

**CONJECTURE ON THE VALUE OF $\pi(10^{26})$, THE NUMBER OF
PRIMES $< 10^{26}$**

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ABSTRACT. Based on the first 25 known values of $\pi(10^n)$, the number of primes less than 10^n with $1 \leq n \leq 25$, we propose a conjectured value range of $\pi(10^{26})$ calculated by using polynomial interpolations with two corrective functions obtained by Thiele interpolations on relative differences of exact and interpolated values of $\pi(10^n)$. The conjectured range value is in agreement with values obtained by the Eulerian logarithmic integral and with the Riemann functions.

1. INTRODUCTION

For $x \in \mathbb{Z}^+$, it is known (see e.g. [1]) that if $\pi(x)$ is the number of primes $\leq x$, then

$$(1.1) \quad \lim_{x \rightarrow \infty} \left(\frac{\pi(x)}{\frac{x}{\log(x)}} \right) = 1$$

meaning in the general sense that the relative error of approximating $\pi(x)$ by $\left(\frac{x}{\log(x)}\right)$ approaches 0 as x approaches infinity. Several better approximations to $\pi(x)$ for $x \lesssim 10^{25}$ are given for example by the offset logarithmic integral or Eulerian logarithmic integral (see e.g. [2])

$$(1.2) \quad Li(x) = li(x) - li(2) = \int_2^x \frac{dt}{\log(t)}$$

in function of the logarithmic integral $li(x) = \int_0^x \frac{dt}{\log(t)}$, or by the Riemann function (see e.g. [3])

$$(1.3) \quad R(x) = \sum_{j=1}^{\infty} \frac{\mu(j)}{j} li\left(x^{\frac{1}{j}}\right)$$

where $\mu(j)$ is the Moebius function and $j \in \mathbb{Z}^+$, or by an even better function

$$(1.4) \quad R(x) - \frac{1}{\log(x)} + \frac{1}{\pi} \arctan\left(\frac{\pi}{\log(x)}\right)$$

Other approximation formulas to $\pi(x)$ include Legendre's [4], Lehmer's [5], and Meissel's [6].

2000 *Mathematics Subject Classification.* 11 Number Theory, 11N05 Distribution of primes .
Key words and phrases. Number Theory, Prime Number Counting Functions, conjectured value of $\pi(10^{26})$.

Values of $\pi(10^n)$ are known for $1 \leq n \leq 25$ (see [7]). The values of $\pi(10^{24})$ and $\pi(10^{25})$ were calculated relatively recently, respectively in 2010 [8] and in June 2013 [9].

In this paper, we propose a simple method based on polynomial interpolations of the known 25 first values of $\pi(10^n)$ introducing two corrective functions to calculate a conjectured value of $\pi(10^{26})$.

2. METHOD

The method we use includes six steps, some involving interpolations¹.

Step 1: First, polynomial functions

$$(2.1) \quad P_n(x) = \sum_{i=0}^{n-1} (a_{i,n} x^i)$$

with coefficients $a_{i,n} \in \mathbb{R}$ and $i, n, x \in \mathbb{Z}^+$, are calculated by polynomial interpolations over the values of $\pi(10^n)$ up to n for $2 \leq n \leq 25$, yielding 23 polynomials, given in Appendix 1, such that the equality

$$(2.2) \quad \pi(10^x) = P_n(x)$$

holds exactly or $1 \leq x \leq n$.

Step 2: For each value of n until $n = 24$, the next value for $x = n+1$ is calculated by (2.1) yielding $P_n(n+1)$ and compared to the value of $\pi(10^{n+1})$. The relative differences

$$(2.3) \quad \delta_{n+1} = \frac{\pi(10^{n+1}) - P_n(n+1)}{\pi(10^{n+1})}$$

are then calculated. As $P_n(n+1) < \pi(10^{n+1})$ for $2 \leq n \leq 24$, δ_{n+1} is such that $0 < \delta_{n+1} < 1$ and decreases with increasing values of n (see Fig. 2.1).

Step 3: For each set value of n , corrective functions Φ_{n+1} in the form of continued fractions

