On the Normality of Arithmetical Constants

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ABSTRACT. Bailey and Crandall [4] recently formulated "Hypothesis A", a general principle to explain the (conjectured) normality of the binary expansion of constants like π and other related numbers, or more generally the base b expansion of such constants for an integer $b \geq 2$. This paper shows that a basic mechanism underlying their principle, which is a relation between single orbits of two discrete dynamical systems, holds for a very general class of representations of numbers. This general class includes numbers for which the conclusion of "Hypothesis A" is not true. The paper also relates the particular class of arithmetical constants treated by Bailey and Crandall to special values of G-functions, and points out an analogy of "Hypothesis A" with Furstenberg's conjecture on invariant measures.

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1. Introduction

Much is known about the irrationality and transcendence of classical arithmetical constants such as π, e, and $\zeta(n)$ for $n \geq 2$. There are general methods which in many cases establish irrationality or transcendence of such numbers. In contrast, almost nothing is known concerning the question of whether arithmetical constants are normal numbers to a fixed base, say $b = 2$. It is unknown whether any algebraic number is normal to any integer base $b \geq 2$. Even very weak assertions in the direction of normality are unresolved. For example, it is not known whether $\sqrt{2}$ has arbitrarily long blocks of zeros appearing in its binary expansion, i.e. whether $\liminf_{n\to\infty}\{\{2^n\sqrt{2}\}\} = 0$.

Recently Bailey and Crandall [4] formulated "Hypothesis A", which provides a hypothetical general principle to explain the (conjectured) normality to base 2 of certain arithmetical constants such as π and $\log 2$.

Hypothesis A. *Given a positive integer* $b \geq 2$ *and a rational function* $R(x) = \frac{p(x)}{q(x)} \in \mathbb{Q}(x)$, *such that* $\deg(p(x)) < \deg(q(x))$, *and* $q(x)$ *has no nonnegative integer roots, define* $\theta = \sum_{n=0}^{\infty} \frac{p(n)}{q(n)} b^{-n}$. *If* $y_0 = 0$ *and*

$$y_{n+1} = by_n + \frac{p(n)}{q(n)} \pmod 1, \tag{1.1}$$

then the sequence $\{y_n : n \geq 1\}$ *either has finitely many limit points or is uniformly distributed* (mod 1).

This hypothesis concerns the behavior of a particular orbit of the discrete dynamical system (1.1). Assuming "Hypothesis A," Bailey and Crandall deduced that the number θ either is rational or else is a normal number to base b; these correspond to the two possible behaviors of the sequence $\{y_n : n \geq 1\}$ allowed by "Hypothesis A", see Theorem 4.1 below. Proving "Hypothesis A" appears intractable, but it seems useful in collecting a number of disparate phenomona together under a single principle. A formulation in terms of dynamical systems is natural, because the property of normality is itself expressable in terms of dynamics of an orbit of another dynamical system, the b-transformation, see §2. The basic mechanism rendering "Hypothesis A" useful is a relation between particular orbits of these two different dynamical systems.

This paper provides some complements to the results of Bailey and Crandall. It shows that the relation between particular orbits of two discrete dynamical systems underlying "Hypothesis A" is valid very generally, in that it applies to expansions of real numbers of the form, $\theta = \sum_{n=1}^{\infty} \epsilon_n b^{-n}$, with ϵ_n arbitrary real numbers with $\epsilon_n \to 0$ as $n \to \infty$, see Theorem 3.1. Every real number has such an expansion, and "Hypothesis A" is not true in such generality. Thus in order to be valid "Hypothesis A" must be restricted to apply only to expansions of some special form. Bailey and Crandall do this, formulating "Hypothesis A" only for a countable class of arithmetical constants which in the sequel we call *BBP-numbers.* It does not seem clear what should be the "optimal" class of arithmetical constants for which "Hypothesis A" might be valid. The remainder of the paper discusses various mathematical topics relevant to this issue. We relate BBP numbers to the theory of G-functions and characterize the subclass of BBP-numbers which are "special values" of G-functions. We also compare "Hypothesis A" to a conjecture of Furstenberg in ergodic theory, and this suggests some further questions to pursue.

We now summarize the contents of the paper in more detail. In §2 and §3 we give the dynamical connection underlying "Hypothesis A." In §2 we review radix expansions to an integer base $b \geq 2$ treated as a discrete dynamical system acting on the interval $[0, 1]$. The radix expansion of a real number θ is described by an orbit of a dynamical system, the b-transformation $T_b(x) = bx \pmod 1$, studied by Renyi [34] and Parry [31]. For a given number θ its b-expansion can be computed from the iterates of this system

$$x_{n+1} = bx_n \pmod 1,$$

with initial condition $x_0 = \theta \pmod 1$. The *b-expansion* of a real number $\theta \in [0,\ 1]$ is

$$\theta = \sum_{j=1}^{\infty} d_j b^{-j},$$

in which the j-th digit $d_j := \lfloor b x_{j-1} \rfloor$. In §3 we suppose the given real number θ is expressed as

$$\theta = \sum_{n=1}^{\infty} \epsilon_n b^{-n}, \tag{1.2}$$

in which ϵ_n is any sequence of real numbers with $\epsilon_n \to 0$ as $n \to \infty$. To this one can associate a *perturbed b-expansion* associated to the *perturbed b-transformation*

$$y_{n+1} = by_n + \epsilon_n \pmod 1, \tag{1.3}$$

starting with an initial condition $y_0 \in [0, 1)$. The recurrence (1.3) is an infinite sequence of maps which change at each iteration. Associated to this recurrence is the *perturbed b-expansion*

$$y_0 + \theta = \sum_{j=0}^{\infty} \tilde{d}_j b^{-j},$$

in which the j-th digit $\tilde{d}_j := \lfloor by_j + \epsilon_{j+1} \rfloor$. Choosing the initial condition $y_0 = 0$ gives a perturbed b-expansion of θ. The mechanism underlying the approach of Bailey and Crandall is that the the b-expansion of θ and the perturbed b-expansion of θ are strongly correlated in the following sense: The orbit $\{y_n \ : \ n \geq 0\}$ of the perturbed b-transformation with initial condition $y_0 = 0$ asymptotically approaches the orbit $\{x_n \ : \ n \geq 0\}$ associated to the b-transformation with initial condition $x_0 = \lfloor \theta \rfloor$. (Theorem 3.1) In particular, the orbits $\{x_n \ : \ n \geq 0\}$ and the orbit $\{y_n \ : \ n \geq 0\}$ have the same set of limit points, and one is uniformly distributed (mod 1) if and only if the other is. This implies that the perturbed b-expansion of θ, although different from the b-expansion of θ, must have similar statistics, in various senses. This connection is quite general, since every real number θ has representations of the form (1.2).

