A Pamphlet on Pi
serving as a
Supplement for the Third Edition
of
Pi: A Source Book

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PREFACE.

Our aim in preparing this pamphlet is to bring the material in the collection of papers in the second edition of our *Pi: A Source Book* [9] up to date. Moreover, several delightful pieces came available and are added.

This substantial supplement to the third addition serves as a stand alone exposition of the recent history of the computation of digits of Pi. It also includes a discussion of the thorny old question of normality of the distribution of the digits. Additional material of historical and cultural interest is included, the most notable being new translations of the two Latin pieces of Viète (Translation of Article 9 (Excerpt 1): Various Responses on Mathematical Matters: Book VII (1593) and (Excerpt 2): Defense for the New Cyclometry or "Anti-Axe"), and a thorough revision of the translation of Huygens's piece (Article 12) published in the second edition.

We should like to thank Prof. Marinus Taisbak of Copenhagen for grappling with Viète's idiosyncratic style to produce the new translations of his work. We should like to thank Karen Aardal for permission to use her photograph of Ludolph's new tombstone in the Pieterskerk in Leiden, the Smithsonian for permission to reproduce a fine photo of ENIAC, and David and Gregory Chudnovsky for providing a "Walk on the digits of pi." We should also like to thank Irving Kaplansky for his gracious permission to include his "A song about pi". Finally, our thanks go to our colleagues whose continued interest in pi has encouraged our publishers to produce this third edition, as well as for the comments and corrections to earlier editions that some of them have sent us.

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Chapter 1

Pi and Its Friends

This chapter and the next are paraphrased from the book *Mathematics by Experiment* [15].

1.1 A Recent History of Pi

The first truly electronic computation of π was performed in 1949 on the original ENIAC. This calculation was suggested by John von Neumann, who wished to study the digits of π and e. Computing 2037 decimal places of π on the ENIAC required 70 hours. A similar calculation today could be performed in a fraction of second on a personal computer.

Later computer calculations were further accelerated by the discovery of advanced algorithms for performing the required high-precision arithmetic operations. For example, in 1965 it was found that the newly-discovered fast Fourier transform (FFT) could be used to perform high-precision multiplications much more rapidly than conventional schemes. These methods dramatically lowered the computer time required for computing π and other mathematical constants to high precision. See also [3] and [19].

In spite of these advances, until the 1970s all computer evaluations of π still employed classical formulas, usually one of the Machin-type formulas. Some new infinite series formulas were discovered by Ramanujan around 1910, but these were not well known until quite recently when his writings were widely published. Ramanujan's related mathematics may be followed in [23, 18, 10]. One of these series is the remarkable formula

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801}\sum_{k=0}^{\infty}\frac{(4k)!(1103+26390k)}{(k!)^4 396^{4k}}. \qquad (1.1.1)$$

Each term of this series produces an additional *eight* correct digits in the result. Bill Gosper used this formula to compute 17 million digits of π in 1985. Gosper also computed the first 17 million terms of the continued fraction expansion of π. At about the same time, David and Gregory Chudnovsky found the following variation of Ramanujan's formula:

$$\frac{1}{\pi} = 12 \sum_{k=0}^{\infty} \frac{(-1)^k \, (6k)! (13591409 + 545140134k)}{(3k)! \, (k!)^3 \, 640320^{3k+3/2}}. \tag{1.1.2}$$

Each term of this series produces an additional 14 correct digits. The Chudnovskys implemented this formula using a clever scheme that enabled them to utilize the results of an initial level of precision to extend the calculation to even higher precision. They used this method in several large calculations of π, culminating with a computation to over four billion decimal digits in 1994.

Along this line, it is interesting to note that the Ramanujan-type series (see [17, pg. 188]

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \left(\frac{\binom{2n}{n}}{16^n} \right)^3 \frac{42\, n + 5}{16} \tag{1.1.3}$$

permits one to compute the billionth binary digit of $1/\pi$ without computing the first half of the series.

While the Ramanujan and Chudnovsky series are considerably more efficient than the classical formulas, they share with them the property that the number of terms one must compute increases linearly with the number of digits desired in the result. In other words, if you want to compute π to twice as many digits, you have to evaluate twice as many terms of the series.

In 1976, Eugene Salamin and Richard Brent independently discovered an algorithm for π based on the arithmetic-geometric mean (AGM) and some ideas originally due to Gauss in the 1800s (although for some reason Gauss never saw the connection to computing π). The Salamin–Brent algorithm may be stated as follows. Set $a_0 = 1, b_0 = 1/\sqrt{2}$ and $s_0 = 1/2$. Calculate

$$\begin{aligned}
a_k &= \frac{a_{k-1} + b_{k-1}}{2} \\
b_k &= \sqrt{a_{k-1} b_{k-1}} \\
c_k &= a_k^2 - b_k^2 \\
s_k &= s_{k-1} - 2^k c_k \\
p_k &= \frac{2a_k^2}{s_k}.
\end{aligned} \tag{1.1.4}$$

Then p_k converges *quadratically* to π: each iteration of this algorithm approximately *doubles* the number of correct digits — successive iterations produce 1, 4, 9, 20, 42, 85, 173, 347 and 697 correct decimal digits of π. Twenty-five iterations are sufficient to compute π to over 45 million decimal digit accuracy. However, each of these iterations must be performed using a level of numeric precision that is at least as high as that desired for the final result.

Beginning in 1985, one of the present authors (Jonathan Borwein) and his brother Peter Borwein discovered some additional algorithms of this type [17]. One is as follows. Set $a_0 = 1/3$ and $s_0 = (\sqrt{3}-1)/2$. Iterate

$$\begin{aligned} r_{k+1} &= \frac{3}{1+2(1-s_k^3)^{1/3}} \\ s_{k+1} &= \frac{r_{k+1}-1}{2} \\ a_{k+1} &= r_{k+1}^2 a_k - 3^k(r_{k+1}^2 - 1). \end{aligned} \tag{1.1.5}$$

Then $1/a_k$ converges *cubically* to π — each iteration approximately triples the number of correct digits. Another algorithm is as follows: Set $a_0 = 6-4\sqrt{2}$ and $y_0 = \sqrt{2}-1$. Iterate

$$\begin{aligned} y_{k+1} &= \frac{1-(1-y_k^4)^{1/4}}{1+(1-y_k^4)^{1/4}} \\ a_{k+1} &= a_k(1+y_{k+1})^4 - 2^{2k+3} y_{k+1}(1+y_{k+1}+y_{k+1}^2). \end{aligned} \tag{1.1.6}$$

Then a_k converges *quartically* to $1/\pi$. This particular algorithm, together with the Salamin–Brent scheme, has been employed by Yasumasa Kanada of the University of Tokyo in several computations of π over the past 15 years or so, including his 1999 computation of π to over 206 billion decimal digits.

Shanks, who in 1961 computed π to over 100,000 digits, once declared that a billion digit computation would be "forever impossible." But both Kanada and the Chudnovskys computed over one billion digits in 1989. Similarly, the intuitionist mathematicians Brouwer and Heyting once asserted the "impossibility" of ever knowing whether the sequence "0123456789" appears in the decimal expansion of π [24]. This sequence was found in 1997 by Kanada, beginning at position 17,387,594,880. Even as late as 1989, British mathematical physicist Roger Penrose, ventured in the first edition of his book *The Emperor's New Mind* that we are not likely to know whether a string of "ten consecutive sevens" occurs in the decimal expansion of π [30, pg. 115]. By the time his book was published, Kanada had already found a string of ten consecutive sixes in his 480-million-digit computation of π. When one of the present authors mentioned this to Penrose in 1990, he replied that he was "startled to learn how far the combination of human mathematical ingenuity with computer technology has enabled the calculation of the decimal expansion of π to be carried out." Accordingly, he changed his text to "twenty consecutive sevens," which appeared in subsequent printings of the book. This was just in time, as a string of ten consecutive sevens was found by Kanada in 1997, beginning at position 22,869,046,249.

In December 2002, Kanada, with a team consisting of Y. Ushiro of Hitachi, H. Kuroda and M. Kudoh of the University of Tokyo, and the assistance of nine others from Hitachi, completed computation of π to over 1.24 *trillion* decimal digits. Kanada and his team first computed π in hexadecimal (base 16) to 1,030,700,000,000 places, using the following two arctangent relations for π:

$$\begin{aligned}\pi &= 48\tan^{-1}\frac{1}{49} + 128\tan^{-1}\frac{1}{57} - 20\tan^{-1}\frac{1}{239} + 48\tan^{-1}\frac{1}{110443}\\ \pi &= 176\tan^{-1}\frac{1}{57} + 28\tan^{-1}\frac{1}{239} - 48\tan^{-1}\frac{1}{682} + 96\tan^{-1}\frac{1}{12943}.\end{aligned} \tag{1.1.7}$$

The first formula was found in 1982 by K. Takano, a high school teacher and song writer. The second formula was found by F. C. W. Störmer in 1896.

Kanada and his team evaluated these formulas using a scheme analogous to that employed by Gosper and the Chudnovskys, in that they were able to avoid explicitly storing the multiprecision numbers involved. This resulted in a scheme that is roughly competitive in efficiency compared to the Salamin-Brent and Borwein quartic algorithms they had previously used, yet with a significantly lower total memory requirement. In particular, they were able to perform their latest computation on a system with 1 Tbyte (10^{12} bytes) main memory, the same as with their previous computation, yet obtain six times as many digits.

After Kanada and his team verified that the hexadecimal digit strings produced by these two computations were in agreement, they performed an additional check by directly computing 20 hexadecimal digits beginning at position 1,000,000,000,001. This calculation employed an algorithm that we shall describe in Section 1.2, and required 21 hours run time, much less than the time required for the first step. The result of this calculation, `B4466E8D21 5388C4E014`, perfectly agreed with the corresponding digits produced by the two arctan formulas. At this point they converted their hexadecimal value of π to decimal, and converted back to hexadecimal as a check. These conversions employed a numerical approach similar to that used in the main and verification calculations. The entire computation, including hexadecimal and decimal evaluations and checks, required roughly 600 hours run time on their 64-node Hitachi parallel supercomputer. The main segment of the computation ran at nearly 1 Tflop/s (i.e., one trillion floating-point operations per second), although this performance rate was slightly lower than the rate of their previous calculation of 206 billion digits. Full details will appear in an upcoming paper [25].

According to Kanada, the ten decimal digits ending in position one trillion are 6680122702, while the ten hexadecimal digits ending in position one trillion are `3F89341CD5`]. Some data on the frequencies of digits in π, based on Kanada's computations, are given in Section 2.1. Additional information of this sort is available from Kanada's website:

`http://www.super-computing.org`

Additional historical background on record-breaking computations of π is available at

`http://www.cecm.sfu.ca/personal/jborwein/pi_cover.html`

A listing of some milestones in the recent history of the computation of π is given in Table 1.1.

In retrospect, one might wonder why in antiquity π was not measured to an accuracy in excess of 22/7. One conjecture is that it reflects not an inability to do so but instead a very different mind set to a modern (Baconian) experimental one.

For those who know *The Hitchhiker's Guide to the Galaxy* it is amusing that 042 occurs at the digits ending at the fifty-billionth decimal place in each of π and $1/\pi$—thereby providing an excellent answer to the ultimate question "What is forty two?"

Much lovely additional material, 'both sensible and silly' can be found in *Pi Unleashed* [2] and in *the Joy of Pi* [13] (`www.joyofpi.com/`).

1.1.1 The ENIAC Integrator and Calculator

ENIAC, built in 1946 at the University of Pennsylvania, had 18,000 vacuum tubes, 6,000 switches, 10,000 capacitors, 70,000 resistors, 1,500 relays, was 10 feet tall, occupied 1,800 square feet and weighed 30 tons. ENIAC could perform 5,000 arithmetic operations per second—1,000 times faster than any earlier machine, but a far cry from today's leading-edge microprocessors, which can perform more than four billion operations per second.