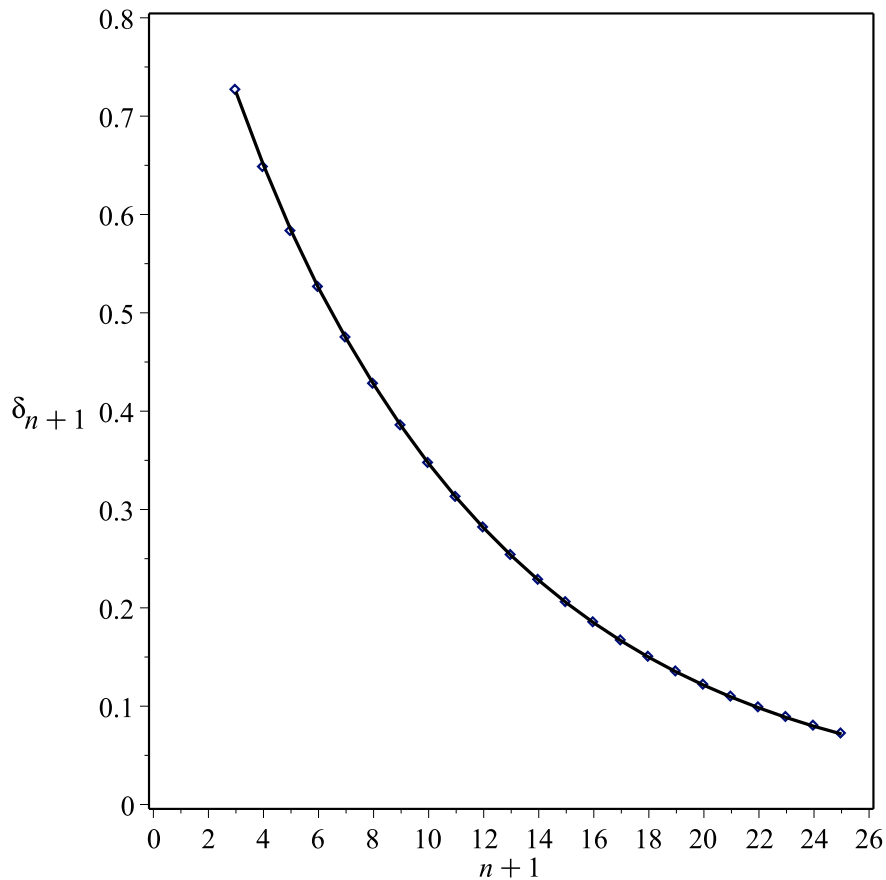
$$(2.4) \quad \Phi_{n+1}(x) = c_1 + \frac{x-3}{x-4} \cfrac{c_2 + \frac{x-(n-3)}{x-(n-2)}}{c_3 + \dots} \dots \cfrac{c_{n-4} + \frac{x-(n-1)}{c_{n-3} + \frac{x-(n-1)}{c_{n-2} + Kx}}}{c_{n-3} + \frac{x-(n-1)}{c_{n-2} + Kx}}$$

with coefficients $c_i, K \in \mathbb{R}$ and $i, n, x \in \mathbb{Z}^+$, $1 \leq i \leq n-2$, are calculated by Thiele interpolations over the $(n-1)$ values of δ_{n+1} for $3 \leq n \leq 24$, yielding 22 corrective functions Φ_{n+1} . The coefficients c_i and K of Φ_{n+1} for $n = 24$ are given in Appendix 2.

Step 4: The predictive power of using the 22 corrective functions Φ_{n+1} to calculate a next interpolated value of $\pi(10^{x+1})$ knowing the first x values of $\pi(10^x)$ with $1 \leq x \leq n$ for all values of n , $1 \leq n \leq 24$, is verified by taking the nearest integer

¹Curve Fitting package of Maple 16.00 with 50 digits precision

FIGURE 2.1. δ_{n+1} in function of $(n + 1)$



to $(P_x(x + 1) / (1 - \Phi_{n+1}(x + 1)))$ and compared to $\pi(10^{x+1})$. It yields that the equality

$$(2.5) \quad \pi(10^{x+1}) = \text{Round} \left(\frac{P_x(x + 1)}{1 - \Phi_{n+1}(x + 1)} \right)$$

holds exactly for all values of $1 \leq x \leq n$ and of n , $1 \leq n \leq 24$, where Φ_4 should be taken for $1 \leq n \leq 3$ and $\text{Round}(X)$ is the nearest integer to X ($X \in \mathbb{R}$).

Step 5: To interpolate the next unknown value for $n = 26$ of $\pi(10^{26})$, as Φ_{26} is obviously unknown, we use a relation similar to (2.5) with $x = 25$ and Φ_{25} instead of Φ_{26} , to obtain a very approximate interpolated value of $\pi(10^{26})$

$$(2.6) \quad \text{Round} \left(\frac{P_{25}(26)}{1 - \Phi_{25}(26)} \right) = 1699246738822618041025224$$

Step 6: To improve this interpolation, we have to find an additional correction to the corrective function Φ_{25} . We performed a similar interpolation in May 2013 (unpublished) on the first 24 values of $\pi(10^n)$ before the announcement of the calculated value of $\pi(10^{25}) = 176846309399143769411680$ by J. Bueth, J. Franke,

TABLE 1. Relative differences δ'_{n+1}

$n + 1$	δ'_{n+1}	$n + 1$	δ'_{n+1}	$n + 1$	δ'_{n+1}
5	3.10676×10^{-2}	12	1.23708×10^{-4}	19	5.44808×10^{-7}
6	-4.15297×10^{-3}	13	-3.31760×10^{-5}	20	1.72692×10^{-7}
7	2.85895×10^{-5}	14	4.84531×10^{-5}	21	-3.64154×10^{-7}
8	7.32975×10^{-4}	15	-1.62333×10^{-5}	22	-7.56044×10^{-8}
9	5.14623×10^{-3}	16	1.11746×10^{-6}	23	-9.45864×10^{-9}
10	-8.04909×10^{-3}	17	-4.32482×10^{-6}	24	2.62139×10^{-8}
11	5.18791×10^{-3}	18	-4.83262×10^{-6}	25	1.34117×10^{-8}

