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Novel Finite Sum Identities Involving Reciprocals of Binomial Coefficients

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Abstract

Finite sums involving reciprocals of binomial coefficients have long intrigued mathematicians, owing to their elegant structure and unexpected identities. In this work, we present a comprehensive general formula that unifies and extends several results previously studied in the literature. To demonstrate the breadth and utility of our approach, we also investigate a variety of significant special cases, shedding light on deeper patterns and connections. This unified perspective aims to appeal to researchers interested in combinatorics, number theory, and special functions.

Keywords: finite binomial sums; parameterized sum; harmonic sum; Bernoulli numbers; gamma function; polygamma function

1. Introduction and Preliminaries

The binomial coefficient $\binom{z}{k}$ is defined for $z \in \mathbb{C}$ and $k \in \mathbb{Z}_{\geqslant 0}$ by

$$\binom{z}{k} = \frac{z(z-1)\cdots(z-k+1)}{k!} = \frac{\Gamma(z+1)}{\Gamma(z-k+1)\,\Gamma(k+1)}, \tag{1}$$

where $\Gamma(z)$ denotes the classical gamma function. In particular, when $z = n \in \mathbb{Z}_{\geqslant 0}$, the binomial coefficient reduces to the familiar form

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad (n, k \in \mathbb{Z}_{\geqslant 0}),$$

with the convention $0! := 1$. For a historical overview of the binomial coefficients and a detailed treatment of related fundamental identities, such as

$$\binom{z+1}{k} = \binom{z}{k} + \binom{z}{k-1}, \quad \text{and} \quad \frac{z+1}{k+1}\binom{z}{k} = \binom{z+1}{k+1}, \tag{2}$$

see Jordan [1] (§§ 22–23).

Using a known identity along with mathematical induction, Sun [2] established the following formula:

$$\sum_{k=1}^{n} \frac{1}{k^2\binom{n+k}{k}} = 3\sum_{k=1}^{n} \frac{1}{k^2\binom{2k}{k}} - \sum_{k=1}^{n} \frac{1}{k^2} \quad (n \in \mathbb{N}). \tag{3}$$

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Throughout this work, we use $\mathbb{C}$, $\mathbb{Z}$, and $\mathbb{N}$ to denote the sets of complex numbers, integers, and positive integers, respectively. Also, for a given $\ell \in \mathbb{Z}$, we use $\mathbb{Z}_{\geqslant \ell}$ (or $\mathbb{Z}_{>\ell}$ and $\mathbb{Z}_{\leqslant \ell}$ (or $\mathbb{Z}_{<\ell}$)) to denote the sets of integers greater than or equal to (or strictly greater than) ℓ, and less than or equal to (or strictly less than) ℓ, respectively. As illustrated by (3), finite or infinite sums involving reciprocals of binomial coefficients have attracted considerable interest due to their rich structure and surprising properties (see, for example, [3] (Equations (2.1)–(2.26), (4.1)–(4.30) and (5.1) and (5.2))).

Gould [3] (Equation (2.12)) presented an infinite analogue of identity (3):

$$\sum_{k=1}^{\infty} \frac{1}{k^2 \binom{n+k}{k}} = \zeta(2) - H_n^{(2)}, \quad (n \in \mathbb{N}), \tag{4}$$

where $\zeta(s)$ denotes the classical Riemann zeta function, defined by

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} \quad (\Re(s) > 1),$$

and $H_n^{(2)}$ is the second-order harmonic number. The generalized harmonic number of order α is defined as

$$H_n^{(\alpha)} := \sum_{k=1}^{n} \frac{1}{k^\alpha} \quad (\alpha \in \mathbb{C},\, n \in \mathbb{N}). \tag{5}$$

By convention, $H_0^{(\alpha)} := 0$, and when $\alpha = 1$, $H_n^{(1)} = H_n$ denotes the nth harmonic number.

Combining (3) with the identity given in [3] (Equation (5.1)) yields the following result for $n \in \mathbb{N}$:

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{1}{k^2 \binom{n+k}{k}^2} &= \binom{2n}{n}\left(\zeta(2) - H_n^{(2)} - \sum_{k=1}^{n} \frac{1}{k^2 \binom{n+k}{k}}\right) = \binom{2n}{n} \sum_{k=n+1}^{\infty} \frac{1}{k^2 \binom{n+k}{k}} \\ &= \binom{2n}{n}\left(\zeta(2) + H_n^2 - H_n^{(2)} + \sum_{k=1}^{n} \frac{(-1)^k}{k} H_{n+k}\right),\end{aligned}$$

where identities (4) and (6) are applied in the second and third equalities, respectively.

Batır and Sofo [4] presented several finite sum identities involving the reciprocals of binomial and central binomial coefficients, along with harmonic numbers, Fibonacci numbers, and Lucas numbers.

Sprugnoli [5] investigated combinatorial sums involving reciprocals of central binomial coefficients, deriving several infinite and finite sum results—some involving the golden ratio—using generating functions and the method of coefficients.

Batır and Chen [6] established a general combinatorial formula involving reciprocals of binomial coefficients and partial sums of arbitrary sequences, from which they derived numerous identities—some previously known and others new—featuring reciprocals of binomial and central binomial coefficients and harmonic numbers.

Apéry [7] used the rapidly converging alternating series

$$\zeta(3) = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3 \binom{2k}{k}}$$

involving reciprocals of central binomial coefficients to prove the irrationality of $\zeta(3)$.

Guillera [8] derived a remarkably fast-converging series for the Catalan constant G, involving reciprocals of central binomial coefficients:

$$G = \sum_{k=0}^{\infty} \frac{(-1)^k (3k+2) 2^{3k-1}}{(2k+1)^3 \binom{2k}{k}^3},$$

a value classically defined by the alternating series

$$G = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} = 0.915965594177219015\ldots.$$

For other similar fast-converging series, see [9].

Sofo and Batır [10] presented an alternative expression for the left-hand side of identity (3). For $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} \frac{1}{k^2 \binom{n+k}{k}} = -H_n^2 - \sum_{k=1}^{n} \frac{(-1)^k}{k} H_{n+k}. \tag{6}$$

It follows from (3) and (6) that

$$\sum_{k=1}^{n} \frac{1}{k^2 \binom{2k}{k}} = \frac{H_n^{(2)} - H_n^2}{3} - \frac{1}{3} \sum_{k=1}^{n} \frac{(-1)^k}{k} H_{n+k} \quad (n \in \mathbb{N}).$$

Batır and Choi [11] investigated the following sums:

$$\sum_{k=1}^{m} \frac{\epsilon^k}{k^p \binom{n+k}{k}} \quad \text{and} \quad \sum_{k=1}^{m} \frac{\epsilon^k H_k}{k^p \binom{n+k}{k}}, \tag{7}$$

where $m, n \in \mathbb{N}$, $\epsilon \in \{-1, 1\}$, and $p \in \{0, 1\}$.

In this paper, we derive an explicit formula for a broad class of generalized sums of the form

$$\sum_{k=1}^{n} \frac{\Gamma_k}{k^p \binom{r+n+k}{k}},$$

where $p \in \mathbb{N}$, $n \in \mathbb{N}$, and $r \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1}$. Here, $\Gamma_n := \gamma_1 + \gamma_2 + \cdots + \gamma_n$, with $\{\gamma_k\}_{k \in \mathbb{N}}$ denoting an arbitrary sequence of complex numbers. Notably, this general expression reduces to known identities under specific parameter choices; for instance, the first sum in (7) becomes (3) when $r = 0$, $p = 2$, $\gamma_1 = 1$, and $\gamma_n = 0$ for all $n \in \mathbb{Z}_{\geqslant 2}$.

In Section 2, we present the main identity (Theorem 1) and its derivation. Section 3 is devoted to examining the three particular cases $p = 1, 2, 3$, as detailed in Corollaries 1–3. In particular, Corollary 1 offers an alternative explicit formula for the case $p = 1$, accompanied by a direct proof.

Section 4 investigates further concrete examples of the identity by selecting specific sequences for $\{\gamma_j\}_{j \in \mathbb{N}}$, thereby illustrating the flexibility of our general framework.

In Section 5, we differentiate the results from Corollaries 1–3 with respect to the parameter r, leading to new identities involving sums of reciprocals of binomial coefficients. These are formalized in Propositions 1–3, and are complemented by several interesting special cases.

Finally, Section 6 concludes the paper by highlighting the depth and adaptability of the central identity presented in Theorem 1.

