TAYLOR SERIES FOR ARCTAN AND BBP-TYPE FORMULAS FOR π

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Dedicated to the memory of Prof. Ratan Prakash Agarwal (1925–2008), President of the Indian Mathematical Society and editor of its periodicals, who encouraged me to carry on with my work in mathematics

Abstract. We give here a neat Taylor series for the arctan function and use that to deduce a dozen BBP-type formulas for π. In addition, an alternative approach for deriving more formulas is also explained.

1. Introduction

In 1995 Bailey, Borwein, and Plouffe [3] discovered a lovely formula for π:

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right).$$

They found it by way of "inspired guessing and extensive searching" using Ferguson's PSLQ integer relation finding algorithm. The "proof" followed the discovery. Earlier, it was believed that if one wanted to determine the n-th digit of π, one had to generate the entire sequence of the first n digits. These authors also found an algorithm for computing individual hexadecimal or binary digits of π. The algorithm is explained by Bailey in a note.[4]

The above-noted *base-2* or binary formula allows us to compute the *nth* hexadecimal or binary digit of π, without computing any of the previous digits. Now a formula of this sort is called as BBP-type formula after the (initials of) three co-discoverers. It may be pointed out that the new algorithm "is not fundamentally faster than best-known schemes for computing all digits of π up to some position."

This notation was introduced by D.H. Bailey and R.E. Crandall later:

$$P(s,\, b,\, n,\, A) = \sum_{k=0}^{\infty} \frac{1}{b^k} \sum_{j=1}^{n} \frac{a_j}{(kn+j)^s},$$

where s, b and n are integers, and $A = (a_1, a_2, \ldots, a_n)$ is a vector of integers. Then the first formula becomes $\pi = P(1, 16, 8,\, (4, 0, 0, -2, -1, -1, 0, 0))$.

Adamchik and Wagon [1] employed *Mathematica* to derive an alternating BBP-type formula. We intend to derive here some such formulas (including that of Adamchik and Wagon) *mathematically* from a beautiful Taylor series expansion of the arctan.

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2. Taylor's Theorem and A Series for Arctan

As is commonly known, a Taylor series is a power series representation of a function as an infinite sum of terms which are evaluated from the function's derivatives at a single point in its domain. The Scottish mathematician James Gregory (1638–75) communicated to John Collins (1625–1683), English mathematician and librarian of the Royal Society, in 1670–71 a number of results on infinite series expansions of various trigonometric functions, including what is now known as Gregory's series for the arctan function. His 1667 book *Vera Circuli et Hyperbolae Quadratura,* reprinted in 1668 with an appendix, *Geometriae Pars,* contained "the earliest enunciation"[6] of the expansions in series of $\sin x$, $\cos x$, $\arcsin x$ and $\arccos x$.

However, it is now an established fact [15, 17] that an enunciation of the inverse tangent series is found in Sanskrit verses attributed to an Indian mathematician Madhava (1340–1425) of Sangamagrama (near Kochi, Kerala) and quoted in the 16th century commentary *Kriyakramkari* on *Lilavati* of Bhaskaracharya (1114-1185). The series for $arctan\, x$ and other trigonometric functions are given in Sanskrit verse in *Tantrasangraha* of Nilakantha (1450–1550, Kerala) and a commentary on this work called *Tantrasangraha-vyakhya* of unknown authorship. *Yuktibhasa* of Jyesthadeva (1500–1610), a commentary in Malalyalam on the *Tantrasangraha* contains a proof of the arctan series. In his *Aryabhatiya-bhasya*, a commentary on Aryabhata's work on astronomy, Nilakantha attributes the sine series to Madhava.[22]

Brook Taylor (1685–1731), an English mathematician, provided the general method for the formation of such series. His method, now known as Taylor's theorem, appears merely as a corollary (Corollary II of Proposition VII, Theorem III) to the corresponding theorem in Finite Differences in his 1715 work *Methodus Incrementorum Directa & Inversa*[24]. But, as Duncan Gregory (1813-1844) notes, he makes no application of it, or remark on its importance.[14]

What Taylor's theorem in effect does is to develop a function of the algebraic sum of two quantities into a series arranged according to the ascending power of one of these quantities, with coefficients depending on the other. While the Binomial Theorem expands $(x+h)^n$ in a series of powers of h, Taylor's Theorem expands *any infinitely differentiable function* of $(x+h)$ in a similar series. The Taylor series of an infinitely differentiable function f is given by: $f(x+h) = \sum_{n=0}^{\infty} \frac{h^n f^{(n)}(x)}{n!}$, where $f^{(n)}(x)$ denotes the nth derivative of f.