A. Jost and T. Kleinjung on 1st June 2013 [9]. This previous interpolation yielded an interpolated value of

$$(2.7) \quad \text{Round} \left(\frac{P_{24}(25)}{1 - \Phi_{24}(25)} \right) = 176846307027334692763889 < \pi(10^{25})$$

giving a relative difference

$$(2.8) \quad \delta'_{n+1} = \frac{\pi(10^{n+1}) - \text{Round} \left(\frac{P_n(n+1)}{1 - \Phi_n(n+1)} \right)}{\pi(10^{n+1})}$$

of $\delta'_{25} \approx 1.34117 \times 10^{-8}$.

To estimate the error made by using Φ_n instead of Φ_{n+1} in (2.5) when interpolating to find a value of $\pi(10^{n+1})$, the relative differences δ'_{n+1} are calculated for $4 \leq n \leq 24$, yielding 21 relative differences δ'_{n+1} (2.8), whose approximate values are reported in Table 1. Their absolute value decreases with increasing n as shown in Fig. 2.2.

Assuming that $|\delta'_{n+1}|$ continues to decrease with n , an interpolated value of $|\delta'_{26}| \lesssim 10^{-8}$ can be expected. A new Thiele interpolation over the last five values of $|\delta'_{n+1}|$ for $20 \leq n \leq 24$ yields a second corrective function $|\psi_{n+1}(n)|$

$$(2.9) \quad |\psi_{n+1}(n)| = d_1 + \frac{n - 20}{d_2 + \frac{n - 21}{d_3 + \frac{n - 22}{d_4 + Mn}}}$$

with $d_i, M \in \mathbb{R}$ and $i, n \in \mathbb{Z}^+$, $1 \leq i \leq 4$. The coefficients d_i and M are given in Appendix 3. For $n = 25$, (2.9) yields $|\psi_{26}| \approx 7.07767 \times 10^{-9}$. Assuming $\psi_{26} \approx \pm 7.1 \times 10^{-9}$ yields a conservative assumption for the value of $\pi(10^{26})$

$$(2.10) \quad \pi(10^{26}) \approx 1699246738822618041025224 \pm 12064651845640588$$

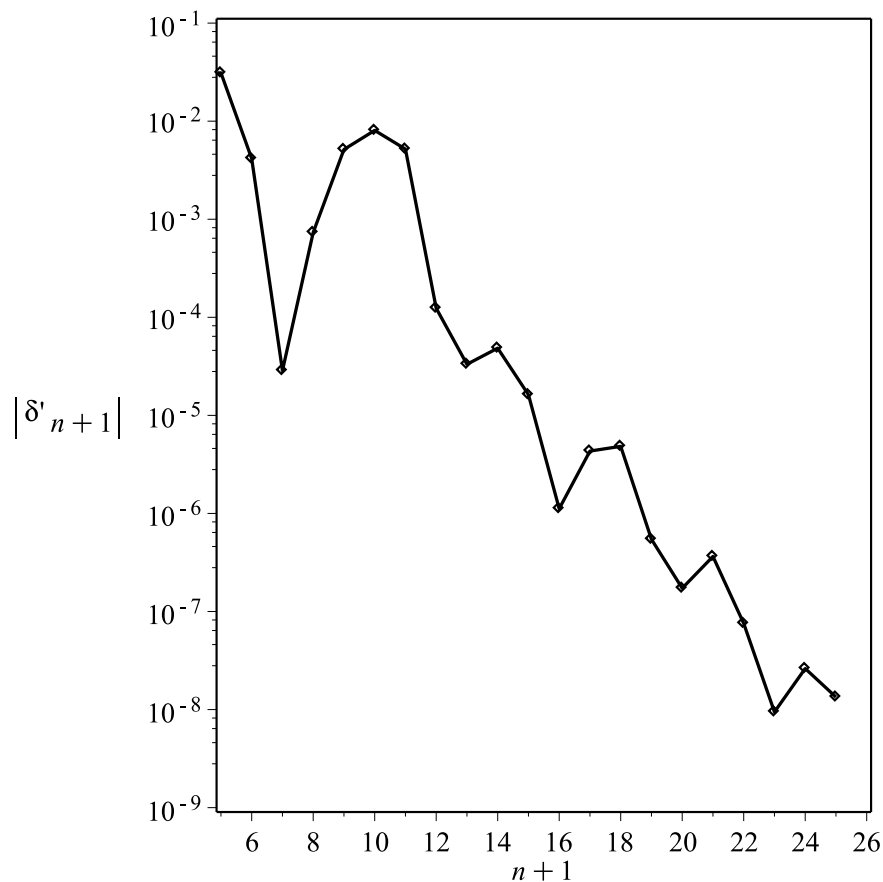
or

$$(2.11) \quad 1699246726757966195384636 \lesssim \pi(10^{26}) \lesssim 1699246750887269886665812$$