In §4 we consider the particular class of arithmetical constants treated in Bailey and Crandall [4], consisting of the countable set of θ given by an expansion (1.2) with $b \geq 2$ an integer and $\epsilon_n = \frac{p(n)}{q(n)}$ where $p(x), q(x) \in \mathbb{Z}(x)$, with $q(n) \neq 0$ for all $n \geq 0$. We call such numbers *BBP-numbers,* and call the associated formula

$$\theta = \sum_{n=1}^{\infty} \frac{p(n)}{q(n)} b^{-n},$$

a *BBP-expansion to base b* of θ. These numbers are named after Bailey, Borwein and Plouffe [5], who demonstrated the usefulness of such representations (when $\deg(p(x)) < \deg(q(x))$) in computing base b radix expansions of such numbers. We consider BPP-numbers having the additional restriction $\deg(p(x)) < \deg(q(x))$, for this condition is necessary and sufficient for $\epsilon_n \to 0$ as $n \to \infty$, so that the results of §3 apply. The number-theoretic character of BPP-numbers is that they are special values (at rational points) of functions satisfying a homogeneous linear differential equation with integer polynomial coefficients. We derive the result of Bailey and Crandall that "Hypothesis A" implies that such θ either are rational or are normal numbers to base b (Theorem 4.1.) This result makes it of interest to find criteria to determine when BBP-numbers are irrational, which we consider next.

In §5 we relate BBP-numbers to the theory of G-functions, and characterize the subclass of BBP-numbers which are "special values" of G-functions. The subject of G-functions has been extensively developed in recent years, see [2], [7], [14], and the special values of such functions can often be proved to be irrational, see [7], [13],[19]. We observe that BBP-numbers satisfy all but one of the properties required to be a special value of a G-function defined over the base field $\mathbb{Q}$. We then show that a BBP-expansion to base b corresponds to a special value of a G-series at $z = \frac{1}{b}$ if and only if the denominator polynomial $q(x)$ (in lowest terms) factors into linear factors over the rationals.(Theorem 5.2.) We show that if all the roots of $q(x)$ are distinct, then such special values are either rational or transcendental, using Baker's results on linear forms in logarithms, in Theorem 5.3, a result also obtained in Adikhari, Saradha, Shorey and Tijdeman [1]. We summarize other known results on irrationality or transcendence of special values of G-functions of the type in Theorem 5.2. It is interesting to observe that every one of the many examples given in [4] is a special value of a G-function. Many other interesting examples of such constants were known earlier, for example in 1975 D. H. Lehmer [26, p. 139] observed that

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)(2n+1)(4n+1)} = \frac{\pi}{3}.$$

In §6 we compare "Hypothesis A" with a conjecture of Furstenberg in ergodic theory, which concerning measures that are ergodic for the joint action of two multiplicatively independent b-transformations. Both conjectures have similar conclusions, though there seems to be no direct relation between their hypotheses. Bailey and Crandall have found examples of arithmetical constants which have the property of being BBP-numbers to two multiplicatively independent bases. This suggests that one should look for further conditions under which the two conjectures are more directly related.

In §7 we make concluding remarks. We describe an empirical taxonomy of various classes of arithmetical constants, and formulate some alternative classes of arithmetical constants as candidates for inclusion in "Hypothesis A."

2. Radix Expansions

We consider radix expansions to an integer base $b \geq 2$. They are obtained by iterating the *b-transformation*

$$T_b(x) = bx \pmod{1} . \tag{2.1}$$

Given a real number $x_0 \in [0, 1)$, as initial condition, we produce the sequence of *remainders*

$$x_{n+1} = bx_n \pmod{1} , \tag{2.2}$$

with $0 \leq x_{n+1} < 1$. That is,

$$x_{n+1} = bx_n - d_{n+1} \tag{2.3}$$

where

$$d_{n+1} = d_{n+1}(x_0) = \lfloor bx_n \rfloor \in \{0, 1, \ldots, b-1\} \tag{2.4}$$

is called the *n-th digit of θ*. The *forward orbit* of x_0 is $\mathcal{O}^+(x_0) = \{x_n : n \geq 0\}$ and we call $\{x_n\}$ the *remainder sequence* of the b-expansion. Iterating (2.3) $n+1$ times yields

$$x_{n+1} = b^{n+1}x_0 - d_{n+1} - bd_n - \cdots - b^n d_1 . \tag{2.5}$$

Dividing by b^{n+1} yields

$$x_0 = \sum_{j=1}^{n} d_j b^{-j} - b^{-n-1} x_{n+1} .$$

Letting $n \to \infty$ yields the *b-expansion* of x_0,

$$x_0 = \sum_{j=1}^{\infty} d_j(x_0) b^{-j} , \tag{2.6}$$

which is valid for $0 \leq x_0 < 1$. For $\theta \in \mathbb{R}$ we take $x_0 = \theta - \lfloor \theta \rfloor$ and $d_0(\theta) = \lfloor \theta \rfloor \in \mathbb{Z}$, thus obtaining the representation

$$\theta = d_0(\theta) + \sum_{j=1}^{\infty} d_j(\theta) b^{-j} , \tag{2.7}$$

which is called the *b-expansion* of *theta*. Note that (2.5) gives

$$x_n \equiv b^n x_0 \pmod 1 \equiv b^n \theta \pmod 1 \tag{2.8}$$

in this case.

The following property than an initial condition θ may have concerns the topological dynamics of the b-transformation for its iterates.

Definition 2.1 A real number $\theta \in [0, 1)$ is *digit-dense to base* b if, for every $m \geq 1$, every legal digit sequence of digits of length m occurs at least once as consecutive digits in the b-expansion $\theta = \sum\limits_{n=1}^{\infty} d_n(\theta)\beta^{-n}$.

The following property that an initial condition θ may have concerns the metric dynamics of the b-transformation for θ. It is well known that the b-transformation T_b has the uniform measure (Lebesgue measure) on $[0, 1]$ as its unique absolutely continuous invariant measure.

Definition 2.2 A real number $\theta \in [0, 1)$ is *normal to base* b if for every $m \geq 1$ every digit sequence $d_1 d_2 \cdots d_m \in \{0, 1, \ldots, d-1\}^m$ occurs with limiting frequency b^{-m}, as given by the invariant measure μ_{Leb}.

Recall that

$$\mu_{Leb}(\{x_0 : d_1(x_0) \cdots d_n(x_0) = d_1 d_2 \cdots d_m\}) = b^{-m} ,$$

where $\mu_{Leb}(S)$ denotes the Lebesgue measure of S. It is well known that, for each $b \geq 2$, the set of $\theta \in [0, 1]$ that is normal to base b has full Lebesgue measure.

The properties of the digit expansion $\{d_n(\theta) : n \geq 1\}$ can be extracted from the remainder sequence $\{x_n\}$. The following result is well known.