The first stored-memory computer, ENIAC could store 200 digits, which again is a far cry from the hundreds of megabytes in a modern personal computer system. Data flowed from one accumulator to the next, and after each accumulator finished a calculation, it communicated its results to the next in line. The accumulators were connected to each other manually. A photo is shown in Figure 4.6. We observe that the photo—obtained digitally—requires orders of magnitudes more data than ENIAC could store.

Ferguson	1946	620
Ferguson	1947	710
Ferguson and Wrench	1947	808
Smith and Wrench	1949	1,120
Reitwiesner et al. (ENIAC)	1949	2,037
Nicholson and Jeenel	1954	3,092
Felton	1957	7,480
Genuys	1958	10,000
Felton	1958	10,021
Guilloud	1959	16,167
Shanks and Wrench	1961	100,265
Guilloud and Filliatre	1966	250,000
Guilloud and Dichampt	1967	500,000
Guilloud and Bouyer	1973	1,001,250
Miyoshi and Kanada	1981	2,000,036
Guilloud	1982	2,000,050
Tamura	1982	2,097,144
Tamura and Kanada	1982	4,194,288
Tamura and Kanada	1982	8,388,576
Kanada, Yoshino and Tamura	1982	16,777,206
Ushiro and Kanada	Oct. 1983	10,013,395
Gosper	Oct. 1985	17,526,200
Bailey	Jan. 1986	29,360,111
Kanada and Tamura	Sep. 1986	33,554,414
Kanada and Tamura	Oct. 1986	67,108,839
Kanada, Tamura, Kubo, et. al	Jan. 1987	134,217,700
Kanada and Tamura	Jan. 1988	201,326,551
Chudnovskys	May 1989	480,000,000
Chudnovskys	Jun. 1989	525,229,270
Kanada and Tamura	Jul. 1989	536,870,898
Kanada and Tamura	Nov. 1989	1,073,741,799
Chudnovskys	Aug. 1989	1,011,196,691
Chudnovskys	Aug. 1991	2,260,000,000
Chudnovskys	May 1994	4,044,000,000
Takahashi and Kanada	Jun. 1995	3,221,225,466
Kanada	Aug. 1995	4,294,967,286
Kanada	Oct. 1995	6,442,450,938
Kanada and Takahashi	Jun. 1997	51,539,600,000
Kanada and Takahashi	Sep. 1999	206,158,430,000
Kanada, Ushiro, Kuroda, Kudoh	Dec. 2002	1,241,100,000,000

Table 1.1: Digital era π calculations

Figure 1.1: The ENIAC "Integrator and Calculator"

1.2 Computing Individual Digits of Pi

An outsider might be forgiven for thinking that essentially everything of interest with regards to π has been discovered. For example, this sentiment is suggested in the closing chapters of Beckmann's 1971 book on the history of π [8, pg. 172]. Ironically, the Salamin–Brent quadratically convergent iteration was discovered only five years later, and the higher-order convergent algorithms followed in the 1980s. In 1990, Rabinowitz and Wagon discovered a "spigot" algorithm for π, which permits successive digits of π (in any desired base) to be computed with a relatively simple recursive algorithm based on the previously generated digits (see [31]).

But even insiders are sometimes surprised by a new discovery. Prior to 1996, almost all mathematicians believed that if you want to determine the d-th digit of π, you have to generate the entire sequence of the first d digits. (For all of their sophistication and efficiency, the schemes described above all have this property.) But it turns out that this is not true, at least for hexadecimal (base 16) or binary (base 2) digits of π. In 1996, Peter Borwein, Simon Plouffe, and one of the present authors (Bailey) found an algorithm for computing individual hexadecimal or binary digits of π [5]. To be precise, this algorithm:

(1) directly produces a modest-length string of digits in the hexadecimal or binary expansion of π, beginning at an arbitrary position, without needing to compute any of the previous digits;

(2) can be implemented easily on any modern computer;

(3) does not require multiple precision arithmetic software;

(4) requires very little memory; and

(5) has a computational cost that grows only slightly faster than the digit position.

Using this algorithm, for example, the one millionth hexadecimal digit (or the four millionth binary digit) of π can be computed in less than a minute on a 2001-era computer. The new algorithm is not fundamentally faster than best known schemes for computing all digits of π up to some position, but its elegance and simplicity are nonetheless of considerable interest. This scheme is based on the following remarkable new formula for π:

Theorem 1.2.1

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i}\left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6}\right). \tag{1.2.8}$$

Proof. First note that for any $k < 8$,

$$\begin{aligned}\int_0^{1/\sqrt{2}} \frac{x^{k-1}}{1-x^8}\,dx &= \int_0^{1/\sqrt{2}} \sum_{i=0}^{\infty} x^{k-1+8i}\,dx \\ &= \frac{1}{2^{k/2}} \sum_{i=0}^{\infty} \frac{1}{16^i(8i+k)}.\end{aligned} \tag{1.2.9}$$

Thus one can write

$$\begin{aligned}&\sum_{i=0}^{\infty} \frac{1}{16^i}\left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6}\right) \\ &\quad = \int_0^{1/\sqrt{2}} \frac{4\sqrt{2} - 8x^3 - 4\sqrt{2}x^4 - 8x^5}{1-x^8}\,dx,\end{aligned} \tag{1.2.10}$$

which on substituting $y = \sqrt{2}x$ becomes

$$\begin{aligned}\int_0^1 \frac{16\,y - 16}{y^4 - 2\,y^3 + 4\,y - 4}\,dy &= \int_0^1 \frac{4y}{y^2-2}\,dy - \int_0^1 \frac{4y-8}{y^2-2y+2}\,dy \\ &= \pi.\end{aligned} \tag{1.2.11}$$

□

However, in presenting this formal derivation we are disguising the actual route taken to the discovery of this formula. This route is a superb example of experimental mathematics in action.

It all began in 1995, when Peter Borwein and Simon Plouffe of Simon Fraser University observed that the following well-known formula for $\log 2$ permits one to calculate isolated digits in the binary expansion of $\log 2$:

$$\log 2 \;=\; \sum_{k=0}^{\infty} \frac{1}{k 2^k}. \tag{1.2.12}$$

This scheme is as follows. Suppose we wish to compute a few binary digits beginning at position $d+1$ for some integer $d > 0$. This is equivalent to calculating $\{2^d \log 2\}$, where $\{\cdot\}$ denotes fractional part. Thus we can write

$$\begin{aligned} \{2^d \log 2\} &= \left\{ \left\{ \sum_{k=0}^{d} \frac{2^{d-k}}{k} \right\} + \sum_{k=d+1}^{\infty} \frac{2^{d-k}}{k} \right\} \\ &= \left\{ \left\{ \sum_{k=0}^{d} \frac{2^{d-k} \bmod k}{k} \right\} + \sum_{k=d+1}^{\infty} \frac{2^{d-k}}{k} \right\}. \end{aligned} \tag{1.2.13}$$

We are justified in inserting "mod k" in the numerator of the first summation, because we are only interested in the fractional part of the quotient when divided by k.

Now the key observation is this: the numerator of the first sum in equation (1.2.13), namely $2^{d-k} \bmod k$, can be calculated very rapidly by means of the binary algorithm for exponentiation, performed modulo k. The binary algorithm for exponentiation is merely the formal name for the observation that exponentiation can be economically performed by means of a factorization based on the binary expansion of the exponent. For example, we can write $3^{17} = ((((3^2)^2)^2)^2) \cdot 3$, thus producing the result in only five multiplications, instead of the usual 16. According to Knuth, this technique dates back at least to 200 BCE [26]. In our application, we need to obtain the exponentiation result modulo a positive integer k. This can be done very efficiently by reducing modulo k the intermediate multiplication result at each step of the binary algorithm for exponentiation. A formal statement of this scheme is as follows:

Algorithm 1 *Binary algorithm for exponentiation modulo k.*

To compute $r = b^n \bmod k$, where r, b, n and k are positive integers: First set t to be the largest power of two such that $t \le n$, and set $r = 1$. Then

A: if $n \ge t$ then $r \leftarrow br \bmod k$; $\quad n \leftarrow n - t$; $\quad$ endif

$t \leftarrow t/2$
if $t \geq 1$ then $r \leftarrow r^2 \bmod k$; go to A; endif □

Note that the above algorithm is performed entirely with positive integers that do not exceed k^2 in size. Thus ordinary 64-bit floating-point or integer arithmetic, available on almost all modern computers, suffices for even rather large calculations. 128-bit floating-point arithmetic (double-double or quad precision), available at least in software on many systems (see Section **??**), suffices for the largest computations currently feasible.

We can now present the algorithm for computing individual binary digits of $\log 2$.

Algorithm 2 *Individual digit algorithm for* $\log 2$.

To compute the $(d+1)$-th binary digit of $\log 2$: Given an integer $d > 0$, (1) calculate each numerator of the first sum in equation (1.2.13), using Algorithm 1, implemented using ordinary 64-bit integer or floating-point arithmetic; (2) divide each numerator by the respective value of k, again using ordinary floating-point arithmetic; (3) sum the terms of the first summation, while discarding any integer parts; (4) evaluate the second summation as written using floating-point arithmetic — only a few terms are necessary since it rapidly converges; and (5) add the result of the first and second summations, discarding any integer part. The resulting fraction, when expressed in binary, gives the first few digits of the binary expansion of $\log 2$ beginning at position $d+1$. □

As soon as Borwein and Plouffe found this algorithm, they began seeking other mathematical constants that shared this property. It was clear that any constant α of the form

$$\alpha = \sum_{k=0}^{\infty} \frac{p(k)}{q(k)2^k}, \tag{1.2.14}$$

where $p(k)$ and $q(k)$ are integer polynomials, with $\deg p < \deg q$ and q having no zeroes at positive integer arguments, is in this class. Further, any rational linear combination of such constants also shares this property. Checks of various mathematical references eventually uncovered about 25 constants that possessed series expansions of the form given by equation (1.2.14).

As you might suppose, the question of whether π also shares this property did not escape these researchers. Unfortunately, exhaustive searches of the mathematical literature did not uncover any formula for π of the requisite form. But given the fact that any rational linear combination of constants with this property also shares this property, Borwein and Plouffe performed integer relation searches to see if a formula of this type existed for π. This was done, using computer programs written by one of the present authors (Bailey), which implement the "PSLQ" integer relation algorithm in high-precision, floating-point arithmetic [21, 4, 15, 14].

In particular, these three researchers sought an integer relation for the real vector $(\alpha_1, \alpha_2, \cdots, \alpha_n)$, where $\alpha_1 = \pi$ and $(\alpha_i,\ 2 \le i \le n)$ is the collection of constants of the requisite form gleaned from the literature, each computed to several hundred decimal digit precision. To be precise, they sought an n-long vector of integers (a_i) such that $\sum_i a_i \alpha_i = 0$, to within a very small "epsilon." After a month or two of computation, with numerous restarts using new α vectors (when additional formulas were found in the literature) the identity (1.2.8) was finally uncovered. The actual formula found by the computation was:

$$\pi \quad = \quad 4F(1/4, 5/4; 1; -1/4) + 2\tan^{-1}(1/2) - \log 5 \tag{1.2.15}$$

where $F(1/4, 5/4; 1; -1/4) = 0.955933837\ldots$ is a hypergeometric function evaluation. Reducing this expression to summation form yields the new π formula:

$$\pi \quad = \quad \sum_{i=0}^{\infty} \frac{1}{16^i} \left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6} \right). \tag{1.2.16}$$

To return briefly to the derivation of the formula (1.2.16), let us point out that it was discovered not by formal reasoning, or even by computer-based symbolic processing, but instead by numerical computations using a high-precision implementation of the PSLQ integer relation algorithm. It is most likely the first instance in history of the discovery of a new formula for π by a computer. We might mention that in retrospect formula (1.2.16) could be found much more quickly, by seeking integer relations in the vector $(\pi, S_1, S_2, \cdots, S_8)$, where

$$S_j \quad = \quad \sum_{k=0}^{\infty} \frac{1}{16^k(8k+j)}. \tag{1.2.17}$$

Such a calculation could be done in a few seconds on a computer, even if one did not know in advance to use 16 in the denominator and nine terms in the search, but instead had to stumble on these parameters by trial and error. But this observation is, as they say, 20-20 hindsight. The process of real mathematical discovery is often far more tortuous and less elegant than the polished version typically presented in textbooks and research journals.