If we put $x = 0$ and change h into x, we get a series named after Colin Maclaurin (1698–1746), who published it in his *A Treatise of Fluxions*[18]. It appears in §751 of Volume II. In the next section, he refers to similar result in Bernoulli's Act. Erud. Lips., 1694. In a letter of Dec. 7, 1728 to the Scottish mathematician James Stirling (1692–1770), Maclaurin refers to "a Theorem in y[our](?)book where a Quantity is expressed by a series whose coefficients are first, second, third fluxions, etc." Stirling had established it on pp.102–103 of his *Methodus Differentialis*[23], but he had used it earlier in a paper entitled 'Methodus Differentialis Newtoniana Illustrata', published in the Philosophical Transactions (1719) of the Royal Society[26]. Augustus De Morgan (1806–71) rightly commented: "...both Maclaurin and Stirling would have been astonished that a particular case of Taylor's theorem would be called by either of their names."[19]

We shall now derive a Taylor series representation for $\arctan(x)$ with simple closed-form coefficients.

Let $f(x) = \arctan(x)$ and $x = \cot\theta = \tan(\frac{\pi}{2} - \theta)$, that is, $\arctan x = \frac{\pi}{2} - \theta = \phi$, meaning that θ and ϕ are complementary angles. Here $|x| < 1$ and $0 < \theta < \frac{\pi}{2}$.

So using a right-angled triangle with opposite side 1 and adjacent side x, we immediately get $\frac{1}{\sqrt{1+x^2}} = \sin\theta$.

The equation $1 + x^2 = (\sin\theta)^{-2}$ leads to $2x\frac{dx}{d\theta} = -2(\sin\theta)^{-3}\cos\theta$.
That is, $\frac{dx}{d\theta} = -\frac{(\sin\theta)^{-3}\cos\theta}{\cot\theta} = -(\sin\theta)^{-2}$.
Hence, $\frac{d\theta}{dx} = -\sin\theta\sin\theta$.

Differentiating it, we obtain

$$\frac{d^2\theta}{dx^2} = -\left(2\sin\theta\cos\theta\right)\frac{d\theta}{dx} = -(\sin 2\theta)(-\sin^2\theta) = \sin^2\theta\sin 2\theta.$$

Further,

$$\begin{aligned} \frac{d}{dx}(\sin^n\theta\,\sin(n\theta)) &= n\sin^{n-1}\theta\cos\theta\frac{d\theta}{dx}\sin(n\theta) + n\sin^n\theta\cos(n\theta)\frac{d\theta}{dx} \\ &= n\sin^{n-1}\theta\frac{d\theta}{dx}\left[\sin n\theta\cos\theta + cos(n\theta)\sin\theta\right] \\ &= n\sin^{n-1}\theta(-\sin^2\theta)\left[\sin(n\theta+\theta)\right] \\ &= -n\sin^{n+1}\theta\,\sin(n+1)\theta. \end{aligned}$$

Applying Taylor' theorem to arctan, we obtain:
$\arctan(x+h) = \arctan x + h\frac{d}{dx}\arctan x + \frac{h^2}{2!}\frac{d^2}{dx^2}\arctan x + \ldots$

Hence, we get:

$$\begin{aligned} \arctan(x+h) &= \left(\frac{\pi}{2} - \theta\right) + h\frac{d}{dx}\left(\frac{\pi}{2} - \theta\right) + \frac{h^2}{2}\frac{d^2}{dx^2}\left(\frac{\pi}{2} - \theta\right) + \ldots \\ &= \left(\frac{\pi}{2} - \theta\right) - h\frac{d\theta}{dx} - \frac{h^2}{2}\frac{d^2\theta}{dx^2} - \frac{h^3}{3!}\frac{d^3\theta}{dx^3} - \frac{h^4}{4!}\frac{d^4\theta}{dx^4}\ldots \end{aligned}$$

Now using all the results that we obtained earlier, we obtain:

$$\begin{aligned} \arctan(x+h) = \left(\frac{\pi}{2} - \theta\right) &+ h\sin\theta\sin\theta - \frac{h^2}{2}\sin^2\theta\sin 2\theta \\ &+ \frac{h^3}{3}\sin^3\theta\sin 3\theta - \frac{h^4}{4}\sin^4\theta\sin 4\theta + - \ldots \end{aligned} \tag{1}$$

We may point out here that inexplicably this *arctan* expansion does not appear in modern calculus textbooks, or indeed in the numerous Taylor series tables that exist on the web. My extensive search led me eventually to an 1841 calculus text[14] by Gregory and [16, 25] where this series appears.

