On a BBP-type formula for π^2 in the golden ratio base

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Abstract

This paper presents a detailed, self-contained proof of a BBP-type formula for π^2 expressed in the golden ratio base, ϕ. The formula was discovered empirically by the author in 2004. The proof presented herein is built upon a fundamental geometric identity connecting ϕ to the fifth roots of unity, offering an intuitive and direct path to the result. The power of the underlying methodology is then demonstrated by extending it to establish a new, computationally efficient Machin-like formula for $\zeta(3)$, expressed through rapidly converging, hierarchical series involving the golden ratio.

1 Introduction and Context

The search for explicit formulas for mathematical constants has long been a source of fascination. A significant breakthrough in this area came with the discovery of "BBP-type" formulas, named after Bailey, Borwein, and Plouffe [1]. A BBP formula for a constant C is a series representation of the form:

$$C = \sum_{k=0}^{\infty} \frac{1}{b^k} \sum_{j=1}^{n} \frac{a_j}{(nk+j)^s}$$

where b is an integer base and a_j are integers. Their primary importance lies in their ability to allow the computation of the d-th digit of C in base b without computing the preceding digits. The archetypal example is the formula for π:

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right)$$

The original discovery for π sparked a flurry of research. This led to many new BBP-type formulas, first for other weight-1 constants, and later for constants of higher weight, such as $\zeta(3)$ or Catalan's constant, as extensively documented in Bailey's compendium [3]. A parallel line of inquiry explored the existence of such formulas in non-integer algebraic bases. The golden ratio base, $\phi = (1+\sqrt{5})/2$, proved to be a particularly fertile ground, yielding formulas for π and other weight-1 constants [4].

This paper addresses a question at the confluence of these two research streams: do BBP-type formulas exist for constants of higher weight in the golden ratio base? It is in this context that in December 2004, while experimenting with the `lindep` algorithm in PARI/GP, I discovered the novel BBP-type formula for π^2—a constant of weight 2—that is the subject of this paper. While, as noted, numerous formulas for weight-1 constants have been found in the golden ratio base, those for constants of higher weight remain exceptionally rare. The formula for π^2 was subsequently featured as an exercise in Experimental Mathematics in Action [2], where the hints provided are sufficient to construct a proof by specializing a general polylogarithmic identity.

In contrast, the demonstration provided herein offers a more fundamental and intuitive perspective. It originates from a single geometric identity (Lemma 2), which, through the

properties of the dilogarithm, establishes a direct connection between a remarkable feature of the regular pentagon and the formula for π^2. The entire proof is thus constructed from first principles, highlighting the underlying algebraic structure in a self-contained way.

Finally, the investigation was extended to $\zeta(3)$, a constant of weight 3. While the experimental search for a pure BBP-type formula for $\zeta(3)$ in the golden ratio base did not succeed, it led to the discovery of a different, noteworthy result: a non-linear Machin-like formula for $\zeta(3)$. This identity, which expresses $\zeta(3)$ as a polynomial of rapidly converging series in terms of ϕ, is established here as a valuable and computationally efficient result in its own right.

2 Mathematical Preliminaries

To make this paper self-contained, this section gathers the key definitions and identities used.

2.1 The Polylogarithm Function

The polylogarithm function, whose study was initiated by Euler and later developed by Landen and Lewin, provides a bridge between series, integrals, and special values of functions like the Riemann zeta function. For instance, $\mathrm{Li}_2(1) = \sum_{k=1}^{\infty} 1/k^2 = \zeta(2) = \pi^2/6$. These functions also appear in various contexts in theoretical physics, such as in the calculation of Feynman diagrams.

The polylogarithm of order s, denoted $\mathrm{Li}_s(z)$, is defined for any complex number z with $|z| < 1$ by the power series:

$$\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}$$

For $s = 2$, this is the *dilogarithm*, and for $s = 3$, the *trilogarithm*.

