying intervals of t.

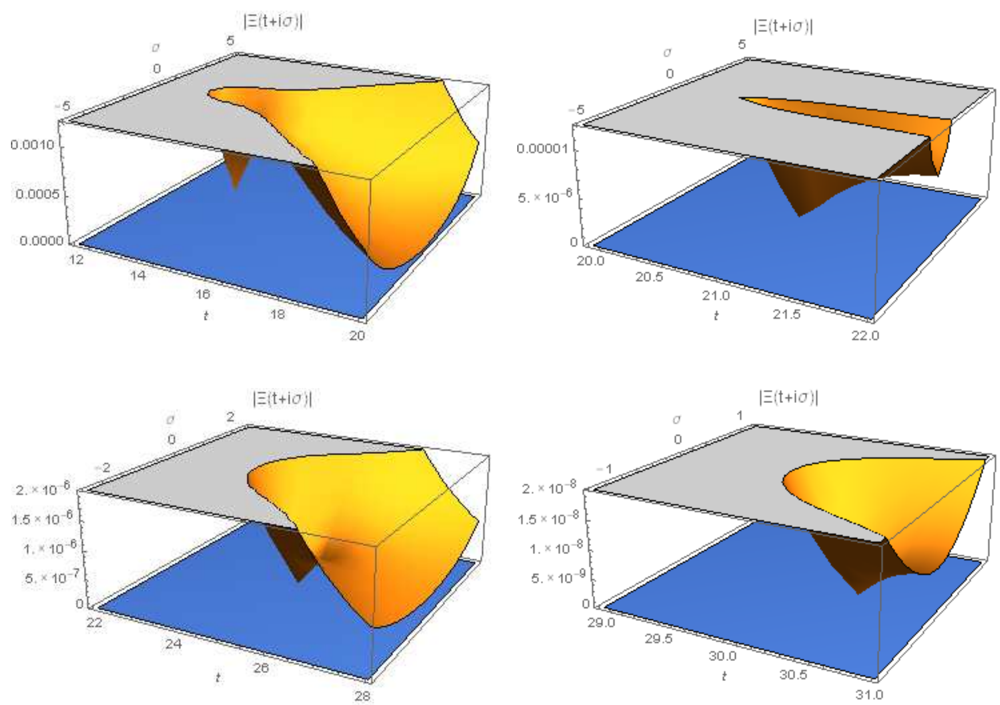

Figure 6.4: Plots of $|\Xi(t+i\sigma)|$ on varying intervals of t, σ.

The function $\Xi(t)$ for $t \in \mathbb{R}$, approaches zero rapidly as $t \to \infty$, as illustrated in the preceding figures. The zeros of $\Xi(t)$ are also the zeros of $\zeta(s)$ along the line $\Re(s) = 1/2$. The first few of these nontrivial zeros are visible in the preceding figures, while values for more nontrivial zeros are given in Appendix A.

6.5 Attempts at proving the Riemann hypothesis

In this section we summarize a few failed attempts at proving the Riemann hypothesis (see, e.g., [17], [50]), which predate the announcement of the *Millenium Problems*. Since the year 2000, with a prize of one million dollars on the line, there have been numerous recent attempts at proof, all without success.

The Prime Number Theorem and Riemann's "conclusion"

In Riemann's celebrated and aforementioned paper *On the number of primes less than a given quantity*, Riemann's hypothesis leads to a conclusion about $\pi(x)$, the function that counts the number of primes less than $x \in \mathbb{R}$. The conclusion, if proved, would be an improvement of the estimate given by the *Prime Number Theorem*, which states

Theorem 6.7. $\dfrac{x}{\ln x}$ *is a good approximation to* $\pi(x)$ *for* x *sufficiently large.*

The improved estimate conjectured by Riemann uses the *offset logarithmic integral*, defined as

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\ln t}. \tag{6.14}$$

At the close of the paper, Riemann states [49]

> "*The known approximating expression $F(x) = Li(x)$ is therefore valid up to quantities of the order $x^{1/2}$ and gives somewhat too large a value... A more regular behaviour than that of $F(x)$ would be exhibited by the function*

$f(x) = F(x)+\frac{1}{2}F(x^{1/2})+\frac{1}{3}F(x^{1/3})+\cdots$, which already in the first hundred is seen very distinctly to agree on average with $Li(x) + \ln\xi(0)$."