3. Series for deriving formulas for π

We now derive some relevant series.

Putting $x = 0$ in (1), that is, $\theta = \frac{\pi}{2}$ and then seting $h = x$ yields the well-known Maclaurin series for $\arctan(x)$:

$$\arctan(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-1}}{2n-1}, \quad -1 < x < 1. \tag{2}$$

If we put $h = -x$ in (1), then $\arctan(x+h) = \arctan(0) = 0$ and so: $0 = \frac{\pi}{2} - \theta - x\sin\theta\sin\theta - \frac{x^2}{2}\sin^2\theta\sin 2\theta - \frac{x^3}{3}\sin^3\theta\sin 3\theta - \ldots$, that is,

$$\begin{aligned} \frac{\pi}{2} - \theta &= \cot\theta\sin\theta\sin\theta + \frac{\cot^2\theta\sin^2\theta\sin 2\theta}{2} + \frac{\cot^3\theta\sin^3\theta\sin 3\theta}{3} + \ldots \\ &= \cos\theta\sin\theta + \frac{\cos^2\theta\sin 2\theta}{2} + \frac{\cos^3\theta\sin 3\theta}{3} + \ldots, \quad 0 < \theta < \frac{\pi}{2} \end{aligned} \tag{3}$$

On using the relation $\frac{\pi}{2} - \theta = \phi$, the equation(3) transforms into an elegant alternating series with two positive terms and two negative terms:

$$\phi = \sin\phi\cos\phi + \frac{\sin^2\phi\sin 2\phi}{2} - \frac{\sin^3\phi\cos 3\phi}{3} - \frac{\sin^4\phi\sin 4\phi}{4} + + - - \ldots, \; 0 < \phi < \frac{\pi}{2} \tag{4}$$

Putting $h = -x + \frac{1}{x} = -\cot\theta + \frac{1}{\cot\theta} = -2\cot 2\theta$ in (1), we get:

$$\arctan\left(\frac{1}{x}\right) = \arctan(x) + (-2\cot 2\theta)\sin\theta\sin\theta$$

$$-\frac{(-2\cot 2\theta)^2}{2}\sin^2\theta\sin 2\theta + \frac{(-2\cot 2\theta)^3}{3}\sin^3\theta\sin 3\theta - \ldots$$

Since $(2\cot 2\theta\sin\theta)^n = \left(\frac{\cos 2\theta}{\cos\theta}\right)^n$, we obtain:

$$\arctan(x) - \arctan\left(\frac{1}{x}\right) = \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{\cos 2\theta}{\cos\theta}\right)^n \sin n\theta.$$

As $x = \cot\theta = \tan(\frac{\pi}{2} - \theta)$ and $\frac{1}{x} = \frac{1}{\cot\theta} = \tan\theta$, we get:

$$\frac{\pi}{2} - 2\theta = \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{\cos 2\theta}{\cos\theta}\right)^n \sin n\theta, \quad 0 < \theta < \frac{\pi}{2}. \tag{5}$$

We see that

$$\begin{aligned} x - \sqrt{1+x^2} &= \frac{\cos\theta}{\sin\theta} - \frac{1}{\sin\theta} = \frac{\cos\theta - 1}{\sin\theta} \\ &= \frac{-2\sin^2(\frac{\theta}{2})}{2\sin(\frac{\theta}{2})\cos(\frac{\theta}{2})} = -\tan\frac{\theta}{2} = \tan\frac{-\theta}{2}. \end{aligned}$$

Similarly, $x + \sqrt{1+x^2} = \cot\frac{\theta}{2} = \tan(\frac{\pi}{2} - \frac{\theta}{2})$.

So on putting $h = -\sqrt{1+x^2}$ in (1), we get after cancellation of $(\sin\theta)^n$:

$$\frac{\pi}{2} - \frac{\theta}{2} = \sum_{n=1}^{\infty} \frac{\sin n\theta}{n}, \quad 0 < \theta < 2\pi. \tag{6}$$

We find this formula in §166, Chapter 6, Part II of Euler's *Foundations of Differential Calculus*[11].

Similarly on putting $h = \sqrt{1+x^2}$ in (1), we obtain:

$$\frac{\theta}{2} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\sin n\theta}{n}, \quad -\pi < \theta < \pi. \tag{7}$$

Since $\sin m\pi = 0$, $\cos(2m+1)\pi = -1$ and $\cos(2m)\pi = 1$ for integer m, we could derive (7) from (6) by simply replacing θ by $\pi - \theta$.