Lemma 1 (Functional Equations for the Dilogarithm)**.** *For $|z| \leq 1$ and z not 0 or 1, the dilogarithm satisfies the following functional equations, which are standard results found in [5]:*

1. ***Landen's Identity (Reflection Formula):***

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(1-z) = \frac{\pi^2}{6} - \ln(z)\ln(1-z) \tag{1}$$

2. ***Inversion Formula:***

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(1/z) = -\frac{\pi^2}{6} - \frac{1}{2}\ln^2(-z) \tag{2}$$

Here, $\ln(z)$ denotes the principal branch of the complex logarithm.

2.2 The Golden Ratio and the Regular Pentagon

The golden ratio, $\phi = (1+\sqrt{5})/2$, is deeply connected to the geometry of the regular pentagon. The vertices of a regular pentagon centered at the origin can be represented by the fifth roots of unity, $\omega^k = e^{2\pi ik/5}$ for $k = 0, 1, 2, 3, 4$. The geometry of this figure dictates the exact trigonometric values needed for our proof. For instance, the ratio of a diagonal to a side in a regular pentagon is ϕ. This property can be used to derive the following standard results [6]:

$$\cos(2\pi/5) = \frac{\phi - 1}{2} = \frac{1}{2\phi} \quad \text{and} \quad \cos(\pi/5) = \frac{\phi}{2}$$

The point $z = \phi^{-1}e^{2\pi i/5}$, which is central to our proof, lies within this geometric context, as shown in Figure 1.

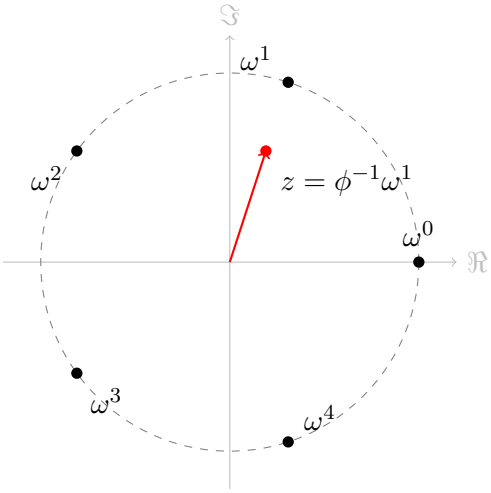


Figure 1: The fifth roots of unity $\omega^k = e^{2\pi ik/5}$ on the unit circle and the key point z in the complex plane.

3 The main results

This paper establishes two main theorems.

Theorem 1 (BBP formula for π^2)**.** *Let $\phi = \frac{1+\sqrt{5}}{2}$ be the golden ratio. The following identity holds:*

$$\frac{\pi^2}{50} = \sum_{k=0}^{\infty} \frac{1}{\phi^{5k}} \left(\frac{\phi^{-2}}{(5k+1)^2} - \frac{\phi^{-1}}{(5k+2)^2} - \frac{\phi^{-2}}{(5k+3)^2} + \frac{\phi^{-5}}{(5k+4)^2} + \frac{2\phi^{-5}}{(5k+5)^2} \right) \tag{3}$$

Theorem 2 (Machin-like formula for $\zeta(3)$)**.** *Let $\phi = \frac{1+\sqrt{5}}{2}$. The constant $\zeta(3)$ can be expressed as the following polynomial of rapidly converging series:*

$$\zeta(3) = \frac{5}{4} M_3(\phi^{-2}) + \Big(M_2(\phi^{-1}) + M_2(\phi^{-2}) \Big) M_1(\phi^{-2}) + \frac{7}{6} \Big(M_1(\phi^{-2}) \Big)^3$$

where the constants $M_s(w)$ are defined by hierarchical sums designed for rapid convergence:

$$M_s(w) = \sum_{j=0}^{\infty} 5^{-sj} \left(\sum_{r=1}^{4} \sum_{k=0}^{\infty} \frac{(w^{5^j})^{5k+r}}{(5k+r)^s} \right)$$

This series converges for $|w| < 1$, which is the case for the arguments ϕ^{-1} and ϕ^{-2} used here.

4 Proof of the π^2 formula (theorem 1)

The proof is presented in two stages: we first prove a core geometric identity and its consequence for a polylogarithm series, and then we demonstrate that the BBP formula is an algebraic rearrangement of this result.

