A New Look at an Old Question: Is π Normal?

James A. Walsh
Department of Mathematics
Oberlin College, Oberlin, OH 44074
jim.a.walsh@oberlin.edu
440-775-8387

The Department of Mathematics at Oberlin College sponsors a series of student/faculty luncheons each semester. Munching on pizza provided by the Department, students interested in mathematics, from first-years to seniors, and faculty interact in an informal and enjoyable setting. Twenty to thirty minutes of each luncheon are reserved for a presentation on something mathematical, ensuring that we all head to our 1:30 classes well-fed in more ways than one.

Gary Kennedy, a mathematician from Ohio State-Mansfield, graciously made such a presentation last year, with his topic the remarkable formula for π discovered by Bailey, P. Borwein and Plouffe [1; also see below]. During his talk Gary vaguely mentioned the discovery of a connection between the normality of π and chaotic dynamical systems. In fact, aware that my field is dynamical systems, he singled me out of the audience as he mentioned this discovery. Now obliged to respond to this challenge, I felt a certain kinship with the character portrayed by Gary Cooper in the classic film *High Noon* as I investigated this connection. Armed with my sheriff's badge, I presented my findings at a subsequent luncheon. That presentation, in which my dear Aunt Phyllis also played a role, is the basis for this article.

We all know that π is irrational (Lambert, 1761) and, moreover, transcendental (Lindemann, 1882). The first 200 billion digits in the decimal expansion of π have been computed, as has the quadrillionth binary digit (it is 0). Apparently, we know π quite well. When it comes to the limiting distribution of digits in the decimal expansion of π, however, the paucity of results is striking. Simply-stated questions such as, "Do any of the digits 0, 1, …, 9 occur infinitely often in the decimal expansion of π?", and, "Are there 1000 consecutive zeros in the digits of π?" remain unanswered to this day.

In 1909, E. Borel, interested in the notion of a real number being random, introduced the concept of normality. A real number μ is *normal to base* $b \geq 2$ if every finite string of k digits appears in the base b-expansion of μ with well-defined limiting frequency b^{-k}. That is, μ is normal to base b if $\lim_{n \to \infty} N(t,n)/n = b^{-k}$ for each string t of length k, where $N(t,n)$ is the number of times t occurs in the first n base b-digits of μ. A number is *normal* if it is normal to every base b.

If π were normal to base 10, the string "3" would occur with limiting frequency 1/10 in its base 10-expansion, the string "58" with limiting frequency 1/100, and so on. In addition, every finite string would occur infinitely often in its base 10 digits, so that the answers to the questions posed above would be a satisfying "yes". Statistical tests support the conjecture that π is normal to base 10 [3].

It is reasonable to ask if there are known examples of numbers normal to a base b. The answer is an unsatisfying "yes". First note that a number normal to a base b must be irrational. Also note that a number may be normal to one base and not normal when expanded in another base (base 2- versus base 3-expansions of certain members of the Cantor middle thirds set provide good examples). In 1933, Champernowne proved that 0.1234567891011…, formed by concatenating the positive integers, is normal to base 10. The Copeland-Erdös number, 0.2357111317…, formed by concatenating the primes, is also normal to base 10.

From a theoretical perspective, most numbers are normal: Borel proved that a real number chosen at random is normal with probability one. This is intuitively plausible if you think of repeatedly rolling a fair b-sided die to generate the base b-digits of a real number. Yet we do not know if any of our favorite constants, such as π, e, $\sqrt{2}$ or $(1+\sqrt{5})/2$, is normal to *any* base b!

Bailey and Crandall presented a connection between base b-normality of constants and discrete dynamical systems in [2]. Suppose μ in [0,1) has base b-expansion $\mu = 0.\mu_1\mu_2\mu_3\ldots = \mu_1/b + \mu_2/b^2 + \mu_3/b^3 + \ldots$, with each μ_i in the set $\{0,1, \ldots, b\text{-}1\}$. If $f : [0,1) \to [0,1)$ is the function which sends x to bx mod 1, then $f(\mu) = 0.\mu_2\mu_3\mu_4\ldots$. The function f, called a *shift map*, is the prototypical chaotic map.

For μ in [0,1), consider the recursively defined sequence

(1) $$x_0 = \mu, \quad x_{n+1} = f(x_n), \quad n \geq 0.$$

Note that $x_n = 0.\mu_{n+1}\mu_{n+2}\mu_{n+3}\ldots = b^n\mu \bmod 1$, so that we are, in a sense, using f to sift through the base b-digits of μ. If μ is rational, the sequence (1) eventually repeats, so that $\{x_n\}$ is a finite set. If μ is irrational, $\{x_n\}$ is an infinite set with an infinite number of limit points. In this case (the one of interest to us), it is reasonable to investigate the *distribution* of the x_n in [0,1). For example, if μ is normal to base b, we would expect the x_n to be uniformly distributed in [0,1).

Let $\mathbf{s}=\{x_n\}$ be the sequence given by (1). Let $0 \leq c < d < 1$, and let $C(\mathbf{s},c,d,i)$ denote the number of times x_n is in $[c,d)$ with $n \leq i$. Then the sequence **s** is *equidistributed* in [0,1) if $\lim_{i\to\infty} C(\mathbf{s},c,d,i)/i = d\text{-}c$ for all such c and d. Perhaps not surprisingly, μ is normal to base b if and only if **s** is equidistributed. Though providing an alternative approach to base b-normality of μ, a potential pitfall is the need for the full base b-expansion of the irrational μ.

This is where the formula for π discovered by Bailey, P. Borwein and Plouffe comes into play. In 1997 they showed that

(2) $$\pi = \sum_{k=1}^{\infty} \frac{1}{16^k}\left(\frac{16(120k^2-89k+16)}{512k^4-1024k^3+712k^2-206k+21}\right) \equiv \sum_{k=1}^{\infty} \frac{1}{16^k}\frac{p(k)}{q(k)}.$$

If, for each $k \geq 1$, the expression $p(k)/q(k)$ in equation (2) was in the set $\{0,1,\ldots,15\}$, equation (2) would provide a base 16-expansion for π. Unfortunately, this is not the case as, for example, $p(1)/q(1)$ is greater than 50!

Not to be deterred, however, remember that π is normal to base 16 if and only if the corresponding sequence (1) is equidistributed in [0,1). Using equation (2), we have

$$16^n\pi \bmod 1 = \left(\sum_{k=1}^{n} 16^{n-k}\frac{p(k)}{q(k)} + \sum_{k=n+1}^{\infty} 16^{n-k}\frac{p(k)}{q(k)}\right) \bmod 1 \equiv (y_n + r_n) \bmod 1.$$

As an exercise, you might show y_n is given recursively by $y_0 = 0$ and $y_n = 16y_{n-1} + p(n)/q(n)$, for $n \geq 1$. Now, since $\deg(p) < \deg(q)$, $r_n \to 0$ as $n \to \infty$. We use the fact that if a sequence $\{a_n\}$ is equidistributed and the sequence $\{b_n\}$ satisfies $b_n \to 0$ as $n \to \infty$, then the sum $\{a_n + b_n\}$ is equidistributed. We conclude: *if* $\{y_n \bmod 1\}$ is equidistributed in [0,1), so is $\{(y_n+r_n) \bmod 1\} = \{16^n\pi \bmod 1\}$, and π is normal to base 16. Pretty neat.

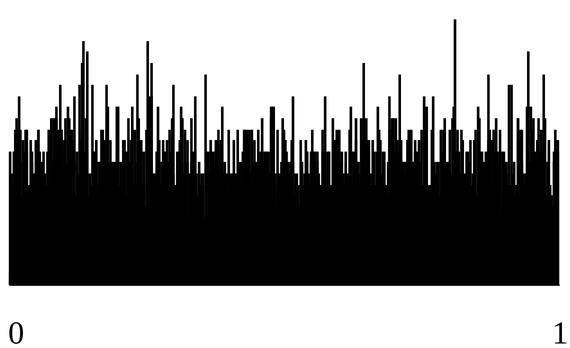
As a second example, you are invited to mimic the above steps, starting with the expression $\log 2 = \sum_{k=1}^{\infty}\frac{1}{2^k}\frac{1}{k}$ (think base 2). The term $2^n \log 2 \bmod 1$ can be written in the form $(y_n+r_n) \bmod 1$, with y_n given recursively by $y_0 = 0$ and $y_n = 2y_{n-1} + 1/n$, for $n \geq 1$. Since the corresponding rational function $p(k)/q(k) = 1/k$, we again have $r_n \to 0$ as $n \to \infty$. Then *if* $\{y_n \bmod 1\}$ is equidistributed in [0,1), so is $\{(y_n+r_n) \bmod 1\} = \{2^n \log 2 \bmod 1\}$, implying log 2 is normal to base 2.

More generally, consider any number defined by a *generalized polylogarithm series* (GPS) $\mu = \sum_{k=1}^{\infty}\frac{1}{b^k}\frac{p(k)}{q(k)}$ with the following stipulations: the functions p and q are polynomials with integer coefficients; q does not vanish at any positive integer; and $\deg(p) < \deg(q)$. As in the two examples above, $b^n\mu \bmod 1$ can be expressed as a sum $(y_n+r_n) \bmod 1$, with y_n given by $y_0 = 0$, $y_n = by_{n-1} + p(n)/q(n)$, $n \geq 1$, and $r_n \to 0$ as $n \to \infty$. Bailey and Crandall prove that μ is rational if and only if there are a finite number of limits of convergent subsequences of $\{y_n \bmod 1\}$, in which case they say the sequence $\{y_n \bmod 1\}$ has a *finite attractor*. For example, since π is irrational, its associated sequence $\{y_n \bmod 1\}$ given above does *not* have a finite attractor.

We can now state an intriguing hypothesis presented in [2]. The authors hypothesize that the sequence $\{y_n \bmod 1\}$ associated with a GPS as defined above either has a finite attractor or is equidistributed in [0,1). So, *assuming the hypothesis is true,* any irrational with a base *b*-GPS is normal to base *b*. The argument follows along the lines of those given earlier for π and log 2. In particular, π is normal to base 16 and log 2 is normal to base 2.

Many GPS are presented in [2]. For example, log(1111111111/387420489) is normal to base 10! More familiar constants are also normal: $\sqrt{2}$ and $\sqrt{e}$ are each normal to base 4. This, of course, assumes the hypothesis.

The histogram below shows the distribution of the first 10^4 elements in the sequence $\{y_n \bmod 1\}$ associated with π. Though numerics will not yield a proof, perhaps numerical investigations with tools more sophisticated than a spreadsheet will provide insight. Also recall that, by Borel's result, the sequence $\{16^n y_0 \bmod 1\}$ is equidistributed for almost all real numbers y_0. It is intriguing to ask if adding the term $p(n)/q(n)$, specified in equation (2), leads to equidistribution of the particular sequence $y_0 = 0$, $y_n = (16y_{n-1} + p(n)/q(n)) \bmod 1$, for $n \geq 1$.



Is π normal to base 16?

In the end I am satisfied with the response I gave to students and faculty regarding Gary Kennedy's "challenge". In fact, as my presentation drew to a close, I could not help but feel a bit like Gary Cooper must have felt at the end of *High Noon*, riding off into the sunset with the beautiful Grace Kelly.

References

[1] Bailey, D., Borwein, P., Plouffe, S., "On the rapid computation of various polylogarithmic constants," *Mathematics of Computation* **66** (218) (1997), 903-913.

[2] Bailey, D., Crandall, R., "On the random character of fundamental constant expansions," *Experimental Mathematics* **10** (2) (2001), 175-190.

[3] Wagon, S., "Is π normal?", *The Mathematical Intelligencer* **7** (1985), 65-67.