Assuming further that ψ_{26} would be positive like δ'_{25} and δ'_{24} and that it would be included in the range $7 \times 10^{-9} \lesssim \psi_{26} \lesssim 7.1 \times 10^{-9}$, the value range for $\pi(10^{26})$ reduces to

$$(2.12) \quad \pi(10^{26}) \approx 1699246738822618041025224^{+12064651845640588}_{+11894727171758326}$$

FIGURE 2.2. $|\delta'_{n+1}|$ in function of $n + 1$



or

$$(2.13) \quad 1699246750717345212783550 \lesssim \pi(10^{26}) \lesssim 1699246750887269886665812$$

3. DISCUSSION

It is interesting to compare this value range to those values obtained with the approximating functions of Section 1. Table 2 shows for $24 \leq n \leq 26$ the values of $\text{Round}\left(\frac{10^n}{\log(10^n)}\right)$, $\text{Round}(Li(10^n))$ (1.2), $\text{Round}(R(10^n))$ (1.3) and the associated relative differences

$$(3.1) \quad \delta_n'' = \frac{\pi(10^n) - \text{Round}(f(n))}{\pi(10^n)}$$

where $f(n)$ is either $\left(\frac{10^n}{\log(10^n)}\right)$, or $Li(10^n)$, or $R(10^n)$, and where $\pi(10^{24}) = 18435599767349200867866$ and $\pi(10^{25}) = 176846309399143769411680$.

Note that in calculating $R(10^n)$, the summation in (1.3) is made for $1 \leq j \leq 1000$, as the value of $R(10^n)$ does not change for higher values of j , due mainly

TABLE 2. Values of $\text{Round}\left(\frac{10^n}{\log(10^n)}\right)$, $\text{Round}(Li(10^n))$ (1.2), $\text{Round}(R(10^n))$ (1.3) for $24 \leq n \leq 26$ and associated δ_n'' (2.12)

n	$f(n)$	$\text{Round}(f(n))$	δ_n''
24	$\frac{10^n}{\log(10^n)}$	18095603412635492818797	1.84424×10^{-2}
24	$Li(10^n)$	18435599767366347775143	-9.30098×10^{-13}
24	$R(10^n)$	18435599767347541878147	8.99884×10^{-14}
25	$\frac{10^n}{\log(10^n)}$	173717792761300731060452	1.76906×10^{-2}
25	$Li(10^n)$	176846309399198930392618	-3.11915×10^{-13}
25	$R(10^n)$	176846309399141934626966	1.03750×10^{-14}
26	$\frac{10^n}{\log(10^n)}$	1670363391935583952504342	-
26	$Li(10^n)$	1699246750872593033005723	-
26	$R(10^n)$	1699246750872419991992147	-

to the operation of rounding to the nearest integer. Similarly, the values obtained by rounding to the nearest integer of the function (1.4) does not differ from those obtained by rounding $R(10^n)$ for the same reason.

It is seen that, for $n = 24$ and 25 , the values obtained with $\frac{10^n}{\log(10^n)}$ are relatively far from the real values of $\pi(10^n)$ with δ_n'' of the order 10^{-2} , while the values of $Li(10^n)$ and $R(10^n)$ gives much better results with δ_n'' of the order $10^{-13} \dots 10^{-14}$. For $n = 26$, the values of $Li(10^{26})$ and $R(10^{26})$ are within the range given by (2.13).

4. CONCLUSION

We proposed a simple method to interpolate a conjectured value of $\pi(10^{26})$, based on polynomial interpolations of the known 25 first values of $\pi(10^n)$ with two corrective functions, the first based on a Thiele interpolation of the relative differences of the exact and interpolated values of $\pi(10^n)$ and the second obtained by a new Thiele interpolation of the last five values of the relative differences of the exact and interpolated values of $\pi(10^n)$ with a previous value of the first corrective function. The range obtained for the conjectured value of $\pi(10^{26})$ contains those values calculated by the Eulerian logarithmic integral and with the Riemann functions.