Theorem 2.1 *Given an integer base $b \geq 2$ and a real number $\theta \in [0, 1]$.*

(1) θ is digit-dense to base b if and only if its remainder sequence $\{x_n(\theta) : n \geq 1\}$ to base b is dense in $[0, 1]$.

(2) θ is normal to base b if and only if its remainder sequence $\{x_n : n \geq 1\}$ to base b is uniformly distributed in $[0, 1]$.

(3) θ has an eventually periodic b-expansion. if and only if its remainder sequence $\{x_n : n \geq 1\}$ to base b has finitely many limit points. This condition holds if and only if $\{x_n : n \geq 1\}$ eventually enters a periodic orbit of the b-transformation, i.e. $x_m = x_{m+p}$ for some m, $p \geq 1$. These equivalent conditions hold if and only θ is rational.

Proof. (1). The set $I(d_1 d_2 \cdots d_m) := \{\theta \in [0,1] : d_1(\theta) \cdots d_m(\theta) = d_1 \cdots d_m\}$ is a half-open interval $[a, a+b^{-m})$ of length b^{-m}, and the b^m intervals partition $[0,1]$. Digit-denseness implies there exists some $x_k \in I(d_1 \cdots d_m)$. This holds for all $m \geq 1$ and generates a dense set of points.

(2). If θ has $\{x_n : n \geq 1\}$ is uniformly distributed (mod 1) then the correct frequency of points occurs in each interval $I(d_1 \cdots d_m)$, and this proves normality of θ. For the converse, one uses the fact that $I(d_1 \cdots d_m)$ is a basis for the Borel sets in $[0,1)$.

(3). The key point to check is that if the limit set of $\{x_n : n \geq 1\}$ is finite, then this finite set forms a single periodic orbit of the b-transformation, and some x_n lies in this orbit. We omit details, cf. Bailey and Crandall [4, Theorem 2.8]. ∎

Remark. Most of the results above generalize to the β-transformation $T_\beta(x) = \beta x \pmod 1$ for a fixed real $\beta > 1$; these maps were studied by Parry [31]. Associated to this map is the notion of a *β-expansion* for any real number θ, in which the allowed digits are $\{0, 1, 2, \ldots, \lfloor\beta\rfloor\}$. Not all digit sequences are allowed in β-expansions, but the set of allowed digit sequences was characterized by Parry [31], see Flatto et al. [17] for other references. One defines a number θ to be *digit-dense to base β* if every allowable finite digit sequence occurs in its β-expansion.. There is a unique absolutely continuous invariant measure $d\mu$ of total mass one for the β-transformation, and one defines a number θ to be *normal to base β* if every finite block of digits occurs in its β-expansion with the limiting frequency prescribed by this invariant measure. With these conventions, Theorem 2.1 remains valid for a general base β, except that Theorem 2.1(3) must be taken only as characterizing eventually periodic orbits of the β-transformation. That is, the final assertion in (3) that θ is rational must be dropped; it does not hold for general β. For results relating normality of numbers in different real bases β, see Brown, Moran and Pollington [12].

3. Perturbed Radix Expansions

Let $b \geq 2$ be an integer, and let $\{\epsilon_n : n \geq 1\}$ be an arbitrary sequence of real numbers satisfying

$$\lim_{n\to\infty} \epsilon_n = 0 \ . \tag{3.1}$$

Set

$$\theta = \theta(b, \{\epsilon_n\}) := \sum_{n=1}^{\infty} \epsilon_n b^{-n} \ . \tag{3.2}$$

We can study the real number θ using a *perturbed b-expansion* associated to the sequence $\{\epsilon_n\}$.

The *perturbed b-transformation* on $[0,1)$ is the recurrence

$$y_{n+1} = by_n + \epsilon_{n+1} \pmod 1 \ , \tag{3.3}$$

with $0 \leq y_{n+1} < 1$ and with given initial condition y_0. That is,

$$y_{n+1} = by_n + \epsilon_{n+1} - \tilde{d}_{n+1}, \tag{3.4}$$

where

$$\tilde{d}_{n+1} = \lfloor by_n + \epsilon_{n+1} \rfloor \in \mathbb{Z} \tag{3.5}$$

is the $(n+1)$-*st digit* of the expansion. The *digit sequence* $\tilde{d}_n = d_n(y_0)$ and *remainder sequence* $\{y_n : n \geq 0\}$ depend on the initial condition y_0. Since $\epsilon_n \to 0$, for all sufficiently large n, one has $\tilde{d}_n \in \{-1, 0, 1, \ldots, b-1, b\}$. Now (3.4) iterated $n+1$ times yields

$$y_{n+1} = \epsilon_{n+1} + b\epsilon_n + \cdots + b^n \epsilon_1 + b^{n+1} y_0 - \sum_{j=0}^{n} \tilde{d}_{n-j} b^j \ . \tag{3.6}$$

Dividing by b^{n+1} yields

$$\sum_{j=1}^{n} \tilde{d}_j b^{-j} = \sum_{j=1}^{n} \epsilon_j b^{-j} + (y_0 - b^{-n-1} y_{n+1}) \ . \tag{3.7}$$

Letting $n \to \infty$ yields the *perturbed b-expansion*

$$y_0 + \theta = \sum_{j=1}^{\infty} \tilde{d}_j(y_0) b^{-j} \ , \tag{3.8}$$

valid for $0 \leq y_0 < 1$. We write $y_n = y_n(y_0)$ for the remainder sequence in (3.4)

The *perturbed b-expansion* $\{d_n^*(\theta) : n \geq 1\}$ for θ given by (3.2) is obtained by choosing the initial condition $y_0 = 0$, i.e. $d_n^*(\theta) := \tilde{d}_n(0)$. We also have the *perturbed remainders* $\{y_n^*(\theta) : n \geq 1\}$ given by $y_n^*(\theta) = y_n(0)$.

The main observation of this section is that the remainders of the perturbed b-expansion of such θ are related to the remainders of their b-expansion.

Theorem 3.1 *Let $b \geq 2$ be an integer and let $\theta := \sum_{n=1}^{\infty} \epsilon_n b^{-n}$, where ϵ_n are real numbers with $\epsilon_n \to 0$ as $n \to \infty$. Let $\{y_n^*(\theta) : n \geq 1\}$ denote the associated perturbed remainder sequence of θ, and $\{x_n(\theta) : n \geq 1\}$ denote the remainder sequence of its b-expansion. If*

$$t_n := \sum_{j=1}^{\infty} \epsilon_{n+j} b^{-j} \ , \tag{3.9}$$

then

$$x_n(\theta) = y_n^*(\theta) + t_n \pmod 1 \ . \tag{3.10}$$

The orbits $\{x_n(\theta) : n \geq 1\}$ and $\{y_n^*(\theta) : n \geq 1\}$ asymptotically approach each other on the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ as $n \to \infty$.