4.1 The Geometric and Polylogarithmic Foundation

Lemma 2 (A key geometric identity)**.** *Let $\phi = (1+\sqrt{5})/2$ and $z = \phi^{-1} e^{2\pi i/5}$. Then, the following identity holds: $1 - z = e^{-i\pi/5}$.*

Proof. We prove the identity by showing that the real and imaginary parts of both sides are equal. First, the real part of $1 - z$:

$$\begin{aligned}\Re(1-z) &= 1 - \phi^{-1}\cos(2\pi/5) = 1 - \phi^{-1}\left(\frac{1}{2\phi}\right) = 1 - \frac{\phi^{-2}}{2} \\ &= 1 - \frac{1-\phi^{-1}}{2} = \frac{2-1+\phi^{-1}}{2} = \frac{1+\phi^{-1}}{2} = \frac{\phi}{2}\end{aligned}$$

This is precisely $\cos(\pi/5) = \Re(e^{-i\pi/5})$. Next, the imaginary part of $1 - z$:

$$\Im(1-z) = -\phi^{-1}\sin(2\pi/5)$$

Using the double-angle formula $\sin(2\theta) = 2\sin\theta\cos\theta$:

$$\begin{aligned}\phi^{-1}\sin(2\pi/5) &= \phi^{-1}\left(2\sin(\pi/5)\cos(\pi/5)\right) \\ &= \phi^{-1}\left(2\sin(\pi/5)\frac{\phi}{2}\right) = (\phi^{-1}\phi)\sin(\pi/5) = \sin(\pi/5)\end{aligned}$$

Thus, $\Im(1-z) = -\sin(\pi/5)$. Since the real and imaginary parts match, the identity is proven. □

Geometric Interpretation. This identity expresses a remarkable metric relationship within the geometry of the pentagon. In the complex plane, the numbers $0, 1, z$ form a triangle. The identity $1 - z = e^{-i\pi/5}$ states that the vector from z to 1 has length 1 and makes an angle of $-\pi/5$ with the real axis. This specific configuration is a direct consequence of the golden ratio's role in the pentagon's construction.

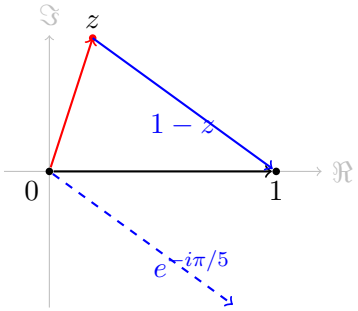


Figure 2: Geometric visualization of the identity $1 - z = e^{-i\pi/5}$.

Lemma 3. *Let $\phi = \frac{1+\sqrt{5}}{2}$. The following identity holds:*

$$\sum_{k=1}^{\infty} \frac{\cos(2\pi k/5)}{\phi^k k^2} = \frac{\pi^2}{100} \tag{4}$$

Proof. Our strategy is to express the target sum using dilogarithms and then use a functional equation to evaluate it. We choose the specific complex number $z = \phi^{-1}e^{2\pi i/5}$ because its real part isolates the desired cosine term, its modulus is ϕ^{-1}, and, crucially, the term $1 - z$ simplifies dramatically due to Lemma 2. This simplification makes Landen's identity, which connects $\mathrm{Li}_2(z)$ and $\mathrm{Li}_2(1-z)$, the ideal tool for the evaluation.