It may be pointed out that popular calculus texts such as [8] use Fourier series to get the expansions (7) and (8). The French mathematician Jean J.B.Fourier (1768–1830) had asserted in his work on the propagation of heat that we can represent "arbitrary" functions (not only odd functions) of a variable into series of sines and cosines of multiple arcs. That is,

$$f(x) = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots = \sum_{k=1}^{\infty} b_k \sin kx.$$

To find the coefficients b_n, he multiplied both sides of the last expression by $\sin nx$ and then integrated over $[0, \pi]$ obtaining

$$\int_0^{\pi} f(x) \sin nx \, dx = \sum_{k=1}^{\infty} b_k \int_0^{\pi} \sin kx \, \sin nx \, dx.$$

Since

$$\int_0^{\pi} \sin kx \, \sin nx \, dx = 0 \text{ for } k \neq n \text{ and } = \frac{\pi}{2} \text{ for } k = n,$$

$$b_k = \frac{2}{\pi} = \int_0^{\pi} f(x) \, \sin nx \, dx.$$

By using a similar analysis, Fourier showed that one could also expand any function whatever into a series of cosines of multiple arcs.

Adding the equation (6) and (7) yields

$$\frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{\sin(2n-1)\theta}{2n-1}, \quad 0 < \theta \leq \frac{\pi}{2}. \tag{8}$$

If we set $h = -2x + \sqrt{1+x^2}$, and $h = -2x - \sqrt{1+x^2}$, in (1), we get:

$$\frac{\pi}{2} - \frac{3\theta}{2} = \sum_{n=1}^{\infty} \frac{1}{n} (2\cos\theta - 1)^n \sin n\theta, \qquad 0 \leq \theta \leq \frac{\pi}{2}. \tag{9}$$

$$\pi - \frac{3\theta}{2} = \sum_{n=1}^{\infty} \frac{1}{n} (2\cos\theta + 1)^n \sin n\theta, \qquad \frac{\pi}{2} \leq \theta \leq \pi. \tag{10}$$

4. BBP-type Formulas for π

We shall now use the expansions to derive formulas for π.

On setting $\theta = \frac{\pi}{4}$ in (3) and on putting the values $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, $\sin\frac{\pi}{4}(4n) = 0$, $\sin\frac{\pi}{4}(8n-6) = 1$, $\sin\frac{\pi}{4}(8n-2) = -1$, $\sin\frac{\pi}{4}(8n-7) = \sin\frac{\pi}{4}(8n-5) = \frac{1}{\sqrt{2}}$, $\sin\frac{\pi}{4}(8n-5) = \sin\frac{\pi}{4}(8n-3) = -\frac{1}{\sqrt{2}}$, we get an alternating series with three positive terms and three negative terms as every fourth term vanishes:

$$\begin{aligned}
\frac{\pi}{4} &= \left(\frac{1}{2}\right)^1 + \frac{1}{2}\left(\frac{1}{2}\right)^1 + \frac{1}{3}\left(\frac{1}{2}\right)^2 - \frac{1}{5}\left(\frac{1}{2}\right)^3 - \frac{1}{6}\left(\frac{1}{2}\right)^3 - \frac{1}{7}\left(\frac{1}{2}\right)^4 + \dots \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{4^n}\left(\frac{2}{4n-3} + \frac{2}{4n-2} + \frac{1}{4n-1}\right),
\end{aligned}$$

which, on shifting the summation index, becomes:

$$\pi = \sum_{n=0}^{\infty}\left(\frac{-1}{4}\right)^n\left(\frac{2}{4n+1} + \frac{2}{4n+2} + \frac{1}{4n+3}\right). \tag{11}$$

Using the standard notation, it becomes: $\pi = P(1, -4, 4, (2, 2, 1, 0))$.
This can be transformed by combining two consecutive terms into one obtaining $\pi = \frac{1}{4}P(1, 16, 8, (8, 8, 4, 0, -2, -2, -1, 0)$.
Or equivalently,

$$\pi = 4\sum_{n=0}^{\infty} \frac{30720n^5 + 90368n^4 + 100064n^3 + 51292n^2 + 11905n + 981}{16^n(8n+1)(8n+2)(8n+3)(8n+5)(8n+6)(8n+7)}.$$

BBP-type formulas can be obtained by assigning Gaussian rational values to the complex number z in the following power series expansion:

$$\ln\frac{1}{1-z} = \sum_{n=1}^{\infty}\frac{z^n}{n}, \qquad |z| < 1.$$

For example, formula (11) can be derived by setting $z = \frac{1+i}{2}$ and taking imaginary parts.