For completeness, we also introduce several auxiliary functions, including extensions of the classical harmonic numbers H_n and generalized harmonic numbers $H_n^{(\alpha)}$, as discussed

in [12,13] (Section 1.3). The psi (or digamma) function, denoted by ψ, is defined as the logarithmic derivative of the gamma function $\Gamma(z)$:

$$\psi(z) := \frac{\Gamma'(z)}{\Gamma(z)} \quad (z \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}). \tag{8}$$

The harmonic number H_n is expressed in term of the digamma function as follows:

$$H_n = \psi(n+1) + \gamma \quad (n \in \mathbb{N}),$$

where $\gamma = \lim_{n\to\infty}(H_n - \log n) = 0.57721\ldots$ is the Euler–Mascheroni constant (see, for example, [13] (Section 1.2) and [14], and the references therein). Since the psi function $\psi(z)$ is defined on $\mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}$, the harmonic number H_n admits the following extension to the complex domain:

$$H_\mu = \psi(\mu+1) + \gamma \quad (\mu \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1}). \tag{9}$$

For non-integer values of the order n, such as a complex number $\mu \in \mathbb{C}$, the generalized harmonic number $H_\mu^{(m+1)}$ can be defined in terms of the polygamma function

$$\psi^{(n)}(z) := \frac{d^n}{dz^n}\psi(z) = \frac{d^{n+1}}{dz^{n+1}}\log\Gamma(z), \quad (n \in \mathbb{Z}_{\geqslant 0}),$$

where $\Gamma(z)$ is the gamma function and $\psi(z)$ is the digamma function. This definition gives

$$H_\mu^{(m+1)} = \zeta(m+1) + \frac{(-1)^m}{m!}\psi^{(m)}(\mu+1), \tag{10}$$

for $m \in \mathbb{N}$ and $\mu \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1}$, where $\zeta(z)$ denotes the Riemann zeta function. In the special case when $m = 0$, (10) reduces to $H_\mu^{(1)} = H_\mu$, where H_μ is the extended harmonic number of order μ in (9).

This method of extending harmonic and generalized harmonic numbers, pioneered by Sofo and Srivastava [12], has proven to be remarkably fruitful. It not only deepens our understanding of these classical objects but also opens the door to a wealth of elegant identities, particularly those related to classical Euler sums.

By applying the logarithmic derivative and invoking identity (8), we obtain the following formula:

$$\begin{aligned} \frac{d}{dz}\binom{z+a}{k}^{-1} &= \binom{z+a}{k}^{-1}\big(\psi(z+a-k+1) - \psi(z+a+1)\big) \\ &= \binom{z+a}{k}^{-1}\big(H_{z+a-k} - H_{z+a}\big), \end{aligned} \tag{11}$$

where (9) is employed for the second equality.

We remark that Pascal's Triangle exhibits symmetry about its central vertical axis. This property arises from the fundamental combinatorial identity

$$\binom{n}{k} = \binom{n}{n-k},$$

which asserts that, in the nth row, the kth entry from the left equals the $(n-k)$th entry from the right. As a result, each row forms a palindromic sequence, reading identically in both directions.

That is, in row n, the kth entry from the left is equal to the $(n-k)$th entry from the right. Consequently, each row of Pascal's Triangle is palindromic, reading identically

from left to right and right to left. The identities and results developed in this paper are deeply connected to this underlying combinatorial symmetry. Many of the recurrence relations and binomial identities we derive inherently reflect this structural balance. Notably, Equation (11) itself exhibits a form of symmetry: differentiating its left-hand side with respect to the parameter a yields a result that mirrors the differentiation on the right-hand side, reinforcing the symmetrical behavior of the underlying functions.

This alignment between structural symmetry in binomial coefficients and the analytic properties of the identities we explore highlights this paper's thematic relevance to the broader concept of symmetry in mathematical analysis and combinatorics.

2. Main Result

This section starts with an identity which links reciprocal of binomial coefficients, weighted sums, and arbitrary sequences. We need the following lemma.

Lemma 1. *Let $x \in \mathbb{C} \setminus \{0\}$ be fixed, and let $a \in \mathbb{C} \setminus \{0, x\}$. For any $\ell \in \mathbb{Z}_{\geqslant 0}$, the following partial fraction decomposition holds:*

$$\frac{1}{x^{\ell}\,(x+a)} = \sum_{\mu=1}^{\ell} \frac{(-1)^{\mu-1}}{a^{\mu} x^{\ell-\mu+1}} + \frac{(-1)^{\ell}}{a^{\ell}\,(x+a)}. \tag{12}$$

By convention, an empty sum is taken to be zero.

Proof. The identity can be established by mathematical induction on ℓ. The proof is straightforward and thus omitted. □

Theorem 1. *Let $\{\gamma_j\}_{j\in\mathbb{N}}$ be an arbitrary complex sequence, and define the partial sums denoted by $\Gamma_n = \gamma_1 + \gamma_2 + \cdots + \gamma_n$ for each $n \in \mathbb{N}$. Also, let $p \in \mathbb{N}$ and $r \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1}$. Then, for all $n \in \mathbb{N}$, the following identity holds:*

$$\begin{aligned}\sum_{k=1}^{n} \frac{\Gamma_k}{k^p \binom{n+r+k}{k}} &= \sum_{\mu=1}^{p-2} \sum_{k=1}^{n} \frac{(-1)^{\mu}}{(k+r)^{\mu}} \sum_{j=1}^{k-1} \frac{\Gamma_j}{j^{p-\mu}\binom{r+k+j-1}{j}} \\ &\quad + \sum_{k=1}^{n} \frac{(-1)^{p-1}}{(k+r)^{p-1}} \left(\sum_{j=1}^{k} \frac{\gamma_j}{j\binom{r+k+j-1}{j}} - \frac{\Gamma_k}{k\binom{r+2k-1}{k}} \right) \\ &\quad + \sum_{k=1}^{n} \frac{\Gamma_k}{k^p \binom{r+2k}{k}}.\end{aligned} \tag{13}$$

Proof. Consider the following sum:

$$\sum_{k=1}^{n} \frac{\Gamma_k}{k^p \binom{n+r+k}{k}}, \quad (p \in \mathbb{N}).$$

We define

$$\Omega_k = \sum_{j=1}^{k} \frac{\Gamma_j}{j^p \binom{k+r+j}{j}}.$$

Then

$$\Omega_{k+1}-\Omega_k=\sum_{j=1}^{k+1}\frac{\Gamma_j}{j^p\binom{k+r+j+1}{j}}-\sum_{j=1}^{k}\frac{\Gamma_j}{j^p\binom{k+r+j}{j}}$$
$$=\sum_{j=1}^{k}\frac{\Gamma_j}{j^p}\left[\frac{1}{\binom{k+r+j+1}{j}}-\frac{1}{\binom{k+r+j}{j}}\right]+\frac{\Gamma_{k+1}}{(k+1)^p\binom{2k+r+2}{k+1}}.$$

Applying (2), we find that

$$\frac{1}{\binom{k+r+j+1}{j}}-\frac{1}{\binom{k+r+j}{j}}=-\frac{j}{(k+r+j+1)\binom{k+r+j}{j}}.$$

Then, we have

$$\Omega_{k+1}-\Omega_k=-\sum_{j=1}^{k}\frac{\Gamma_j}{j^{p-1}(k+r+j+1)\binom{k+r+j}{j}}+\frac{\Gamma_{k+1}}{(k+1)^p\binom{2k+r+2}{k+1}}$$
$$=\sum_{\mu=1}^{p-1}\frac{(-1)^\mu}{(k+r+1)^\mu}\sum_{j=1}^{k}\frac{\Gamma_j}{j^{p-\mu}\binom{r+k+j}{j}}+\frac{(-1)^p}{(k+r+1)^{p-1}}\sum_{j=1}^{k}\frac{\Gamma_j}{(j+k+r+1)\binom{r+k+j}{j}} \tag{14}$$
$$+\frac{\Gamma_{k+1}}{(k+1)^p\binom{r+2k+2}{k+1}},$$

where the identity (12) is used for the second equality.

Using the second identity in (2), we get

$$(r+k+j+1)\binom{r+k+j}{j}=(j+1)\binom{r+k+j+1}{j+1}.$$

Applying this formula, we find that

$$\sum_{j=1}^{k}\frac{\Gamma_j}{(r+k+j+1)\binom{r+k+j}{j}}=\sum_{j=1}^{k}\frac{\Gamma_j}{(j+1)\binom{r+k+j+1}{j+1}}=\sum_{j=2}^{k+1}\frac{\Gamma_{j-1}}{j\binom{r+k+j}{j}}.$$

Since Γ_0 is an empty sum and is regarded as zero, with the aid of $\Gamma_{j-1}=\Gamma_j-\gamma_j$, we derive

$$\sum_{j=1}^{k}\frac{\Gamma_j}{(r+k+j+1)\binom{r+k+j}{j}}=\sum_{j=1}^{k+1}\frac{\Gamma_{j-1}}{j\binom{r+k+j}{j}}$$
$$=\sum_{j=1}^{k+1}\frac{\Gamma_j}{j\binom{r+k+j}{j}}-\sum_{j=1}^{k+1}\frac{\gamma_j}{j\binom{r+k+j}{j}}$$
$$=\sum_{j=1}^{k}\frac{\Gamma_j}{j\binom{r+k+j}{j}}-\sum_{j=1}^{k+1}\frac{\gamma_j}{j\binom{r+k+j}{j}}+\frac{\Gamma_{k+1}}{(k+1)\binom{r+2k+1}{k+1}}.$$