Proof. Since $y_0 = 0$, formula (3.6) gives

$$y_{n+1} = \sum_{j=1}^{n} b^{n-j} \epsilon_j \pmod 1 \ . \tag{3.11}$$

Now

$$b^n\theta = \sum_{y=1}^{\infty} b^{n-j}\epsilon_j = \sum_{j=1}^{n} b^{n-j}\epsilon_j + t_n \ .$$

Thus

$$b^n\theta = y_n + t_n \pmod 1 \ . \tag{3.12}$$

For the b-expansion, (2.8) gives $b_n\theta \equiv x_n \pmod 1$, and combining this with (3.12) yields (3.10).

Since $\epsilon_n \to 0$ as $n \to \infty$, we have $t_n \to 0$ as $n \to \infty$. Thus $|x_n(\theta) - y_n^*(\theta)| \to 0$ on $\mathbb{T}$ as $n \to \infty$. Note that on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ the points ϵ and $1-\epsilon$ are close. ∎

Lemma 3.2 *Let $\{x_n : n \geq 1\}$ and $\{y_n : n \geq 1\}$ be any two sequences in $[0,1]$ with $x_n = y_n + \delta_n \pmod 1$ with $\delta_n \to 0$ as $n \to \infty$.*

(1) The sequences $\{x_n : n \geq 1\}$ and $\{y_n : n \geq 1\}$ have the same sets of limit points, provided the endpoints 0 and 1 are identified.

(2) The sequence $\{x_n : n \geq 1\}$ is uniformly distributed $\pmod 1$ if and only if $\{y_n : n \geq 1\}$ is uniformly distributed $\pmod 1$.

Proof. (1) This is clear since $x_{n_j} \to \psi$ implies $y_{n_j} \to \psi$ and vice-versa, except at the endpoints $\psi = 0$ or 1, which, by convention, we identify as the same point.

(2) This is well known, see Kuipers and Niederreiter [25, Theorem 1.2, p. 3]. ∎

One can compare the b-expansion $\{d_n(\theta) : n \geq 1\}$ and the perturbed b-expansion $\{d_n^*(\theta) : n \geq 1\}$ of such θ. We have

$$\begin{aligned} d_n(\theta) &= \lfloor b x_{n-1} \rfloor \\ d_n^*(\theta) &= \lfloor b_{n-1} + \epsilon_n \rfloor = \lfloor b(x_{n-1} + t_{n-1} (\mathrm{mod}\ 1)) + \epsilon_n \rfloor \ . \end{aligned}$$

Since $t_n \to 0$ and $\epsilon_n \to 0$ as $n \to \infty$, one expects that "most" digit values of the two expansions will agree[1], i.e. $d_n(\theta) = d_n^*(\theta)$ for "most" sufficiently large values of n. However there is still room for there to be infinitely many n where $d_n(\theta) \neq d_n^*(\theta)$.

We next consider perturbed b-expansions having a finite number of limit points, and show that they correspond to rational θ.

Theorem 3.3 *Let $b \geq 2$ be an integer and let $\theta = \sum_{n=1}^{\infty} \epsilon_n b^{-n}$ with ϵ_n a sequence of real numbers with $\epsilon_n \to 0$ as $n \to \infty$. The following conditions are equivalent.*

(i) $\theta \in \mathbb{Q}$.

(ii) The remainders $\{y_n^*(\theta) : n \geq 1\}$ of the perturbed b-expansion of θ have finitely many limit points in $[0,1]$.

[1] This is an unproved heuristic statement. It is an open problem to prove that a natural density one proportion of all n have $d_n(\theta) = d_n^*(\theta)$.

(iii) The orbit $\{y_n^*(\theta) : n \geq 1\}$ of the perturbed b-transformation asymptotically approaches a periodic orbit $\{x_k : 0 \leq k \leq p\}$ of the b-transformation, with $T_b(x_k) = x_{k+1}$ and $T_b(x_p) = x_0$ and for $0 \leq j \leq p-1$, such that

$$y_n = x_j + \delta_n \pmod 1 \quad \text{if} \quad n \equiv j \pmod p \tag{3.13}$$

with $\delta_n \to 0$ as $n \to \infty$.

Proof. (i)⇒(ii). By Theorem 2.1 if $\theta \in \mathbb{Q}$ the remainders $\{x_n(\theta) : n \geq 1\}$ of the b-transformation have finitely many limit points. By Theorem 3.1 and Lemma 3.2 we conclude that $\{y_n^*(\theta) : n \geq 1\}$ has the same set of limit points.
(ii)⇒(iii). By Theorem 3.1 and Lemma 3.2 the limit points of $\{y_n^*(\theta) : n \geq 1\}$ are the same as $\{x_n(\theta) : n \geq 1\}$. By Theorem 2.1 such limit points must form a periodic orbit of the b-transformation.
(iii)⇒(i). The values $\{y_n^*(\theta) : n \geq 1\}$ have limit points the periodic orbit $\{x_j : 1 \leq j \leq n\}$ of T_b. By Theorem 2.1, it follows that $\theta \in \mathbb{Q}$. ■

Remarks. (1). Any real number θ has some perturbed b-expansion satisfying the hypotheses of Theorem 3.1, so in a sense these expansions are completely general. It follows from Theorem 3.3 that Hypothesis A cannot be valid for all such θ, since there exist irrational θ that are not normal numbers.

(2). The rationality criterion of Theorem 3.3 is not directly testable computationally, unless all $\epsilon_n = 0$ for $n \geq n_0$; the latter case essentially is the same as that of a b-transformation. When infinitely many $\epsilon_n \neq 0$, then the points $\{y_n^*(\theta) : n \geq 1\}$ stay outside the periodic orbit for infinitely many values of n, and the role of the $\{\epsilon_n\}$ is to compensate for the expanding nature of the map $T(x) = bx \pmod 1$ by providing negative feedback to push the iterates closer and closer to the periodic orbit.

(3). Theorem 3.1 does not extend to β-expansions for non-integer β. One can consider

$$\theta = \theta(\beta, \{\epsilon_n\}) := \sum_{n=1}^{\infty} \epsilon_n \beta^{-n} \ . \tag{3.14}$$

and define an associated perturbed β-transformation in the obvious way. However when β is not an integer the analogue of Theorem 3.1 fails to hold, since (3.12) is no longer valid. In particular, Theorem 3.1 does not extend to rational $\beta = \frac{b}{a} > 1$, with $a > 1$.

4. BBP-Numbers and Hypothesis A

We consider expansions of the following special form.