Further, a BBP-type formula can also be obtained by computing integrals obtained in the following manner where $|\frac{1}{b}| < 1$ and permissible swapping of sigma and integral signs is resorted to:

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{1}{b^n(kn+c)} &= b^{c/k}\sum_{n=0}^{\infty}\frac{1}{kn+c}\left(\frac{1}{b^{1/k}}\right)^{kn+c} = b^{c/k}\sum_{n=0}^{\infty}\left[\frac{x^{kn+c}}{bn+c}\right]_0^{\frac{1}{b^{1/k}}} \\
&= b^{c/k}\sum_{n=0}^{\infty}\int_0^{\frac{1}{b^{1/k}}} x^{kn+c-1}\,dx = b^{c/k}\int_0^{\frac{1}{b^{1/k}}}\sum_{n=0}^{\infty}(x^k)^n\,x^{c-1}\,dx \\
&= b^{c/k}\int_0^{\frac{1}{b^{1/k}}}\frac{x^{c-1}}{1-x^k}\,dx.
\end{aligned}$$

Thus on putting $b^{1/k}x = y$, we obtain

$$\sum_{n=0}^{\infty} \frac{1}{b^n(kn+c)} = b\int_0^1 \frac{y^{c-1}}{b-y^k}\,dy.$$

This procedure, adapted from [1, 2, 7], has been used by Prof. S.K. Lucas to derive such formulas. Adamchik and Wagon derive, on page 854 in [2], an integral $g(i)$ for the case $k = 4$, taking variable z and constant i in place of c. On setting $z = \frac{x}{\sqrt{2}}$ and replacing i by c, their integral becomes

$$\sum_{n=0}^{\infty} \left(\frac{-1}{4}\right)^n \frac{1}{4n+c} = 4\int_0^1 \frac{x^{c-1}}{4+x^4}\,dx.$$

We find in a standard table of integrals [10, 27]:

$$\int \frac{dx}{a^4+x^4} = \frac{1}{4a^3\sqrt{2}} \ln \frac{x^2+ax\sqrt{2}+a^2}{x^2-ax\sqrt{2}+a^2} + \frac{1}{2a^3\sqrt{2}} \arctan \frac{ax\sqrt{2}}{a^2-x^2},$$

$$\int \frac{x\,dx}{a^4+x^4} = \frac{1}{2a^2} \arctan \frac{x^2}{a^2},$$

$$\int \frac{x^2\,dx}{a^4+x^4} = -\frac{1}{4a\sqrt{2}} \ln \frac{x^2+ax\sqrt{2}+a^2}{x^2-ax\sqrt{2}+a^2} + \frac{1}{2a\sqrt{2}} \arctan \frac{ax\sqrt{2}}{a^2-x^2}.$$

Computing these integrals from 0 to 1 with $a = \sqrt{2}$, we arrive at

$$\sum_{n=0}^{\infty} \left(\frac{-1}{4}\right)^n \left(\frac{2}{4n+1} + \frac{2}{4n+2} + \frac{1}{4n+3}\right) = 2\left(\arctan 2 + \arctan \frac{1}{2}\right) = \pi.$$

5. Formulas involving convergents of π

Further, we observe that

$$\sum_{n=0}^{\infty} \left(\frac{-1}{4}\right)^n \frac{1}{4n+6} = -4\sum_{n=1}^{\infty} \left(\frac{-1}{4}\right)^n \frac{1}{4n+2},$$

$$\sum_{n=0}^{\infty} \left(\frac{-1}{4}\right)^n \frac{1}{4n+7} = -4\sum_{n=1}^{\infty} \left(\frac{-1}{4}\right)^n \frac{1}{4n+3}.$$

So it becomes clear that

$$\begin{aligned}&\sum_{n=0}^{\infty} \left(\frac{-1}{4}\right)^n \left(\frac{2}{4n+1} - \frac{6}{4n+2} + \frac{5}{4n+3} - \frac{2}{4n+6} + \frac{1}{4n+7}\right)\\ &\quad = \frac{2}{3} + \sum_{n=1}^{\infty} \left(\frac{-1}{4}\right)^n \left(\frac{2}{4n+1} + \frac{2}{4n+2} + \frac{1}{4n+3}\right) = \pi - \frac{8}{3}\end{aligned}$$

Hence,

$$\sum_{n=1}^{\infty}\left(\frac{-1}{4}\right)^n\left(\frac{2}{4n+1}-\frac{6}{4n+2}+\frac{5}{4n+3}-\frac{2}{4n+6}+\frac{1}{4n+7}\right)$$
$$=\pi-\frac{8}{3}-\left(2-3+\frac{5}{3}-\frac{1}{3}+\frac{1}{7}\right)=\pi-\frac{22}{7}.$$