APPENDIX 1: POLYNOMIALS $P_n(x)$ FOR $2 \leq n \leq 25$

$$P_2(x) = 21x - 17$$

$$P_3(x) = 61x^2 - 162x + 105$$

$$P_4(x) = (398/3)x^3 - 735x^2 + (3892/3)x - 691$$

$$P_5(x) = (1397/6)x^4 - (6587/3)x^3 + (44485/6)x^2 - (31033/3)x + 4897$$

$$P_6(x) = (41269/120)x^5 - (118219/24)x^4 + (648877/24)x^3 - (1679165/24)x^2 + (1677731/20)x - 36372$$

$$P_7(x) = (63053/144)x^6 - (2124317/240)x^5 + (10324961/144)x^4 - (14150231/48)x^3 + (46161541/72)x^2 - (6885127/10)x + 278893$$

$$P_8(x) = (1230899/2520)x^7 - (1059103/80)x^6 + (106869757/720)x^5 - (42511909/48)x^4 + (2168305201/720)x^3 - (692786273/120)x^2 + (199310224/35)x - 2182905$$

$$P_9(x) = (1087519/2240)x^8 - (12232463/720)x^7 + (8059007/32)x^6 - (295746589/144)x^5 + (9612782119/960)x^4 - (5337867053/180)x^3 + (17329512589/336)x^2 - (190033621/4)x + 17392437$$

$$P_{10}(x) = (8766223/20160)x^9 - (1526557/80)x^8 + (3642052523/10080)x^7 - (61717169/16)x^6 + (73322957917/2880)x^5 - (6425876741/60)x^4 + (119710398973/420)x^3 - (11000451139/24)x^2 + (13960806758/35)x - 140399577$$

$$P_{11}(x) = (1285948381/3628800)x^{10} - (13829848163/725760)x^9 + (2713678729/6048)x^8 - (734294140229/120960)x^7 + (8994784761253/172800)x^6 - (10167707046167/34560)x^5 + (80107171523893/72576)x^4 - (488994253144639/181440)x^3 + (204682947179773/50400)x^2 - (1697281382717/504)x + 1145548804$$

$$P_{12}(x) = (1762482209/6652800)x^{11} - (62163411143/3628800)x^{10} + (356291415727/725760)x^9 - (496320428783/60480)x^8 + (1914173704139/21600)x^7 - (111749346412919/172800)x^6 + (261253448575099/80640)x^5 - (4021281732287857/362880)x^4 + (22852595048399971/907200)x^3 - (1810394687049689/50400)x^2 + (113126344733737/3960)x - 9429344450$$

$$P_{13}(x) = (87613503863/479001600)x^{12} - (1117825763711/79833600)x^{11} + (4178914904089/8709120)x^{10} - (671147586733/69120)x^9 + (1870673383324673/14515200)x^8 - (2850595750775761/2419200)x^7 + (66036047077439107/8709120)x^6 - (50008128480417883/1451520)x^5 + (1188002313137637187/10886400)x^4 - (2241074007266561/9600)x^3 + (105572714697948823/332640)x^2 - (562059733555247/2310)x + 78184159413$$

$$P_{14}(x) = (730505748461/6227020800)x^{13} - (418827227947/39916800)x^{12} + (202948195226041/479001600)x^{11} - (37036465604101/3628800)x^{10} + (338543894892361/2073600)x^9 - (2209549040385367/1209600)x^8 + (638513890320991673/43545600)x^7 - (309191577104894683/3628800)x^6 + (3885522866329964983/10886400)x^5 - (1921510055977553161/1814400)x^4 + (1532239610437270573/712800)x^3 - (517946743840607861/184800)x^2 + (749473155761448241/360360)x - 652321589048$$

$$P_{15}(x) = (68044412311/968647680)x^{14} - (434610313091/59875200)x^{13} + (54461331244787/159667200)x^{12} - (1154924881829021/119750400)x^{11} + (2655624146858351/14515200)x^{10} - (1274125884612191/518400)x^9 + (814648425163663999/33868800)x^8 - (1890972821539241113/10886400)x^7 + (3363912482966950097/3628800)x^6 - (39446931450255536567/10886400)x^5 + (202795315040803498879/19958400)x^4 - (28079184621772453729/1425600)x^3 + (416132103978867941789/16816800)x^2 - (70617762061263719/3960)x + 5471675518942$$

$$P_{16}(x) = (51581436427121/1307674368000)x^{15} - (29037678164927/6227020800)x^{14} + (2356528303232491/9340531200)x^{13} - (3963130920435319/479001600)x^{12} + (188475479592111883/1026432000)x^{11} - (42323388417591161/14515200)x^{10} + (15614861323300656127/457228800)x^9 - (13050538335127845887/43545600)x^8 + (2588220260396190960353/1306368000)x^7 - (53527147712466477239/5443200)x^6 + (232250648536987989971/6415200)x^5 - (214241913350759156389/2217600)x^4 + (544191268377690363241837/3027024000)x^3 - (524844387326399456539/2402400)x^2 + (6906580916276240896/45045)x - 46109760908179$$

$$\begin{aligned}
P_{17}(x) = & (17450310541093/836911595520)x^{16} - (7313219107110283/2615348736000)x^{15} \\
& + (180531219567405389/1046139494400)x^{14} - (121365437052931943/18681062400)x^{13} \\
& + (3833898578147855953/22992076800)x^{12} - (44388621420665486663/14370048000)x^{11} \\
& + (311968634799119772881/7315660800)x^{10} - (408405984743715765221/914457600)x^9 \\
& + (104689503078054721538951/29262643200)x^8 - (57304152503501292804619/2612736000)x^7 \\
& + (587617527762868915556291/5748019200)x^6 - (256560216571537768707173/718502400)x^5 \\
& + (981414623224315552474067/1076275200)x^4 - (29710845425841425154523003/18162144000)x^3 \\
& + (389276944856756983375399/201801600)x^2 - (105829165042439513371/80080)x \\
& + 390148002619146
\end{aligned}$$