It is known ([12], [17]), that as $x \to \infty$, if

$$\pi(x) - \mathrm{Li}(x) = \mathcal{O}(x^{1/2}\ln x), \tag{6.15}$$

then the Riemann hypothesis is true. This, of course, has yet to be shown.

Hardy-Littlewood zeta-function conjectures

In 1914, Godfrey Harold Hardy proved that the function $\zeta(1/2+it)$ has infinitely many real zeros [30]. Then in 1921, Hardy and John Edensor Littlewood published a paper containing the *Hardy-Littlewood zeta-function conjectures*, which are two conjectures concerning the density and distance between zeros of $\zeta(1/2+it)$. Let $N(T)$ be the total number of real zeros of $\zeta(1/2+it)$, and $N_0(T)$ the total number of zeros of odd order of the function $\zeta(1/2+it)$ in the interval $(0,T]$. Then the two conjectures are

1. For any $\varepsilon > 0$, there exists $T_0(\varepsilon) > 0$ such that when $T \geq T_0(\varepsilon)$ and $H = T^{1/4+\varepsilon}$, the interval $(T, T+H]$ contains a zero of odd order.

2. For any $\varepsilon > 0$, there exists a $T_0(\varepsilon) > 0$ and $c_\varepsilon > 0$ such that $N_0(T+H)-N_0(T) \geq c_\varepsilon H$ holds when $T \geq T_0(\varepsilon)$ and $H = T^{0.5+\varepsilon}$.

Neither of these has been proven or disproven.

Lindelöf's function and hypothesis

The Lindelöf function ([25], §9), denoted by $\mu(\sigma)$, is defined as the infimum of numbers A such that $|\zeta(\sigma+it)|t^{-A}$ is bounded as $t \to \infty$. *Lindelöf's hypothesis* states that $\mu(\sigma)$ is zero for $\sigma \geq 1/2$ and $1/2 - \sigma$ for $\sigma \leq 1/2$. Since it is known that $\mu(\sigma)$ is convex (i.e. concave up), this statement is equivalent to the conjecture that $\mu(1/2) = 0$. This hypothesis is implied by the Riemann hypothesis, but also has yet to be proved. Currently it is known that $\mu(1/2) < 53/342$. (Note: this function μ is not related to the aforementioned *Möbius function.*)

Other failed proofs

In [50], a story is given about the famous mathematician John Nash, who is the subject of the book *A Beautiful Mind*, and film of the same name. In 1959, the thirty-year-old Nash gave a talk announcing his proof of the Riemann hypothesis. His idea, previously shared with colleagues, used *pseudoprimes*, which are numbers that are not prime, but behave like primes. However, Nash's speech was a dreadful argument that made no sense, and it turns out that this period of time coincided with the height of Nash's struggle with schizophrenia.

In 1943, an article in *Time* magazine [17], [50] announced a (failed) *disproof* of the Riemann hypothesis that was found to be a "false alarm." An excerpt from the April 30 issue reads:

> "*One day last month electrifying news arrived at the University of Chicago office of Dr. Adrian A. Albert, editor of the Transactions of the American Mathematical Society. A wire from the society's secretary, University of Pennsylvania professor John R. Kline, asked editor Albert to stop the presses; a paper disproving the Riemann Hypothesis was on the way. Its author: Professor Hans Adolf Rademacher, a refugee German mathematician now at Penn.*
>
> *On the heels of the telegram came a letter from Professor Rademacher himself reporting that his calculations had been checked and confirmed by famed mathematician Carl Siegel of Princeton's Institute for Advanced Study. Editor Albert got ready to publish the historic paper in the May issue. US mathematicians, hearing the wildfire rumor, held their breath. Alas for drama, last week the issue went to press without the Rademacher article. At the last moment the professor wired meekly that it was all a*

> *mistake; on rechecking, mathematician Siegel has discovered a flaw (undisclosed) in the Rademacher reasoning. US mathematicians felt much like the morning after a phoney armistice celebration. Said editor Albert, 'The whole thing certainly raised a lot of false hopes.'* "

It was subsequently found that Rademacher's error stemmed from the fact that the complex logarithm does not have a uniquely defined value.