Luckily, collecting all the terms into one leaves only a constant in the numerator yielding a neat formula for the second convergent of π:

$$\pi=\frac{22}{7}+120\sum_{n=1}^{\infty}\left(\frac{-1}{4}\right)^n\frac{1}{(4n+1)(4n+2)(4n+3)(4n+6)(4n+7)},$$

which can be transformed into a series with only positive terms:

$$\pi=\frac{22}{7}-15\sum_{n=1}^{\infty}\frac{768n^3+1984n^2+836n+87}{\left(\begin{array}{c}16^{n-1}(8n-3)(8n-2)(8n-1)(8n+1)\\(8n+2)(8n+3)(8n+6)(8n+7)\end{array}\right)}. \tag{12}$$

Let me touch upon the formulas involving the convergents of π. Dalzell [9] found in 1944 the following integral and series

$$\pi=\frac{22}{7}-\int_0^1\frac{t^4(1-t)^4}{1+t^2}. \tag{13}$$

$$\pi=\frac{22}{7}+\sum_{n=1}^{\infty}\left(\frac{-1}{4}\right)^n\left[\frac{3(4n)!^2}{(8n+1)!}+\frac{(4n+1)!^2}{(8n+3)!}-\frac{1}{2}\frac{(4n+2)!^2}{(8n+5)!}-\frac{(4n+3)!^2}{(8n+7)!}\right]. \tag{14}$$

By partial fractions decomposition and simplification, I was able to transform it into a 'cousin' of BBP-type formulas with a rapidly rising additional factor in the denominator

$$\pi=\frac{1}{4^5}\sum_{n=0}^{\infty}\left(\frac{-1}{4}\right)^n\frac{1}{\binom{8n}{4n}}\left(\frac{3183}{8n+1}+\frac{117}{8n+3}-\frac{15}{8n+5}-\frac{5}{8n+7}\right) \tag{15}$$

which may be written in the form

$$\frac{\pi}{2}=\sum_{n=0}^{\infty}\left(\frac{-1}{4}\right)^n\left[\frac{820n^3+1533n^2+902n+165}{\binom{8n}{4n}(8n+1)(8n+3)(8n+5)(8n+7)}\right].$$

or

$$\pi=\frac{22}{7}-\sum_{n=1}^{\infty}\frac{P(n)}{n(4n-1)(8n-1)(8n-3)(16n+1)(16n+3)(16n+5)(16n+7)}\frac{1}{2^{4n-1}\binom{16n}{8n}}, \tag{16}$$

$P(n)=1717985280n^7+747324416n^6-238713984n^5-103697680n^4+6801420n^3+2995544n^2-21201n-10080,$

We do not know whether such 'natural' formulas exist for the further convergents of π, but I could derive this 'artificial' formula:

$$\pi = \frac{355}{113} - \frac{1}{8\cdot 19\cdot 113}\sum_{n=2}^{\infty}\frac{(9546n^3 - 147727n^2 - 55625n + 409063)}{(2n+1)(2n+2)(2n+3)(2n+4)(2n+5)}\frac{\binom{2n}{n}}{2^{4n}} \tag{17}$$

and some more formulas like this for $\frac{22}{7}$

$$\pi = \frac{22}{7} - \frac{1}{7}\sum_{n=1}^{\infty}\frac{n(102n+231)\binom{2n}{n}}{(2n+1)(2n+2)(2n+3)(2n+4)(2n+5)(2n+6)2^{4n-2}}. \tag{18}$$

6. BBP-type Formulas for $\pi\sqrt{3}$

If we set $\theta = \frac{\pi}{3}$ in the expansion in (3), we get a series with two positive terms followed by two negative terms as every third term vanishes:

$$\pi\sqrt{3} = 18\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{8^n}\left(\frac{2}{3n-2} + \frac{1}{3n-1}\right),$$

which, on shifting the summation index, becomes:

$$\pi\sqrt{3} = \frac{9}{4}\sum_{n=0}^{\infty}\left(\frac{-1}{8}\right)^n\left(\frac{2}{3n+1} + \frac{1}{3n+2}\right). \tag{19}$$

Using the standard notation, it becomes: $\pi\sqrt{3} = \frac{9}{4}P(1,-8,3,(2,1,0))$.
This can be transformed into $\pi\sqrt{3} = \frac{9}{32}P(1,64,6,(16,8,0,-2,-1,0))$.
Or equivalently,

$$\pi\sqrt{3} = \frac{9}{8}\sum_{n=0}^{\infty}\frac{1134n^3 + 2097n^2 + 1188n + 193}{64^n(6n+1)(6n+2)(6n+4)(6n+5)}.$$

Formula (11), discovered by Adamchik and Wagon [1, 2], finds place as entry (16) with associated entry (15) in the compendium of such formulas [5] while (19) in the associated form is entry (18).