It should be clear at this point that the scheme for computing individual hexadecimal digits of π is very similar to Algorithm 2. For completeness we state it as follows:

Algorithm 3 *Individual digit algorithm for π.*

To compute the $(d+1)$-th hexadecimal digit of π: Given an integer $d > 0$, we can write

$$\{16^d \pi\} = \{4\{16^d S_1\} - 2\{16^d S_4\} - \{16^d S_5\} - \{16^d S_6\}\}, \quad (1.2.18)$$

using the S_j notation of equation (1.2.17). Now apply Algorithm 2, with

$$\begin{aligned} \{16^d S_j\} &= \left\{ \left\{ \sum_{k=0}^{d} \frac{16^{d-k}}{8k+j} \right\} + \sum_{k=d+1}^{\infty} \frac{16^{d-k}}{8k+j} \right\} \\ &= \left\{ \left\{ \sum_{k=0}^{d} \frac{16^{d-k} \bmod 8k+j}{8k+j} \right\} + \sum_{k=d+1}^{\infty} \frac{16^{d-k}}{8k+j} \right\} \quad (1.2.19) \end{aligned}$$

instead of equation (1.2.13), to compute $\{16^d S_j\}$ for $j = 1, 4, 5, 6$. Combine these four results, discarding integer parts, as shown in (1.2.18). The resulting fraction, when expressed in hexadecimal notation, gives the hex digit of π in position $d+1$, plus a few more correct digits. □

As with Algorithm 2, multiple-precision arithmetic software is not required—ordinary 64-bit or 128-bit floating-point arithmetic suffices even for some rather large computations. We have omitted here some numerical details for large computations—see [5]. Sample implementations in both C and Fortran-90 are available from the web site `http://www.nersc.gov/~dhbailey`.

One mystery that remains unanswered is why the formula (1.2.8) was not discovered long ago. As you can see from the above proof, there is nothing very sophisticated about its derivation. There is no fundamental reason why Euler, for example, or Gauss or Ramanujan, could not have discovered it. Perhaps the answer is that its discovery was a case of "reverse mathematical engineering." Lacking a motivation to find such a formula, mathematicians of previous eras had no reason to derive one. But this still doesn't answer the question of why the algorithm for computing individual digits of $\log 2$ had not been discovered before—it is based on a formula, namely equation (1.2.12), that has been known for centuries.

Needless to say, Algorithm 3 has been implemented by numerous researchers. In 1997, Fabrice Bellard of INRIA computed 152 binary digits of π starting at the trillionth position. The computation took 12 days on 20 workstations working in parallel over the Internet. His scheme is actually based on the following variant of 1.2.8:

$$\begin{aligned} \pi &= 4 \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k (2k+1)} \\ &\quad - \frac{1}{64} \sum_{k=0}^{\infty} \frac{(-1)^k}{1024^k} \left(\frac{32}{4k+1} + \frac{8}{4k+2} + \frac{1}{4k+3} \right). \quad (1.2.20) \end{aligned}$$

Position	Hex Digits Beginning At This Position
10^6	26C65E52CB4593
10^7	17AF5863EFED8D
10^8	ECB840E21926EC
10^9	85895585A0428B
10^{10}	921C73C6838FB2
10^{11}	9C381872D27596
1.25×10^{12}	07E45733CC790B
2.5×10^{14}	E6216B069CB6C1

Table 1.2: Computed hexadecimal digits of π

This formula permits individual hex or binary digits of π to be calculated roughly 43% faster than (1.2.8).

A year later, Colin Percival, then a 17-year-old student at Simon Fraser University, utilized a network of 25 machines to calculate binary digits in the neighborhood of position five trillion, and then in the neighborhood of 40 trillion. In September, 2000, he found that the quadrillionth binary digit is '0,' based on a computation that required 250 CPU-years of run time, carried out using 1734 machines in 56 countries. The table in Table 1.2 gives some results known as of this writing.

One question that immediately arises in the wake of this discovery is whether or not there is a formula of this type and an associated computational scheme to compute individual *decimal* digits of π. Searches conducted by numerous researchers have been unfruitful. Now it appears that there is no non-binary formula of this type—this is ruled out by a new result co-authored by one of the present authors (see Section 1.3) [16]. However, none of this removes the possibility that there exists some completely different approach that permits rapid computation of individual decimal digits of π. Also, there do exist formulas for certain other constants that admit individual digit calculation schemes in various non-binary bases (including base ten).

1.3 Does Pi Have a Non-Binary BBP Formula?

As mentioned above, from the day that the BBP-formula for π was discovered, researchers have wondered whether there exist BBP-type formulas that would permit computation of individual digits in bases other than powers of two (such as base ten). This is not such a far-fetched possibility, because both base-2 and base-3 formulas are known for π^2, as well as for $\log 2$. But extensive computations failed to find any non-binary formulas for π.

Recently it has been shown that there are no non-binary Machin-type arctangent formulas for π. We believe that if there is no non-binary Machin-type arctangent formula for π, then there is no non-binary BBP-type formula of any form for π. We summarize this result here. Full details and other related results can be found in [16].

We say that the integer $b > 1$ is not a proper power if it cannot be written as c^m for any integers c and $m > 1$. We will use the notation $\mathrm{ord}_p(z)$ to denote the p-adic order of the rational $z \in Q$. In particular, $\mathrm{ord}_p(p) = 1$ for prime p, while $\mathrm{ord}_p(q) = 0$ for primes $q \neq p$, and $\mathrm{ord}_p(wz) = \mathrm{ord}_p(w) + \mathrm{ord}_p(z)$. The notation $\nu_b(p)$ will mean the order of the integer b in the multiplicative group of the integers modulo p. We will say that p is a primitive prime factor of $b^m - 1$ if m is the least integer such that $p|(b^m - 1)$. Thus p is a primitive prime factor of $b^m - 1$ provided $\nu_b(p) = m$. Given the Gaussian integer $z \in Q[i]$ and the rational prime $p \equiv 1 \pmod 4$, let $\theta_p(z)$ denote $\mathrm{ord}_{\mathfrak{p}}(z) - \mathrm{ord}_{\overline{\mathfrak{p}}}(z)$, where $\mathfrak{p}$ and $\overline{\mathfrak{p}}$ are the two conjugate Gaussian primes dividing p, and where we require $0 < \Im(\mathfrak{p}) < \mathrm{R}(\mathfrak{p})$ to make the definition of θ_p unambiguous. Note that

$$\theta_p(wz) \quad = \quad \theta_p(w) + \theta_p(z). \tag{1.3.21}$$

Given $\kappa \in R$, with $2 \leq b \in Z$ and b not a proper power, we say that κ has a Z-linear or Q-linear *Machin-type BBP formula* to the base b if and only if κ can be written as a Z-linear or Q-linear combination (respectively) of generators of the form

$$\arctan\left(\frac{1}{b^m}\right) \quad = \quad \Im \log\left(1 + \frac{i}{b^m}\right) \quad = \quad b^m \sum_{k=0}^{\infty} \frac{(-1)^k}{b^{2mk}(2k+1)}. \tag{1.3.22}$$

We will also use the following theorem, first proved by Bang in 1886:

Theorem 1.3.1 *The only cases where $b^m - 1$ has no primitive prime factor(s) are when $b = 2$, $m = 6$, $b^m - 1 = 3^2 \cdot 7$; and when $b = 2^N - 1$, $N \in Z$, $m = 2$, $b^m - 1 = 2^{N+1}(2^{N-1} - 1)$.*

We can now state the main result of this section:

Theorem 1.3.2 *Given $b > 2$ and not a proper power, then there is no Q-linear Machin-type BBP arctangent formula for π.*

Proof. It follows immediately from the definition of a Q-linear Machin-type BBP arctangent formula that any such formula has the form

$$\pi \quad = \quad \frac{1}{n} \sum_{m=1}^{M} n_m \Im \log(b^m - i) \tag{1.3.23}$$

where $n > 0 \in Z$, $n_m \in Z$, and $M \geq 1$, $n_M \neq 0$. This implies that

$$\prod_{m=1}^{M} (b^m - i)^{n_m} \in e^{ni\pi} Q^\times \quad = \quad Q^\times \tag{1.3.24}$$

For any $b > 2$ and not a proper power we have $M_b \leq 2$, so it follows from Bang's Theorem that $b^{4M} - 1$ has a primitive prime factor, say p. Furthermore, p must be odd, since $p = 2$ can only be a *primitive* prime factor of $b^m - 1$ when b is odd and $m = 1$. Since p is a primitive prime factor, it does not divide $b^{2M} - 1$, and so p must divide $b^{2M} + 1 = (b^M + i)(b^M - i)$. We cannot have both $p|b^M + i$ and $p|b^M - i$, since this would give the contradiction that $p|(b^M + i) - (b^M - i) = 2i$. It follows that $p \equiv 1 \pmod 4$, and that p factors as $p = \mathfrak{p}\overline{\mathfrak{p}}$ over $Z[i]$, with exactly one of $\mathfrak{p}$, $\overline{\mathfrak{p}}$ dividing $b^M - i$. Referring to the definition of θ, we see that we must have $\theta_p(b^M - i) \neq 0$. Furthermore, for any $m < M$ neither $\mathfrak{p}$ nor $\overline{\mathfrak{p}}$ can divide $b^m - i$ since this would imply $p \mid b^{4m} - 1$, $4m < 4M$, contradicting the fact that p is a primitive prime factor of $b^{4M} - 1$. So for $m < M$ we have $\theta_p(b^m - i) = 0$. Referring to equation (1.3.23), using equation (1.3.21) and the fact that $n_M \neq 0$, we get the contradiction

$$0 \;\neq\; n_M \theta_p(b^M - i) \;=\; \sum_{m=1}^{M} n_m \theta_p(b^m - i) \;=\; \theta_p(Q^\times) = 0. \tag{1.3.25}$$

Thus, our assumption that there was a b-ary Machin-type BBP arctangent formula for π must be false. □

Chapter 2

Normality of Numbers

2.1 Normality: A Stubborn Question

Given a real number α and an integer $b > 2$, we say that α is *b-normal* or *normal base b* if *every* sequence of k consecutive digits in the base-b expansion of α appears with limiting frequency b^{-k}. In other words, if a constant is 10-normal, then the limiting frequency of '3' (or any other single digit) in its decimal expansion is 1/10, the limiting frequency of '58' (or any other two-digit pair) is 1/100, and so forth. We say that a real number α is *absolutely normal* if it is b-normal for all integers $b > 1$ simultaneously.

In spite of these strong conditions, it is well-known from measure theory that the set of absolutely normal real numbers in the unit interval has measure one, or in other words that almost all real numbers are absolutely normal [28]. Further, from numerous analyses of computed digits, it appears that all of the fundamental constants of mathematics are normal to commonly used number bases. By "fundamental constants" we include π, e, $\sqrt{2}$, the golden mean $\tau = (1+\sqrt{5})/2$, as well as $\log n$ and the Riemann zeta function $\zeta(n)$ for positive integers $n > 1$, and many others. For example, it is a reasonable conjecture that *every* irrational algebraic number is absolutely normal, since there is no known example of an irrational algebraic number whose decimal expansion (or expansion in any other base) appears to have skewed digit-string frequencies.