$$\begin{aligned}
P_{18}(x) = & (1851587512783157/177843714048000)x^{17} - (32892317466569501/20922789888000)x^{16} \\
& + (256329638772689/2335132800)x^{15} - (4918740790637408989/1046139494400)x^{14} \\
& + (103257725801778476353/747242496000)x^{13} - (339479305160332664821/114960384000)x^{12} \\
& + (636260413866908181391/13412044800)x^{11} - (170936402957313265801/292626432)x^{10} \\
& + (58215415330924617167453/10450944000)x^9 - (6039147285766985737916771/146313216000)x^8 \\
& + (1362372951548947987220519/5748019200)x^7 - (6008945224011701433565219/5748019200)x^6 \\
& + (2278512605057461714112724299/653837184000)x^5 \\
& - (1243038724423425067320005221/145297152000)x^4 \\
& + (2568389927503727128176139/172972800)x^3 - (3437947478393805445387967/201801600)x^2 \\
& + (139868165483953060330259/12252240)x - 3313027022947168
\end{aligned}$$

$$\begin{aligned}
P_{19}(x) = & (2628028871421403/533531142144000)x^{18} - (10567575582731201/12703122432000)x^{17} \\
& + (21003369878917183/321889075200)x^{16} - (8251376607822726667/2615348736000)x^{15} \\
& + (7058709267627854321/67060224000)x^{14} - (137168476397716450849/53374464000)x^{13} \\
& + (22944546189331191060451/482833612800)x^{12} - (271196454331659216727/399168000)x^{11} \\
& + (556013030981016791428933/73156608000)x^{10} - (2446752331753149321520481/36578304000)x^9 \\
& + (1840753991498633195715463/3973939200)x^8 - (72170805307845647867807051/28740096000)x^7 \\
& + (82740150719665463161579783529/7846046208000)x^6 \\
& - (98402301347766134600009057/2918916000)x^5 \\
& + (107115515185413881741729737/1341204480)x^4 \\
& - (814304007652526714435690759/6054048000)x^3 \\
& + (119147362970798190946403539/791683200)x^2 - (1210609739820985493184649/12252240)x \\
& + 28223319434109668
\end{aligned}$$

$$\begin{aligned}
P_{20}(x) = & (26934382640709253/12164510040883200)x^{19} \\
& - (166368869850866779/400148356608000)x^{18} \\
& + (2987733695853594407/82081714176000)x^{17} \\
& - (3538361726199431459/1793381990400)x^{16} \\
& + (211578849172220030591/2853107712000)x^{15} \\
& - (32187947021596169826761/15692092416000)x^{14} \\
& + (313223082576559198263053/7242504192000)x^{13} \\
& - (1028851538290603359043297/1448500838400)x^{12} \\
& + (44456420445740629307608553/4828336128000)x^{11} \\
& - (212467505628141678941887/2239488000)x^{10} \\
& + (3760078141213234941907434283/4828336128000)x^9 \\
& - (4906343664090048465680852441/965667225600)x^8 \\
& + (4812537157115295300593562547/183891708000)x^7 \\
& - (2475493108822130612914163579363/23538138624000)x^6 \\
& + (986767217403394741887962341/3048192000)x^5 \\
& - (5547876667800341701953704701/7472424960)x^4
\end{aligned}$$

$$\begin{aligned}
&+ (125190395324618857278613349737/102918816000)x^3 \\
&- (41062191060690947895024739489/30875644800)x^2 \\
&+ (15342006664059599043795083/17907120)x - 241120506972982862
\end{aligned}$$

$$\begin{aligned}
P_{21}(x) &= (768801874165960931/810967336058880000)x^{20} \\
&- (47895830419641353593/243290200817664000)x^{19} \\
&+ (19593696496463403689/1024379792916480)x^{18} \\
&- (448205658674858604491/388022648832000)x^{17} \\
&+ (8712644329419234113383/179338199040000)x^{16} \\
&- (189707965396142126979173/125536739328000)x^{15} \\
&+ (1809328805921664964890847/50214695731200)x^{14} \\
&- (253665818771705300538215719/376610217984000)x^{13} \\
&+ (527355591384145604585438497/52672757760000)x^{12} \\
&- (460925183825799784732199789/3862668902400)x^{11} \\
&+ (252657778262904116233611461/220723937280)x^{10} \\
&- (170656737803291291159227967873/19313344512000)x^9 \\
&+ (1906512687609116773290392565521/34871316480000)x^8 \\
&- (2302249554842995226554178587199/8559323136000)x^7 \\
&+ (19560002409333744035028716869169/18830510899200)x^6 \\
&- (199482007364920203714291611479/64576512000)x^5 \\
&+ (54603156665489883803994128245399/7939451520000)x^4 \\
&- (753730761427013331958837118357/68612544000)x^3 \\
&+ (156740472492042038497972499701/13332664800)x^2 \\
&- (866115476410575028425322373/116396280)x + 2065285115524899931
\end{aligned}$$