Let $z = \phi^{-1}e^{2\pi i/5}$ and its conjugate $\bar{z} = \phi^{-1}e^{-2\pi i/5}$. We want to evaluate the sum $S = \sum_{k=1}^{\infty} \frac{\cos(2\pi k/5)}{\phi^k k^2}$. This sum can be expressed using the dilogarithm as:

$$2S = \sum_{k=1}^{\infty} \frac{2\cos(2\pi k/5)}{\phi^k k^2} = \sum_{k=1}^{\infty} \frac{z^k + \bar{z}^k}{k^2} = \mathrm{Li}_2(z) + \mathrm{Li}_2(\bar{z})$$

Our goal is to show that this sum equals $\pi^2/50$. We use Landen's Identity (1) and our key result from Lemma 2, $1 - z = e^{-i\pi/5}$ (which implies $1 - \bar{z} = e^{i\pi/5}$):

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(\bar{z}) = \left(\frac{\pi^2}{6} - \mathrm{Li}_2(1-z) - \ln(z)\ln(1-z)\right) + \left(\frac{\pi^2}{6} - \mathrm{Li}_2(1-\bar{z}) - \ln(\bar{z})\ln(1-\bar{z})\right)$$

$$2S = \frac{\pi^2}{3} - \left(\mathrm{Li}_2(e^{-i\pi/5}) + \mathrm{Li}_2(e^{i\pi/5})\right) - \left(\ln(z)\ln(1-z) + \ln(\bar{z})\ln(1-\bar{z})\right)$$

We evaluate the two parenthesized terms separately. First, the dilogarithm term is $2\Re(\mathrm{Li}_2(e^{i\pi/5}))$. The real part of $\mathrm{Li}_2(e^{i\theta})$ is the Clausen function, which has the known series representation $\sum \frac{\cos(k\theta)}{k^2} = \frac{\pi^2}{6} - \frac{\pi|\theta|}{2} + \frac{\theta^2}{4}$ for $0 \le |\theta| \le 2\pi$. For $\theta = \pi/5$, this term becomes:

$$2 \times \left(\frac{\pi^2}{6} - \frac{\pi(\pi/5)}{2} + \frac{(\pi/5)^2}{4}\right) = 2\pi^2\left(\frac{1}{6} - \frac{1}{10} + \frac{1}{100}\right) = 2\pi^2\left(\frac{50-30+3}{300}\right) = \frac{23\pi^2}{150}$$

Second, for the logarithm term, we use $\ln(z) = -\ln\phi + i\frac{2\pi}{5}$, $\ln(1-z) = -i\frac{\pi}{5}$, and their conjugates. The sum is:

$$\left(-\ln\phi + i\frac{2\pi}{5}\right)\left(-i\frac{\pi}{5}\right) + \left(-\ln\phi - i\frac{2\pi}{5}\right)\left(i\frac{\pi}{5}\right) = \left(\frac{2\pi^2}{25} + i\frac{\pi\ln\phi}{5}\right) + \left(\frac{2\pi^2}{25} - i\frac{\pi\ln\phi}{5}\right) = \frac{4\pi^2}{25}$$

Substituting these results back into the expression for $2S$:

$$2S = \frac{\pi^2}{3} - \frac{23\pi^2}{150} - \frac{4\pi^2}{25} = \frac{50\pi^2 - 23\pi^2 - 24\pi^2}{150} = \frac{3\pi^2}{150} = \frac{\pi^2}{50}$$

Therefore, $S = \pi^2/100$, which completes the proof. □

4.2 Derivation of the BBP formula

Let $\mathcal{T} = \sum_{k=1}^{\infty} \frac{\cos(2\pi k/5)}{\phi^k k^2} = \frac{\pi^2}{100}$, as proven in Lemma 3. To transform this cosine series into a BBP-type formula, we use a standard dissection technique. We partition the sum based on the index k modulo 5. This is the natural choice because the term $\cos(2\pi k/5)$ is periodic with period 5. This dissection isolates the powers of ϕ^5, revealing the structure required for a BBP formula.

We set $k = 5j + r$ for $j \ge 0$ and $r \in \{1, 2, 3, 4, 5\}$:

$$\mathcal{T} = \sum_{j=0}^{\infty}\sum_{r=1}^{5} \frac{\cos(2\pi(5j+r)/5)}{\phi^{5j+r}(5j+r)^2} = \sum_{j=0}^{\infty}\frac{1}{\phi^{5j}}\sum_{r=1}^{5}\frac{\cos(2\pi r/5)}{\phi^r(5j+r)^2}$$

The values of $\cos(2\pi r/5)$ are given in Table 1.