Decimal values are given for a variety of well-known mathematical constants in Table 2.1 [20, 22]. In addition to the widely recognized constants such as π and e, we have listed Catalan's constant (G), Euler's constant (γ), an evaluation of the elliptic integral of the first kind $K(1/\sqrt{2})$, an evaluation of an elliptic integral of the second kind $E(1/\sqrt{2})$, Feigenbaum's α and δ constants, Khintchine's constant $\mathcal{K}$ and Madelung's constant $\mathcal{M}_3$. Binary values for some of these constants, as well as Chaitin's Ω constant (from the field of computational complexity) [20], are given in Table 2.2. As you can see, none of the expansions

Figure 2.1: A random walk based on a million digits of π

in either table exhibits any evident "pattern."

The digits of π have been studied more than any other single constant, in part because of the widespread fascination with π. Along this line, Yasumasa Kanada of the University of Tokyo has tabulated the number of occurrences of the ten decimal digits '0' through '9' in the first one trillion decimal digits of π. These counts are shown in Table 2.3. For reasons given in Section 1.2, binary (or hexadecimal) digits of π are also of considerable interest. To that end Kanada has also tabulated the number of occurrences of the 16 hexadecimal digits '0' through '`F`,' as they appear in the first one trillion hexadecimal digits. These counts are shown in Table 2.4. As you can see, both the decimal and hexadecimal single-digit counts are entirely reasonable.

Some readers may be amused by the PiSearch utility, which is available at:

`http://pi.nersc.gov`

This online tool permits one to enter one's name (or any other modest-length alphabetic string, or any modest-length hexadecimal string) and see if it appears encoded in the first four billion binary digits of π (i.e., the first one billion hexadecimal digits of π). Along this line, a graphic based on a random walk of the first million decimal digits of π, courtesy of David and Gregory Chudnovsky, is shown in Figure 2.1. It maps the digit stream to a surface in ways similar to those used by Mandelbrot and others.

The question of whether π, in particular, or, say, $\sqrt{2}$ is normal or not has intrigued mathematicians for centuries. But in spite of centuries of effort, not a single one of the fundamental constants of mathematics has ever been proven to be b-normal for any integer b, much less for all integer bases simultaneously.

Constant	Value
$\sqrt{2}$	1.4142135623730950488...
$\sqrt{3}$	1.7320508075688772935...
$\sqrt{5}$	2.2360679774997896964...
$\phi = \frac{\sqrt{5}-1}{2}$	0.61803398874989484820...
π	3.1415926535897932385...
$1/\pi$	0.31830988618379067153...
e	2.7182818284590452354...
$1/e$	0.36787944117144232160...
e^{π}	23.140692632779269007...
$\log 2$	0.69314718055994530942...
$\log 10$	2.3025850929940456840...
$\log_2 10$	3.3219280948873623478...
$\log_{10} 2$	0.30102999566398119522...
$\log_2 3$	1.5849625007211561815...
$\zeta(2)$	1.6449340668482264365...
$\zeta(3)$	1.2020569031595942854...
$\zeta(5)$	1.0369277551433699263...
G	0.91596559417721901505...
γ	0.57721566490153286061...
$\Gamma(1/2) = \sqrt{\pi}$	1.7724538509055160273...
$\Gamma(1/3)$	2.6789385347077476337...
$\Gamma(1/4)$	3.6256099082219083121...
$K(1/\sqrt{2})$	1.8540746773013719184...
$E(1/\sqrt{2})$	1.3506438810476755025...
α_f	4.669201609102990...
δ_f	2.502907875095892...
$\mathcal{K}$	2.6854520010653064453...
$\mathcal{M}_3$	1.7475645946331821903...

Table 2.1: Decimal values of various mathematical constants

Constant	Value
π	11.001001000011111101101010100010001000010110100011000010001 ...
e	10.101101111110000101010001011000101000101011101101001010100 ...
$\sqrt{2}$	1.0110101000001001111001100110011111110011101111001100100100 ...
$\sqrt{3}$	1.1011101101100111101011101000010110000100110010101010011100 ...
$\log 2$	0.1011000101110010000101111111011111010001110011110111100110 ...
$\log 3$	1.0001100100111110101001111010101011010000001100001010100101 ...
Ω	0.0000001000000100001000001000011101110011001001111000100100 ...

Table 2.2: Binary values of various mathematical constants

Digit	Occurrences
0	99999485134
1	99999945664
2	100000480057
3	99999787805
4	100000357857
5	99999671008
6	99999807503
7	99999818723
8	100000791469
9	99999854780
Total	1000000000000

Table 2.3: Statistics for the first trillion decimal digits of π

Digit	Occurrences
0	62499881108
1	62500212206
2	62499924780
3	62500188844
4	62499807368
5	62500007205
6	62499925426
7	62499878794
8	62500216752
9	62500120671
A	62500266095
B	62499955595
C	62500188610
D	62499613666
E	62499875079
F	62499937801
Total	1000000000000

Table 2.4: Statistics for the first trillion hexadecimal digits of π

And this is not for lack of trying—some very good mathematicians have seriously investigated this problem, but to no avail. Even much weaker results, such as "the digit '1' appears with nonzero limiting frequency in the binary expansion of π" and "the digit '5' appears infinitely often in the decimal expansion of $\sqrt{2}$" have heretofore remained beyond the reach of modern mathematics.

One result in this area is the following. Let $f(n) = \sum_{1 \le j \le n} \lfloor \log_{10} j \rfloor$. Then the Champernowne number

$$\sum_{N=0}^{\infty} \frac{n}{10^{n+f(n)}} \quad = \quad 0.12345678910111213141516171819202122232425\ldots,$$

(i.e., where the positive integers are concatenated) is known to be 10-normal, with a similar form and normality result for other bases (the authors are indebted to Richard Crandall for the formula above). However no one, to the authors' knowledge, has ever argued that this number is a "natural" or "fundamental" constant of mathematics.

Consequences of a proof in this area would definitely be interesting. For starters, such a proof would immediately provide an inexhaustible source of provably reliable pseudorandom numbers for numerical or scientific experimentation. We also would obtain the mind-boggling but uncontestable consequence that if π for example is shown to be 2-normal, then the entire text of the Bible, the Koran and the works of William Shakespeare, as well as the full LaTeXsource text for this book, must all be contained somewhere in the binary expansion of π, where consecutive blocks of eight bits (two hexadecimal digits) each represent one ASCII character. Unfortunately, this would not be much help to librarians or archivists, since every conceivable misprint of each of these books would also be contained in the binary digits of π.

2.2 BBP Constants and Normality

Until recently, the BBP formulas mentioned in Sections 1.2 and **??** were assigned by some to the realm of "recreational" mathematics—interesting but of no serious consequence. But the history of mathematics has seen many instances where results once thought to be idle curiosities were later found to have significant consequences. This now appears to be the case with the theory of BBP-type constants.

What we shall establish below, in a nutshell, is that the 16-normality of π (which of course is equivalent to the 2-normality of π), as well as the normality of numerous other irrational constants that possess BBP-type formulas, can be reduced to a certain plausible conjecture in the theory of chaotic sequences. We do not know at this time what are the full implications of this result. It may be the first salvo in the resolution of this age-old mathematical question, or it may be merely a case of reducing one very difficult mathematical problem to another. But at the least, this result appears to lay out a structure—a "roadmap" of sorts—for the analysis of this question. Thus it seems worthy of investigation.

We shall also establish that a certain well-defined class of real numbers, uncountably infinite in number, *is* indeed b-normal for certain bases b, which result is not dependent on any unproven conjecture. We will also present some results on the digit densities of algebraic irrationals. All of these recent results are direct descendants of the theory of BBP-type constants that we have presented in Sections 1.2 and [15].

The results for BBP-type constants derive from the following observation, which was given in a recent paper by one of the present authors and Richard Crandall [6]. Here we define the norm $||\alpha||$ for $\alpha \in [0, 1)$ as $||\alpha|| = \min(\alpha, 1-\alpha)$. With this definition, $||\alpha - \beta||$ measures the shortest distance between α and β on the unit circumference circle in the natural way. Suppose α is given by a BBP-type formula, namely

$$\alpha \;=\; \sum_{k=0}^{\infty} \frac{p(k)}{b^k q(k)} \qquad (2.2.1)$$

where p and q are polynomials with integer coefficients, with $0 \le \deg p < \deg q$, and with q having no zeroes at positive integer arguments. Now define the recursive sequence (x_n) as $x_0 = 0$, and

$$x_n = \left\{ b x_{n-1} + \frac{p(n)}{q(n)} \right\} \tag{2.2.2}$$

where the notation $\{\cdot\}$ denotes fractional part as before. Recall from Section 1.2 that we can write the base-b expansion of α beginning at position $n+1$, which we denote α_n, as

$$\begin{aligned} \alpha_n &= \{b^n \alpha\} = \left\{ \sum_{k=0}^{\infty} \frac{b^{n-k} p(k)}{q(k)} \right\} \\ &= \left\{ \left\{ \sum_{k=0}^{n} \frac{b^{n-k} p(k)}{q(k)} \right\} + \sum_{k=n+1}^{\infty} \frac{b^{n-k} p(k)}{q(k)} \right\}. \end{aligned} \tag{2.2.3}$$

Now observe that the sequence (x_n) generates the first part of this expression. In particular, given $\epsilon > 0$, assume that n is sufficiently large such that $p(k)/q(k) < \epsilon$ for all $k \ge n$. Then we can write, for all sufficiently large n,

$$\begin{aligned} ||x_n - \alpha_n|| &= \left| \sum_{k=n+1}^{\infty} \frac{b^{n-k} p(k)}{q(k)} \right| \\ &\le \epsilon \sum_{k=n+1}^{\infty} b^{n-k} = \frac{\epsilon}{b-1} < \epsilon. \end{aligned} \tag{2.2.4}$$

With this argument, we have established the following, which we observe is also true if the expression $p(k)/q(k)$ is replaced by any more general sequence $r(k)$ that tends to zero for large k:

Theorem 2.2.1 *Let α be a BBP-type constant as defined above, with α_n the base-b expansion of α beginning at position $n+1$, and (x_n) the BBP sequence associated with α, as given in (2.2.2) above. Then $|x_n - \alpha_n| \to 0$ as $n \to \infty$.*

In other words, the *BBP sequence* associated with α (as given in formula (2.2.2) is a close approximation to the sequence (α_n) of shifted digit expansions, so much so that we might expect that if one has a property such as equidistribution in the unit interval, then the other does also. We now state a hypothesis, which is believed to be true, based on experimental evidence, but which is not yet proven:

Hypothesis A (Bailey-Crandall). Let $p(x)$ and $q(x)$ be polynomials with integer coefficients, with $0 \le \deg p < \deg q$, and with q having no zeroes for positive integer arguments. Let $b \geq 2$ be an integer, and let $r_n = p(n)/q(n)$. Then the sequence $x = (x_0, x_1, x_2, \ldots)$ determined by the iteration $x_0 = 0$, and

$$x_n = \{bx_{n-1} + r_n\} \tag{2.2.5}$$

either has a finite attractor or is equidistributed in $[0, 1)$.

The terms "equidistributed" and "finite attractor" are defined in [6, 15]. Here we rely on intuition.