Substituting this final identity in (14), we obtain

$$\Omega_{k+1}-\Omega_k=\sum_{\mu=1}^{p-2}\frac{(-1)^\mu}{(k+r+1)^\mu}\sum_{j=1}^{k}\frac{\Gamma_j}{j^{p-\mu}\binom{r+k+j}{j}}$$
$$+\frac{(-1)^p}{(k+r+1)^{p-1}}\left(-\sum_{j=1}^{k+1}\frac{\gamma_j}{j\binom{r+k+j}{j}}+\frac{\Gamma_{k+1}}{(k+1)\binom{r+2k+1}{k+1}}\right)$$
$$+\frac{\Gamma_{k+1}}{(k+1)^p\binom{r+2k+2}{k+1}},$$

Summing both sides from $k=0$ to $k=n-1$, the left-hand side telescopes to

$$\Omega_n-\Omega_0=\Omega_n,$$

since Ω_0 is an empty sum. Thus, we arrive at

$$\Omega_n=\sum_{\mu=1}^{p-2}\sum_{k=1}^{n}\frac{(-1)^\mu}{(k+r)^\mu}\sum_{j=1}^{k-1}\frac{\Gamma_j}{j^{p-\mu}\binom{r+k+j-1}{j}}$$
$$+\sum_{k=1}^{n}\frac{(-1)^{p-1}}{(k+r)^{p-1}}\left(\sum_{j=1}^{k}\frac{\gamma_j}{j\binom{r+k+j-1}{j}}-\frac{\Gamma_k}{k\binom{r+2k-1}{k}}\right)$$
$$+\sum_{k=1}^{n}\frac{\Gamma_k}{k^p\binom{r+2k}{k}}.$$

This completes the proof. □

3. Particular Cases

This section explores the three specific cases of the main identity (13), corresponding to $p=1,2,3$, as presented in the following corollaries.

Corollary 1. *Let $\{\gamma_k\}_{k\in\mathbb{N}}$ be an arbitrary complex sequence, and define the partial sums denoted by $\Gamma_n=\gamma_1+\gamma_2+\cdots+\gamma_n$ for each $n\in\mathbb{N}$. Suppose $r\in\mathbb{C}\setminus\mathbb{Z}_{\leqslant -1}$. Then, for all $n\in\mathbb{N}$, the following identities hold:*

$$\sum_{k=1}^{n}\frac{\Gamma_k}{k\binom{n+r+k}{k}}=\sum_{k=1}^{n}\sum_{j=1}^{k}\frac{\gamma_j}{j\binom{r+k+j-1}{j}}-\sum_{k=1}^{n}\frac{\Gamma_k}{k\binom{r+2k-1}{k}}+\sum_{k=1}^{n}\frac{\Gamma_k}{k\binom{r+2k}{k}}. \tag{15}$$

$$\sum_{k=1}^{n}\frac{\Gamma_k}{k\binom{r+n+k}{k}}=\sum_{k=1}^{n}\frac{\gamma_k}{k\binom{r+n+k-1}{k}}-\frac{\Gamma_n}{(n+1)\binom{2n+r}{n+1}}. \tag{16}$$

Proof. The identity (15) immediately follows by setting $p=1$ in the main identity (13).

The proof of (16) is as follows: We set

$$P_n=\sum_{k=1}^{n}\frac{\Gamma_k}{k\binom{r+n+k}{k}}.$$

Then,

$$\begin{aligned} P_{n+1} - P_n &= \sum_{k=1}^{n+1} \frac{\Gamma_k}{k\binom{r+n+k+1}{k}} - \sum_{k=1}^{n} \frac{\Gamma_k}{k\binom{r+n+k}{k}} \\ &= \sum_{k=1}^{n} \frac{\Gamma_k}{k\binom{r+n+k+1}{k}} - \sum_{k=1}^{n} \frac{\Gamma_k}{k\binom{r+n+k}{k}} + \frac{\Gamma_{n+1}}{(n+1)\binom{r+2n+2}{n+1}} \\ &= -\sum_{k=1}^{n} \frac{\Gamma_k \binom{r+n+k}{k-1}}{k\binom{r+n+k}{k}\binom{r+n+k+1}{k}} + \frac{\Gamma_{n+1}}{(n+1)\binom{r+2n+2}{n+1}}. \end{aligned}$$

By (2) we have

$$\binom{r+n+k}{k-1} = \frac{k}{r+n+k+1}\binom{r+n+k+1}{k},$$

so that

$$P_{n+1} - P_n = -\sum_{k=1}^{n} \frac{\Gamma_k}{(r+n+k+1)\binom{r+n+k}{k}} + \frac{\Gamma_{n+1}}{(n+1)\binom{r+2n+2}{n+1}}.$$

Applying (2) again, a reindexing argument gives

$$\begin{aligned} P_{n+1} - P_n &= -\sum_{k=2}^{n+1} \frac{\Gamma_{k-1}}{k\binom{r+n+k}{k}} + \frac{\Gamma_{n+1}}{(n+1)\binom{r+2n+2}{n+1}} \\ &= -\sum_{k=2}^{n} \frac{\Gamma_{k-1}}{k\binom{r+n+k}{k}} + \frac{\Gamma_{n+1}}{(n+1)\binom{r+2n+2}{n+1}} - \frac{\Gamma_n}{(n+1)\binom{r+2n+1}{n+1}} \\ &= -\sum_{k=2}^{n} \frac{\Gamma_k - \gamma_k}{k\binom{r+n+k}{k}} + \frac{\Gamma_{n+1}}{(n+1)\binom{r+2n+2}{n+1}} - \frac{\Gamma_n}{(n+1)\binom{r+2n+1}{n+1}}. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} P_{n+1} &= \sum_{k=1}^{n} \frac{\gamma_k}{k\binom{r+n+k}{k}} + \frac{\Gamma_{n+1}}{(n+1)\binom{r+2n+2}{n+1}} - \frac{\Gamma_n}{(n+1)\binom{r+2n+1}{n+1}} \\ &= \sum_{k=1}^{n+1} \frac{\gamma_k}{k\binom{r+n+k}{k}} + \frac{\Gamma_{n+1}}{(n+1)\binom{r+2n+2}{n+1}} - \frac{\Gamma_{n+1}}{(n+1)\binom{r+2n+1}{n+1}}. \end{aligned}$$

Applying (2), we get

$$P_{n+1} = \sum_{k=1}^{n+1} \frac{\gamma_k}{k\binom{r+n+k}{k}} - \frac{\Gamma_{n+1}}{(n+2)\binom{r+2n+2}{n+2}}.$$

Finally, replacing n with $n-1$ in the last identity, we prove the desired identity (16). □

Corollary 2. *Let $\{\gamma_j\}_{j\in\mathbb{N}}$ be an arbitrary complex sequence, and define the partial sums denoted by $\Gamma_n = \gamma_1 + \gamma_2 + \cdots + \gamma_n$ for each $n \in \mathbb{N}$. Suppose $r \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1}$. Then, for all $n \in \mathbb{N}$, the following identity holds:*

$$\begin{aligned} \sum_{k=1}^{n} \frac{\Gamma_k}{k^2\binom{n+r+k}{k}} = &-\sum_{k=1}^{n} \frac{1}{r+k} \sum_{j=1}^{k} \frac{\gamma_j}{j\binom{r+k+j-1}{j}} + \sum_{k=1}^{n} \frac{\Gamma_k}{k(r+k)\binom{r+2k-1}{k}} \\ &+ \sum_{k=1}^{n} \frac{\Gamma_k}{k^2\binom{r+2k}{k}}. \end{aligned} \tag{17}$$

Proof. The identity (17) immediately follows by setting $p = 2$ in the main identity (13). □

Corollary 3. *Let $\{\gamma_j\}_{j\in\mathbb{N}}$ be an arbitrary complex sequence, and define the partial sums denoted by $\Gamma_n = \gamma_1 + \gamma_2 + \cdots + \gamma_n$ for each $n \in \mathbb{N}$. Suppose $r \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1}$. Then, for all $n \in \mathbb{N}$, the following identity holds:*

$$\begin{aligned}\sum_{k=1}^{n} \frac{\Gamma_k}{k^3\binom{n+r+k}{k}} = &-\sum_{k=1}^{n} \frac{1}{k+r} \sum_{j=1}^{k-1} \frac{\Gamma_j}{j^2\binom{k+r+j-1}{j}} + \sum_{k=1}^{n} \frac{1}{(k+r)^2} \sum_{j=1}^{k} \frac{\gamma_j}{j\binom{k+r+j-1}{j}} \\ &- \sum_{k=1}^{n} \frac{\Gamma_k}{k\,(k+r)^2\binom{r+2k-1}{k}} + \sum_{k=1}^{n} \frac{\Gamma_k}{k^3\binom{r+2k}{k}}.\end{aligned} \tag{18}$$

Proof. The identity (18) immediately follows by setting $p = 3$ in the main identity (13). □

4. Further Particular Cases

This section explores further specific cases of the identities presented in Section 3 with concrete examples of the identity by selecting specific sequences for $\{\gamma_j\}_{j\in\mathbb{N}}$, as outlined in the following corollaries.