Definition 4.1 A *BBP-number to base b* is a real number θ with a representation

$$\theta = \sum_{n=1}^{\infty} \frac{p(n)}{q(n)} b^{-n} \ , \tag{4.1}$$

in which $b \geq 2$ is an integer and $p(x)$, $q(x) \in \mathbb{Z}[x]$ are relatively prime polynomials, with $q(n) \neq 0$ for each $n \in \mathbb{Z}_{\geq 0}$. We call (4.1) a *BBP-expansion to base b.*

Bailey, Borwein and Plouffe [5, p. 904] introduced this class of numbers, proving that for them the d-th digit is computable[2] in time at most $O(d\log^{O(1)} d)$ using space at most $O(\log^{O(1)} d)$, which is the complexity class SC^*, a subclass of SC, see [24, p. 127].

We mainly consider BBP-numbers that satisfy the extra condition

$$\deg(q(x)) > \deg(p(x)) \ . \tag{4.2}$$

This condition guarantees that $\epsilon_n = \frac{p(n)}{q(n)} \to 0$ as $n \to \infty$, which allows Theorem 3.1 to be applicable. We now formulate two hypotheses, whose conclusions are in terms of topological dynamics and metric dynamics, respectively. The second of these is "Hypothesis A" of Bailey and Crandall [4].

Weak Dichotomy Hypothesis. *Let there be given a perturbed b-transformation with $\epsilon_n = \frac{p(n)}{q(n)}$ with $p(x),\ q(x) \in \mathbb{Z}[x]$ and $\deg(q(x)) > \deg(p(x))$. Then the orbit $\{y_n : n \geq 1\}$ for $\theta(b, \{\epsilon_n\})$ either has finitely many limit points or else is dense in $[0,1]$.*

Strong Dichotomy Hypothesis *Let there be given a perturbed b-transformation with $\epsilon_n = \frac{p(n)}{q(n)}$ with $p(x),\ q(x) \in \mathbb{Z}[x]$ and $\deg(q(x)) > \deg(p(x))$. Then the orbit $\{y_n : n \geq 1\}$ for $\theta(b, \{\epsilon_n\})$ either has finitely many limit points or is uniformly distributed on $[0,1]$. Equivalently, in measure theoretic terms, the measures $\mu_N = \frac{1}{N}\sum_{k=1}^{N} \delta_{y_k}$ converge in the vague topology as $N \to \infty$ to a limit measure μ, which is an invariant measure for the b-transformation, and which is either a measure supported on a finite set or else is Lebesgue measure on $[0,1]$.*

Bailey and Crandall [4] essentially established the following result.

Theorem 4.1 *Let θ be a BBP-number to base b whose associated BBP expansion satisfies*

$$\deg(q(x)) > \deg(p(x)) \ . \tag{4.3}$$

Then the following conditional results hold.

(1) The Weak Dichotomy Hypothesis implies that θ is either rational or digit-dense to base b.

(2) The Strong Dichotomy Hypothesis implies that θ is either rational or a normal number to base b.

Proof. The condition (4.3) guarantees that $\epsilon_n = \frac{p(n)}{q(n)} \to 0$ as $n \to \infty$. Thus Theorem 3.1 applies to the BBP-number

$$\theta = \sum_{n=1}^{\infty} \frac{p(n)}{q(n)} b^{-n} \ .$$

(1) By the Weak Dichotomy Hypothesis the limit set of $\{y_n^*(\theta) : n \geq 1\}$ is dense in $[0,1]$. Now Lemma 3.2 (1) implies that b-expansion remainders $\{x_n(\theta) : n \geq 1\}$ are dense in $[0,1]$. Theorem 2.1 (1) then shows that θ is digit-dense.

[2] Bailey, Borwein and Plouffe use the convention that "computing the d-th digit" means computing is an approximation to $b^d\theta \pmod 1$ that is guaranteed to be within a specified distance to it (mod 1). Usually this determines the d-th digit, but it may not, near the endpoints of the digit interval.

(2) By the Strong Dichotomy Hypothesis $\{y_n^*(\theta) : n \geq 1\}$ is uniformly distributed in $[0,1]$. Now Lemma 3.2 (2) implies that $\{x_n(\theta) : n \geq 1\}$ is uniformly distributed in $[0,1]$. Now θ is normal to base b by Theorem 2.1 (2). ▪

Bailey, Borwein and Plouffe [5] and Bailey and Crandall [4] give many examples of BBP-numbers satisfying (4.2) where the associated real number θ is known to be irrational. For example for various b one can obtain π, $\log 2$, $\zeta(3)$ etc. They also observe that $\zeta(5)$ is a BBP-number, to base $b = 2^{60}$, but it remains an open problem to decide if $\zeta(5)$ is irrational. All the examples they give of BBP-numbers are actually of a special form: they are "special values" of G-functions defined over $\mathbb{Q}$, as we discuss next.

5. Special Values of G-Functions

The notion of G-function was introduced by Siegel [37] in 1929.

Definition 5.1 A power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{5.1}$$

defines a *G-series over the base field* $\mathbb{Q}$ if the following conditions hold.

(i) *Rational coefficients condition.* All $a_n \in \mathbb{Q}$ so we may write $a_n = \frac{p_n}{q_n}$, with $p_n, q_n \in \mathbb{Z}$ with $(p_n, q_n) = 1$ and $q_n \geq 1$.

(ii) *Local analyticity condition.* The power series $f(z)$ has positive radius of convergence r_∞, and for each prime p the p-adic function $f_p(z) := \sum_{n=0}^{\infty} a_n z^n$ viewing $a_n \in \mathbb{Q} \subseteq \mathbb{Q}_p$ has positive radius of convergence r_p in $\mathbb{C}_p$, where $\mathbb{C}_p = \hat{\bar{\mathbb{Q}}}_p$ is the completion of the algebraic closure of $\mathbb{Q}_p$.

(iii) *Linear differential equation condition.* The power series $f(z)$ formally satisfies a homogeneous linear differential equation in $D = \frac{d}{dz}$ with coefficients in the polynomial ring $\mathbb{Q}[z]$.

(iv) *Growth condition.* There is a constant $C < \infty$ such that

$$g_n := \mathrm{lcm}(q_1, q_2, \ldots, q_n) < C^n \tag{5.2}$$

for all $n \geq 1$.

There is an extensive theory of G-functions, see Bombieri [7], André [2] and Dwork, Geroth and Sullivan [14]. For the general definition of a G-function over an algebraic number field K see André [2, p. 14], or Dwork et al. [14]. G-functions have an important role in arithmetic algebraic geometry, where it is conjectured that G-functions are exactly the set of solutions over $\bar{\mathbb{Q}}[z]$ of a geometric differential equation over $\bar{\mathbb{Q}}$, as defined in Andre [2, p. 2]. In any case it is known that the (minimal) homogeneous linear differential equation satisfied by a G-series is of a very restricted kind: it must have regular singular points, and these must all have rational exponents, by a result of Katz, cf. Bombieri [7, p. 46] and Bombieri and Sperber [8]. (The growth condition (iv) plays a crucial role in obtaining this result.) It follows that a G-series analytically continues to a multi-valued function on $\mathbb{P}^1(\mathbb{C})$ minus a finite number of singular points, cf. [14, p. xiv]. We call this multi-valued function a *G-function.*

It is known that the set $\mathcal{G}_K$ of G-series defined over a number field K form a ring over K, under addition and multiplication, which is also closed under the Hadamard product

$$f \boxtimes g(z) = \sum_{n=0}^{\infty} a_n b_n z^n \ , \tag{5.3}$$

see [2, Theorem D, p. 14].