Theorem 2.2.2 *Assuming Hypothesis A, any constant α given by a formula of the type $\alpha = \sum_k p(k)/(b^k q(k))$, with $p(k)$ and $q(k)$ polynomials as given in Hypothesis A, is either rational or normal base b.*

We should note here that even if a particular instance of Hypothesis A could be established, it would have significant consequences. For example, if it could be established that the simple iteration given by $x_0 = 0$ and

$$x_n = \left\{2x_{n-1} + \frac{1}{n}\right\} \tag{2.2.6}$$

is equidistributed in $[0, 1)$, then it would follow from Theorem 2.2.2 that $\log 2$ is 2-normal. Observe that this sequence is simply the BBP sequence for $\log 2$. In a similar vein, if it could be established that the iteration given by $x_0 = 0$ and

$$x_n = \left\{16x_{n-1} + \frac{120n^2 - 89n + 16}{512n^4 - 1024n^3 + 712n^2 - 206n + 21}\right\} \tag{2.2.7}$$

is equidistributed in $[0, 1)$, then it would follow that π is 16-normal (and so is 2-normal also). This is the BBP sequence for π. The fractional term here is obtained by combining the four fractions in the BBP formula for π, namely equation (1.2.8), into one fraction, and then shifting the index by one.

Before continuing, we wish to mention a curious phenomenon. Suppose we compute the binary sequence $y_n = \lfloor 2x_n \rfloor$, where (x_n) is the sequence associated with $\log 2$ as given in equation 2.2.6. In other words, (y_n) is the binary sequence defined as $y_n = 0$ if $x_n < 1/2$ and $y_n = 1$ if $x_n \geq 1/2$. Theorem 2.2.1 tells us, in effect, that (y_n) eventually should agree very well with the true sequence of binary digits of $\log 2$. In explicit computations, we have found that the sequence (y_n) disagrees with 15 of the first 200 binary digits of $\log 2$, but in only one position over the range 5000 to 8000.

As noted above, the BBP sequence for π is $x_0 = 0$, and x_n as given in equation (2.2.7). In a similar manner as with $\log 2$, we can compute the hexadecimal digit sequence $y_n = \lfloor 16x_n \rfloor$. In other words, we can divide the unit interval into 16 equal subintervals, labeled $(0, 1, 2, 3, \ldots, 15)$, and set y_n to be the label of the subinterval in which x_n lies. When this is done, a remarkable phenomenon occurs: the sequence (y_n) appears to perfectly (not just approximately) produce the hexadecimal expansion of π. In explicit computations, the first 1,000,000 hexadecimal digits generated by this sequence are identical with the first 1,000,000 hexadecimal digits of $\pi - 3$. (This is a fairly difficult computation, as it must be performed to very high precision and is not easily performed on a parallel computer system.)

Conjecture. The sequence $(\lfloor 16x_n \rfloor)$, where (x_n) is the sequence of iterates defined in equation (2.2.7), precisely generates the hexadecimal expansion of $\pi - 3$.

Evidently this phenomenon arises from the fact that in the sequence associated with π, the perturbation term $r_n = p(n)/q(n)$ is summable, whereas the corresponding expression for $\log 2$, namely $1/n$, is not summable. In particular, note that expression (2.2.4) for $\alpha = \pi$ is

$$\begin{aligned} ||\alpha_n - x_n|| &= \sum_{k=n+1}^{\infty} \frac{120k^2 - 89k + 16}{16^{j-n}(512k^4 - 1024k^3 + 712k^2 - 206k + 21)} \\ &\approx \frac{120(n+1)^2 - 89(n+1) + 16}{16(512(n+1)^4 - 1024(n+1)^3 + 712(n+1)^2 - 206(n+1) + 21)} \end{aligned} \tag{2.2.8}$$

so that

$$\sum_{n=1}^{\infty} ||\alpha_n - x_n|| \approx 0.01579 \ldots. \tag{2.2.9}$$

For the sake of heuristic argument, let us assume for the moment that the α_n are independent, uniformly distributed random variables in $(0, 1)$, and let $\delta_n = ||\alpha_n - x_n||$. Note that an error (i.e. an instance where x_n lies in a different subinterval of the unit interval than α_n) can only occur when α_n is within δ_n of one of the points $(0, 1/16, 2/16, \cdots, 15/16)$. Since $x_n < \alpha_n$ for all n (where $<$ is interpreted in the wrapped sense when x_n is slightly less than one), this event has probability $16\delta_n$. Then the fact that the sum (2.2.9) has a finite value implies, by the first Borel-Cantelli lemma, that there can only be finitely many errors [12, pg. 153]. The comparable figure for $\log 2$ is infinite, which implies by the second Borel-Cantelli lemma that discrepancies can be expected to appear indefinitely, but with decreasing frequency. Further, the small value of the sum (2.2.9) suggests that it is unlikely that any errors will be observed. If instead of summing (2.2.9) from one to infinity, we instead sum from 1,000,001 to infinity (since we have computationally verified that there are no errors in the first 1,000,000 elements), then we obtain 1.465×10^{-8}, which suggests that it is very unlikely that any errors will ever occur.

2.3 A Class of Provably Normal Constants

We now summarize an intriguing recent development in this arena, due to one of the present authors and Richard Crandall, which offers additional hope that the BBP approach may eventually yield the long-sought proof of normality for π, $\log 2$ and other BBP-type constants [7]. In the previous section, we noted that the 2-normality of

$$\log 2 \;=\; \sum_{n=1}^{\infty} \frac{1}{n 2^n} \tag{2.3.10}$$

rests on the (unproven) conjecture that the iteration given by $x_0 = 0$ and $x_n = \{2x_{n-1} + 1/n\}$ is equidistributed in the unit interval. We now consider the class of constants where the summation defining $\log 2$, namely (2.2.6), is taken over a certain subset of the positive integers:

$$\alpha_{b,c} \;=\; \sum_{n=c^k>1} \frac{1}{n b^n} \;=\; \sum_{k=1}^{\infty} \frac{1}{c^k b^{c^k}}, \tag{2.3.11}$$

where $b > 1$ and $c > 2$ are integers. The simplest instance of this class is

$$\begin{aligned} \alpha_{2,3} &= \sum_{n=3^k>1} \frac{1}{n 2^n} \;=\; \sum_{k=1}^{\infty} \frac{1}{3^k 2^{3^k}} && (2.3.12) \\ &= 0.041883680831502985071252898624571682426096758465485 7\ldots_{10} \\ &= \texttt{0.0AB8E38F684BDA12F684BF35BA781948B0FCD6E9E06522C3F35B}\ldots_{16}. \end{aligned}$$

We first prove the following interesting fact:

Theorem 2.3.1 *Each of the constants $\alpha_{b,c}$, where $b > 1$ and $c > 2$ are integers, is transcendental.*

Proof. A famous theorem due to Roth states [32] that if $|P/Q - \alpha| < 1/Q^{2+\epsilon}$ admits infinitely many rational solutions P/Q (i.e. if α is approximable to degree $2 + \epsilon$ for some $\epsilon > 0$), then α is transcendental. We show here that $\alpha_{b,c}$ is approximable to degree $c - \delta$. Fix a k and write

$$\alpha_{b,c} \;=\; P/Q + \sum_{n>k} \frac{1}{c^n b^{c^n}}, \tag{2.3.13}$$

where $\gcd(P, Q) = 1$ and $Q = c^k b^{c^k}$. The sum over n gives

$$|\alpha_{b,c} - P/Q| \;<\; \frac{2}{c^{k+1}(Q/c^k)^c} \;<\; \frac{c^{kc}}{Q^c}. \tag{2.3.14}$$

Now $c^k \log b + k \log c = \log Q$, so that $c^k < \log Q / \log b$, and we can write

$$c^{kc} \ < \ (\log Q / \log b)^c \ = \ Q^{c(\log\log Q - \log\log b)/\log Q}. \tag{2.3.15}$$

Thus for any fixed $\delta > 0$,

$$|\alpha_{b,c} - P/Q| \ < \ \frac{1}{Q^{c(1+\log\log b/\log Q - \log\log Q/\log Q)}} \ < \ \frac{1}{Q^{c-\delta}}, \tag{2.3.16}$$

for all sufficiently large k. □

Consider now the BBP sequence associated with $\alpha_{2,3}$, namely the sequence defined by $x_0 = 0$, and

$$x_n \ = \ \{2x_{n-1} + r_n\} \tag{2.3.17}$$

where $r_n = 1/n$ if $n = 3^k$, and $r_n = 0$ otherwise. Successive iterates of this sequence are:

$$\begin{array}{l}
0,\ 0,\ 0, \frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \frac{2}{3}, \\
\frac{4}{9}, \frac{8}{9}, \frac{7}{9}, \frac{5}{9}, \frac{1}{9}, \frac{2}{9}, \frac{4}{9}, \frac{8}{9}, \frac{7}{9}, \frac{5}{9}, \frac{1}{9}, \frac{2}{9}, \frac{4}{9}, \frac{8}{9}, \frac{7}{9}, \frac{5}{9}, \frac{1}{9}, \frac{2}{9}, \\
\frac{13}{27}, \frac{26}{27}, \frac{25}{27}, \frac{23}{27}, \frac{19}{27}, \frac{11}{27}, \frac{22}{27}, \frac{17}{27}, \frac{7}{27}, \frac{14}{27}, \frac{1}{27}, \frac{2}{27}, \frac{4}{27}, \frac{8}{27}, \frac{16}{27}, \frac{5}{27}, \frac{10}{27}, \frac{20}{27}, \\
\frac{13}{27}, \frac{26}{27}, \frac{25}{27}, \frac{23}{27}, \frac{19}{27}, \frac{11}{27}, \frac{22}{27}, \frac{17}{27}, \frac{7}{27}, \frac{14}{27}, \frac{1}{27}, \frac{2}{27}, \frac{4}{27}, \frac{8}{27}, \frac{16}{27}, \frac{5}{27}, \frac{10}{27}, \frac{20}{27}, \\
\frac{13}{27}, \frac{26}{27}, \frac{25}{27}, \frac{23}{27}, \frac{19}{27}, \frac{11}{27}, \frac{22}{27}, \frac{17}{27}, \frac{7}{27}, \frac{14}{27}, \frac{1}{27}, \frac{2}{27}, \frac{4}{27}, \frac{8}{27}, \frac{16}{27}, \frac{5}{27}, \frac{10}{27}, \frac{20}{27},
\end{array} \tag{2.3.18}$$

A pattern is clear: the sequence consists of a concatenation of triply-repeated segments, each consisting of fractions whose denominators are successively higher powers of 3, and whose numerators range over all integers less than the denominator that are coprime to the denominator. Indeed, the successive numerators in each subsequence are given by the simple linear congruential pseudorandom number generator $z_n = 2z_{n-1} \mod 3^j$ for a fixed j.

What we have observed is that the question of the equidistribution of the sequence (x_n) (and, hence, the question of the normality of $\alpha_{2,3}$) reduces to the behavior of a concatenation of normalized pseudorandom sequences of a type (namely linear congruential) that have been studied in mathematical literature, and which in fact are widely implemented for use by scientists and engineers.

These observations lead to a rigorous proof of normality for many of these constants. In particular, we obtain the result that each of the constants

$$\alpha_{b,c} = \sum_{n=c^k>1} \frac{1}{nb^n} = \sum_{k=1}^{\infty} \frac{1}{c^k b^{c^k}}, \tag{2.3.19}$$

where $b > 1$, and c is odd and coprime to b, is b-normal. This result was first given in [7]. One may present significantly simpler proof, although it requires a modest excursion into measure theory and ergodic theory.

Chapter 3

Historia Cyclometrica

3.1 1 Kings, 2 Chronicles, and Maimonides

We quote two versions of the famous and controversial biblical text suggesting that setting Pi equal to three sufficed for the Old Testament. These are:

> Then he [Solomon] made the molten sea: it was round, ten cubits from brim to brim, and five cubits high. A line of thirty cubits would encircle it completely. [29, 1 Kings 7:23]

and:

> Then he made the molten sea: it was round, ten cubits from rim to rim, and five cubits high. A line of thirty cubits would encircle it completely. [29, 2 Chronicles 4:2]

Several millennia later the great Rabbi Moses ben Maimon Maimonedes (1135-1204) is translated by Tzvi Langermann, in "The 'true perplexity' " [27, p. 165] as clearly asserting the irrationality of Pi.