Table 1: Values of $\cos(2\pi r/5)$ used in the dissection.

r	$2\pi r/5$	$\cos(2\pi r/5)$
1	$2\pi/5$	$(\phi-1)/2 = \phi^{-1}/2$
2	$4\pi/5$	$-(\phi)/2$
3	$6\pi/5$	$-(\phi)/2$
4	$8\pi/5$	$(\phi-1)/2 = \phi^{-1}/2$
5	2π	1

Substituting these values into the inner sum gives:

$$\sum_{r=1}^{5}\frac{\cos(2\pi r/5)}{\phi^r(5j+r)^2} = \frac{\phi^{-1}/2}{\phi^1(5j+1)^2} + \frac{-\phi/2}{\phi^2(5j+2)^2} + \frac{-\phi/2}{\phi^3(5j+3)^2} + \frac{\phi^{-1}/2}{\phi^4(5j+4)^2} + \frac{1}{\phi^5(5j+5)^2}$$

$$= \frac{1}{2}\left(\frac{\phi^{-2}}{(5j+1)^2} - \frac{\phi^{-1}}{(5j+2)^2} - \frac{\phi^{-2}}{(5j+3)^2} + \frac{\phi^{-5}}{(5j+4)^2} + \frac{2\phi^{-5}}{(5j+5)^2}\right)$$

This is exactly half the term in the BBP formula (3). Let $\mathcal{S}$ be the sum in that formula. Then we have shown that $\mathcal{T} = \mathcal{S}/2$. Since we proved $\mathcal{T} = \pi^2/100$, it follows that $\mathcal{S} = 2\mathcal{T} = \pi^2/50$. This completes the proof of Theorem 1.

Remark. The final term in the BBP polynomial, corresponding to $r = 5$, can be written as a series involving the dilogarithm. The full sum associated with this term is $\sum_{j=0}^{\infty} \frac{2\phi^{-5}}{\phi^{5j}(5j+5)^2} = \frac{2\phi^{-5}}{25} \sum_{j=0}^{\infty} \frac{(\phi^{-5})^j}{(j+1)^2} = \frac{2}{25}\mathrm{Li}_2(\phi^{-5})$. This highlights that every component of the formula is structurally a polylogarithm value.

5 Proof of the $\zeta(3)$ formula (theorem 2)

Lemma 4. *For any complex number w such that $|w| < 1$ and any integer $s \geq 1$, we have $\mathrm{Li}_s(w) = M_s(w)$.*

Proof. We begin with the definition of the polylogarithm, $\mathrm{Li}_s(w) = \sum_{n=1}^{\infty} \frac{w^n}{n^s}$. Any positive integer n can be uniquely written as $n = m \cdot 5^j$, where $j \geq 0$ is an integer and m is an integer not divisible by 5. We can thus re-index the sum by splitting it over j and m:

$$\mathrm{Li}_s(w) = \sum_{j=0}^{\infty} \sum_{\substack{m=1 \\ 5 \nmid m}}^{\infty} \frac{w^{m \cdot 5^j}}{(m \cdot 5^j)^s} = \sum_{j=0}^{\infty} \frac{1}{(5^s)^j} \sum_{\substack{m=1 \\ 5 \nmid m}}^{\infty} \frac{(w^{5^j})^m}{m^s}$$

The inner sum is over all integers m not divisible by 5. This means m can be written as $5k + r$ for $r \in \{1, 2, 3, 4\}$. The sum becomes:

$$\sum_{\substack{m=1 \\ 5 \nmid m}}^{\infty} \frac{(w^{5^j})^m}{m^s} = \sum_{r=1}^{4} \sum_{k=0}^{\infty} \frac{(w^{5^j})^{5k+r}}{(5k+r)^s}$$

Substituting this back into the expression for $\mathrm{Li}_s(w)$ gives exactly the definition of $M_s(w)$. □

Lemma 5 (Key Polylogarithm Identities)**.** *The proof relies on the following known evaluations of polylogarithm constants, found in sources such as [5].*