Definition 5.2 A *special value of a G-function* defined over K is a value $f(b)$, where $b \in K$, which is obtained by analytically continuation along some path from 0 to b that avoids singular points.

Siegel [37] introduced G-functions and observed that irrationality results could be proved for their "special values", but did not give any details. Bombieri [7] developed the theory of G-functions and gave explicit irrationality criteria in specific cases (his Theorem 6) for points close to the center of the circle of convergence of the G-series, as a by-product of very general results.

It is easy to show that each BBP-number is a special value of a power series on $\mathbb{Q}$ that satisfies conditions (i)–(iii) of a G-series. They do not always satisfy the growth condition (iv), however, and in a subsequent result we give necessary and sufficient conditions for the condition (iv) to hold.

Theorem 5.1 *Let $R(x) = \frac{p(x)}{q(x)} \in \mathbb{Q}(x)$ with $p(x),\ q(x) \in \mathbb{Q}[x]$ with $(p(x), q(x)) = 1$ and with $q(n) \neq 0$ for all $n \geq 0$, and set*

$$f(z) = \sum_{n=0}^{\infty} \frac{p(n)}{q(n)} z^n \ . \tag{5.4}$$

Let $f_p(z)$ be the p-adic power series obtained by interpreting $\frac{p(n)}{q(n)} \in \mathbb{Q} \subseteq \mathbb{Q}_p$. Then the power series $f(z)$ satisfies a homogeneous linear differential equation in $\frac{d}{dz}$ with coefficients in $\mathbb{Q}[z]$, and $f(z)$ has positive radius of convergence in $\mathbb{C}$ and $f_p(z)$ has a positive radius of convergence in $\mathbb{C}_p = \widehat{\bar{\mathbb{Q}}_p}$ for all primes p.

Proof. For the first assertion, let $p(x) = \sum_{j=0}^{l} a_j x^j$ and $q(x) = \sum_{j=0}^{m} b_j x^j$. Then the operator

$$D := \frac{d^{l+1}}{dz^{l+1}}(1-z)^{l+1}(\sum_{j=0}^{m} b_j (z\frac{d}{dz})^j) \in \mathbb{Q}[z, \frac{d}{dz}] \tag{5.5}$$

has the property that

$$Df(z) = 0. \tag{5.6}$$

Indeed one has

$$q(z\frac{d}{dz})f(z) = \sum_{n=0}^{\infty} p(n) z^n = \sum_{j=0}^{l} a'_j (\frac{1}{1-z})^{j+1},$$

where a'_j are defined by the polynomial identity

$$\sum_{j=0}^{l} a_j x^j = \sum_{j=0}^{l} a'_j \binom{x}{j}.$$

Multiplying this rational function by $(1-z)^{l+1}$ yields a polynomial of degree l in z, which is annihilated by $\frac{d^{l+1}}{dz^{l+1}}$, and this verifies (5.6).

For the second assertion, the power series expansion of $f(z)$ clearly has radius of convergence 1 in $\mathbb{C}$. It is easy to establish that the the p-adic series $f_p(z)$ has a positive radius of convergence on some p-adic disk around zero since $|q(n)| \le cn^d$ cannot contain more than $cd \log n$ factors of p. ∎

We now give necessary and sufficient conditions for a power series arising from a BBP-number to be a G-series.

Theorem 5.2 *Let $R(x) = \frac{p(x)}{q(x)} \in \mathbb{Q}(x)$ with $p(x)$, $q(x) \in \mathbb{Q}[x]$ with $(p(x), q(x)) = 1$ and with $q(n) \neq 0$ for all $n \geq 0$, and set $f(z) = \sum_{n=0}^{\infty} \frac{p(n)}{q(n)} z^n$. Then the power-series $f(z)$ is a G-series (necessarily defined over $\mathbb{Q}$) if and only if $q(x)$ factors into linear factors in $\mathbb{Q}[x]$.*

Proof. Suppose first that $q(x)$ factors into linear factors over $\mathbb{Q}$, say

$$q(x) = A \prod_{j=1}^{r} L_j(x)$$

with $L_j(x) = l_j x + m_j$ with $l_j, m_j \in \mathbb{Z}$ with $(l_j, m_j) = 1$. To show $f(z)$ is a G-series, by Theorem 5.1 it suffices to we check the growth condition (iv). Now

$$\begin{aligned} \mathrm{lcm}(q_1, q_2, \ldots, q_n) &\leq \mathrm{lcm}(q(1), q(2), \ldots, q(n)) \\ &\leq |A| \prod_{j=1}^{r} \mathrm{lcm}(L_j(1), \ldots, L_j(n)) \end{aligned} \tag{5.7}$$

where $L_j(n) = l_j x + m_j$. It is well-known that

$$\begin{aligned} \log(\mathrm{lcm}[1, 2, \ldots, m]) &= \sum_{\{p,j : p^j \leq m\}} \log p \\ &= \sum_{n=1}^{m} \Lambda(n) = m + O(m) \end{aligned} \tag{5.8}$$

by the prime number theorem. This yields

$$\mathrm{lcm}[1, 2, \ldots, m] = e^{m(1+o(1))} \tag{5.9}$$

as $m \to \infty$. This gives a bound

$$\begin{aligned} \mathrm{lcm}(L_j(1), \ldots, L_j(n)) &\leq \mathrm{lcm}(1, 2, \ldots, |l_j|n + |m_j|) \\ &\leq e^{(|l_j|n + |m_j|)(1+o(1))} . \end{aligned}$$

Substituting this in (5.7) implies condition (iv).