> You ought to know that the ratio of the diameter of the circle to its circumference is unknown, nor will it ever be possible to express it precisely. This is not due to any shortcoming of knowledge on our part, as the ignorant think. Rather, this matter is unknown due to its nature, and its discovery will never be attained.

3.2 Francois Viète. Book VIII, Chapter XVIII.

3.2.1 Ratio of Regular Polygons, Inscribed in a Circle, to the Circle

Archimedes squared the parabola by continuously inscribing triangles that are IN A RATIONAL RATIO[1]. For, having inscribed the greatest possible triangle in the parabola, he further inscribed triangles in continuous proportion to the greatest, namely in the constant ratio 1 to 4, infinitely. And so he concluded that the parabola is four thirds of that greatest triangle. But Antiphon could not square the circle in that way, since triangles inscribed continuously in a circle are in an IRRATIONAL RATIO and constantly changing. But will it not be possible to square the circle, then? For if a figure, composed of triangles that are constructed successively and infinitely in the ratio 1 to 4 to the given greatest triangle, is made four thirds of the same, then there is a certain knowledge of the infinitely many. And it is possible to compose a plane figure of triangles that are infinitely and continuously inscribed in a circle in A RATIO, albeit IRRATIONALS and constantly changing. And this composed figure will have a certain ratio to the greatest inscribed figure. The Euclideans, however, will maintain with authority that an angle greater than an acute and smaller than an obtuse is not a right angle. About that I propose the following so that it is possible to philosophize more freely about the uncertain and changing [ratio] of any regular polygon, inscribed in a circle, to a polygon with an infinite number of sides, or a circle if you will.

Proposition 1. *If two regular polygons be inscribed in the same circle, and if furthermore the number of sides or angles of the first one is half of the sides or angles of the second one, then the first polygon will be to the second as the apotome of the first side is to the diameter.* ('Apotome of a side' is my name for the cord which subtends the arc of the semicircle that supplements the arc subtended by the side.)

[**Proof of Proposition 1.**] Thus, in a circle with centre A, diameter BC, let any regular polygon be inscribed, whose side is BD. And let the arc BD, bisected at E, be subtended by BE [and ED]. That is to say, let another polygon be inscribed whose side is BE. So the number of sides or angles of the first polygon will be half of the number of sides or angles of the second. Let DC be joined. I say that the first polygon with side BD is to the second polygon

[1]Greek in original is rendered in small capitals.

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with side BE or ED as DC to BC. For let DA, ED be joined. The first polygon consists of as many triangles BAD as there are sides or angles in the first polygon. And the second polygon consists of as many trapezia BEDA. Therefore the first polygon is to the second as the triangle BAD is to the trapezium BEDA. But the trapezium BEDA is divided into two triangles BAD, BED, whose common base is BD. And triangles whose base is the same are [to each other] as the heights. Therefore, let the half diameter AE be drawn intersecting BD in F. Thus, since the arc BD is bisected in E, AE intersects BD at right angles. Therefore AF is the height of the triangle BDA and FE is the height of the triangle BED. And so the triangle BAD is to the triangle BED as AF to EF, and *componendo* the triangle BAD to the triangles BAD, BED together, that is the trapezium BEDA, as AF to AE. And the first polygon will be to the second in that ratio, too. But, AF is to AE or AB as DC is to BC; for, the angle BDC is right, as is the angle BFA, and therefore AF and DC are parallel. Thus the first polygon, whose side is BD, is to the second polygon, whose side is BE or ED, as DC to BC. Which was to be shown.

Proposition 2. *If in one and the same circle infinitely many regular polygons are inscribed, and the number of sides of the first is 1/2 of the sides of the second, and 1/4 of the number of sides of the third, 1/8 of the fourth, 1/16 of the fifth, and so on in continuous halvings, then the first polygon will be to the third as the product of [lit. 'the rectangle contained by'] the apotomes of the sides of the first and the second is to the square on the diameter. To the fourth it will be as the product of [lit. 'the solid made of'] the apotomes of the sides of the first, the second, and the third is to the cube on the diameter. To the fifth it will be as the product of the four lengths [lit. 'plano-planum'] of the apotomes of the sides of the first, the second, the third, and the fourth is to the fourth power of [lit 'quadrato-quadratum on'] the diameter. To the sixth it will be as the product of the five lengths [lit. 'plano-solidum'] of the apotomes of the sides of the first, the second, the third, the fourth, and the fifth is to the fifth power of [lit. 'quadrato-cubum on'] the diameter. To the seventh it will be as the product of the six lengths [lit. 'solido-solidum'] of the apotomes of the sides of the first, the second, the third, the fourth, the fifth, and the sixth is to the sixth power of [lit. 'cubo-cubum on'] the diameter. And so on in continuous progression* ad infinitum. [Having noted Viète's terms for various kinds of products and powers

[2]These page numbers refer to those of the original printed Latin text of the work reprinted in this volume.

of geometrical magnitudes we henceforth translate them by modern terminology for products.]

[**Proof.**] Let B be the apotome of the side of the first polygon, C of the second, D of the third, F of the fourth, G of the fifth, H of the sixth. And let the diameter of the circle be Z. According to the first proposition the first polygon will be to the second as B to Z; therefore the product of B and the second polygon will be equal to the product of Z and the first polygon. But the second polygon will be to the third as C to Z; consequently the product of the second polygon and B, that is the product of the first polygon and Z, will be to the product of the third polygon and B as C to Z. Therefore the product of the first polygon and Z squared equals the product of the third polygon and the product of B,C. Therefore the first polygon is to the third as the product of B,C to Z squared. And the product of the third polygon and the product B×C is equal to the product of the first polygon and Z squared.¿ Again, according to the same previous proposition, as the third polygon is to the fourth, so is D to Z. And consequently, the product of the third and the rectangle B times C, that is the product of the first and Z squared, is to the product of the fourth and the rectangle B times C, as D to Z. Therefore, the product of the first [polygon] and Z to the third power is equal to the product of the fourth [polygon] and B times C times D. Therefore the first polygon is to the fourth as B times C times D to Z cubed. By the same method of demonstration it [the first polygon] will be to the fifth as B times C times D times F to Z to the fourth power. To the sixth as B times C times D times F times G to Z to the fifth power. To the seventh as B times C times D times F times G times H to Z to the sixth power. And so forth in this constant progression ad infinitum.

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Corollary. *Therefore the square inscribed in the circle will be as the side of this square to the highest power of the diameter divided by the continuous product of the apotomes of the sides of the octagon, the sixteen-gon, the polygon with 32 sides, with 64 sides, with 128 sides, with 256 sides, and of all the others in the half ratio of angles and sides.*

[**Proof.**] For, let the square be the first polygon inscribed in the circle; then the octagon will be the second, the sixteen-gon the third, the thirty-two-gon the fourth, and so on in continuous order. Thus the square inscribed in the circle will have the same ratio to the extreme polygon–with infinitely many sides–as the product made by the apotomes of the sides of the square, the octagon, the sixteen-gon, and all the others in the half ratio *ad infinitum* has to the highest power of the diameter. And by a common division [the square inscribed in the circle will have the same ratio to the extreme polygon] as the apotome of the side of the square has to the highest power of the diameter divided by the product of the apotomes of the sides of the octagon, sixteen-gon, and the others in double ratio *ad infinitum.* But the apotome of the side of the square inscribed in a circle is the side itself, and the polygon with infinitely many sides is the circle itself.

Let the circle's diameter be 2, and the side of the inscribed square be $\sqrt{2}$. The apotome of the 8-gon is $\sqrt{2+\sqrt{2}}$. The apotome of the 16-gon is $\sqrt{2+\sqrt{2+\sqrt{2}}}$. The apotome of the 32-gon is $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}$. The apotome of the 64-gon is $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{\sqrt{2}+2}}}}$. And so on in that progression.

But then, let the diameter be 1 and the [area of the] circle N. [Vi'ete writes, in the manner of Diophantus '1N'.] 1/2 [the area of the inscribed square] will be to N as $\sqrt{1/2}$ to the unit divided by

$\sqrt{1/2+\sqrt{1/2}}$, times $\sqrt{1/2+\sqrt{1/2+\sqrt{1/2}}}$,

times $\sqrt{1/2+\sqrt{1/2+\sqrt{1/2+\sqrt{1/2}}}}$,

times $\sqrt{1/2+\sqrt{1/2+\sqrt{1/2+\sqrt{1/2+\sqrt{1/2}}}}}$,

times $\sqrt{1/2+\sqrt{1/2+\sqrt{1/2+\sqrt{1/2+\sqrt{1+\sqrt{1/2}}}}}}$.

Let the diameter be X and the circle (equal to) the plane area A. The half square on X will be to the area A as the side of the half square on X to the greatest power of X divided by the product of the binomial square root [i.e. a square root of the sum of two terms] of ($1/2X^2$ the square root of $1/2X^4$) times the binomial square root of ($1/2X^2$+ the binomial square root of ($1/2X^4$+ the square root of $1/2X^8$))) times the binomial square root of ($1/2X^2$+ the binomial square root of ($1/2X^4$+ the binomial square root of ($1/2X^8$+the square root of $1/2X^{16}$)))$^{\cdots}$ times ... etc. *ad infinitum* while observing this uniform method.

3.2.2 Defense Against the New Cyclometry or ANTI-AXE.

Those who have tried to set the circle equal to thirty-six segments of the hexagon by means of their figures which they call hatchets, unluckily are wasting their efforts. For how can determined results be obtained from splitting completely undetermined magnitudes? If they add or subtract equals, if they divide or multiply by equals, if they invert, permute and at last increase or decrease by arbitrary degrees of proportion, they will not advance one inch in their research, but will make the mistake that logicians call BEGGING THE QUESTION, and Diophantines [call] NON-QUANTITIES. Or they delude themselves by false calculations, though they could have foreseen it if any light from the true analytic doctrine had enlightened them. Others, however, who are terrified by these ONE-EDGED double bladed weapons and already are lamenting that Archimedes is wounded by them, are quite unfit for war. But Archimedes lives, nor do THE FALSE WRITINGS ABOUT THE TRUTH, THE FALSE RECKONINGS, the Non-proofs, the magnificent words, shock him. But in order that they may live more secure, I bring them "Shields decorated with clouds and weapons untouched by slaughter," but HARD TO AXE, wherewith they can fortify themselves to begin with, ready to reinforce them with means FOR WAR if the enemies' impudence be too fierce.

Proposition 1. *The circumference of a dodecagon inscribed in a circle has a ratio to the diameter (that is) less than triple plus one-eighth.*

With centre A and an arbitrary radius AB let the circle BCD be described, in which let BC be taken as arc of the hexagon [i.e. an arc equal to 1/6 of the circumference], which is bisected at D, and let DB be subtended. Thus DB is the side of the dodecagon; and if it be extended twelve times to E, DE will be equal to the circumference of the dodecagon inscribed in the circle BDC. Let the diameter DF be drawn. I say that DE to DF has a ratio less than triple plus one-eighth.

For, let BC and BA be joined, and let the diameter DF intersect BC at G. Therefore it will bisect it at right angles. And let the triangle DEH be constructed similar to the triangle DBG.

Since the line BC is subtended under an arc of the hexagon, BA or DA is equal to BC. Therefore, if AC or BC are composed of eight (equal) parts, BG is four of the same (parts). So the square on AG is $[64 - 16 =]48$, and so AG is greater than $6\,12/13$ [since $48 = 7(7 - 1/7)$, and the arithmetic mean of the two factors is $7 - 1/14$.