1. $\mathrm{Li}_3(\phi^{-2}) = \frac{4}{5}\zeta(3) + \frac{2}{3}(\ln\phi)^3 - \frac{2}{15}\pi^2 \ln\phi$
2. $\ln\phi = \mathrm{Li}_1(\phi^{-2})$
3. $\pi^2 = 6\Big(\mathrm{Li}_2(\phi^{-1}) + \mathrm{Li}_2(\phi^{-2}) + 2(\ln\phi)^2\Big)$

Identity 3 is proven in Appendix A using standard dilogarithm evaluations (Eqs. 5-6). For further details on all identities, see Appendix A.

Proof of theorem 2. The proof follows a "bootstrapping" strategy. By Lemma 4, we know that $\mathrm{Li}_s(w) = M_s(w)$ for $|w| < 1$, so we can use these expressions interchangeably in the following identities. We begin by rearranging the first identity in Lemma 5 to isolate $\zeta(3)$:

$$\zeta(3) = \frac{5}{4}\mathrm{Li}_3(\phi^{-2}) - \frac{5}{6}(\ln\phi)^3 + \frac{1}{6}\pi^2 \ln\phi$$

Now, we substitute the expressions for $\ln\phi$ (identity 2) and π^2 (identity 3) into this equation.

$$\zeta(3) = \frac{5}{4}\mathrm{Li}_3(\phi^{-2}) - \frac{5}{6}\big(\mathrm{Li}_1(\phi^{-2})\big)^3 + \frac{1}{6}\left[6\Big(\mathrm{Li}_2(\phi^{-1}) + \mathrm{Li}_2(\phi^{-2}) + 2(\mathrm{Li}_1(\phi^{-2}))^2\Big)\right]\mathrm{Li}_1(\phi^{-2})$$

Distributing the last term and combining the cubic terms yields the desired identity in terms of polylogarithms:

$$\zeta(3) = \frac{5}{4}\mathrm{Li}_3(\phi^{-2}) + \Big(\mathrm{Li}_2(\phi^{-1}) + \mathrm{Li}_2(\phi^{-2})\Big)\mathrm{Li}_1(\phi^{-2}) + \frac{7}{6}\Big(\mathrm{Li}_1(\phi^{-2})\Big)^3$$

To obtain the final formula stated in Theorem 2, we substitute each $\mathrm{Li}_s(w)$ term with its equivalent hierarchical series $M_s(w)$. □

Numerical illustration

The efficiency of the formula for $\zeta(3)$ comes from the factor 5^{-sj} in the hierarchical sums $M_s(w)$. We know $\zeta(3) \approx 1.202056903159...$

- The **j=0** term alone gives an approximation: $\zeta(3) \approx 1.202041...$
- Including the **j=1** term yields a much better result: $\zeta(3) \approx 1.202056902...$
- The **j=2** term adds a correction of order 10^{-33}, yielding an accuracy of over 30 decimal places.[1]

6 Conclusion and Perspectives

We have presented a self-contained, geometrically-motivated proof of a BBP-type formula for π^2 in the golden ratio base. By building the proof from a specific identity tied to the regular pentagon, our approach offers an intuitive perspective on the formula's origin.

Furthermore, we have proved a new Machin-like identity for $\zeta(3)$. This work opens several avenues for reflection:

- **Generalizations:** The success of this method is intimately tied to the specific geometric and algebraic properties of ϕ within the field of the fifth roots of unity. It seems unlikely that this approach could be generalized to other irrational algebraic numbers that do not possess such a remarkable geometric counterpart. The formula appears to be a "beautiful accident" of the pentagon's properties. One might wonder if analogous properties for other regular polygons (e.g., the octagon, related to $\sqrt{2}$) could yield similar formulas, but this remains a challenging open question.
- **Higher-order constants:** It is significant that our method yields a Machin-like formula for $\zeta(3)$ and not a pure BBP formula. While BBP formulas for $\zeta(3)$ in integer bases have been found [3], the structure of polylogarithm identities related to ϕ seems to naturally produce the mixed terms characteristic of a Machin-like formula. This highlights a structural difference between representations in integer bases versus certain algebraic bases.
- **Computational aspects:** The formula for $\zeta(3)$ exhibits linear convergence. The number of correct digits obtained is roughly proportional to the number of terms computed in the outer sum (j). Specifically, each new term in $M_s(w)$ adds approximately $s \log_{10}(5)$ digits of precision. This is computationally effective for high-precision calculations, although its rate of linear convergence is lower than that of some known hypergeometric series for $\zeta(3)$. A detailed complexity analysis comparing this formula with other state-of-the-art algorithms/series would be a valuable next step.

