# EVEN LATIN SQUARES OF ORDER 10

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Abstract. The Alon-Tarsi Conjecture says that the number of even latin squares of even order is not equal to the number of odd latin squares of the same order. The conjecture is known to be true for $n \leq 8$ and for all $n = 2^r \cdot p$, where $p$ is prime. In the current paper we show that the number of even latin squares is greater than the number of odd latin squares for $n = 10$.

## 1. Introduction and Notation

Let $n \in \mathbb{N}$, we denote the number of all latin squares of order $n$ by $L(n)$, the number of even latin squares of order $n$ by $Els(n)$ and the number of odd latin squares of order $n$ by $Ols(n)$.

Alon and Tarsi [4] conjectured that $Els(n) \neq Ols(n)$ for all even $n$. Drisko [6] proved that $Els(n) \neq Ols(n)$, when $n = p \cdot 2^r$ for any prime $p$. Further, it is known that $Els(n) > Ols(n)$ for $n \in \{2, 4, 6, 8\}$ ([1], [2]). It is clear that $L(n) = Els(n) + Ols(n)$ for all $n \in \mathbb{N}$. In the current paper, we show that $Els(10) > \frac{1}{2}L(10)$ and thus $Els(10) > Ols(10)$.

The number of latin squares is known for $n \in \{1, \ldots, 11\}$ (A002860 in OEIS [3] ) The number of even latin squares and odd latin squares are known for $n \in \{1, \ldots, 8\}$ (A114628 and A114629 in OEIS [1],[2])

**Remark 1.** *It is known that $Els(n) = Ols(n)$ for odd $n$ (see e.g. [7]) thus*

$$Els(9) = Ols(9) = \frac{1}{2}L(9) = 2762375748078446421265612800$$

*and*

$$Els(11) = Ols(11) = \frac{1}{2}L(11) =$$
$$388483418085885072053722173367115341155532800000.$$

**Definition 1.** *A latin square of order $n$ is a table of $n$ rows and $n$ columns, each of them having the entries $\{1, \ldots, n\}$, none of them having an entry twice. A latin square is called symmetric, if its rows coincide with its columns. A latin square is called reduced symmetric, if it is symmetric and its first row and first column are $(1, 2, \ldots, n)$.*

We denote the number of symmetric latin squares of order $n$ by $S(n)$ and the number of reduced symmetric latin squares of order $n$ by $R(n)$. Then we know [9] that $S(n) = n! \cdot R(n)$.

**Definition 2.** *Given a latin square $\Lambda$, we can consider its rows and columns as permutations on the set $\{1, \ldots, n\}$. We denote the row permutations by $\sigma_1, \ldots, \sigma_n$ and the column permutations by $\tau_1, \ldots, \tau_n$. A permutation is called even, if the number of its inversions is even, otherwise it is called odd. For more details on permutations see for example* [8].

| $\Lambda$ | $\tau_1$ | $\tau_2$ | $\ldots$ | $\tau_n$ |
|---|---|---|---|---|
| $\sigma_1$ | | | $\ldots$ | |
| $\sigma_2$ | | | $\ldots$ | |
| $\vdots$ | | | $\vdots$ | |
| $\sigma_n$ | | | $\ldots$ | |

**Definition 3.** *A latin square $\Lambda$ is called even, if $\prod_{i=1}^{n} \sigma_i \cdot \tau_i$ is even. Otherwise, $\Lambda$ is called odd.*

**Lemma 1.** *Let $\Lambda$ be a symmetric latin square. Then $\Lambda$ is even.*

*Proof.* Since $\Lambda$ is symmetric, we have $\sigma_i = \tau_i$ for each $i \in \{1, \ldots, n\}$. Thus $\prod_{i=1}^{n} \sigma_i^2$ contains an even number of odd permutations, i.e. the product itself is even. $\square$

## 2. Transformations

Let $\Lambda$ be a reduced symmetric latin square of order 10. Then by Lemma 1 $\Lambda$ is even. Our main goal is to find transformations on $\Lambda$ which keep the parity, but do not keep reduced symmetry.