If we put $\theta = \frac{\pi}{6}$ in (3), we obtain a sluggish alternating series having five positive terms followed by five negative terms as every sixth term vanishes and we get a formula with non-integral base:

$$\pi = \frac{3\sqrt{3}}{64}\sum_{n=0}^{\infty}\left(\frac{-27}{64}\right)^n\left[\frac{16}{6n+1} + \frac{24}{6n+2} + \frac{24}{6n+3} + \frac{18}{6n+4} + \frac{9}{6n+5}\right]. \tag{20}$$

We now deduce some formulas which have unit-base though Bailey's compendium includes only those formulas with $b > 1$ for exponential rather than linear rate of convergence.

Putting $\theta = \frac{\pi}{3}$ in (5) in leads to:

$$\pi = 3\sqrt{3}\left[\sum_{n=1}^{\infty}\left(\frac{1}{3n-2} - \frac{1}{3n-1}\right)\right]. \tag{21}$$

Putting $\theta = \frac{\pi}{2}$ and $\theta = \frac{\pi}{3}$ in (6), we get the celebrated Madhava-Leibnitz-Gregory series

$$\frac{\pi}{4} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{2n-1}. \tag{22}$$

$$\pi = \frac{3\sqrt{3}}{2}\left[\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{3n+1} + \frac{1}{3n+2}\right)\right]. \tag{23}$$

Putting $\theta = \frac{\pi}{3}$, $\theta = \frac{\pi}{4}$ and $\theta = \frac{\pi}{6}$ in (8), we get:

$$\pi = 2\sqrt{3}\left[\sum_{n=0}^{\infty}\left(\frac{1}{6n+1} - \frac{1}{6n+5}\right)\right]. \tag{24}$$

$$\pi = 2\sqrt{2}\left[\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{4n+1} + \frac{1}{4n+3}\right)\right]. \tag{25}$$

$$\pi = 2\left[\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{6n+1} + \frac{2}{6n+3} + \frac{1}{6n+5}\right)\right]. \tag{26}$$

If we add the equations (22) and (25), we obtain

$$\pi = 8(\sqrt{2}-1)\left[\sum_{n=0}^{\infty}\left(\frac{1}{8n+1} - \frac{1}{8n+7}\right)\right]. \tag{27}$$

Formulas (21) and (24) appear in §176 and §177 respectively of Chapter X, Vol. I of Euler's *Analysis*[12]. Formulas (25) and (27) occur in §179 there.

Putting $\theta = \frac{\pi}{4}$ in (9) or $\theta = \frac{3\pi}{4}$ in (10) yields an interesting formula:

$$\frac{\pi}{4} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(1+\sqrt{2})^{4n+2}}\left(\frac{2+\sqrt{2}}{4n+1} + \frac{2}{4n+2} + \frac{2-\sqrt{2}}{4n+3}\right). \tag{28}$$

Finally, we give an efficient alternating *base-3* or ternary formula deduced by putting $\theta = \frac{\pi}{6}$ in (5):

$$\pi\sqrt{3} = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{-1}{27}\right)^n\left(\frac{9}{6n+1} + \frac{9}{6n+2} + \frac{6}{6n+3} + \frac{3}{6n+4} + \frac{1}{6n+5}\right). \tag{29}$$

To use the standard notation: $\pi\sqrt{3} = \frac{1}{3}P(1, -27, 6,\ (9, 9, 6, 3, 1, 0))$.
The alternating formula (29) transforms into
$\pi\sqrt{3} = \frac{1}{81}P(1, 3^6, 12,\ (243, 243, 162, 81, 27, 0, -9, -9, -6, -3, -1, 0))$.
Neither of the two forms is in Bailey's compendium which has entry (66) as:
$\pi\sqrt{3} = \frac{1}{9}P(1, 3^6, 12,\ (81, -54, 0, -9, 0, -12, -3, -2, 0, -1, 0, 0))$.