Since the geometric mean of the two factors (which is the square root of their product) is less than the arithmetic mean $\sqrt{48} < 6\,13/14$. A good guess therefore is that $\sqrt{48} > 6\,12/13$, which is easily verified.] Therefore DG is less than $1\,1/13$. And since DE is composed of twelve times DB, EH will also be twelve times

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BG, and DH twelve times DG. Therefore EH is 48 of the same parts. And DH will be less than 13, namely less than $12\,12/13$. The square on the side 48 is

2304, and on 13 it is 169. Those two square added make 2473, not quite 2500, the square on the side 50. Therefore the line DE, whose square is equal to the sum of the squares on EH and DH, is less than 50. And the ratio of 50 to 16 is triple and one-eighth exactly. Therefore the ratio of DE to DF is less than triple and one-eighth. Which was to be proved.

Arithmetic is absolutely as much science as geometry [is]. Rational magnitudes are conveniently designated by rational numbers, and irrational [magnitudes] by irrational [numbers]. If someone measures magnitudes with numbers and by his calculation get them different from what they really are, it is not the reckoning's fault but the reckoner's.

Rather, says Proclus, ARITHMETIC IS MORE EXACT THAN GEOMETRY. To an accurate calculator, if the diameter is set to one unit, the circumference of the inscribed dodecagon will be the side of the binomial [i.e. square root of the difference] $72 - \sqrt{3888}$. Whoever declares any other result, will be mistaken, either the geometer in his measurements or the calculator in his numbers.

That the ratio of the circumference of the circle to its diameter is greater than triple and one-eighth as well as less than triple and one seventh has not been doubted up till now by the school of mathematicians, for Archimedes proved that convincingly. So one should not, by a false calculation, have induced a MANIFEST ABSURDITY, that a straight line is longer than the circular arc terminating at the same endpoints, since Archimedes assumes the contrary FROM THE COMMON NOTION and Eutocius demonstrates the same, defining generally that OF ALL LINES HAVING THE SAME EXTREMITIES, THE STRAIGHT LINE IS THE SHORTEST.

Proposition 2. *If the half diameter of the circle be divided by the quadratrix, the part from the centre to the quadratrix is greater than the mean proportional between the half diameter and two fifths of the half diameter.*

Let ABC be a quadrant of a circle, and BD the quadratrix; let AE be taken equal to two fifths of the half diameter AB or AC; and let AF be made the mean proportional of AE, AC. I say that AD is greater than AF. For, from what was proved by Pappus about the quadratrix, the half diameter AB or

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AC is the mean proportional of the arc BC and AD. Let AB be 7 parts; then the arc BC, which is the a quarter of the perimeter, will be less than 11, for the diameter is 14, and the perimeter is less than 44. But then, let AB be 35 parts; then the arc BC will be less than 55. But the [area] contained by AD and [the arc] BC is equal to the square on AB. Therefore AD is greater than 22 [+] 3/11. And of those units of which AB, that is AC, is 35, AE is 14, but AF is less than

22 [+] 3/22. [See bracketed explanation of a similar claim near end of p. 437.] Therefore AD is greater than AF. Which was to be proved.

Therefore, if from the diameter AB be subtracted the line AG equal to AF, and if the parallelogram GHDA be completed, it will be an oblong rectangle, not a square. And when the square BC is completed, the diagonal BK will not go through H, but through some point I further from D. It was important to notice this to avoid a false diagram.

Proposition 3. *The square on the circumference of the circle is less than ten times the square on the diameter.*

Let the diameter be 7, the square on the diameter will be 49, and ten times that 490. But the circumference of the circle will be less than 22, and consequently its square less than 484. The Arabs' opinion that 'the square on the circumference of the circle is equal to ten times the square on the diameter' has since long been rejected. He is not to be tolerated who CONTRADICTORILY proposes what Archimedes proves to be unprovable.

Proposition 4. *The circle has a greater ratio to the hexagon inscribed in it than six to five.*

Let the hexagon BCDEFG be inscribed in the circle with centre A. I say that the circle with centre A has a greater ratio to the hexagon BCDEFG than six to five. When AB, AC BC are joined, let the perpendicular AZ fall on BC. Then, since in triangle ABC the legs AB, AC are equal, the base is bisected in Z, and BZ and ZC are equal. But the triangle ABC is equilateral, for its legs are both half diameters, and the base - since it is the side of the hexagon - is also equal to the half diameter. Now, if the half diameter BA or AC be set to 30 [units], BZ or ZC becomes 15, and AZ becomes less than 26, whose square is 676. But the difference between the squares [on] AB, AZ is only 675. Further, the rectangle contained by BZ, ZA is equal to the triangle BAC. So, let 15 be multiplied by 26, they become 390. Therefore, of such units of which the square [on] AB is 900, of the same [units] the triangle ABC will be less than 390, or - if all be divided by 30 - if the square AB is 30, the triangle ABC will be less than 13. Let AD, AE, AF, AG be joined; the hexagon BCDEFG consists of six triangles equal to BAC. Therefore, of such [units] of which the square [on] AB becomes 30, of the same [units] the hexagon will be less than 78. Or, of such units of which the square [on] AB becomes five, of the same [units] the hexagon will be less than thirteen.

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But as the perimeter of the circle is to the diameter, so is the rectangle made by the perimeter of the circle and a quarter of the diameter to the area made by the diameter and a quarter of the diameter. But the rectangle made by the perimeter of the circle and a quarter of the diameter is equal to the circle. And the area made by the diameter and a quarter of the diameter is the square on the half diameter. Therefore, as the perimeter is to the diameter, so is the circle to the square on the half diameter. Now, of such units as the diameter is 1, the perimeter is greater than 3[+]10/71, and so evidently greater than 3[+]10/8 or 3[+]1/8. And of such units as the square on the diameter is 5 (as above) the circle is greater than 15[+]5/8. But the hexagon was less than 13 of the same units. Therefore the circle will have to the hexagon inscribed in it a greater ratio than 15[+]5/8 to 13, that is, as 125 to 104, or as [750 to 624, that is] 6 to 4[+]124/125, and so evidently greater than six to five. Which was to be proved.

Therefore, those who set the circle equal to the hexagon and a fifth part of the hexagon do not square it PROPERLY, since it is greater according to the limits set by Archimedes FROM HIS OWN PRINCIPLES. Our schools are Platonic: Oh splendid professors; therefore, do not fight against the geometric principles. And just as these AXE-SWINGERS have truncated the circle, they may now - as a kind of compensation for the damage done - themselves be shortened at the pointed end of their swallow tail.

Proposition 5. *Thirty-six segments of the hexagon are greater than the circle.*

For, since the circle has a greater ratio to the hexagon inscribed in it than six to five, or as 1 to 5/6, therefore the difference between the circle and the hexagon will be greater than one-sixth of the circle. But the circle differs from the hexagon by six segments of the hexagon. Thus six segments of the hexagon are greater than one-sixth of the circle, and so thirty-six segments will be greater than one, that is: the circle. What was to be proved.

Proposition 6. *Any segment of the circle is greater than the sixth of the similar segment similarly drawn in a circle whose radius is equal to the base of the segment set out.*

In the circle BDC described around the centre A let a cord be subtended under an arbitrary arc BD, and let a straight line BE touch the circle, and with B as centre and BD as radius let another circle DEF be drawn.

Then the arc ED will be similar to half of the arc BD. And then, let the arc DF be taken double of DE, and let BF and AD be joined. Thus the sectors BAD and FBD will be similar. I say that the segment of the circle BDC contained by the line BD and the arc under which it is suspended, is greater than one-sixth of the sector FBD.

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For, let a spiral line whose origin is B be drawn, crossing [the circle] at D so that BD is the same part of the first revolution BEZ [the point Z is not labeled in the diagram] as the angle EBD is of four right angles. Thus the area contained by the straight line BD and the spiral is the third part of the sector EBD, for that has Pappus proved after Archimedes in proposition xxii in Book IV of the *Mathematical Collection*. Therefore the [sector] FBD will be six times the same area, for the sector FBD is double of EBD by construction. The spiral does not coincide with the circle, for that would be absurd, nor does the spiral in its course get out of the circle before reaching point D. For, let the angle EBD be divided arbitrarily by the straight line BGH, which intercepts the spiral in G and the circumference in H. Then the line BD will be to the line BG as the angle EBD to the angle EBG; that is, as the arc BD to the arc BH, according to the conditions of spirals . But the ratio of the arc BD to the arc BH is greater than the ratio of the cord BD to the cord BH. For greater arcs have to lesser arcs a greater ratio than the straight lines to the straight lines that subtend those same arcs. Therefore the line BH is greater than BG, and the same will happen to any straight lines that divide the angle EBD. Therefore the spiral will proceed under the arc BD and will leave some area between itself and the arc. By that area the segment of the circle contained between the line BD and the circumference exceeds the area which is enclosed between the same line and the spiral, and which is proved equal to one-sixth of the sector FBD. Therefore that segment will be greater than one-sixth of the sector FBD. Which was to be proved.

Corollary. *And from this it is also obvious that thirty-six segments of the hexagon are greater than the circle.*

For when BD happens to be a segment of the hexagon, the sectors FBD and BAD will be equal since their circles' half diameters BD [and] AD will be equal. Thus six segments of the hexagon will be greater than the sector BAD, and therefore thirty-six segments greater than six sectors, that is, the whole circle. It is possible to propose a no less general theorem to be proved by parabolas, or rather by the same geometrical methods through which the parabola is squared: Any segment of the circle is greater than four-thirds of the isosceles triangle inscribed in the segment with the same base. By that, it will soon appear that the ratio of thirty-six segments of the hexagon to the circle is greater than 48 to 47. But to an even more accurate calculator only thirty four segments and an area little greater than two-thirds but less than three-quarters of a segment can be found to complete the circle. It is possible in this way to make known to the eyes that the excess is greater than one-third of a twelfth.

Proposition 7. *In a given circle to cut off the thirty-sixth part of the circle itself from a segment of the hexagon.*

Let a circle be given with centre A, diameter BC and a segment of the hexagon BD. It is required, in the given circle BDC from the segment of the hexagon contained by the cord BD and the arc under which it is suspended, to cut off the thirty-sixth part of the circle itself. Let the straight line BE be tangent to the circle, and let a spiral line be described, with origin B and passage through D, so that BD is the same part of the first revolution BEZ as the angle EBD is of four right angles, and with centre B and radius BD let the circle DE be described. Then the third part of the sector EBD is the area contained by the line BD and the spiral. But BD is equal to the half diameter BA, for BD is the side of the hexagon by hypothesis, and the angle EBD is one-third of a right angle because the size of arc BD is two-thirds of a right angle. Thus the sector EBD is a twelfth of the circle, and consequently the spiral area BD is a third of the twelfth, that is a thirty-sixth part of the circle. The spiral will pass through the segment and will not meet the circle, (nor will it cut off a circular area

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on its way from B) until it reaches the point D, as has been proved. Thus, in the given circle BDC the thirty-sixth part of the circle itself has been cut off from the hexagon's segment BD. Which was to be done.

And by this sevenfold shield let the blade of the soft and blunt axe have been weakened enough.

If some should want a sketch of THE FIGHT OF THE AXE itself, lest they miss it let them study it in a few short pages.

ANALYSIS OF THE CIRCLE, according to the AXE-SWINGERS.

1. The circle consists of six scalpels of the hexagon.

2. The scalpel of the hexagon consists of [i.e. is equal to] the segment of the hexagon and the triangle of the hexagon, or the (so-called) ''major'.

