A relative of the Thue-Morse Sequence

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Abstract

We study a sequence, **c**, which encodes the lengths of blocks in the Thue-Morse sequence. In particular, we show that the generating function for **c** is a simple product.

Consider the sequence

$$\mathbf{c} : c_0, c_1, c_2, c_3, \ldots = 1, 3, 4, 5, 7, 9, 11, 12, 13, \ldots$$

defined to be the lexicographically least sequence of positive integers satisfying $n \in \mathbf{c}$ implies $2n \notin \mathbf{c}$. In fact, the lexicographic minimality of $\mathbf{c}$ makes it possible to replace the previous "implies" with "if and only if." Equivalently, $\mathbf{c}$ is defined inductively by $c_0 = 1$ and

$$c_{k+1} = \begin{cases} c_k + 1 & \text{if } (c_k + 1)/2 \notin \mathbf{c} \\ c_k + 2 & \text{otherwise} \end{cases} \tag{1}$$

for $k \geq 0$. This sequence was the focus of a problem of C. Kimberling in the *American Mathematical Monthly* [6]. (In fact, he looked at the sequence $4c_0, 4c_1, 4c_2, \ldots$) The solution was given by D. Bloom [4]. Our Corollary 7 answers essentially the same question. Related results have recently been announced by J. Tamura [9].

At the 4è Colloque Séries Formelles et Combinatoire Algébrique (Montréal, June 1992) S. Plouffe and P. Zimmermann [8] posed the following problem. Show that the generating function for $\mathbf{c}$ is

$$\sum_{k \geq 0} c_k x^k = \frac{1}{1-x} \prod_{j \geq 1} \frac{1 - x^{2e_j}}{1 - x^{e_j}} = \frac{1}{1-x} \prod_{j \geq 1} (1 + x^{e_j}) \tag{2}$$

the sequence of exponents being

$$\mathbf{e} : e_1, e_2, e_3, e_4, \ldots = 1, 1, 3, 5, 11, 21, 43, \ldots$$

where $e_1 = 1$ and

$$e_{j+1} = \begin{cases} 2e_j + 1 & \text{if } j \text{ is even} \\ 2e_j - 1 & \text{if } j \text{ is odd} \end{cases} \tag{3}$$

for $j \geq 1$. They found this conjecture by using a method that goes back to Euler. First they assumed that the generating function was of the form

$$\prod_{j \geq 0} \frac{1 - x^{a_j}}{1 - x^{b_j}}$$

for a certain pair of sequences a_j, b_j. Then they took the logarithm to convert the product into a sum. Finally they used Möbius inversion to determine the candidate sequences. Details of this procedure can be found in the text of G. Andrews [2, Theorem 10.3].

The purpose of this note is to prove (2). Before doing this, however, we will show that $\mathbf{c}$ has a number of other interesting properties. Chief among these is the fact that $\mathbf{c}$ is closely related to the famous Thue-Morse sequence, $\mathbf{t}$. See the survey article of J. Berstel [3] for more information about $\mathbf{t}$.

First we need to have a characterization of the integers in the sequence $\mathbf{c}$.

Proposition 1 *If n is any positive integer then $n \in \mathbf{c}$ if and only if $n = 2^{2i}(2j+1)$ for some nonnegative integers i and j.*

Proof. Every positive integer n can be uniquely written in the form $n = 2^k(2j+1)$ where $k, j \geq 0$. We will proceed by induction on k.

If $k = 0$, then n is odd. But then $n/2$ is not an integer, and so n is in the sequence by definition (1).

Now assume that $k \geq 1$ and that the proposition holds for all powers less than k of 2. If $k = 2i$ is even, then by induction we have $2^{2i-1}(2j+1) \notin \mathbf{c}$. So $n = 2^{2i}(2j+1) \in \mathbf{c}$ by (1). On the other hand, if $k = 2i+1$ is odd, then induction implies $2^{2i}(2j+1) \in \mathbf{c}$. Thus $n = 2^{2i+1}(2j+1) \notin \mathbf{c}$ as desired. ■

Let χ be the characteristic function of $\mathbf{c}$, i.e.,

$$\chi(n) = \begin{cases} 1 & \text{if } n \in \mathbf{c} \\ 0 & \text{otherwise.} \end{cases}$$

Restating the previous proposition in terms of χ yields the next result.