4.1. Choice of the Sequence: $\gamma_1 = 1$ and $\gamma_j = 0$ for All $j \in \mathbb{Z}_{\geqslant 2}$

Setting $\gamma_1 = 1$ and $\gamma_j = 0$ for all $j \in \mathbb{Z}_{\geqslant 2}$ in the results in Corollaries 1–3, we obtain the following identities, as stated in Corollaries 4–6.

Corollary 4. *Let $r \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1}$. Then, for all $n \in \mathbb{N}$, the following identities hold:*

$$\sum_{k=1}^{n} \frac{1}{k\binom{n+r+k}{k}} = \sum_{k=1}^{n} \frac{1}{r+k} - \sum_{k=1}^{n} \frac{1}{k\binom{r+2k-1}{k}} + \sum_{k=1}^{n} \frac{1}{k\binom{r+2k}{k}}. \tag{19}$$

$$\sum_{k=1}^{n} \frac{1}{k\binom{r+n+k}{k}} = \frac{1}{n+r} - \frac{1}{(n+1)\binom{2n+r}{n+1}}. \tag{20}$$

In particular,

$$\sum_{k=1}^{n} \frac{1}{k\binom{n+k}{k}} = H_n - \sum_{k=1}^{n} \frac{1}{k\binom{2k}{k}} = \frac{1}{n} - \frac{1}{(n+1)\binom{2n}{n+1}}. \tag{21}$$

Proof. Setting $r = 0$ in (19) and (20) yields the first and second equalities of (21), respectively. □

Corollary 5. *Let $r \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1}$. Then, for all $n \in \mathbb{N}$, the following identity holds:*

$$\sum_{k=1}^{n} \frac{1}{k^2\binom{n+r+k}{k}} = -\sum_{k=1}^{n} \frac{1}{(r+k)^2} + \sum_{k=1}^{n} \frac{1}{k(r+k)\binom{r+2k-1}{k}} + \sum_{k=1}^{n} \frac{1}{k^2\binom{r+2k}{k}}. \tag{22}$$

Also,

$$\sum_{k=1}^{n} \frac{1}{k^2\binom{n+k}{k}} = 3\sum_{k=1}^{n} \frac{1}{k^2\binom{2k}{k}} - H_n^{(2)}, \tag{23}$$

which agrees with Sun's identity (3).

Proof. Setting $r = 0$ in (22) leads to (23). □

Corollary 6. *Let $r \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1}$. Then, for all $n \in \mathbb{N}$, the following identities hold:*

$$\sum_{k=1}^{n} \frac{1}{k^3\binom{n+r+k}{k}} = -\sum_{k=1}^{n} \frac{1}{k+r} \sum_{j=1}^{k-1} \frac{1}{j^2\binom{k+r+j-1}{j}} + \sum_{k=1}^{n} \frac{1}{(k+r)^3} + \sum_{k=1}^{n} \frac{1}{k^3\binom{r+2k}{k}} - \sum_{k=1}^{n} \frac{1}{k\,(k+r)^2\binom{r+2k-1}{k}}. \tag{24}$$

In particular,

$$\sum_{k=1}^{n} \frac{1}{k^3\binom{n+k}{k}} = H_n^{(3)} - \sum_{k=1}^{n} \frac{1}{k} \sum_{j=1}^{k-1} \frac{1}{j^2\binom{k+j-1}{j}} - \sum_{k=1}^{n} \frac{1}{k^3\binom{2k}{k}}. \tag{25}$$

Also,

$$\sum_{k=1}^{n} \frac{1}{k^3\binom{n+r+k}{k}} = \sum_{k=1}^{n} \frac{1}{k+r} \sum_{j=1}^{k} \frac{1}{(r+j)^2} - \sum_{k=1}^{n} \frac{1}{k+r} \sum_{j=1}^{k-1} \frac{3j^2+3rj+r^2}{j^2(r+j)^2\binom{r+2j}{j}} + \sum_{k=1}^{n} \frac{1}{k^3\binom{r+2k}{k}} - \sum_{k=1}^{n} \frac{1}{k(k+r)^2\binom{r+2k-1}{k}}. \tag{26}$$

In particular,

$$\sum_{k=1}^{n} \frac{1}{k^3\binom{n+k}{k}} = \sum_{k=1}^{n} \frac{H_k^{(2)}}{k} - 3\sum_{k=1}^{n} \frac{1}{k} \sum_{j=1}^{k-1} \frac{1}{j^2\binom{2j}{j}} - \sum_{k=1}^{n} \frac{1}{k^3\binom{2k}{k}}. \tag{27}$$

Proof. Particularly, setting $r = 0$ in (24) yields (25).

A proof of (26) is provided: We define

$$G_k = \sum_{j=1}^{k} \frac{1}{j^3\binom{r+k+j}{j}}.$$

Then,

$$\begin{aligned} G_{k+1} - G_k &= \sum_{k=1}^{k+1} \frac{1}{j^3\binom{r+k+1+j}{j}} - \sum_{j=1}^{k} \frac{1}{j^3\binom{r+k+j}{j}} \\ &= \sum_{j=1}^{k} \left[\frac{1}{j^3\binom{r+k+1+j}{j}} - \frac{1}{j^3\binom{r+k+j}{j}} \right] + \frac{1}{(k+1)^3\binom{r+2k+2}{k+1}} \\ &= \sum_{j=1}^{k} \frac{\binom{r+k+j}{j} - \binom{r+k+j+1}{j}}{j^3\binom{r+k+j}{j}\binom{r+k+j+1}{j}} + \frac{1}{(k+1)^3\binom{r+2k+2}{k+1}}. \end{aligned} \tag{28}$$

Using (2), we find

$$\binom{r+k+j}{j} - \binom{r+k+j+1}{j} = -\frac{j}{r+k+j+1}\binom{r+k+j+1}{j},$$

which is applied to Equation (28) to yield

$$G_{k+1} - G_k = -\sum_{j=1}^{k} \frac{1}{j^2(r+k+j+1)\binom{r+k+j}{j}} + \frac{1}{(k+1)^3\binom{r+2k+2}{k+1}}. \tag{29}$$

An elementary computation gives

$$\frac{1}{j^2(r+k+j+1)} = \frac{1}{(r+k+1)j^2} - \frac{1}{(r+k+1)^2 j} + \frac{1}{(k+r+1)^2(r+k+j+1)}.$$

Using this identity in (29), we find that

$$\begin{aligned} G_{k+1} - G_k = &-\tfrac{1}{r+k+1}\sum_{j=1}^{k}\tfrac{1}{j^2\binom{r+k+j}{j}} + \tfrac{1}{(k+1)^3\binom{r+2k+2}{k+1}} \\ &+\tfrac{1}{(r+k+1)^2}\left[\sum_{j=1}^{k}\tfrac{1}{j\binom{r+k+j}{j}} - \sum_{j=1}^{k}\tfrac{1}{(r+k+j+1)\binom{r+k+j}{j}}\right]. \end{aligned} \tag{30}$$

Employing the second identity given in (2), we see that

$$\begin{aligned} \sum_{j=1}^{k}\tfrac{1}{(r+k+j+1)\binom{r+k+j}{j}} &= \sum_{j=1}^{k}\tfrac{1}{(j+1)\binom{r+k+j+1}{j+1}} = \sum_{j=2}^{k+1}\tfrac{1}{j\binom{r+k+j}{j}} \\ &= \sum_{j=1}^{k}\tfrac{1}{j\binom{r+k+j}{j}} + \tfrac{1}{(k+1)\binom{r+2k+1}{k+1}} - \tfrac{1}{r+k+1}. \end{aligned} \tag{31}$$

Substituting (31) into (30), we obtain

$$\begin{aligned} G_{k+1} - G_k = &-\frac{1}{r+k+1}\sum_{j=1}^{k}\frac{1}{j^2\binom{r+k+j}{j}} + \frac{1}{(k+1)^3\binom{r+2k+2}{k+1}} \\ &+\frac{1}{(r+k+1)^3} - \frac{1}{(k+1)(r+k+1)^2\binom{r+2k+1}{k+1}}. \end{aligned} \tag{32}$$

Applying Corollary 5 to the summation in (32), we get

$$\begin{aligned} G_{k+1} - G_k = &-\frac{1}{r+k+1}\left[\sum_{j=1}^{k}\frac{1}{j^2\binom{r+2j}{j}} + \sum_{j=1}^{k}\frac{1}{j(r+j)\binom{r+2j-1}{j}}\right] \\ &+\frac{1}{r+k+1}\sum_{j=1}^{k}\frac{1}{(r+j)^2} + \frac{1}{(k+1)^3\binom{r+2k+2}{k+1}} \\ &+\frac{1}{(r+k+1)^3} - \frac{1}{(k+1)(r+k+1)^2\binom{r+2k+1}{k+1}}. \end{aligned}$$