$$\begin{aligned}
P_{22}(x) &= (54386159471042293/140359731240960000)x^{21} \\
&- (11339777731555550441/128047474114560000)x^{20} \\
&+ (6933583440707809527311/729870602452992000)x^{19} \\
&- (16277845641786522178019/25609494822912000)x^{18} \\
&+ (173069291634903343162099/5820339732480000)x^{17} \\
&- (185313460297730292870869/179338199040000)x^{16} \\
&+ (31253190030180758167460353/1129830653952000)x^{15} \\
&- (439746760244290352119569907/753220435968000)x^{14} \\
&+ (27854308762263416011926122669/2824576634880000)x^{13} \\
&- (7087441471779667149333596851/52672757760000)x^{12} \\
&+ (17272738385198816463545470907/11588006707200)x^{11} \\
&- (74078609744831895304816687447/5518098432000)x^{10} \\
&+ (85271649858214871482393447550287/869100503040000)x^9 \\
&- (544996898468027472985563167444741/941525544960000)x^8 \\
&+ (70149627027741059244452076836923/25677969408000)x^7 \\
&- (957972246454320768062382326886223/94152554496000)x^6 \\
&+ (39105801276089863543963319222962243/1333827855360000)x^5 \\
&- (504262702012069575930635459206387/7939451520000)x^4 \\
&+ (71182532364843711432355416393311/718331328000)x^3 \\
&- (36474259397560749064143053993/350859600)x^2 \\
&+ (7533688390536925014719660137/116396280)x - 17731276931934494721
\end{aligned}$$

$$\begin{aligned}
P_{23}(x) &= (626047699915867799/4132355616829440000)x^{22} \\
&- (32308010721489584677/851515702861824000)x^{21} \\
&+ (2511103473848270746499/561438924963840000)x^{20}
\end{aligned}$$

$$\begin{aligned}
& - (990899107137455767879/3003582726144000)x^{19} \\
& + (2186295775571121711474881/128047474114560000)x^{18} \\
& - (255701609845973121847979/388022648832000)x^{17} \\
& + (119722559289831616224557863/6083703521280000)x^{16} \\
& - (1227534164901745056978902207/2636271525888000)x^{15} \\
& + (100182659337689089936042566167/11298306539520000)x^{14} \\
& - (2154242664974814138465530011/15692092416000)x^{13} \\
& + (3693244759563863089340159781247/2124467896320000)x^{12} \\
& - (348679419716275796227603114639/19313344512000)x^{11} \\
& + (24313863183439316679673602226004513/158176291553280000)x^{10} \\
& - (225569651273130061058657092854493/210901722071040)x^9 \\
& + (187595874338792344136694981328643/31039303680000)x^8 \\
& - (3134444555753150961218198128157/114124308480)x^7 \\
& + (46582581480572090585766685040140853/470762772480000)x^6 \\
& - (24628252584588755381459107318208291/88921857024000)x^5 \\
& + (332629942234296028174359960597644779/568586874240000)x^4 \\
& - (73343429897691545576959449692313389/82129215168000)x^3 \\
& + (741767456331546132245359568800763/806626220400)x^2 \\
& - (65620214252713224536332366223/116396280)x + 152553697445181546607
\end{aligned}$$

$$\begin{aligned}
P_{24}(x) &= (97841530496558649293/1723467782592331776000)x^{23} \\
& - (335407509903912323681/21615398611107840000)x^{22} \\
& + (454501108157980384073/227070854096486400)x^{21} \\
& - (107504314873384627683143/663518729502720000)x^{20} \\
& + (8983776081040688437620797/973160803270656000)x^{19} \\
& - (50404505281634117870005549/128047474114560000)x^{18} \\
& + (1168666259954948513855852177/89633231880192000)x^{17} \\
& - (2471024795530257064602656791/7189831434240000)x^{16} \\
& + (51511683265529304575508645967/7030057402368000)x^{15} \\
& - (721535570132495990554385229809/5649153269760000)x^{14} \\
& + (30370138820999258718781973500777/16570849591296000)x^{13} \\
& - (46160207442983253249607104925813/2124467896320000)x^{12} \\
& + (408425601328540276858577344312109/1917288382464000)x^{11} \\
& - (1907340749270059838043121409492039/1106127912960000)x^{10} \\
& + (10084417668013132593085446666361763/878757175296000)x^9 \\
& - (898532173217897912413387346270287/14411105280000)x^8 \\
& + (876114367621403488143688018877125577/3201186852864000)x^7 \\
& - (695775343904264963827651396107198739/727542466560000)x^6 \\
& + (26751505111131816927397223600866889/10266151896000)x^5 \\
& - (79477104547597150222616625089859530191/14783258730240000)x^4 \\
& + (403589896543329394729142860419921947/50190075936000)x^3 \\
& - (52511426579628409245398898318755171/6453009763200)x^2 \\
& + (26325492596728861363119710723467/5354228880)x - 1315069260003198192788
\end{aligned}$$