Lemma 2 *The function χ is uniquely determined by the equations*

$$\begin{aligned} \chi(2n+1) &= 1 \\ \chi(4n+2) &= 0 \\ \chi(4n) &= \chi(n). \quad \blacksquare \end{aligned}$$

Another way of obtaining the sequence $\chi(n)$ for $n \geq 1$ is as follows. Starting from the sequence

$$101 \bullet 101 \bullet 101 \bullet 101 \bullet \ldots$$

defined on the alphabet $\{0, 1, \bullet\}$, fill in the sucessive holes with the sucessive terms of the sequence itself, obtaining:

$$101110101011101 \bullet \ldots$$

Iterating this process infinitely many times (by inserting the initial sequence into the holes at each step), one gets a "Toeplitz transform" which is nothing but our sequence χ. The proof of this fact is easily obtained using Lemma 2. See the article of J.-P. Allouche and R. Bacher [1] for more information about Toeplitz transformations.

The connection with the Thue-Morse sequence can now be obtained. This sequence is

$$\mathbf{t} : t_0, t_1, t_2, t_3, \ldots = 0, 1, 1, 0, 1, 0, 0, 1, \ldots$$

defined by the conditions

$$\begin{aligned} t_0 &= 0 \\ t_{2n+1} &\equiv t_n + 1 \pmod 2 \\ t_{2n} &= t_n. \end{aligned}$$

We will need a lemma relating $\mathbf{t}$ and χ. All congruences in this and any future results will be modulo 2.

Lemma 3 *For every positive integer, n, we have*

$$\chi(n) \equiv t_n + t_{n-1}.$$

Proof. This is a three case induction based on Lemma 2 and the definitions of χ and $\mathbf{t}$. We will only do one of the cases as the others are similar.

$$\begin{aligned} t_{4n} + t_{4n-1} &\equiv t_{2n} + t_{2n-1} + 1 \\ &\equiv t_n + t_{n-1} + 2 \\ &\equiv \chi(n) \\ &= \chi(4n). \quad \blacksquare \end{aligned}$$

Define d_k to be the first difference sequence of c_k, i.e., $d_k = c_k - c_{k-1}$, for $k \geq 0$ ($c_{-1} = 0$). So $\mathbf{d}$ is the sequence

$$d_0, d_1, d_2, d_3, d_4, \ldots = 1, 2, 1, 1, 2, 2, 2, 1, 1, 2, 1, \ldots$$

Note that from the definition of $\mathbf{c}$ in (1), the value of d_k is either 1 or 2. Write the Thue-Morse sequence in term of its blocks

$$\mathbf{t} = 011010011\ldots = 0^{d'_0}1^{d'_1}0^{d'_2}1^{d'_3}\ldots$$

defining a sequence d'_k. It is this sequence that is related to our original one via the difference operator.

Theorem 4 *For all $k \geq 0$ we have $d_k = d'_k$.*

Proof. Since both sequences consist of 1's and 2's, we need only verify that the 1's appear in the same places in both. It will be convenient to let $c'_k = \sum_{i \leq k} d'_i$. We now proceed by induction on k, assuming that $d_i = d'_i$ for $i \leq k$. Then, from the definitions,

$$d_{k+1} = 1 \Leftrightarrow \chi(c_k + 1). \tag{4}$$

But by the induction hypothesis, $c_k = \sum_{i\le k} d_i = \sum_{i\le k} d'_i = c'_k$. So, from equation (4),

$$\begin{aligned} d_{k+1} = 1 &\Leftrightarrow \chi(c'_k + 1) = 1 \\ &\Leftrightarrow t_{c'_k+1} + t_{c'_k} \equiv 1 \quad \text{(Lemma 3)} \\ &\Leftrightarrow t_{c'_k+1} \neq t_{c'_k} \\ &\Leftrightarrow d'_{k+1} = 1 \qquad\quad \text{(definitions).} \quad \blacksquare \end{aligned}$$

S. Brlek [5] used the sequence $\mathbf{d}$ in calculating the number of factors of $\mathbf{t}$ of given length. The paper of A. de Luca and S. Varricchio [7] attacks the same problem in a different way.