On a pedagogical level, we hope this self-contained proof serves as an effective gateway for aspiring mathematicians, illustrating how a surprising numerical discovery can lead to the exploration of deep connections between analysis, number theory, and geometry.

Acknowledgements

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[1] The high-order correction arises from the polynomial structure of the formula. While the correction from any single M_s term is modest (e.g., the $j = 2$ term in $M_3(\phi^{-2})$ is of order 10^{-15}), the interplay between terms in the full polynomial for $\zeta(3)$ leads to significant cancellations, producing the much smaller final correction.

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A On the Polylogarithm Identities in Lemma 5

To enhance the self-contained nature of this paper, we provide context for the three identities involving polylogarithm constants related to ϕ. These are deep results from the theory of polylogarithms, with detailed proofs found in specialized literature such as Lewin's treatise [5].

A.1 Identity 2: $\ln\phi = \mathrm{Li}_1(\phi^{-2})$

This is the most straightforward identity. By definition, $\mathrm{Li}_1(z) = -\ln(1-z)$. We also have the algebraic property $\phi^{-2} = 1 - \phi^{-1}$. Therefore:

$$\mathrm{Li}_1(\phi^{-2}) = -\ln(1-\phi^{-2}) = -\ln(\phi^{-1}) = -(-\ln\phi) = \ln\phi$$

A.2 Identity 3: $\pi^2 = 6\Big(\mathrm{Li}_2(\phi^{-1}) + \mathrm{Li}_2(\phi^{-2}) + 2(\ln\phi)^2\Big)$

This identity can be derived from two well-known evaluations for the golden ratio:

$$\mathrm{Li}_2(\phi^{-2}) = \frac{\pi^2}{15} - (\ln\phi)^2 \tag{5}$$

$$\mathrm{Li}_2(\phi^{-1}) = \frac{\pi^2}{10} - (\ln\phi)^2 \tag{6}$$

Adding these two equations gives:

$$\mathrm{Li}_2(\phi^{-1})+\mathrm{Li}_2(\phi^{-2}) = \left(\frac{\pi^2}{15}+\frac{\pi^2}{10}\right)-2(\ln\phi)^2 = \frac{2\pi^2+3\pi^2}{30}-2(\ln\phi)^2 = \frac{5\pi^2}{30}-2(\ln\phi)^2 = \frac{\pi^2}{6}-2(\ln\phi)^2$$

Rearranging this result yields the desired identity:

$$\mathrm{Li}_2(\phi^{-1}) + \mathrm{Li}_2(\phi^{-2}) + 2(\ln\phi)^2 = \frac{\pi^2}{6} \implies \pi^2 = 6\Big(\mathrm{Li}_2(\phi^{-1}) + \mathrm{Li}_2(\phi^{-2}) + 2(\ln\phi)^2\Big)$$

The proofs of (5) and (6) themselves rely on intricate manipulations of functional equations for the dilogarithm, such as the inversion formula and the five-term relation, evaluated at special points related to ϕ.

A.3 Identity 1: The $\mathrm{Li}_3(\phi^{-2})$ evaluation

The evaluation of $\mathrm{Li}_3(\phi^{-2})$ in terms of $\zeta(3)$, π^2, and powers of $\ln\phi$ is a much deeper result. Its proof requires advanced techniques, including functional equations for the trilogarithm and connections to the theory of modular forms. A full derivation is well beyond the scope of this paper but can be found in advanced resources on polylogarithms [5, Chapter 6].