Summing both sides of this identity from $k = 0$ to $k = n-1$, and letting $k+1 = k'$ and dropping the prime on k, and using $G_0 = 0$, we get

$$\begin{aligned} G_n = &-\sum_{k=1}^{n}\frac{1}{r+k}\sum_{j=1}^{k-1}\frac{1}{j^2\binom{r+2j}{j}} - \sum_{k=1}^{n}\frac{1}{r+k}\sum_{j=1}^{k-1}\frac{1}{j(r+j)\binom{r+2j-1}{j}} \\ &+\sum_{k=1}^{n}\frac{1}{r+k}\sum_{j=1}^{k}\frac{1}{(r+j)^2} + \sum_{k=1}^{n}\frac{1}{k^3\binom{r+2k}{k}} \\ &-\sum_{k=1}^{n}\frac{1}{k(r+k)^2\binom{r+2k-1}{k}}. \end{aligned}$$

Finally, using the identity

$$\binom{r+2j-1}{j} = \frac{r+j}{r+2j}\binom{r+2j}{j}$$

to unify the first two summations, we derive the desired result (26). Particularly, setting $r = 0$ in (26) yields (27). □

4.2. Generalized Harmonic Numbers

Setting $\gamma_j = \frac{1}{j^\alpha}$ for all $j \in \mathbb{N}$ and some $\alpha \in \mathbb{C}$ in Corollaries 1, 2, and 3, we obtain the following identities, as stated in Corollaries 7, 8, and 9, respectively.

Corollary 7. *Let $\alpha \in \mathbb{C}$. Suppose $r \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1}$. Then, for all $n \in \mathbb{N}$, the following identity holds:*

$$\sum_{k=1}^{n} \frac{H_k^{(\alpha)}}{k\binom{r+n+k}{k}} = \sum_{k=1}^{n} \frac{1}{k^{1+\alpha}\binom{r+n+k-1}{k}} - \frac{H_n^{(\alpha)}}{(n+1)\binom{2n+r}{n+1}}. \tag{33}$$

In particular,

$$\sum_{k=1}^{n} \frac{H_k^{(\alpha)}}{k\binom{n+k}{k}} = \sum_{k=1}^{n} \frac{1}{k^{1+\alpha}\binom{n+k-1}{k}} - \frac{H_n^{(\alpha)}}{(n+1)\binom{2n}{n+1}}. \tag{34}$$

Proof. Setting $r = 0$ in (33) leads to (34). □

Corollary 8. *Let $\alpha \in \mathbb{C}$. Suppose $r \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1}$. Then, for all $n \in \mathbb{N}$, the following identity holds:*

$$\begin{aligned}\sum_{k=1}^{n} \frac{H_k^{(\alpha)}}{k^2\binom{n+r+k}{k}} = &- \sum_{k=1}^{n} \frac{1}{r+k} \sum_{j=1}^{k} \frac{1}{j^{1+\alpha}\binom{r+k+j-1}{j}} \\ &+ \sum_{k=1}^{n} \frac{H_k^{(\alpha)}}{k(r+k)\binom{r+2k-1}{k}} + \sum_{k=1}^{n} \frac{H_k^{(\alpha)}}{k^2\binom{r+2k}{k}}.\end{aligned} \tag{35}$$

In particular,

$$\sum_{k=1}^{n} \frac{H_k^{(\alpha)}}{k^2\binom{n+k}{k}} = - \sum_{k=1}^{n} \frac{1}{k} \sum_{j=1}^{k} \frac{1}{j^{1+\alpha}\binom{k+j-1}{j}} + 3 \sum_{k=1}^{n} \frac{H_k^{(\alpha)}}{k^2\binom{2k}{k}}. \tag{36}$$

Proof. Setting $r = 0$ in (35) yields (36). □

Corollary 9. *Let $\alpha \in \mathbb{C}$. Suppose $r \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1}$. Then, for all $n \in \mathbb{N}$, the following identity holds:*

$$\begin{aligned}\sum_{k=1}^{n} \frac{H_k^{(\alpha)}}{k^3\binom{n+r+k}{k}} = &- \sum_{k=1}^{n} \frac{1}{k+r} \sum_{j=1}^{k-1} \frac{H_j^{(\alpha)}}{j^2\binom{k+r+j-1}{j}} + \sum_{k=1}^{n} \frac{1}{(k+r)^2} \sum_{j=1}^{k} \frac{1}{j^{1+\alpha}\binom{k+r+j-1}{j}} \\ &- \sum_{k=1}^{n} \frac{H_k^{(\alpha)}}{k\,(k+r)^2\binom{r+2k-1}{k}} + \sum_{k=1}^{n} \frac{H_k^{(\alpha)}}{k^3\binom{r+2k}{k}}.\end{aligned} \tag{37}$$

In particular,

$$\sum_{k=1}^{n} \frac{H_k^{(\alpha)}}{k^3\binom{n+k}{k}} = - \sum_{k=1}^{n} \frac{1}{k} \sum_{j=1}^{k-1} \frac{H_j^{(\alpha)}}{j^2\binom{k+j-1}{j}} + \sum_{k=1}^{n} \frac{1}{k^2} \sum_{j=1}^{k} \frac{1}{j^{1+\alpha}\binom{k+j-1}{j}} - \sum_{k=1}^{n} \frac{H_k^{(\alpha)}}{k^3\binom{2k}{k}}. \tag{38}$$

Proof. Setting $r = 0$ in (37) yields (38). □

4.3. Bernoulli Numbers

The Bernoulli numbers $B_n := B_n(0)$ are are defined by the generating function (see [13] (Section 1.7)):

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} \quad (|z| < 2\pi). \tag{39}$$

Using the well-known facts:

$$B_1 = -\frac{1}{2} \quad \text{and} \quad B_{2n+1} = 0 \quad (n \in \mathbb{N}),$$

the right-hand side of Equation (39) can be rewritten as:

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{j=1}^{\infty} B_{2j} \frac{z^{2j}}{(2j)!} \quad (|z| < 2\pi).$$

The first few of the Bernoulli numbers are listed:

$$\begin{aligned} &B_0 = 1,\ B_1 = -\tfrac{1}{2},\ B_2 = \tfrac{1}{6},\ B_4 = -\tfrac{1}{30},\ B_6 = \tfrac{1}{42},\ B_8 = -\tfrac{1}{30},\ B_{10} = \tfrac{5}{66},\\ &B_{12} = -\tfrac{691}{2730},\quad B_{14} = \tfrac{7}{6},\quad B_{16} = -\tfrac{3617}{510},\quad B_{18} = \tfrac{43867}{798},\quad B_{20} = -\tfrac{174611}{330},\\ &B_{22} = \tfrac{854513}{138},\quad B_{24} = -\tfrac{236364091}{2730},\quad B_{26} = \tfrac{8553103}{6}, \cdots. \end{aligned}$$

Recall the following identity (see [15] (p. 1, Entry 0.121)): For any $q \in \mathbb{N}$ and $n \in \mathbb{N}$,

$$\sum_{j=1}^{n} j^q = \frac{n^{q+1}}{q+1} + \frac{n^q}{2} + \sum_{\ell=1} \frac{1}{2\ell} \binom{q}{2\ell - 1} B_{2\ell}\, n^{q-(2\ell-1)} =: F(n, q, \mathrm{Ber}), \tag{40}$$

where $B_{2\ell}$ are Bernoulli numbers, and the last term involves either n or n^2 depending on q.

The first three specific cases of (40) are

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2}; \tag{41}$$

$$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}; \tag{42}$$

$$\sum_{j=1}^{n} j^3 = \left(\frac{n(n+1)}{2}\right)^2. \tag{43}$$

By assigning $\gamma_j = j^q$ for all $j \in \mathbb{N}$ and some fixed $q \in \mathbb{N}$,

Corollaries 1, 2, and 3 yield the results stated in Corollaries 10, 11, and 12, respectively. Each corollary includes the specific case when $r = 0$, within which the particular identities (41)–(43) are also utilized.