Now if $n \in \mathbf{c}$ then we will consider its *rank*, $r(n)$, which is the function satisfying $c_{r(n)} = n$. Note that $r(n)$ is not defined for all positive integers n. In order to obtain a formula for $r(n)$, we will need a definition. Let the base 2 expansion of n be

$$n = \sum_{i\ge 0} \epsilon_i 2^i$$

with the $\epsilon_i \in \{0,1\}$ for all i. Define a function s by

$$s(n) = \sum_{i\ge 0} (-1)^i \epsilon_i.$$

In other words, $s(n)$ is the alternating sum of the binary digits of n.

Theorem 5 *If $n \in \mathbf{c}$ then*

$$r(n) = (2n + s(n))/3 - 1. \tag{5}$$

Proof. The proof will be by induction. From Proposition 1, $n \in \mathbf{c}$ if and only if n is odd or $n = 2^{2i}(2j+1)$ where $i > 0$ and $j \ge 0$. To facilitate the induction, it will be convenient to split the odd numbers into two groups depending upon whether the highest power of 2 dividing $n+1$ is even or odd. So there will be three cases

1. $n = 2^{2i}(2j+1)$
2. $n = 2^{2i}(2j+1) - 1$
3. $n = 2^{2i-1}(2j+1) - 1$

where $i > 0$ and $j \ge 0$. The arguments are similar, so we will only do the first case.

So suppose n is even (remember that $i > 0$). Thus $n+1$ is odd and, by Proposition 1, we have $n+1 \in \mathbf{c}$. Since both n and $n+1$ are in $\mathbf{c}$, the left side of equation (5) satisfies $r(n+1) = r(n)+1$. So, by induction, it suffices to show that $r'(n+1) = r'(n)+1$ where $r'(n)$ is the right side of this equation. Moreover, n is a multiple of 4, hence $s(n+1) = s(n)+1$ (write down their binary expansions). Thus

$$\begin{aligned} r'(n+1) &= (2n+2+s(n+1))/3 - 1 \\ &= (2n+2+s(n)+1)/3 - 1 \\ &= (2n+s(n))/3 \\ &= r'(n)+1. \quad \blacksquare \end{aligned}$$

As straightforward corollaries we have the next two results.

Corollary 6 *If $n \in \mathbf{c}$ then*

$$r(n) = 2n/3 + O(\log\ n)$$

and $r(n)$ takes the value $2n/3$ infinitely often. $\blacksquare$

Corollary 7 *For any nonnegative integer k*

$$c_k = 3k/2 + O(\log\ k)$$

and $c_k = 3k/2$ infinitely often. $\blacksquare$

We shall now prove the identity (2). First we note a property of the exponents e_j which is a simple consequence of their definition (3).

Lemma 8 *For $k \geq 2$, let $f_k = \sum_{2 \leq j \leq k} e_j$. Then*

$$f_k = \begin{cases} e_{k+1} - 2 & \text{if } k \text{ is even} \\ e_{k+1} - 1 & \text{if } k \text{ is odd.} \end{cases} \quad \blacksquare$$

Finally, we come to the proof. We restate the generating function here for easy reference.

Theorem 9 *The generating function for $\mathbf{c}$ is*

$$\sum_{k \geq 0} c_k x^k = \frac{1}{1-x} \prod_{j \geq 1} (1 + x^{e_j}).$$

Proof. It suffices to show that if $k \geq 2$ then

$$g_k(x) = \frac{1}{1-x}(1+x^1)(1+x^1)(1+x^3)\cdots(1+x^{e_k})$$

is the generating function for the sequence

$$1, 3, 4, 5, 7, \ldots, c_{f_k}, 2^k, 2^k, 2^k, \ldots$$

with $c_{f_k} = 2^k - 1$. The proof is an induction, breaking up into two parts depending on the parity of k. We will do the case where k is odd. (Even k is similar.) Now, by Lemma 8, $g_k(x)(1+x^{e_{k+1}})$ is the generating function for the sequence

$$1, 3, \ldots, c_{f_k}, 2^k+1, 2^k+3, \ldots, 2^k+c_{f_k}, 2^{k+1}, 2^{k+1}, \ldots$$