$$\begin{aligned}
P_{25}(x) &= (1810237126686092753419/88635485961891348480000)x^{24} \\
& - (62771250251033894473907/10340806695553990656000)x^{23} \\
& + (11511991109112409288411081/13488008733331292160000)x^{22} \\
& - (308610477223222584935011/4087275373736755200)x^{21} \\
& + (1651947699182756157926795881/350337889177436160000)x^{20} \\
& - (430292416333562341190138281/1946321606541312000)x^{19}
\end{aligned}$$

$$\begin{aligned}
 &+ (1693062186931273696473576313/209532230369280000) x^{18} \\
 &- (2440242314982415470323718869/10342295986176000) x^{17} \\
 &+ (42496596088997036408436172662149/7592461994557440000) x^{16} \\
 &- (13805367401638213159287611947619/126541033242624000) x^{15} \\
 &+ (10501406884609102136813341938733721/5965505852866560000) x^{14} \\
 &- (260923334002982532857433489649357/11047233060864000) x^{13} \\
 &+ (3149961803456717214034882917658324207/11931011705733120000) x^{12} \\
 &- (311007895030981777959409995030791881/126541033242624000) x^{11} \\
 &+ (36072334629973139334722245407290958851/1898115498639360000) x^{10} \\
 &- (3842017793050978223673061999182697157/31635258310656000) x^9 \\
 &+ (9526409805672264569123818314621843047/14966587883520000) x^8 \\
 &- (17325638483889432743363938986730301171/6402373705728000) x^7 \\
 &+ (5590963217656815416105678129476360121221/608225502044160000) x^6 \\
 &- (4938656200253742430432451463654529907/202164221952000) x^5 \\
 &+ (16036374727879322218852223186722596976589/325231692065280000) x^4 \\
 &- (43579059908227428615914106531732753781/602280911232000) x^3 \\
 &+ (3563764005018304979715944223158462749/49473074851200) x^2 \\
 &- (114930909252025411165097498628379/2677114440) x + 11356590626799451081145
 \end{aligned}$$

APPENDIX 2: COEFFICIENTS c_i AND K OF THE CORRECTIVE FUNCTION Φ_{25}

$$\begin{aligned}
 c_1 &= 0.72619047619047619047619047619047619047619048 \\
 c_2 &= -12.737322640345465761875385564466378778531770512029 \\
 c_3 &= -0.84164269999731510129175214910348073427241614034768 \\
 c_4 &= -17.908330903689032603895485760334710528008413393816 \\
 c_5 &= -21.793051968869308781494503792492773700351128057503 \\
 c_6 &= 0.052225391819545299875787414478370692540986477002 \\
 c_7 &= 22.867256875351930467366237953412010183478716371242 \\
 c_8 &= -1.707611319924344890707228312842658677594869606318 \\
 c_9 &= -1.2751632549654008209020770141453274057116453048774 \\
 c_{10} &= 92.538545141732012060260360179516606702270761634364 \\
 c_{11} &= 0.15088559448367433870497668805212911805725048971437 \\
 c_{12} &= 12.296035579537995708862927240169729240327162730035 \\
 c_{13} &= 0.47119594517891782795780492658781237198369447024440 \\
 c_{14} &= 103.88603190326434306252340497028079649950656584049 \\
 c_{15} &= 0.00206293819355223802469982625483902305678068096840 \\
 c_{16} &= -187.13045591233740512137368315625509082987760574938 \\
 c_{17} &= -0.27349669744110325871269828279318826062620906588370 \\
 c_{18} &= 60.84802398138390071856628905328286526305777316700 \\
 c_{19} &= 0.038078160763329325175635232207793882248458501401699 \\
 c_{20} &= -375.10551758837650646142692040134675795455506670138 \\
 c_{21} &= 0.12511512946150416165180535229285204189299499111780 \\
 c_{22} &= 71.229908224807213192533908002847675276237637077608 \\
 K &= -2.4774348845851811138845060795239470850970682780195
 \end{aligned}$$

APPENDIX 3: COEFFICIENTS d_i AND M OF THE CORRECTIVE FUNCTION $|\psi_{n+1}|$

$$\begin{aligned}
 d_1 &= -3.6415421544257471159022604244391523071564117009437 \times 10^{-7} \\
 d_2 &= -3485608.6856471123162187714001336338709185526509134 \\
 d_3 &= -4.6018631805033582153977197516014111986555627816497 \times 10^{-7}
 \end{aligned}$$

$$d_4 = -294118428.93353607774712567420242866941568199754406$$

$$M = 13267560.424903927928414820656294875945590431439845$$

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