Corollary 10. *Let $q \in \mathbb{N}$. $r \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1}$. Then, for all $n \in \mathbb{N}$, the following identity holds:*

$$\sum_{k=1}^{n} \frac{F(k, q, \mathrm{Ber})}{k\binom{r+n+k}{k}} = \sum_{k=1}^{n} \frac{1}{k^{1-q}\binom{r+n+k-1}{k}} - \frac{F(n, q, \mathrm{Ber})}{(n+1)\binom{2n+r}{n+1}}.$$

In particular,

$$\sum_{k=1}^{n} \frac{F(k, q, \mathrm{Ber})}{k\binom{n+k}{k}} = \sum_{k=1}^{n} \frac{1}{k^{1-q}\binom{n+k-1}{k}} - \frac{F(n, q, \mathrm{Ber})}{(n+1)\binom{2n}{n+1}}.$$

Furthermore,

$$\sum_{k=1}^{n} \frac{(k+1)(2k+1)}{\binom{n+k}{k}} = 6 \sum_{k=1}^{n} \frac{k}{\binom{n+k-1}{k}} - \frac{(n+1)(2n+1)}{\binom{2n}{n}};$$

$$\sum_{k=1}^{n} \frac{k(k+1)^2}{\binom{n+k}{k}} = 4 \sum_{k=1}^{n} \frac{k^2}{\binom{n+k-1}{k}} - \frac{n(n+1)^2}{\binom{2n}{n}}.$$

Corollary 11. *Let $q \in \mathbb{N}$. Suppose $r \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1}$. Then, for all $n \in \mathbb{N}$, the following identity holds:*

$$\sum_{k=1}^{n} \frac{F(k,q,\mathrm{Ber})}{k^2\binom{n+r+k}{k}} = -\sum_{k=1}^{n} \frac{1}{r+k} \sum_{j=1}^{k} \frac{1}{j^{1-q}\binom{r+k+j-1}{j}} + \sum_{k=1}^{n} \frac{F(k,q,\mathrm{Ber})}{k(r+k)\binom{r+2k-1}{k}} + \sum_{k=1}^{n} \frac{F(k,q,\mathrm{Ber})}{k^2\binom{r+2k}{k}}.$$

In particular,

$$\sum_{k=1}^{n} \frac{F(k,q,\mathrm{Ber})}{k^2\binom{n+k}{k}} = -\sum_{k=1}^{n} \frac{1}{k} \sum_{j=1}^{k} \frac{1}{j^{1-q}\binom{k+j-1}{j}} + 3\sum_{k=1}^{n} \frac{F(k,q,\mathrm{Ber})}{k^2\binom{2k}{k}}.$$

Furthermore,

$$\sum_{k=1}^{n} \frac{k+1}{k\binom{n+k}{k}} = -2\sum_{k=1}^{n} \frac{1}{k} \sum_{j=1}^{k} \frac{1}{\binom{k+j-1}{j}} + 3\sum_{k=1}^{n} \frac{k+1}{k\binom{2k}{k}};$$

$$\sum_{k=1}^{n} \frac{(k+1)(2k+1)}{k\binom{n+k}{k}} = -6\sum_{k=1}^{n} \frac{1}{k} \sum_{j=1}^{k} \frac{j}{\binom{k+j-1}{j}} + 3\sum_{k=1}^{n} \frac{(k+1)(2k+1)}{k\binom{2k}{k}};$$

$$\sum_{k=1}^{n} \frac{(k+1)^2}{\binom{n+k}{k}} = -4\sum_{k=1}^{n} \frac{1}{k} \sum_{j=1}^{k} \frac{j^2}{\binom{k+j-1}{j}} + 3\sum_{k=1}^{n} \frac{(k+1)^2}{\binom{2k}{k}}.$$

Corollary 12. *Let $q \in \mathbb{N}$. Suppose $r \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1}$. Then, for all $n \in \mathbb{N}$, the following identity holds:*

$$\sum_{k=1}^{n} \frac{F(k,q,\mathrm{Ber})}{k^3\binom{n+r+k}{k}} = -\sum_{k=1}^{n} \frac{1}{k+r} \sum_{j=1}^{k-1} \frac{F(j,q,\mathrm{Ber})}{j^2\binom{k+r+j-1}{j}} + \sum_{k=1}^{n} \frac{1}{(k+r)^2} \sum_{j=1}^{k} \frac{1}{j^{1-q}\binom{k+r+j-1}{j}} - \sum_{k=1}^{n} \frac{F(k,q,\mathrm{Ber})}{k\,(k+r)^2\binom{r+2k-1}{k}} + \sum_{k=1}^{n} \frac{F(k,q,\mathrm{Ber})}{k^3\binom{r+2k}{k}}.$$

In particular,

$$\sum_{k=1}^{n} \frac{F(k,q,\mathrm{Ber})}{k^3\binom{n+k}{k}} = -\sum_{k=1}^{n} \frac{1}{k} \sum_{j=1}^{k-1} \frac{F(j,q,\mathrm{Ber})}{j^2\binom{k+j-1}{j}} + \sum_{k=1}^{n} \frac{1}{k^2} \sum_{j=1}^{k} \frac{1}{j^{1-q}\binom{k+j-1}{j}} - \sum_{k=1}^{n} \frac{F(k,q,\mathrm{Ber})}{k^3\binom{2k}{k}}.$$

Furthermore,

$$\sum_{k=1}^{n} \frac{k+1}{k^2\binom{n+k}{k}} = -\sum_{k=1}^{n} \frac{1}{k} \sum_{j=1}^{k-1} \frac{j+1}{j\binom{k+j-1}{j}} + 2\sum_{k=1}^{n} \frac{1}{k^2} \sum_{j=1}^{k} \frac{1}{\binom{k+j-1}{j}} - \sum_{k=1}^{n} \frac{k+1}{k^2\binom{2k}{k}};$$

$$\sum_{k=1}^{n} \frac{(k+1)(2k+1)}{k^2\binom{n+k}{k}} = -\sum_{k=1}^{n} \frac{1}{k} \sum_{j=1}^{k-1} \frac{(j+1)(2j+1)}{j\binom{k+j-1}{j}} + 6\sum_{k=1}^{n} \frac{1}{k^2} \sum_{j=1}^{k} \frac{j}{\binom{k+j-1}{j}} - \sum_{k=1}^{n} \frac{(k+1)(2k+1)}{k^2\binom{2k}{k}};$$

$$\sum_{k=1}^{n} \frac{(k+1)^2}{k\binom{n+k}{k}} = -\sum_{k=1}^{n} \frac{1}{k} \sum_{j=1}^{k-1} \frac{(j+1)^2}{\binom{k+j-1}{j}} + 4\sum_{k=1}^{n} \frac{1}{k^2} \sum_{j=1}^{k} \frac{j^2}{\binom{k+j-1}{j}} - \sum_{k=1}^{n} \frac{(k+1)^2}{k\,\binom{2k}{k}}.$$

4.4. Arithmetic Progression

Recall the arithmetic progression identity: For some fixed a, $d \in \mathbb{C}$ and each $n \in \mathbb{N}$,

$$\sum_{j=1}^{n} (a + (j-1)d) = \frac{n}{2}[2a + (n-1)d]. \tag{44}$$

The particular case of (44) when $a = 1$ and $d = 2$ gives

$$\sum_{j=1}^{n}(2j-1) = n^2. \tag{45}$$

By setting $\gamma_j = a + (j-1)d$ for all $j \in \mathbb{N}$ and fixed parameters $a, d \in \mathbb{C}$, Corollaries 1, 2, and 3 lead to the results given in Corollaries 13, 14, and 15, respectively. Each corollary includes the special case $r = 0$, wherein the identity (45) is also applied.

Corollary 13. *Let $a, d \in \mathbb{C}$. Suppose $r \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1}$. Then, for all $n \in \mathbb{N}$, the following identities hold:*

$$\sum_{k=1}^{n}\frac{2a+(k-1)d}{\binom{n+r+k}{k}} = 2\sum_{k=1}^{n}\sum_{j=1}^{k}\frac{a+(j-1)d}{j\binom{r+k+j-1}{j}} - \sum_{k=1}^{n}\frac{2a+(k-1)d}{\binom{r+2k-1}{k}} + \sum_{k=1}^{n}\frac{2a+(k-1)d}{\binom{r+2k}{k}}.$$

$$\sum_{k=1}^{n}\frac{2a+(k-1)d}{\binom{r+n+k}{k}} = 2\sum_{k=1}^{n}\frac{a+(k-1)d}{k\binom{r+n+k-1}{k}} - \frac{n[2a+(n-1)d]}{(n+1)\binom{2n+r}{n+1}}.$$

In particular,

$$\sum_{k=1}^{n}\frac{2a+(k-1)d}{\binom{n+k}{k}} = 2\sum_{k=1}^{n}\sum_{j=1}^{k}\frac{a+(j-1)d}{j\binom{k+j-1}{j}} - \sum_{k=1}^{n}\frac{2a+(k-1)d}{\binom{2k}{k}};$$

$$\sum_{k=1}^{n}\frac{2a+(k-1)d}{\binom{n+k}{k}} = 2\sum_{k=1}^{n}\frac{a+(k-1)d}{k\binom{n+k-1}{k}} - \frac{2a+(n-1)d}{\binom{2n}{n}}.$$

Furthermore,

$$\sum_{k=1}^{n}\sum_{j=1}^{k}\frac{2j-1}{j\binom{k+j-1}{j}} = \sum_{k=1}^{n}\frac{k}{\binom{n+k}{k}} + \sum_{k=1}^{n}\frac{k}{\binom{2k}{k}};$$

$$\sum_{k=1}^{n}\frac{k}{\binom{n+k}{k}} = \sum_{k=1}^{n}\frac{2k-1}{k\binom{n+k-1}{k}} - \frac{n}{\binom{2n}{n}}.$$

$$\sum_{k=1}^{n}\frac{k}{\binom{n+k}{k}} = 2\sum_{k=1}^{n}\frac{1}{\binom{n+k-1}{k}} - \sum_{k=1}^{n}\frac{1}{k\binom{n+k-1}{k}} - \frac{n}{\binom{2n}{n}}.$$

Corollary 14. *Let $a, d \in \mathbb{C}$. Suppose $r \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1}$. Then, for all $n \in \mathbb{N}$, the following identities hold:*

$$\sum_{k=1}^{n}\frac{2a+(k-1)d}{k\binom{n+r+k}{k}} = -2\sum_{k=1}^{n}\frac{1}{r+k}\sum_{j=1}^{k}\frac{a+(j-1)d}{j\binom{r+k+j-1}{j}} + \sum_{k=1}^{n}\frac{2a+(k-1)d}{(r+k)\binom{r+2k-1}{k}} + \sum_{k=1}^{n}\frac{2a+(k-1)d}{k\binom{r+2k}{k}}.$$

In particular,

$$\sum_{k=1}^{n}\frac{2a+(k-1)d}{k\binom{n+k}{k}} = -2\sum_{k=1}^{n}\frac{1}{k}\sum_{j=1}^{k}\frac{a+(j-1)d}{j\binom{k+j-1}{j}} + 3\sum_{k=1}^{n}\frac{2a+(k-1)d}{k\binom{2k}{k}}.$$

Furthermore,

$$\sum_{k=1}^{n}\frac{1}{\binom{n+k}{k}} = -\sum_{k=1}^{n}\frac{1}{k}\sum_{j=1}^{k}\frac{2j-1}{j\binom{k+j-1}{j}} + 3\sum_{k=1}^{n}\frac{1}{\binom{2k}{k}}.$$

Corollary 15. *Let $a, d \in \mathbb{C}$. Suppose $r \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1}$. Then, for all $n \in \mathbb{N}$, the following identities hold:*

$$\sum_{k=1}^{n} \frac{2a+(k-1)d}{k^2\binom{n+r+k}{k}} = -\sum_{k=1}^{n} \frac{1}{k+r} \sum_{j=1}^{k-1} \frac{2a+(j-1)d}{j\binom{k+r+j-1}{j}} + 2\sum_{k=1}^{n} \frac{1}{(k+r)^2} \sum_{j=1}^{k} \frac{a+(j-1)d}{j\binom{k+r+j-1}{j}} - \sum_{k=1}^{n} \frac{2a+(k-1)d}{(k+r)^2\binom{r+2k-1}{k}} + \sum_{k=1}^{n} \frac{2a+(k-1)d}{k^2\binom{r+2k}{k}}.$$

In particular,

$$\sum_{k=1}^{n} \frac{2a+(k-1)d}{k^2\binom{n+k}{k}} = -\sum_{k=1}^{n} \frac{1}{k} \sum_{j=1}^{k-1} \frac{2a+(j-1)d}{j\binom{k+j-1}{j}} + 2\sum_{k=1}^{n} \frac{1}{k^2} \sum_{j=1}^{k} \frac{a+(j-1)d}{j\binom{k+j-1}{j}} - \sum_{k=1}^{n} \frac{2a+(k-1)d}{k^2\binom{2k}{k}}.$$

Furthermore,

$$\sum_{k=1}^{n} \frac{1}{k\binom{n+k}{k}} = -\sum_{k=1}^{n} \frac{1}{k} \sum_{j=1}^{k-1} \frac{1}{\binom{k+j-1}{j}} + \sum_{k=1}^{n} \frac{1}{k^2} \sum_{j=1}^{k} \frac{2j-1}{j\binom{k+j-1}{j}} - \sum_{k=1}^{n} \frac{1}{k\binom{2k}{k}}.$$

5. Further Results

By differentiating the identities established in Corollaries 1–3 with respect to r, we derive several identities involving sums of reciprocals of binomial coefficients, as stated in Propositions 1–3. We further explore notable special cases arising from these results.

Proposition 1. *Let $\{\gamma_k\}_{k\in\mathbb{N}}$ be an arbitrary complex sequence, and define the partial sums denoted by $\Gamma_n = \gamma_1 + \gamma_2 + \cdots + \gamma_n$ for each $n \in \mathbb{N}$. Suppose $r \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1}$. Then, for all $n \in \mathbb{N}$, the following identity holds:*

$$\sum_{k=1}^{n} \frac{\Gamma_k(H_{r+n} - H_{r+n+k})}{k\binom{r+n+k}{k}} = \sum_{k=1}^{n} \frac{\gamma_k}{k\binom{r+n+k-1}{k}}(H_{r+n-1} - H_{r+n+k-1}) + \frac{\Gamma_n(H_{2n+r} - H_{n+r-1})}{(n+1)\binom{2n+r}{n+1}}. \tag{46}$$

In particular,

$$\sum_{k=1}^{n} \frac{\Gamma_k(H_n - H_{n+k})}{k\binom{n+k}{k}} = \sum_{k=1}^{n} \frac{\gamma_k}{k\binom{n+k-1}{k}}(H_{n-1} - H_{n+k-1}) + \frac{\Gamma_n(H_{2n} - H_{n-1})}{(n+1)\binom{2n}{n+1}}. \tag{47}$$

Proof. We used (9) and (11). Identity (47) is the special case of (46) when $r = 0$. □

Proposition 2. *Let $\{\gamma_j\}_{j\in\mathbb{N}}$ be an arbitrary complex sequence, and define the partial sums denoted by $\Gamma_n = \gamma_1 + \gamma_2 + \cdots + \gamma_n$ for each $n \in \mathbb{N}$. Suppose $r \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1}$. Then, for all $n \in \mathbb{N}$, the following identity holds:*

$$\begin{aligned}
&\sum_{k=1}^{n} \frac{\Gamma_k(H_{n+r} - H_{n+r+k})}{k^2\binom{n+r+k}{k}} \\
&= \sum_{k=1}^{n} \frac{1}{r+k} \sum_{j=1}^{k} \frac{\gamma_j}{j\binom{r+k+j-1}{j}} \left(H_{r+k+j-1} - H_{r+k-1}\right) \\
&\quad - \sum_{k=1}^{n} \frac{\Gamma_k(H_{r+2k-1} - H_{r+k-1})}{k(r+k)\binom{r+2k-1}{k}} + \sum_{k=1}^{n} \frac{\Gamma_k(H_{r+k} - H_{r+2k})}{k^2\binom{r+2k}{k}} \\
&\quad + \sum_{k=1}^{n} \frac{1}{(r+k)^2} \sum_{j=1}^{k} \frac{\gamma_j}{j\binom{r+k+j-1}{j}} - \sum_{k=1}^{n} \frac{\Gamma_k}{k(r+k)^2\binom{r+2k-1}{k}}.
\end{aligned} \tag{48}$$

In particular,

$$\begin{aligned}\sum_{k=1}^{n}\frac{\Gamma_k(H_n-H_{n+k})}{k^2\binom{n+k}{k}}&=\sum_{k=1}^{n}\frac{1}{k}\sum_{j=1}^{k}\frac{\gamma_j}{j\binom{k+j-1}{j}}\Big(H_{k+j-1}-H_{k-1}\Big)\\&\quad-3\sum_{k=1}^{n}\frac{\Gamma_k(H_{2k}-H_k)}{k^2\binom{2k}{k}}-3\sum_{k=1}^{n}\frac{\Gamma_k}{k^3\binom{2k}{k}}+\sum_{k=1}^{n}\frac{1}{k^2}\sum_{j=1}^{k}\frac{\gamma_j}{j\binom{k+j-1}{j}}.\end{aligned}\tag{49}$$

Proof. We used (9) and (11). Identity (49) is the special case of (48) when $r=0$. □

Proposition 3. *Let $\{\gamma_j\}_{j\in\mathbb{N}}$ be an arbitrary complex sequence, and define the partial sums denoted by $\Gamma_n=\gamma_1+\gamma_2+\cdots+\gamma_n$ for each $n\in\mathbb{N}$. Suppose $r\in\mathbb{C}\setminus\mathbb{Z}_{\leqslant -1}$. Then, for all $n\in\mathbb{N}$, the following identity holds:*

$$\begin{aligned}\sum_{k=1}^{n}\frac{\Gamma_k(H_{n+r}-H_{n+r+k})}{k^3\binom{n+r+k}{k}}&=\sum_{k=1}^{n}\frac{1}{(k+r)^2}\sum_{j=1}^{k-1}\frac{\Gamma_j}{j^2\binom{k+r+j-1}{j}}-\sum_{k=1}^{n}\frac{1}{k+r}\sum_{j=1}^{k-1}\frac{\Gamma_j\big(H_{k+r-1}-H_{k+r+j-1}\big)}{j^2\binom{k+r+j-1}{j}}\\&\quad-2\sum_{k=1}^{n}\frac{1}{(k+r)^3}\sum_{j=1}^{k}\frac{\gamma_j}{j\binom{k+r+j-1}{j}}+\sum_{k=1}^{n}\frac{1}{(k+r)^2}\sum_{j=1}^{k}\frac{\gamma_j\big(H_{k+r-1}-H_{k+r+j-1}\big)}{j\binom{k+r+j-1}{j}}\\&\quad+\sum_{k=1}^{n}\frac{\Gamma_k}{k\,(k+r)^3\binom{r+2k-1}{k}}-\sum_{k=1}^{n}\frac{\Gamma_k\big(H_{r+k-1}-H_{r+2k-1}\big)}{k\,(k+r)^2\binom{r+2k-1}{k}}+\sum_{k=1}^{n}\frac{\Gamma_k\big(H_{r+k}-H_{r+2k}\big)}{k^3\binom{r+2k}{k}}.\end{aligned}\tag{50}$$