Beginning in 1986, Louis de Branges [23], who had earlier proved the *Bieberbach conjecture* in 1985 [50], began publishing papers announcing his proof of the Riemann hypothesis. However, most mathematicians who have read them, feel that De Branges's papers seem to have no actual proofs. As an example, Conrey and Li [21], in 1998, proved a counterexample to de Branges's approach, essentially showing that the theory developed in his papers is not feasible.

The Riemann hypothesis is true for finite fields

In 2008, Jasbir Chahal and Brian Osserman proved that the Riemann hypothesis is true for finite fields on elliptic curves [19]. An outline of the proof is as follows.

First, take a finite field $\mathbb{F}_q$ which has $q = p^r$ elements with p a fixed prime (but not 2 or 3), and $r \in \mathbb{N}$. Then take numbers $\{a, b\} \in \mathbb{F}_q$ such that $y^2 = x^3 + ax + b$ is a non-singular elliptic curve, E, meaning $4a^3 + 27b^2 \neq 0$. Then the zeta function for E is

$$Z_E(t) = \frac{1 - a_q(E)t + qt^2}{(1-t)(1-qt)}, \tag{6.16}$$

where $a_q = q - N_q$, with N_q the number of solutions of E in F_q. It is noted in [19] that the formula (6.16) is the generalized zeta function for E by first writing

$$\zeta_K(s) = \exp\left(\sum_{m=1}^{\infty} N_m(C)\frac{q^{-ms}}{m}\right), \tag{6.17}$$

where K is a function field over an integral domain for a curve C.

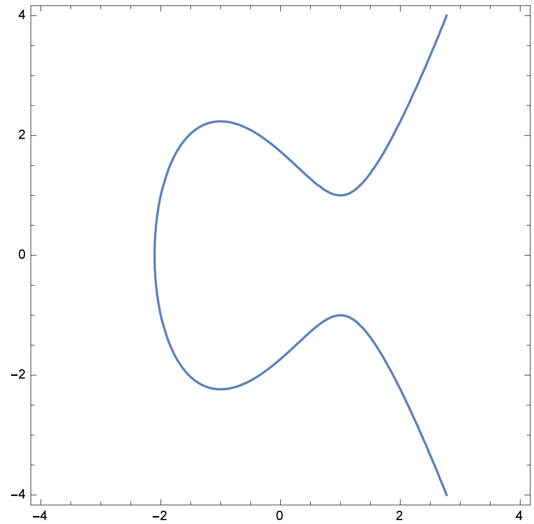

Figure 6.5: A non-singular elliptic curve has no cusps, self-intersections, or isolated points.

The Riemann hypothesis for E is the statement that $Z_E(q^{-s}) = 0$ implies $\Re(s) = 1/2$. The proof involves the claim that *Hasse's Theorem*, first proved in [32],

$$|N_q - q| \leq 2\sqrt{q} \tag{6.18}$$

is equivalent to the Riemann hypothesis for E. In fact, taking $u = qt$ in (6.16) we can write $f(u) = u^2 - a_q u + q$, where it is noted that if $Z_E(q^{-s}) = 0$, then q^s must be a root of $f(u)$. If $f(u)$ has its discriminant $a_q^2 - 4q \leq 0$, then the two roots u_1, u_2 of $f(u)$ are either equal or complex with $|u_1| = |u_2|$. Since the constant term q in $f(u)$ is the product $u_1 u_2$, it follows that (6.18) is true if and only if $|u_1| = |u_2| = \sqrt{q}$. So $Z_E(q^{-s}) = 0$ if and only if $|q^s| = \sqrt{q}$, which implies that $\Re(s) = 1/2$.

We conclude this chapter with a poem by Tom Apostol, found in [29].