**Lemma 2.** *Let $\Lambda$ be an even latin square of even order $n$. Then a latin square gained by any permutation of rows and/or columns of $\Lambda$ is also even.*

*Proof.* Let $\omega$ be a permutation of the columns of $\Lambda$. We denote the latin square gained by $\omega$ as $\Lambda^*$ and its row and column permutations by $\sigma_1^*, \ldots, \sigma_n^*$ and $\tau_1^*, \ldots, \tau_n^*$ respectively. It is clear that $\prod_{i=1}^{n} \tau_i = \prod_{i=1}^{n} \tau_i^*$. Although $\omega$ is moving different symbols in each row, its parity is the same. If $\omega$ is an odd permutation and $\sigma_i$ is even, then $\sigma_i^*$ is odd. If $\sigma_i$ is odd, then $\sigma_i^*$ is even. If $\omega$ is even, then the parity of $\sigma_i$ does not change. Since $n$ is even we have that the parity of $\prod_{i=1}^{n} \sigma_i$ equals the parity of $\prod_{i=1}^{n} \sigma_i^*$. The proof is similar if $\omega$ is a permutation of the rows. $\square$

**Remark 2.** *The number of all permutations of rows and columns in a latin square of order $n$ is $(n!)^2$.*

**Lemma 3.** *Let $\Lambda$ be a reduced symmetric latin square. Then there exists a row permutation $\sigma_i$ which contains a transposition $(1, i)$.*

*Proof.* We denote the nodes of $\Lambda$ by $\Lambda(i, j)$ for $i, j \in \{1, \ldots, n\}$.

Let $\Lambda$ be a reduced symmetric latin square. We assume indirectly that there is no transposition containing 1. Since $\Lambda(1, 1) = 1$ the non existence of such a transposition means that there is no other index $i$, such that $\Lambda(i, i) = 1$. Since $\Lambda$ is symmetric, we have $\Lambda(i, j) = 1$ implies $\Lambda(j, i) = 1$ for any $i, j \in \{2, \ldots, n\}$. Thus we have an even number $+1$ of 1-s in $\Lambda$. But $\Lambda$ is a latin square, therefore for even $n$ there is an even number of 1-s in $\Lambda$, which is a contradiction. Thus there exists at least one $i \neq 1$, such that $\Lambda(i, i) = 1$ and thus $\sigma_i$ contains the transposition $(1, i)$. $\square$

We define a map $\chi : \Lambda \mapsto \Lambda^*$ in the following way: Choose the row permutation containing a transposition $(1, i)$. Then we fix 1 and $i$ in the $1^{st}$ and $i^{th}$ row and interchange all other elements of the affected two rows, i.e. we switch the two affected rows except for the elements $\{1, i\}$. We demonstrate the map $\chi$ in the picture below.

$$
\begin{array}{cccccccccc}
\multicolumn{10}{l}{\Lambda} \\
\boxed{1} & 2 & 3 & 4 & 5 & \boxed{6} & 7 & 8 & 9 & 10 \\
2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 1 \\
3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 1 & 2 \\
4 & 5 & 6 & 7 & 8 & 9 & 10 & 1 & 2 & 3 \\
5 & 6 & 7 & 8 & 9 & 10 & 1 & 2 & 3 & 4 \\
\boxed{6} & 7 & 8 & 9 & 10 & \boxed{1} & 2 & 3 & 4 & 5 \\
7 & 8 & 9 & 10 & 1 & 2 & 3 & 4 & 5 & 6 \\
8 & 9 & 10 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
9 & 10 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
10 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9
\end{array}
\quad \mapsto \quad
\begin{array}{cccccccccc}
\multicolumn{10}{l}{\Lambda^*} \\
\boxed{1} & 7 & 8 & 9 & 10 & \boxed{6} & 2 & 3 & 4 & 5 \\
2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 1 \\
3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 1 & 2 \\
4 & 5 & 6 & 7 & 8 & 9 & 10 & 1 & 2 & 3 \\
5 & 6 & 7 & 8 & 9 & 10 & 1 & 2 & 3 & 4 \\
\boxed{6} & 2 & 3 & 4 & 5 & \boxed{1} & 7 & 8 & 9 & 10 \\
7 & 8 & 9 & 10 & 1 & 2 & 3 & 4 & 5 & 6 \\
8 & 9 & 10 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
9 & 10 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
10 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9
\end{array}
$$

**Lemma 4.** *Let $\Lambda$ be a reduced symmetric latin square. Then $\chi(\Lambda)$ is not reduced symmetric.*

*Proof.* Since the first row of $\Lambda$ is $1, 2, \ldots, i, \ldots, n$ we have in $\chi(\Lambda)$ only 1 and $i$ at their natural place (place in natural order), all other $n - 2$ numbers are not at their natural place. $\square$

**Lemma 5.** *For any reduced symmetric latin square $\Lambda$ of order 10 we have that $\chi(\Lambda)$ is even.*

*Proof.* We denote the row and column permutations of $\Lambda^*$ by $\sigma_1^*, \ldots, \sigma_{10}^*$ and $\tau^*, \ldots, \tau_{10}^*$ respectively. We have $\sigma_j^* = \sigma_j$ for all $j \neq 1, j \neq i$.