Using Proposition 1 and the fact that k is odd, we see that $2^k + 1 = c_{f_k+1}$ and $2^k + c_{f_k} = 2^{k+1} - 1 = c_{f_{k+1}}$. So we want to show that

$$c_{f_k+1}, c_{f_k+2}, \ldots, c_{f_{k+1}} = 2^k + c_0, 2^k + c_1, \ldots, 2^k + c_{f_k}.$$

But if $n < 2^k$, then the highest power of 2 dividing n is equal to the highest power dividing $2^k + n$. Thus, by Proposition 1 again, $n \in \mathbf{c}$ if and only if $2^k + n \in \mathbf{c}$. This gives us the desired equality of the two sequences. ∎

One possible generalization of $\mathbf{c}$ is the sequence $\mathbf{c}^{(\alpha)}$ defined by $n \in \mathbf{c}^{(\alpha)}$ if and only if $\alpha n \notin \mathbf{c}^{(\alpha)}$. Thus $\mathbf{c}$ is the special case $\alpha = 2$.

The following observation is a direct consequence of our definitions.

Proposition 10 *If $\chi^{(\alpha)}(n)$ is the characteristic function of $\mathbf{c}^{(\alpha)}$, then the sequence $(\chi^{(\alpha)}(n))$ is the unique fixed point of the morphism*

$$\begin{array}{ccc} 1 & \rightarrow & 1^{\alpha-1}0 \\ 0 & \rightarrow & 1^{\alpha-1}1 \end{array}$$

which begins with 1. ∎

One can also see that $\mathbf{c}^{(\alpha)}$ satisfies analogs of many of our previous theorems. For example, if one defines $e_1^{(\alpha)} = 1$ and

$$e_{j+1}^{(\alpha)} = \begin{cases} \alpha e_j^{(\alpha)} + 1 & \text{if } j \text{ is even} \\ \alpha e_j^{(\alpha)} - 1 & \text{if } j \text{ is odd} \end{cases}$$

for $j \geq 1$, then the following result is a generalization of Theorem 9 and has an analogous proof.

Theorem 11 *The generating function for $\mathbf{c}^{(\alpha)}$ is*

$$\frac{1}{1-x}\prod_{j\geq 1}\frac{1-x^{\alpha e_j^{(\alpha)}}}{1-x^{e_j^{(\alpha)}}}. \quad ∎$$

References

[1] J.-P. Allouche and R. Bacher, Toeplitz sequences, paperfolding, towers of Hanoi and progression-free sequences of integers, *Ens. Math.*, 2e serie, **38**, fascicule 3-4, (1992), 315–327.

[2] G. Andrews, "q-Series: Their Development and Application in Analysis, Number Theory, Combinatorics, Physics, and Computer Algebra," Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, number 66, Amer. Math. Soc., Providence, RI, 1986.

[3] J. Berstel, Axel Thue's work on repetitions in words, in "Séries Formelles et Combinatoire Algébrique," P. Leroux and C. Reutenauer eds., Publications du LACIM, Vol. 11, Université du Québec à Montréal, 1992, 65–80.

[4] D. M. Bloom, Solution to problem E 2850, *Amer. Math. Monthly* **89** (1982), 599–600.

[5] S. Brlek, Enumeration of factors in the Thue-Morse word, *Discrete Appl. Math.* **24** (1989), 83–96.

[6] C. Kimberling, Problem E 2850, *Amer. Math. Monthly* **87** (1980), 671.

[7] A. de Luca and S. Varricchio, Some combinatorial properties of the Thue-Morse sequence, *Theoret. Comput. Sci.* **63** (1989), 333–348.

[8] S. Plouffe and P. Zimmermann, Quelques conjectures, preprint (1992).

[9] J. Tamura, Some problems and results having their origin in the power series $\sum_{n=1}^{+\infty} z^{[\alpha n]}$ sum from 1 to infinity of z to the integral part of alpha times n, in "Reports of the Meeting on Analytic Theory of Numbers and Related Topics," Gakushuin Univ., 1992, 190–212.