Where Are the Zeros of Zeta of s?
by Tom Apostol

Where are the zeros of zeta of s?
G.F.B. Riemann has made a good guess;
They're all on the critical line, saith he,
And their density's one over 2pi log t.

This statement of Riemann's has been like a trigger
And many good men, with vim and with vigor,
Have attempted to find, with mathematical rigor,
What happens to zeta as mod t gets bigger.

The efforts of Landau and Bohr and Cramer,
And Littlewood, Hardy and Titchmarsh are there,
In spite of their efforts and skill and finesse,
In locating the zeros there's been little success.

In 1914 G.H. Hardy did find,
An infinite number that lay on the line,
His theorem however won't rule out the case,
There might be a zero at some other place.

Let P be the function pi minus li,
The order of P is not known for x high,
If square root of x times log x we could show,
Then Riemann's conjecture would surely be so.

Related to this is another enigma,
Concerning the Lindelof function mu (sigma)
Which measures the growth in the critical strip,
On the number of zeros it gives us a grip.

But nobody knows how this function behaves,
Convexity tells us it can have no waves,
Lindelof said that the shape of its graph,
Is constant when sigma is more than one-half.

Oh, where are the zeros of zeta of s?
We must know exactly, we cannot just guess,
In order to strengthen the prime number theorem,
The integral's contour must not get too near 'em.

Chapter 7: Conclusion

We do not claim to have written an exhaustive exposition of the Riemann zeta function. However, what is written here represents an introduction of some of the deep mathematics surrounding its study. Our investigation into the history of the Riemann zeta function began as the study of the harmonic series in the 13th and 14th centuries, and later progressed to the study of p-series during the time of Euler in the 18th century. In the next century, the Riemann zeta became so-named after Bernhard Riemann, and still fascinates mathematicians today. We have journeyed from Baroque architecture, through Euler's solution of the *Basel problem*, and then into the groundbreaking paper of Riemann where the function we write about was proved to have its analytic continuation to the complex numbers.

In this work, we have shown other methods of analytic continuation of $\zeta(s)$, besides Riemann's, and we have also outlined the calculation of zeta constants. Additionally, we have introduced the reader to some generalizations of the function and included references for further study. A list of identities related to the zeta function was selected, and their significance was noted. Furthermore, we have given a brief discussion of the Riemann hypothesis and its importance in the study of the distribution of prime numbers. In our discussion, we highlighted equivalent statements of the Riemann hypothesis that are seemingly unrelated.

The study of Riemann's zeta function has bridged many areas of mathematics and science including: complex analysis, number theory, statistics, group theory, and physics. Before Riemann's work *On the number of prime numbers less than a given quantity*, some of these areas seemed to be unrelated. Perhaps mathematicians must make even more connections among seemingly unrelated topics before someone finally finds the answer to the million-dollar question "*Where are the zeros of zeta of s?*"

Appendix A: The first 50 nontrivial zeros of $\zeta(s)$.

For $s = \sigma + it$, the first 50 nontrivial zeros of $\zeta(s)$ occur on the real line $\sigma = 1/2$ with values of t given in the table below, rounded to 6 decimal places.

n	t_n	n	t_n
1	14.134725	26	92.491899
2	21.022040	27	94.651344
3	25.010858	28	95.870634
4	30.424876	29	98.831194
5	32.935062	30	101.317851
6	37.586178	31	103.725538
7	40.918719	32	105.446623
8	43.327073	33	107.168611
9	48.005151	34	111.029536
10	49.773832	35	111.874659
11	52.970321	36	114.320221
12	56.446248	37	116.226680
13	59.347044	38	118.790783
14	60.831779	39	121.370125
15	65.112544	40	122.946829
16	67.079811	41	124.256819
17	69.546402	42	127.516684
18	72.067158	43	129.578704
19	75.704691	44	131.087689
20	77.144840	45	133.497737
21	79.337375	46	134.756510
22	82.910381	47	138.116042
23	84.735493	48	139.736209
24	87.425275	49	141.123707
25	88.809111	50	143.111846

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