3. The triangle of the hexagon consists of a segment of the hexagon and a hatchet.

4. The hatchet consists of two segments of the hexagon and the complement of the hatchet.

5. The complement of the hatchet consists of one segment of the hexagon and the remainder of a segment.

6. But on the other hand, the complement of the hatchet consists of the minor triangle and the remainder of the minor triangle. The minor triangle is [by definition] the fifth part of the hexagon, the so-called 'major'.

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TWO TRUE LEMMAS, First: *Ten minor triangles are equal to six segments of the hexagon and two complements of the hatchet.* For, since [“Quorum” in text should be “Quoniam”, a common misreading] the hexagon’s triangle consists of a segment and a hatchet, and the hatchet of two segments and a complement, therefore two triangles of the hexagon consist of six segments and two complements. But two triangles of the hexagon, or the major, are equal to ten minor [triangles] (by definition). Therefore ten minor triangles will be equal to six segments and two complements. What was to be proved.
Second: *Forty minor triangles are equal to the triangle and two complements of the hatchet.*

For, since the circle equals six scalpels of the hexagon, but six scalpels are equal to six triangles of the hexagon and six segments, and further six triangles of the hexagon make thirty minor triangles, therefore the circle equals thirty minor triangles and six segments. Let two complements of the hatchet be added to both sides.

Thus the circle plus two complements of the hatchet will be equal to thirty minor triangles and six segments and two complements. But six segments and two complements make ten minor triangles, according to the previous lemma. Therefore forty minor triangles are equal to six segments of the hexagon and two complements of the hatchet. Which was to be proven.

FALLACY: I say that the minor triangle is equal to its remainder. By way of proof: Since the circle plus two complements of the hatchet (the latter being equal to two minor triangles plus two remainders of the minor triangle) is equal to thirty six minor triangles and another four, let from both sides [of the equation] be taken away two minor triangles.

When from this side they are subtracted from two complements, two remainders of the [minor] triangle are left. When from that side they are subtracted from four triangles, two triangles are left. Therefore two remainders equal two triangles.

Refutation of the FAULTINESS OF LOGIC. Equals must be subtracted from equal wholes, not from equal parts, to make the remainders equal. To subtract something from a part of equals is to assume that the remainder[s] of the whole[s] are equal, as here the circle is set equal to thirty-six minor triangles. But that is flatly denied and totally false. To grant oneself what should be proved looks as if one wants to show how to make an error.

FOR ANOTHER FALLACY, TWO TRUE LEMMAS. First: *Twenty-four quarters of the hexagon’s triangle plus six segments are equal to twenty-four segments and six complements of the hatchet.*

For, since the triangle of the hexagon consists of three segments and a complement of the hatchet, but the circle is composed of six triangles and six segments, therefore twenty-four segments and six complements are equal to the circle.

And since four quarters make one whole, also twenty-four quarters of the hexagon 's triangle plus six segments are equal to the circle. But things that are equal to one [and the same] are equal to one another. Therefore twenty-four quarters of the hexagon's triangle plus six segments are equal to twenty-four segments and six complements of the hatchet. Which was to be proved.

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Second. *If there are three unequal magnitudes, of which the middle taken twenty-four times and added to the least taken six times produces the same magnitude as the least taken twenty-four times and added to the greatest taken six times, then the difference between four times the middle and thrice the least will be equal to the greatest.* For, let B be the least, D the middle, A the greatest one. By hypothesis, then, 6 B plus 24 D = 6 A plus 24 B. Let 24 B be subtracted from both sides. Then 24 D minus 18 B = 6 A. If all be divided by 6, 4 D minus 3 B = A. That is exactly what was stated.

UNPROVED THEOREM. *There are three unequal plane figures that are commensurable between them; the smallest is the segment of the hexagon; the middle is a quarter of the triangle of the hexagon; the greatest is the complement of the hatchet of the hexagon.* Inequality and the degree of inequality could be proved, but nobody will ever prove commensurability and incommensurability unless he first has compared the hexagon's triangle or another rectilineal figure with the circle. But that comparison is unknown hitherto, and IF IT IS FEASIBLE, IT IS IN THE LAP OF THE GODS.

PSEUDO-PORISM. Such parts of which a quarter of the hexagon's triangle will be five, of the same parts the segment must necessarily be four.

BY WAY OF PROOF. For, let B be a segment of the hexagon, and D a quarter of the hexagon's triangle, and Z the complement of the hatchet. Now, since there are three unequal magnitudes, of which B is the least, D the middle, and Z the greatest, [therefore] they will have to one another [a ratio] as a number to a number. Suppose that D is five parts, and that B is three or four parts, and no more. Let it be, if it is possible, three parts; according to the first and second lemma, Z will be eleven. So the complement will consist of two segments and a twelfth of a segment, but that contradict our senses. Therefore B is four.

Refutation of the FAULTINESS OF LOGIC. If the magnitude D is five parts, it can be proved that B is greater than three parts. But will B therefore be four, even admitted-what is unknown-that B is to D as a number to a number? That conclusion is totally invalid. For, what if B is said to be four parts plus some rational fraction? Is not four and a half to five as a number to a number, that is, as 9 to 10? Nobody but an NON-LOGICIAN OR NON-GEOMETER will deny that. In fact, if D is set to 11 parts, B becomes a bit greater than 9, and Z a little less than 17, according to Archimedes' limits. However, from these two FALLACIES were spread the other AXEFIGHTERS' ERRORS CONCERNING THE AREA OF THE CIRCLE AND THE SURFACE OF THE SPHERE.

END OF THE FIGHT AGAINST THE AXE-SWINGERS.

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The second FIGHT AGAINST THE AXE-SWINGERS. An outline, FROM THE ADDENDUM.

In a circle with centre A let an arc of the hexagon BCD be taken, and let AB, AD, BD be joined. Further, from AB let a line segment be cut off whose square is to the square on AB as one to five. Let it be BE, and through E let a parallel to AD be drawn, intersecting BD in F. Then the triangle BEF will be equiangular with BAD and one-fifth of it.

Lemma 1. True. *Thirty-seven triangles BEF are greater than the circle BCD.*

For, in the comments to the *Canon Mathematicus,* the circle is shown to have a ratio to the square on the half diameter that is very close to 31,415,926,536 to 10,000,000,000. If, therefore, the side AB, that is the half diameter, is set to 100,000, the height of the equilateral triangle ABD is $86,602[+]54,038/100,000$. Therefore:

The triangle ABD becomes 4,330,127,019. The triangle BEF becomes 866,025,404. Thirty-seven triangles BEF 32,042,939,948 [The circle] 31,415,926,536 [The triangles] exceed the circle by 617,013,412.

Lemma 2. True. *The circle BCD is not greater than thirty-six segments BCDF.*

Even more, the circle BCD is far less than thirty-six segments BCDF. For the sector BAD is the sixth part of the whole circle. Therefore

The circle is 31,415,926,536. The sector BAD is 5,235,987,756. Let the triangle ABD be subtracted from it 4,330,127,019. There remains the hexagon sector or the *mixtiline* space BCDF 905,860,737. But three dozens of such segments are 32,610,986,532 exceeding the circle by 1,195,059,996.

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PSEUDO-PORISM. Consequently thirty-seven triangles BEF are greater than thirty-six segments BCDF. Refutation of the false conclusion. In grammar, to give to the ships the south winds, and to give the ships to the south winds, mean the same thing. But in geometry, it is one thing to assume that the circle BCD is not greater than thirty-six segments BCDF and another to assume that thirty-six segments are not greater than the circle BCD. The first is true, the second is false. Then, when I argue in this way:

Thirty-seven triangles are greater than the circle, but thirty-six segments are not greater than the circle, therefore thirty-seven triangles are greater than thirty-six segments,

I conclude syllogistically, but wrongly, because the assumption is false. But I sin against the laws of logic when I establish the syllogism in this formula:

The circle is smaller than thirty-seven triangles. The circle is not greater than thirty-six segments. Therefore thirty-seven triangles are greater than thirty-six segments.

But this is AN ERROR OF THE EYES, not of the INTELLECT. For, when at the beginning the Circle-measurers had proposed that the circle is not greater than thirty-six segments of the hexagon, they read it in the light of subsequent events as 'not less than', and thus extracted the false corollary.

END

3.3 Christian Huygens. Problem IV, Proposition XX.

3.3.1 Determining the Magnitude of the Circle

To find the ratio between the circumference and the diameter; and, given chords in a given circle, to find the lengths of the arcs that they subtend.

Consider a circle of center D, with CB as a diameter, and let AB be an arc one-sixth of the circumference, for which we draw the chord AB and the sine AM. If we suppose then that the half-diameter DB is 100,000 parts, the chord BA will contain the same number. But AM will be made of 86,603 parts and not one less (which means that if we should take away one part or one unit of the 86,603 we would have less than what it should be), since it is half of the side of the equilateral triangle inscribed in the circle.

From there, the excess of AB over AM becomes 13397, less than the true value. One third of it is $4{,}465\frac{2}{3}$, which, added to the 100,000 of AB, gives $104{,}465\frac{2}{3}$, which is less than arc AB. And this is a first lower limit; in the following, we will find another one, closer to the real value. But first we must also find an upper limit, according to the same theorem.

Then a fourth proportional is to be found for three numbers. The first equals the double parts of AB and the triple of AM. It will then be 459,809, less than the real value, (since we also have to make sure that this number here is less;

and in the same way with the other, as we shall specify) the second is equal to the quadruple of AB and AM, which is 486,603, more than the real value. And the third is one-third of the excess of AB over AM, 4,466, more than the real value which, added to AB or 100,000 gives 104 ,727, larger than the number of parts that arc AB, a sixth of the periphery, contains [according to the above]. Then we already found the length of arc AB within an upper and lower limit, of which two the last is far closer to the real value because the number 104, 719 is closer to the real value.

But through these two, we will obtain another lower limit, more exact than the first one, using the following precept, which results from a more precise examination of the center of gravity .

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Add four-thirds of the difference of the above limits to the double of the chord and the triple of the sine, and the same ratio as between the line made this way and three and one-third, or $\frac{10}{3}$ times the sum of the sine and the chord, also exists between the excess of the chord over the sine and another line. This last one added to the sine will be a line smaller than the arc.

The lower limit was 104,465$\frac{2}{3}$; the upper one is 104,727; their difference is 261$\frac{1}{3}$. Again we need to find a fourth proportional to three numbers. The first one is the double of the parts of AB increased by the triple of AM and by four-thirds of the difference of the limits. We find 460,158, larger than the real value. The second is the $\frac{10}{3}$ of AB and AM taken together, 622,008, smaller than the real value. Last the third is the excess of AB over AM, 13,397, smaller than the real value. The fourth proportional to these numbers is 18,109, smaller than the real value.

Then if we add this to the number of parts of AM, 86,602$\frac{1}{2}$, less than the real value, we get 104, 711$\frac{1}{2}$, less than arc AB. Thus the sextuple of these parts, 628,269, will be less than the whole circumference. But because 104,727 of these parts were found larger than arc AB, their sextuple will be larger than the circumference. Thus the ratio of the circumference to the diameter is smaller than that of 628,362 to 200,000 and larger than that of 628,268 to 200,000, or smaller than that of 314,181 [to 100,000] and larger than that of 314,135 to 100,000. From that, the ratio is certainly smaller than $3\frac{1}{7}$ and larger than $3\frac{10}{71}$. From there also is refuted Longomontanus's mistake, who wrote that the periphery is larger than 314,185 parts, when the radius contains 100,000.

Let us suppose that arc AB is $\frac{1}{8}$ of the circumference; then AM, half of the side of the square inscribed in the circle, will measure 7,071,068 parts, and not one less, of which the radius DB measures 10,000,000. On the other hand, AB, side of the octagon, measures 7,653,668 parts and not one more. Through this data, we will find, in the same manner as above, as first lower limit of the length of arc AB 7,847,868. