An interaction with π

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Abstract. In this article, we present a brief history of π with a minimal prerequisites of high school mathematics. We present some major milestones in its journey from mathematicians to modern-day computer scientists. All this is done in a very playful manner, as a fictional dialogue between π and a layman.

Keywords. Rational number, Decimal expansion, Algebraic number.

Mathematics Subject Classification. 00A05, 15A06.

1. Introduction

Since antiquity, circles and squares have been the most important shapes, and thus so are the numbers π and $\sqrt{2}$. The Pythagoreans established that the latter is not a fraction, and the proof is quite elementary. Mathematicians wondered for more than two thousand years whether π is also a fraction or not. This led to a quest for the decimal expansion of this marvellous number, which still continues for different reasons. It includes approximations through Archimedean polygons, Newtonian calculus, and computer algorithms.

In this article, we will briefly outline the history of π, along with various other interesting ideas associated to it. We will discuss its journey from mathematicians to modern computer scientists. It will be done in a playful interactive manner.

In the next section, we are going to present a teenager-level chirpy dialogue between π and a layman Alpha. The target readership is anyone with a basic high school level understanding of π and having an open mind to learn more about it. A common high school student above 15 years will understand it easily.

One of the aims of this write-up is to explain the essence of questioning; and that sometimes it takes decades and even centuries to answer some simple looking questions of mathematics. With a few examples presented in this article, a layperson will also glimpse the nature of research in pure mathematics.

2. Alpha Meets Pi

There are often many faces whom we know but do not recognize. Alpha also believed that he knew Pi, due to which he never bothered to interact with this marvellous character. Once on the international Pi day, which is celebrated on March 14 every year due to the approximation 3.14 of π, he happened to spend some time with π. Here we present some of the highlights of their conversation.

ALPHA: (attempts to initiate a conversation) I think I have seen you somewhere?

PI: That's very likely. Maybe in some conference/workshop on Mathematics, Engineering or Science. Most of the new and old formulas can't survive without me.

ALPHA: In my eagerness of knowing more about you, once I went to watch a movie 'Life of Pi'. But that turned out to be some sea boat adventures of a teenager named Pi with a tiger.

PI: (mockingly) Very funny!

ALPHA: Are you equal to 22/7? (Pi does not respond and makes an angry face. In surprise and excitement, Alpha continues) But I always believed it.

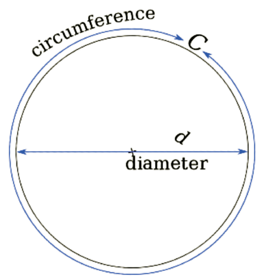
PI: (responds bluntly) Then it is your problem, not mine. It is indeed extremely unfortunate that almost everyone claims to know me, but only a few question about this basic approximation 22/7.

ALPHA: (trying to control his embarrassment) Today I want to listen to you only. Tell me something about your life. How should anyone approach you?

PI: (notoriously) To meet me, divide the circumference of a circle with its diameter. I'll be there.

ALPHA: You are simply rephrasing the formula about the circumference of a circle.

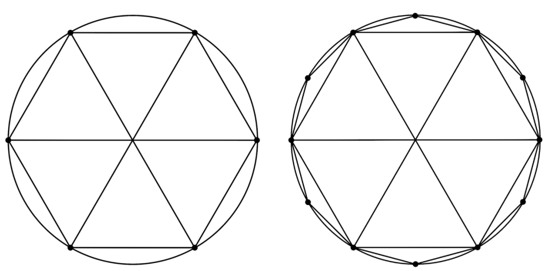
Pi: My fans know me as this ratio since ages. This ratio is same for all circles, whether it is the size of a coin or a circular playground. That incited their keen interest in me.



Alpha: Okay. Do you have some early memories to share?

Pi: For most of the people, I am still 22/7 or 3.14, whereas around 250 BC, Archimedes had already established that my value lies strictly between these two numbers. He considered a regular hexagon inside a circle of unit radius. The perimeter of this hexagon provided the first approximation to the circumference of the circle. Hence

$$\pi \sim \frac{(1 \times 6)}{2} = 3.$$



Then he doubled the number of sides and approximated again. He continued this process up to 96 sides and established that my value lies between 223/71 and 22/7.

Alpha: (getting excited), Interesting! Wasn't there anyone interested in you before Archimedes? When is your birthday?

Pi: (thoughtfully). No one has any clue about my birthday. Around 2000 BC, Babylonians were content with 25/8 as my value. Around the same time or a bit earlier, Egyptians were assuming that a circle with diameter 9 units has the same area as the square with side 8 units. So they used the fraction 256/81 for me. Most in the antiquity were content with the approximation 3. The same approximation has been mentioned in the Old Testament.

Alpha: (asks with curiosity) Then what are you among these; 22/7, oh sorry, I mean 3, 25/8, or 256/81?

Pi: (ignores the question and further presents an age old question) Ancient Greeks wondered whether I am even a fraction; a ratio of two integers?

Alpha: Are you?

Pi: (proudly) This mystery couldn't be revealed for the next twenty-four centuries.

Alpha: Twenty four centuries? (Alpha continues shockingly) A single question remained unanswered for a mega span of twenty four centuries! Unbelievable!

Pi: (philosophizes) Often a serious query and its solutions do not emerge from a vacuum. Every question and its answers come along a lot of paradigms, for which sometimes one has to wait for centuries.

Such questions are not like the thorns of an acacia in the backyard, which must be removed immediately, or the acacia itself needs to be uprooted. Rather, they are seeds which should be stored until the ground is ready for them to germinate. Scattering seeds on the roads is of no use.

Instead of being satisfied with the ambiguous or incomplete answers, a genuine seeker first carves out the question to strike better. In case of no satisfactory answers, an inquisitive seeker passes on his experience to the future generations. This is the real character of true scholars.

Alpha: (Alpha thought that we never think twice while issuing categorical statements about the big questions like origin of the universe, and here geeks spent twenty-four centuries on a single question. He asked cautiously) Keep it simple. Didn't anyone ask this question before the Greeks?

Pi: From the writings of the Babylonians and Egyptians about over four thousand years ago, it appears that they too were struggling with this question. However, this question was not explicitly stated before Greeks. Look, it took almost twenty centuries for a clear presentation of this query!

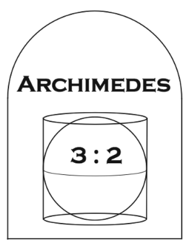
ALPHA: Then what was the answer?

PI: It became clear to Archimedes that I was not 22/7. He never claimed to know me completely. An important aspect of his method through polygonal approximations was that it could be used to get arbitrary closer to me, by obtaining any number of decimal digits though polygonal approximations. This was the only way to get closer to me for about eighteen centuries after Archimedes.

ALPHA: Is there anyone else who used this method to approach you?

PI: Many. There are two remarkable ones. About 7 centuries after Archimedes, Tsu Chung Shih approximated me as 355/113. Then, about 11 centuries later, in 1610, Ludolph van Ceulen got to know me till 34 decimal places. He devoted his entire life to obtain this approximation upto 34 decimal digits, which were later used to adorn his tombstone. His grave was once lost, but was found again in 2000.

PI: Do you know that two tombstones are very special in the history of mathematics?



ALPHA: (curiously) Really? Who's the second one?

PI: Archimedes.

ALPHA: What is so special about his tombstone?

PI: (explains) It has a sphere inscribed in a cylinder. Archimedes had proved that such a cylinder is one and a half times as big as a sphere, in volume as well as surface area.

ALPHA: (Alpha gets lost for these lovers of mathematics, controls himself, and then asks) Do all mathematicians use that Archimedean method?

PI: (responds in a confident tone) Absolutely not. The second phase of my life began with the advent of Newtonian Calculus. It would not be an exaggeration to say that within a few years after this, various series came into existence for my service, which made it easier to know my decimal digits. For example,

$$\frac{\pi}{4} = \tan^{-1}(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

$$\begin{aligned}\frac{\pi}{4} &= \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} \\ &= \left(\frac{1}{2} - \frac{1}{3.2^3} + \frac{1}{5.2^5} - \dots\right) \\ &\quad + \left(\frac{1}{3} - \frac{1}{3.3^3} + \frac{1}{5.3^5} - \dots\right).\end{aligned}$$

Indian mathematician Madhava presented the first series above in his Yuktibhasa, and it is now called the Madhava-Leibniz series.

(Pi paused a while and asked) Tell me one thing. Given a choice, which series would you use to approach me?

ALPHA: (responds immediately) The first series appears easy. Why to struggle with bigger denominators?

PI: and don't you have to face big denominators in the first series eventually...by large numbers? In that too, you have to divide one by every big odd number!

ALPHA: (nods his head in agreement and asserts) That's right. Even with the first series, if I can't escape the burden of big denominators, then why bother with unnecessary calculations, and so the second series will be better.

PI: (sets the ball rolling again and asserts) These denominators are nothing as compared to the series for π obtained by Machin's formula

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{239}.$$

It was used to obtain my first hundred digits in 1706. In 1873, Shank obtained my first 707 digits using it; which had an error after 527th decimal place. This feat took him over 15 years.

Newton too once sheepishly admitted that he, too, did not know how many tactics he had adopted to approach me, having no other business at that time.

ALPHA: (gets irritated by so much of self-praise by Pi and mocks) What's so special about your decimal digits?

PI: (answers sarcastically) Ask this to my lovers, who have been longing for these approximations for centuries.

ALPHA: (expressing his irresistible curiosity) But still ...

PI: (becomes serious and responds) You must be knowing $\sqrt{2}$, length of the diagonal of the unit square. Since circles and squares are the most important geometric shapes, myself and $\sqrt{2}$ have been in the limelight for ages. A student of Pythagoras, Hippasus (530BC-450BC), proved that $\sqrt{2}$ is not a fraction. It was against the beliefs of the Pythagorean sect. As a result, Hippasus became the first martyr in the documented history.

(Pi paused a while and continued) The followers of Pythagoras believed that every number could be written as a fraction. But after the proclamation of Hippasus, $\sqrt{2}$ was called irrational, which literally means illogical in English. Now the question arises: Am I also irrational?

ALPHA: Are you? If yes, can it be proved like $\sqrt{2}$.

PI: (pauses for a while and explains) The method that worked for $\sqrt{2}$ could not be used for this question. If some block of my decimal digits had a repetition after a certain stage, then it would have been possible to write me as a fraction; such as

$$0.333\ldots. = \frac{1}{3} \text{ and } 0.64646464\cdots = \frac{64}{99}, \text{ etc.}$$

That's why for centuries geeks have been struggling with my decimal digits to find an exact fraction for me instead of 22/7 and 355/113.

ALPHA: (curiously) Could they find it?

PI: Of course not.... Lambert and Legendre proved in 1768 that I am not equal to any fraction. Then, in 1882, Lindemann proved that I am non-algebraic; also called transcendental.

This also answered the age old question of Greek geeks that circle can't be squared by ruler and compass.

ALPHA: (getting a bit irritated with the term transcendental, asks) A transcendental number? What's so transcendental about you?

PI: (responds in a sneering tone and mischievous smile) Independent transcendental numbers like me are rare; can be counted on the finger-tips.... even though every algebraic number becomes transcendental by joining me.

ALPHA: (trying to change the topic) How did you get this name 'Pi'?

PI: (with a little seriousness) The modern symbol π was first used for me by William Jones in 1706. This Greek letter was chosen as it is pronounced like 'P' in English, and the word perimeter had been used for thousands of years for the circumference of a circle. Surprisingly, in both Greek and English, it is the sixteenth letter.

(Saying this, Pi again sneers and then becomes serious) Euler used this symbol for me in 1737, and since then this has been my identity.

ALPHA: Did the advent of computer make any difference to you?

PI: (in a cheerful excitement, Pi explains) With this I entered the third phase of my glorious life. In 1949, for the first time with the help of a computer, my first 2037 decimal digits were found in 70 hours. In a few years, the available number of my digits reached millions and then trillions. A variety of Mathematics and Logic was used for these calculations. Principles like Machin's Formula continued to be used until 1970.

ALPHA: Is there any idea better than the Machin's Formula, to calculate your digits?

PI: (responds thoughtfully) Srinivasa Ramanujan presented a series for me in 1910.

$$\frac{1}{\pi} = \frac{\sqrt{8}}{99^2} \sum_{n=0}^{\infty} \frac{(4n)!}{(4^n n!)^4} \frac{1103 + 26390n}{99^{4n}}. \tag{1}$$

Just the first four terms of this series provide my first 14 decimal digits. Each successive term adds roughly eight more correct digits.

ALPHA: Really?

PI: Yes. Gosper (1985) used it to compute my first 17 million digits. A similar and better series was provided by Chudnowsky brothers in 1988.

$$\frac{1}{\pi} = 12 \sum_{n=0}^{\infty} \frac{((-1)^n (6n)!)}{(3n)!(n!)^3} \times \frac{[13591409 + 545140134n]}{640320^{3n+3/2}}.$$

The first seven terms of this series produce my 13, 27, 26, 55, 69, 84 and 98 decimal digits, respectively. Using this series, many world records on my digits are being made till today.

Suppose you want to get my 100th decimal digit, would it be necessary to first obtain my first 99 digits? (Pi asks in a mischievous tone.)

ALPHA: It seems so. Isn't it?

PI: (proudly) In 1997, Plouffe established that any digit in my decimal representation can be computed without knowing the preceding digits. Various iterative algorithms have emerged in the last decade to obtain my decimal digits.

ALPHA: What is the latest world record on your digits?

PI: Recently in August 2021, Team DAViS with its 1 TB RAM, 510 TB Hard Disk supercomputer achieved more than my 62.8 Trillion decimal digits in 108 days.

ALPHA: (astonishes at this and suddenly remembers something, and asks) Wait a second! What could be the purpose of finding so many decimal digits, when it has already been proved that you are not a fraction? Then why still…

PI: (interrupts and counter questions) Do you know how many decimal digits are required to measure the circumference of our Milky Way galaxy?

(Alpha nods and she continues) My first 40 decimal digits are sufficient to compute this circumference to an error less than the size of a proton.

ALPHA: Then why the fuss? (Alpha gets irritated).

PI: (replies sneeringly) I am honoured to test the integrity of computer hardware and software. Without my billions and trillions of digits, how will you check the efficiency of your modern machines?…How will you test a supercomputer without my billions and trillions of 'random' numbers?

(noticing Alpha's restlessness, Pi asks) When is your birthday?

ALPHA: I am not like you, to have gone through ages of the Archimedean polygons, Newtonian calculus, and computer algorithms. You may suppose that I was born on February 11, 2009.

PI: (searched for something on her mobile phone and shows) In my decimal digits, your birthday 110209 is at the 79th place. Anyone can see their birthday in my decimal digits through the link:

https://www.facade.com/legacy/amiinpi/

ALPHA: Really? Anyone's birthday? Even if it is true, what's the use of that?

PI: It means that my decimal digits contain every six digit number. But whether these contain every eight digit number, no one knows yet. No one knows whether all the decimal digits $0, 1, \ldots, 9$ appear equally often in my decimal representation? Many such questions are still a mystery!

ALPHA: Equally often? What does it mean?

PI: A number N is called normal in base b if in the b base expansion of N, every finite string of digits $0, 1, \ldots, b-1$ of a given length occurs with the same frequency. From all the computations so far, I am expected to be a normal number in any base; but not one is yet sure about it.

ALPHA: (feels dizzy and thinks it best to change the topic)

Tell me something else. Can you have some other class of admirers, other than scientists and engineers?

PI: (smiles and answers) Many of my followers write songs, poems, stories, and now even plays for me. Micheal Kieth wrote a whole book using my first 10,000 digits in 2010. In 2015, Two Rajasthani Indian boys Rajveer Meena and Suresh Kumar Sharma made world records by memorizing my first 70000 and 70030 digits, respectively.

ALPHA: (in a slightly naughty tone) Any political experiences?

PI: (getting serious) Absolutely! A few years before World War II, Edmund Landau of the University of Gottingen, presented me unconventionally in his textbook. He defined me as double the value of x between 1 and 2 for which $\cos x$ vanishes. This is correct and is still popular today. But due to his non-German ethnicity, Landau had to face racism and was dismissed from his chair.

ALPHA: (Pi is getting emotional and is offered a glass of water by Alpha. He tries to change the subject)

I hear you have blessed even the most beautiful equation?

PI: (Pi's face lits up, and says) Of course, the equation

$$e^{\pi i} + 1 = 0$$

is absolutely the most beautiful one, as it amazingly connects five most important constants 0, 1, π, e, and i with three most important operations addition, multiplication, and powers.

ALPHA: What can be expected from you in the future?

PI: (thoughtfully) I have already done a lot of wonders in history. The randomness of my numbers has kept a lot of secrets. Which secret will be revealed next, is still in the womb of the future.

I have repeatedly surprised the humans as well as computers, as at many points in the history, it was generally believed that almost everything significant about me has been found. My digits have passed all statistical tests for randomness so far. Who knows what the future will hold for my magical characteristics?

ALPHA: (enforces his question) But still, tell me something...

PI: (in a mischievous style, leaving the glass from her hand)

Wonder if I may break some of your profound beliefs like this!

Alpha's dream shatters by a falling glass in his sleep; he wakes up and starts writing about it.

Alpha is wonderstruck by the unlimited mystery and charms of π; and recalls Shakespeare's words about the irresistible beauty and charm of Cleopatra who had mesmerized the mighty Romans emperor:

"Age cannot wither her, nor custom stale her infinite variety,"

Act 2, Scene 2, Antony and Cleopatra.

3. Postscript

In a Ramanujan centenary conference volume, J. M. Borwein and P. B. Borwein assert that the partial sums in the series (1) converge to the true value more rapidly than any other calculation of until the 1970's. Each successive term adds roughly eight more correct digits. The Borweins improved Ramanujan's result in 1987. In [4], they say:

"Iterative algorithms (where the output of one cycle is taken as the input for the next) which rapidly converge to pi were, in many respects, anticipated by Ramanujan, although he knew nothing of computer programming. Indeed, computers not only have made it possible to apply Ramanujan's work but have also helped to unravel it. Sophisticated algebraic manipulations software has allowed further exploration of the road Ramanujan travelled alone and unaided 75 years ago."

The following are some of the several amazing formulas about π :

- Newton's Algebra teacher John Wallis (1616–1703), invented the following infinite product formula for π (see [7], for an amazing 'geometric' proof).

$$\frac{\pi}{2} = \frac{2}{1}.\frac{2}{3}.\frac{4}{3}.\frac{4}{5}.\frac{6}{5}.\frac{6}{7}.\frac{8}{7}.\frac{8}{9}\dots$$

- Viéte's formula (1593):

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2}.\frac{\sqrt{2+\sqrt{2}}}{2}.\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\dots.$$

- Madhava (c. 1340 – c. 1425) from Kerala presented

$$\pi = \sqrt{12}\left(\frac{1}{1.3^0} - \frac{1}{3.3^1} + \frac{1}{5.3^2} - \frac{1}{7.3^3} + \dots\right).$$

- For every positive integer n, the series

$$1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \frac{1}{5^{2n}} + \dots$$

converges to π^{2n} times some rational number. In particular,

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots$$
$$\frac{\pi^4}{90} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \dots.$$

- The following formulas for π can be used to generate successive rational approximations of π :

$$\pi = 3 + \cfrac{1^2}{6 + \cfrac{3^2}{6+\cfrac{5^2}{6+\cfrac{7^2}{6+\cfrac{9^2}{6+\cfrac{11^2}{6+\dots}}}}}} \text{ and } \frac{\pi}{2} = 1 - \cfrac{1}{3 - \cfrac{2.3}{1-\cfrac{1.2}{3-\cfrac{4.5}{1-\cfrac{3.4}{3-\cfrac{6.7}{1-\cfrac{5.6}{3-\dots}}}}}}}.$$

- **Putnam Integral Problem 1968:** Putnam Competition is an annual mathematics competition for undergraduate college students enrolled at institutions of higher learning in the United States and Canada. It featured the following in 1968.

$$0 < \int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx = \frac{22}{7} - \pi.$$

The same appeared multiple times in the entrance examinations for the Indian Institutes of Technology.

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Ramanujan in the 21st Century

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Abstract. We give a brief survey of the impact of Ramanujan's work in the 20th century and indicate the new lines of research that it has inspired in the present century.

Keywords. Ramanujan, Modular forms, Mock modular forms.

Mathematics Subject Classification. 00A05, 15A06.

1. Introduction

If one had to point to a single mathematical idea that transformed 20th century mathematics, it would have to be the concept of a modular form. Ramanujan did not discover this idea because it already existed in the 19th century. He brought it to the foreground. He connected it to number theory and made it the center piece of his attention. His 1916 paper [20] modestly titled "On certain arithmetical functions" and published in the Transactions of the Cambridge Philosophical Society had a remarkable impact on number theory. It inspired the development of what is now called the Hecke theory of modular forms developed by Erich Hecke twenty years later. It sowed the seeds of the theory of automorphic forms initiated by Atle Selberg, John Tate, Harish-Chandra, Robert Langlands and others in the 1950's and 1960's. It led to an expansion of our idea of a zeta function and brought about the marriage between number theory and algebraic geometry as evinced in the solution of the celebrated conjectures of André Weil by Pierre Deligne in the 1970's (who was building on the earlier work of Alexander Grothendieck). It even brought about the classification of finite simple groups through "monstrous moonshine" of the 1980's and a complete solution of the notorious conjecture of Fermat by Andrew Wiles in the 1990's. So modular forms appear everywhere and Ramanujan singled them out as the essential idea to develop.

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