Formula (29) can be written as:

$$\pi\sqrt{3} = \frac{16}{9}\sum_{m=0}^{\infty} \frac{\begin{pmatrix} 26085556224m^9 + 126116020224m^8 \\ +262617292800m^7 + 307980610560m^6 \\ +223165269504m^5 + 103065777216m^4 \\ +30145041120m^3 + 5342703940m^2 \\ +515537612m + 20359495 \end{pmatrix}}{\begin{pmatrix} 729^n(12m+1)(12m+2)(12m+3) \\ (12m+4)(12m+5)(12m+7)(12m+8) \\ (12m+9)(12m+10)(12m+11) \end{pmatrix}}$$

To compute the five relevant integrals, we can use the following general formula for $m < 2n$ given in [13]:

$$\int \frac{x^{m-1}dx}{1+x^{2n}} = -\frac{1}{2n}\sum_{k=1}^{n} \cos\frac{m\pi(2k-1)}{2n} \ln(1 - 2x\cos\frac{\pi(2k-1)}{2n} + x^2)$$
$$+ \frac{1}{n}\sum_{k=1}^{n} \sin\frac{m\pi(2k-1)}{2n} \arctan\frac{x - \cos\frac{\pi(2k-1)}{2n}}{\sin\frac{\pi(2k-1)}{2n}}.$$

Motivated by the formula (29), Prof. S.K. Lucas discovered a formula with a general parameter:

$$\pi = \sum_{n=0}^{\infty}\left(\frac{-1}{27}\right)^n \left(\frac{9e}{6n+1} + \frac{2\sqrt{3}-9e}{6n+2} + \frac{2\sqrt{3}-12e}{6n+3} + \frac{2\sqrt{3}-9e}{3(6n+4)} + \frac{e}{6n+5}\right).$$

The choices of $e = 2\sqrt{3}/9$ and $e = \sqrt{3}/6$ eliminate fractions, giving

$$\pi = \frac{2\sqrt{3}}{9}\sum_{n=0}^{\infty}\left(\frac{-1}{27}\right)^n \left(\frac{9}{6n+1} - \frac{3}{6n+3} + \frac{1}{6n+5}\right) = 2\sqrt{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{3^n(2n+1)},$$

which is Abraham Sharp's formula deduced by putting $x = \frac{1}{\sqrt{3}}$ in the Maclaurin's series for $\arctan x$. In fact on combining two consecutive terms, Sharp's series leads to another BBP-type of formula:

$$\pi\sqrt{3} = 2\sum_{n=0}^{\infty}\left(\frac{1}{9}\right)^n \left(\frac{3}{4n+1} - \frac{1}{4n+3}\right).$$

Two infinite classes of formulas like that of Sharp have been derived by the author in his recent paper[21].

More formulas can be deduced by combining the various results. I am giving only one of these — a 'balanced formula', sum of whose coefficients is zero:
$\pi\sqrt{3} = P(1, -3^3, 6, (9, -3, -6, -1, 1, 0))$,
$\pi\sqrt{3} = \frac{1}{27}P(1, 3^6, 12, (243, -81, -162, -27, 27, 0, -9, 3, 6, 1, -1, 0))$.
This may be combined with the one given in the compendium to deduce:
$\pi\sqrt{3} = \frac{1}{27}P(1, 3^6, 12, (243, 0, -324, -27, 54, 36, -9, 12, 12, 5, -2, 0))$.

Bailey records this formula for $\log 7$ as the entry (70) in his compendium:
$\ln 7 = \frac{2}{3^5}P(1, 3^6, 6, (405, 81, 72, 9, 5, 0))$.

We discovered this pair of formulas using the integrals approach:

$$\ln 7 = \frac{2}{3^2} P(1, -3^3, 6(9, 0, 0, 0, -1, 0)), \tag{30}$$

$$\ln 7 = \frac{2}{3^5} P(1, 3^6, 12(243, 0, 0, 0, -27, 0, -9, 0, 0, 0, 1, 0)). \tag{31}$$

Put in sigma notation, our 4-term formula is

$$\ln 7 = \frac{2}{243} \sum_{n=0}^{\infty} \frac{1}{729^n} \left(\frac{243}{12n+1} - \frac{27}{12n+5} - \frac{9}{12n+7} + \frac{1}{12n+11} \right).$$

We may mention that the Japanese team led by Prof. Y. Kanada of Tokyo University used Machin like arctangent identities for computing π. But while giving similar exponential convergence, these formulas mix bases and thus are not as good as Ramanujan type series or BBP-type formulas.

Our discovery of these BBP-type formulas shows that results *can* be discovered *mathematically* which a sophisticated software may fail to find. However, given the advantage that a computer has in terms of speed, memory and precision in computation, there is no denying the fact that advanced computing technology, in the shape of software such as *Mathematica* and *Maple*, can be used as an adjunct in mathematical research for discovering new mathematical results as well as for validation of empirically obtained conjectures. The author himself used a computer program to discover Machin-like arctangent identities for π[20].

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