In particular,

$$\begin{aligned}\sum_{k=1}^{n}\frac{\Gamma_k(H_n-H_{n+k})}{k^3\binom{n+k}{k}}&=\sum_{k=1}^{n}\frac{1}{k^2}\sum_{j=1}^{k-1}\frac{\Gamma_j}{j^2\binom{k+j-1}{j}}-\sum_{k=1}^{n}\frac{1}{k}\sum_{j=1}^{k-1}\frac{\Gamma_j\big(H_{k-1}-H_{k+j-1}\big)}{j^2\binom{k+j-1}{j}}\\&\quad-2\sum_{k=1}^{n}\frac{1}{k^3}\sum_{j=1}^{k}\frac{\gamma_j}{j\binom{k+j-1}{j}}+\sum_{k=1}^{n}\frac{1}{k^2}\sum_{j=1}^{k}\frac{\gamma_j\big(H_{k-1}-H_{k+j-1}\big)}{j\binom{k+j-1}{j}}\\&\quad+2\sum_{k=1}^{n}\frac{\Gamma_k}{k^4\binom{2k}{k}}-2\sum_{k=1}^{n}\frac{\Gamma_k\big(H_{k-1}-H_{2k-1}\big)}{k^3\binom{2k}{k}}+\sum_{k=1}^{n}\frac{\Gamma_k\big(H_k-H_{2k}\big)}{k^3\binom{2k}{k}}.\end{aligned}\tag{51}$$

By setting $\gamma_j=\frac{1}{j}$ for all $j\in\mathbb{N}$ in the identities in Propositions 1–3, we obtain the corresponding identities, as given in the following corollary.

Corollary 16. *Let $r\in\mathbb{C}\setminus\mathbb{Z}_{\leqslant -1}$. Then, for all $n\in\mathbb{N}$, the following identities hold:*

$$\begin{aligned}\sum_{k=1}^{n}\frac{H_k(H_{n+r}-H_{n+r+k})}{k^2\binom{n+r+k}{k}}&=\sum_{k=1}^{n}\frac{1}{r+k}\sum_{j=1}^{k}\frac{H_{r+k+j-1}-H_{r+k-1}}{j^2\binom{r+k+j-1}{j}}\\&\quad-\sum_{k=1}^{n}\frac{H_k(H_{r+2k-1}-H_{r+k-1})}{k(r+k)\binom{r+2k-1}{k}}+\sum_{k=1}^{n}\frac{H_k(H_{r+k}-H_{r+2k})}{k^2\binom{r+2k}{k}}\\&\quad-\sum_{k=1}^{n}\frac{H_k}{k(r+k)^2\binom{r+2k-1}{k}}+\sum_{k=1}^{n}\frac{1}{(r+k)^2}\sum_{j=1}^{k}\frac{1}{j^2\binom{r+k+j-1}{j}};\end{aligned}$$

$$\begin{aligned}&\sum_{k=1}^{n}\frac{H_k(H_n-H_{n+k})}{k^2\binom{n+k}{k}}\\&\quad=\sum_{k=1}^{n}\frac{1}{k}\sum_{j=1}^{k}\frac{\big(H_{k+j-1}-H_{k-1}\big)}{j^2\binom{k+j-1}{j}}-3\sum_{k=1}^{n}\frac{H_k(H_{2k}-H_k)}{k^2\binom{2k}{k}}\\&\qquad-3\sum_{k=1}^{n}\frac{H_k}{k^3\binom{2k}{k}}+\sum_{k=1}^{n}\frac{1}{k^2}\sum_{j=1}^{k}\frac{1}{j^2\binom{k+j-1}{j}};\end{aligned}$$

$$\sum_{k=1}^{n}\frac{H_k(H_{r+n}-H_{r+n+k})}{k\binom{r+n+k}{k}}$$
$$=\sum_{k=1}^{n}\frac{(H_{r+n-1}-H_{r+n+k-1})}{k^2\binom{r+n+k-1}{k}}-\frac{H_n(H_{n+r-1}-H_{2n+r})}{(n+1)\binom{2n+r}{n+1}};$$

$$\sum_{k=1}^{n}\frac{H_k(H_n-H_{n+k})}{k\binom{n+k}{k}}=\sum_{k=1}^{n}\frac{H_{n-1}-H_{n+k-1}}{k^2\binom{n+k-1}{k}}-\frac{H_n(H_{n-1}-H_{2n})}{(n+1)\binom{2n}{n+1}};$$

$$\sum_{k=1}^{n}\frac{H_k(H_{n+r}-H_{n+r+k})}{k^3\binom{n+r+k}{k}}=\sum_{k=1}^{n}\frac{1}{(k+r)^2}\sum_{j=1}^{k-1}\frac{H_j}{j^2\binom{k+r+j-1}{j}}-\sum_{k=1}^{n}\frac{1}{k+r}\sum_{j=1}^{k-1}\frac{H_j\left(H_{k+r-1}-H_{k+r+j-1}\right)}{j^2\binom{k+r+j-1}{j}}$$
$$-2\sum_{k=1}^{n}\frac{1}{(k+r)^3}\sum_{j=1}^{k}\frac{1}{j^2\binom{k+r+j-1}{j}}+\sum_{k=1}^{n}\frac{1}{(k+r)^2}\sum_{j=1}^{k}\frac{H_{k+r-1}-H_{k+r+j-1}}{j^2\binom{k+r+j-1}{j}}$$
$$+\sum_{k=1}^{n}\frac{H_k}{k\,(k+r)^3\binom{r+2k-1}{k}}-\sum_{k=1}^{n}\frac{H_k\left(H_{r+k-1}-H_{r+2k-1}\right)}{k\,(k+r)^2\binom{r+2k-1}{k}}+\sum_{k=1}^{n}\frac{H_k\left(H_{r+k}-H_{r+2k}\right)}{k^3\binom{r+2k}{k}};$$

$$\sum_{k=1}^{n}\frac{H_k(H_n-H_{n+k})}{k^3\binom{n+k}{k}}=\sum_{k=1}^{n}\frac{1}{k^2}\sum_{j=1}^{k-1}\frac{H_j}{j^2\binom{k+j-1}{j}}-\sum_{k=1}^{n}\frac{1}{k}\sum_{j=1}^{k-1}\frac{H_j\left(H_{k-1}-H_{k+j-1}\right)}{j^2\binom{k+j-1}{j}}$$
$$-2\sum_{k=1}^{n}\frac{1}{k^3}\sum_{j=1}^{k}\frac{1}{j^2\binom{k+j-1}{j}}+\sum_{k=1}^{n}\frac{1}{k^2}\sum_{j=1}^{k}\frac{H_{k-1}-H_{k+j-1}}{j^2\binom{k+j-1}{j}}$$
$$+2\sum_{k=1}^{n}\frac{H_k}{k^4\binom{2k}{k}}-\sum_{k=1}^{n}\frac{H_k\left(H_{k-1}-H_{2k-1}\right)}{k^3\binom{2k}{k}}+\frac{1}{2}\sum_{k=1}^{n}\frac{H_k}{k^4\binom{2k}{k}}.$$

6. Concluding Remarks

In this work, we have explored a variety of special cases stemming from our main result, illustrating the richness and flexibility of the underlying identities. Thanks to the generality of Theorem 1, many further instances can be generated. In particular, whenever a closed-form expression for the partial sums of a sequence is available, our framework enables the derivation of explicit identities, as evidenced by the corollaries presented.

Moreover, the identities in these corollaries admit further refinement, leading to even more concrete formulas. As an example, Formula (40) reproduces the classical identity

$$\sum_{j=1}^{n}j^4=\frac{1}{30}n(n+1)(2n+1)\left(3n^2+3n-1\right)\quad(n\in\mathbb{N}).\tag{52}$$

Similarly, by applying Equation (52), one can derive additional identities that extend the special cases discussed in Corollaries 10–12.

We hope that this line of investigation not only contributes to a deeper understanding of summation identities involving reciprocals of binomial coefficients but also lays a foundation for discovering new analytic results with potential applications in number theory, combinatorics, and mathematical analysis.

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