Since $10 = 2 \cdot 5$, we know that $\sigma_i$ is the product of five transpositions, one of them is $(1, i)$ by Lemma 3. Then $\sigma_1^* = \sigma_i \cdot (1, i)$, which is the product of four transpositions. Thus $\sigma_1^*$ is an even permutation as $\sigma_1 = 1_{id}$. Further, $\sigma_i^* = (1, i) \cdot \sigma_1 = (1, i)$, i.e. $\sigma_i^*$ is an odd permutation as $\sigma_i$. Therefore $\prod_{i=1}^{10} \sigma_i^*$ has the same parity as $\prod_{i=1}^{10} \sigma_i$. Now, we check the column permutations. We have $\tau_1^* = \tau_1$ and $\tau_i^* = \tau_i$. All other $\tau_j$ are multiplied by one transposition. Thus $\prod_{k=1}^{10} \tau_k^*$ has the same parity as $\prod_{k=1}^{10} \tau_k \cdot ((j_1, j_2)(j_3, j_4)(j_5, j_6)(j_7, j_8))^2$, where $j_1, \dots, j_8 \in \{2, 3, 4, 5, 6, 7, 8, 9, 10\} \setminus \{i\}$. □

**Theorem 1.** $Els(10) \geq R(10) \cdot (10!)^4$

*Proof.* Let $\Lambda$ be a reduced symmetric latin square of order 10. By Lemma 1 $\Lambda$ is even. Further, Lemma 3 makes it possible to create a map $\chi : \Lambda \mapsto \Lambda^*$, such that $\Lambda^*$ is even by Lemma 5 and not reduced symmetric by Lemma 4. Further, we can permute all rows and columns of $\Lambda$ and $\chi(\Lambda)$, combine these in any order and get again an even latin square. □

## 3. Reduced symmetric latin squares of order 10

We compute $R(10)$ with Algorithm 1. We generate every possible reduced symmetric latin square of order 10 recursively. We build a function that is given an incomplete square as an input, and yields all possible squares that can be obtained from filling out that square as an output. This function will systematically generate every possible reduced symmetric square when the input square is an incomplete square only filled out with the first row and the first column ($1, 2, \dots, 10$ for both).

Algorithm 1 shows a high-level description of how the function works. In the description, $\{\sigma_i\}$ represents the set of elements of $\sigma_i$ and $p_j$ represents the $j$th element of a permutation $p$.

Algorithm 1 was implemented in Python, and executed on a server provided by the University of Debrecen. The implementation can be found at $https://arato.inf.unideb.hu/major.sandor/research-latin.py$. The computation is single-threaded, running on a 3.1 GHz Intel(R) Xeon(R) Gold 6254 CPU. The code is essentially a Python generator function, yielding, counting and discarding reduced symmetric latin squares of order 10 at an average speed of $\sim 4036$ squares per second.

**Algorithm 1** Generate reduced symmetric latin squares of order $n$

**Input** $\Lambda$: a possibly incomplete square, $n$: the order of $\Lambda$
**Output** all reduced symmetric latin squares of order $n$ that can be obtained by filling out $\Lambda$

**function** GENERATE($\Lambda$, $n$)
    **if** $\Lambda$ has no incomplete lines **then**
        **yield** $\Lambda$
    **else**
        Find the smallest $1 < i \leq n$ for which $\sigma_i$ is an incomplete line of $\Lambda$.
        Let $E = \{1, \ldots, n\} \setminus \{\sigma_i\}$ be the set of permissible elements in $\sigma_i$. [1]
        Let $P$ be the set of permissible permutations of $E$. [2]
        **for each** $p \in P$ **do**
            Let $\Lambda'$ be a copy of $\Lambda$.
            Fill out $\sigma_i'$ and $\tau_i'$ with $p$.
            **for each** $\Lambda^r \in Generate(\Lambda', n)$ **do**
                **yield** $\Lambda^r$
            **end for**
        **end for**
    **end if**
**end function**

1: By the construction of the algorithm, $|E| = n - i + 1$
2: A $p \in P$ is permissible if for each $j \in \{1, \ldots, |E|\}$, $p_j \notin \tau_{j+i-1}$. In other words, filling out $\sigma_i$ with $p$ would not violate the rules of a latin square.

Although the computation has not ended yet, on the 87th day of computation (125106 minutes) we got 30337000000 reduced symmetric latin squares. Thus

$$R(10) \geq 30337000000$$

We know by [3] and [5] the number of all latin squares of order 10 is

$$L(10) = 9982437658213039871725064756920320000.$$

We have by Theorem 1

$$Els(10) \geq R(10) \cdot (10!)^4 \geq 5260472602655869580083200000000000000 >$$
$$4991218829106519935862532378460160000 = \frac{1}{2} \cdot L(10).$$

Thus

$$Els(10) > Ols(10).$$

## References

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