For the opposite direction, we will show that if $q(x)$ does not factor into linear factors over $\mathbb{Q}$ then condition (iv) does not hold. Nagell [28] showed that if $q(x) \in \mathbb{Z}[x]$ is an irreducible polynomial of degree $d \geq 2$, then there is a positive constant $c(d)$ with the property that for any $\epsilon > 0$ there is a positive constant $C(\epsilon)$ such that

$$\mathrm{lcm}(q(1), q(2), \ldots, q(n)) > C(\epsilon) n^{(c(d)-\epsilon)n} \tag{5.10}$$

holds for all $n \geq 1$. One can prove this result with $c(d) = \frac{d-1}{d^2}$. Such a lower bound applies to any denominator $q(x)$ that does not split into linear factors over $\mathbb{Q}$. To complete the argument one must bound the possible cancellation between the numerators $p(n)$, and denominators $q(n)$. If $(p(x), q(x)) = 1$ over $\mathbb{Z}[x]$, then

$$\prod_{j=1}^{n} \gcd(p(j), q(j)) \leq C^n, \tag{5.11}$$

for a finite constant $C = C(p(x), q(x))$. This follows since

$$\gcd(p(n), q(n)) \leq C$$

holds for all n, for a suitable C. To see this, factor $p(x) = \prod(x - \alpha_i)$ and $q(x) = \prod(x - \beta_j)$, with $\alpha_i \neq \beta_j$ for all i, j. Then one has, over the number field K spanned by these roots,

$$\mathrm{ideal} - \gcd((n - \alpha_i), (n - \beta_j)) \mid (\alpha_i - \beta_j). \tag{5.12}$$

Taking a norm from $K/\mathbb{Q}$ of the product of all these ideals gives the desired constant C. ∎

Remarks. (1) It is an interesting open question to determine what is the largest value of $c(d)$ allowed in (5.10). One can prove that it cannot be larger than $d - 1$.

(2) There are many more G-functions defined over $\mathbb{Q}$ than those given in Theorem 5.2. The set of G-functions defined over $\mathbb{Q}$ is closed under multiplication, so that $(\log(1-z))^2)$ is a G-function, but its power series coefficients around $z = 0$ are not given by a rational function. Also, for rational a, b, c the Gaussian hypergeometric function

$${}_2F_1(a, b, c, z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n,$$

is a G-function which is not of the above kind for "generic" a, b, c, see André [3].

According to the results of §4, the conclusion of "Hypothesis A" is really a statement about irrational BBP-numbers. There is a good deal known about the irrationality or transcendence of the special values of the G-series covered in Theorem 5.2, a topic which we now address.

Theorem 5.3 *Let $R(x) = \frac{p(x)}{q(x)} \in \mathbb{Q}(x)$ with $p(x)$, $q(x) \in \mathbb{Q}[x]$ with $(p(x), q(x)) = 1$ and with $q(n) \neq 0$ for all $n \geq 0$, and set*

$$f(z) = \sum_{n=0}^{\infty} \frac{p(n)}{q(n)} z^n \ .$$

If $q(z)$ factors into distinct linear factors over $\mathbb{Q}$, then for each rational r in the open disk of convergence of $q(z)$ around $z = 0$ the special value $f(r)$ is either rational or transcendental. Furthermore there is an effective algorithm to decide whether $f(r)$ is rational or transcendental.

Proof. We only sketch the details, since a similar result has been obtained by Adhikari, Saradha, Shorey and Tijdeman [1], see also Tijdeman [38, Theorem 6].

By expanding $R(x)$ in partial fractions, under the hypothesis that $q(x)$ splits in linear factors over $\mathbb{Q}$ one obtains an expansion of the form

$$R(x) = p_0(x) + \sum_{j=1}^{s} \frac{c_j}{x - r_j},$$

in which $p_0(x) \in \mathbb{Q}[x]$, and each $c_j, r_j \in \mathbb{Q}$. In fact $r_j \notin \mathbb{Z}_{\geq 0}$, so all denominators $q(n) \neq 0$. Now if $r_j = \frac{p_j}{q_j}$ then one has a decomposition,

$$\sum_{n=0}^{\infty} \frac{1}{n - r_j} z^j = p_j(z) + \sum_{k=1}^{q_j} \beta_{j,k} \log(1 - \exp(\frac{2\pi i k}{q_j})z),$$

in which $p_j(z)$ is a polynomial with rational coefficients, while the coefficients β_j are effectively computable algebraic numbers in the field $\mathbb{Q}(exp(\frac{2\pi i}{q_j}))$. It follows from this that one can express the function $f(z)$ as a finite sum of terms of the form $\frac{a_j}{(1-z)^j}$ with rational coefficients plus a finite sum of terms of the form $-\beta_{j,k} \log(1 - \alpha_j z)$, with β_j, α_j effectively computable algebraic numbers. The non-logarithmic terms all combine to give a rational function $R_0(z)$ with coefficients in $\mathbb{Q}$. Given a rational r with $0 < |r| < 1$, it follows that $f(r)$ is a finite sum of linear forms in logarithms with algebraic coefficients, evaluated at algebraic points. Using Baker's transcendence result on linear forms in logarithms (Baker [6, Theorem 2.1]), $f(r)$ is transcendental if and only if the sum of all the logarithmic terms above is nonzero. There is also an effective decision procedure to tell whether this sum is zero or not. If the logarithmic terms do sum to zero, then the remaining rational function terms sum up to the rational number $f(r) = R_0(r)$. ∎

The case where $q(x)$ factors into linear factors over $\mathbb{Q}$ but has repeated factors is not covered in the result above. This case includes the polylogarithm $\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}$ of order k, for each $k \geq 2$. Various results are known concerning the irrationality of such numbers. For example, $\mathrm{Li}_k(\frac{1}{b})$ is irrational for all sufficiently large integers b, see Bombieri [7]. In fact it is known that the set of numbers $1, \mathrm{Li}_1(\frac{p}{q}), ..., \mathrm{Li}_n(\frac{p}{q})$, with $\mathrm{Li}_1(z) = \log(1 - z)$, are linearly independent over the rationals whenever $|p| \geq 1$ and $|q| \geq (4n)^{n(n-1)}|p|^n$, according to Nikishin [30]. For polylogarithms one has $\mathrm{Li}_k(1) = \zeta(k)$, also on the boundary of the disk of convergence. It is not known whether $\zeta(k)$ is irrational for odd $k \geq 5$, although a very recent result of T. Rivoal [35] shows that an infinite number of $\zeta(k)$ for odd k must be irrational.

6. Invariant Measures and Furstenberg's Conjecture

It is well known that for single expanding dynamical system, such as the b-transformation T_b , there always exist chaotic orbits exhibiting a wide range of pathology. For example, there exist uncountably many $\theta \in [0, 1]$ whose 2-transformation iterates $\{x_n\}$ satisfy

$$\frac{1}{25} < x_n < \frac{24}{25} \quad \text{for all} \quad n \geq 0 \ ,$$

see Pollington [33]. One can obtain ergodic invariant measures of T_b supported on the closure of suitable orbits, which for example may form Cantor sets of measure zero.

If one considers instead two b-transformations, say T_{b_1} and T_{b_2}, with multiplicatively independent values, i.e. which generate a non-lacunary commutative semigroup $\mathcal{S} = \langle T_{b_1}, T_{b_2} \rangle$, then the set of ergodic invariant measures for the whole semigroup is apparently of an extremely restricted form. Furstenberg has proposed the following conjecture, suggested as an outgrowth of his work on topological dynamics, cf. Furstenberg [18, Sect. IV]. It is explicitly stated in Margulis [27, Conjecture 4].

Furstenberg's Conjecture. *Let $a, b \geq 2$ be multiplicatively independent integers. The only Borel measures on $[0, 1]$ that are simultaneously invariant ergodic measures for $T_a(x) = ax \pmod 1$ and $T_b(x) = bx \pmod 1$ are Lebesgue measure and measures supported on finite sets which are periodic orbits of both T_a and T_b.*

Various results concerning this conjecture appear in Rudolph [36], Parry [32], Host [22] and Johnson [23]. In particular, if there is any exceptional invariant measure violating the conjecture, it must have entropy zero with respect to Lebesgue measure.

Furstenberg's conjecture involves some ingredients similar to "Conjecture A", and its conclusion involves a dichotomy similar to that in "Conjecture A." This makes it natural to ask if there is any relation between the two conjectures. At present none is known, in either direction.

One may look for BBP-numbers $\theta \notin \mathbb{Q}$ which have properties similar to that expressed in the hypothesis of Furstenberg's conjecture , i.e. which possess BBP-expansions to two multiplicatively independent bases. It is known that there exist irrational BBP-numbers $\theta = \sum_{n=1}^{\infty} R(n) b^{-n}$ which do possess BBP-expressions to two multiplicatively independent bases. For example, Bailey and Crandall [4] observe that $\theta = \log 2$ has this property, on taking

$$b = 2 \qquad \text{and} \qquad R(x) = \frac{1}{x},$$

and

$$b = 3^2 \qquad \text{and} \qquad R(x) = \frac{6}{2x - 1},$$

see [4, eqn (4), and (10)]. They also observe that $\theta = \pi^2$ has this property, as it possesses BBP-expansions to bases $b = 2$ and $b = 3^4$, the latter one found by Broadhurst [11, eqn. (212), p. 35].

Question. Do all BBP-numbers which are special values of G-functions have BBP-expansions in two multiplicatively independent bases?

To make tighter a possible connection between the two conjectures, one can ask for which numbers does the following weaker version of "Hypothesis A" hold.

Invariant Measure Hypothesis *Every BBP-number to base b has b-transformation iterates $\{x_n\}$ that are asymptotically distributed according to a limiting measure on $[0, 1]$.*

It would be interesting to find extra hypotheses on a class of arithmetical constants under which a precise connection can be established between "Hypothesis A" and Furstenberg's conjecture.

7. Concluding Remarks

Many of the examples of arithmetical constants arise as special values of G-functions defined over the rationals, or at least "special values" of functions satisfying linear differential equations with polynomial coefficients in $\mathbb{Q}[x]$. Based on the known results, one may empirically group these constants into three classes, of apparently increasing order of difficulty of establishing irrationality or transcendence results.

(1). special values of G-functions $f(\frac{p}{q})$ defined over the rationals, with $\frac{p}{q}$ inside the disk of convergence of the G-series.

(2). "singular values" $f(1)$ of such a G-function, which are values taken at a singular point of the associated (minimal order) linear differential equation, on the boundary of the disk of convergence of a G-series, at which the G-expansion converges absolutely.

(3). "renormalized singular values," which are the constant terms in an asymptotic expansion of a G-function around a singular point.

In this hierarchy, an arithmetical constant may occur as more than one type. For example, $\frac{\pi^2}{6} = \zeta(2) = \mathrm{Li}_2(1)$ occurs as a number of type (2), but it is also realized as a number of type (1), which falls in the class of constants considered in this paper. It is a nontrivial problem to determine what is the lowest level in the hierarchy a given constant belongs.

Various constants of types (1) and (2) appear in the renormalization of massive Feynman diagrams, see Broadhurst [11] and Groote, Körner and Pivovarov [21], who cite $\mathrm{Li}_4(\frac{1}{2})$ as such a constant. Multiple zeta values and polylogarithms give many examples of type (2), see Borwein et al. [9], [10]. Many of the most interesting arithmetical constants naturally arise as constants of type (2) and (3). For examples, the values $\zeta(k) = \mathrm{Li}_k(1)$ appear as constants of type (2), while Euler's constant appears as a type (3) "renormalized" value at $z = 1$ of $\mathrm{Li}_1(z)$. The problem of showing the linear independence of all odd zeta values $\zeta(2n+1)$ over the rationals has recently been of great interest from connections with various conjectures in arithmetical algebraic geometry, see Goncharov [20]. Many other examples of type (2) and (3) constants appear in Lehmer [26] and Flajolet and Salvy [16]. I am not aware of any irrationality or transcendence results proved for a constant of type (3).

One can extend the hierarchy above outside the class of G-functions. E. Bombieri observes that the power series

$$h(z) = \sum_{n=1}^{\infty} \frac{1}{n(n^2+1)} z^n$$

of BBP-type, which is not a G-series, has special value at $z = 1$ given by

$$h(1) = \frac{1}{2}\Re(\frac{\Gamma'(i)}{\Gamma(i)}).$$

The value $z = 1$ lies on the boundary of the disk of convergence of the power series for this function, and corresponds to type (2) above. Another example is

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1} = \frac{2\pi}{e^{\pi} - e^{-\pi}} - 1,$$

see Flajolet and Salvy[16, p. 18], who give many other interesting examples.

The relevant special values of a rational power series for the approach of Bailey and Crandall to apply are $z = \frac{1}{b}$ for integer $b \geq 2$, where the disk of convergence of the associated power series has radius 1. One observes that the theory of G-functions provides irrationality results for rational values $z = \frac{a}{b}$, without regard for whether $a = 1$ or not. This suggests the following question.

Question. Given a rational value $z = \frac{a}{b}$, with $1 < |a| < |b|$, is there an associated dynamical system (possibly higher dimensional) for which an analogue of Theorem 3.3 holds, relating the dynamics of one orbit to the β-expansion of θ, with $\beta = \frac{a}{b}$.?

At present there seems to be no evidence that strongly favors a particular class of arithmetical constants for which "Hypothesis A" might be expected to hold. The discussions of §5 and §6 suggest that one might consider the following classes.

(1). The largest class is the set of "special values" of power series $f(z)$ defined over $\mathbb{Q}$ at $z = \frac{1}{b}$, arising from solutions of $Df(z) = 0$ for some $D \in \mathcal{W} := \mathbb{Q}[z, \frac{d}{dz}]$, whose power-series coefficients $a_n \to 0$ as $n \to \infty$. This class includes all BBP-numbers.

(2) One could restrict to the subclass of special values $z = \frac{1}{b}$ of G-functions defined over the rationals. However we know of no compelling reason to restrict to special values of G-functions.

(3) The smallest class consists of a class of arithmetical constants which satisfy extra conditions analogous to the hypotheses of Furstenberg's conjecture. These consist of those constants which are BBP-numbers to at least two multiplicatively independent bases. One might add the further restriction that they also be special values of G-functions. As noted in §6, this class includes π^2 and $\log 2$.

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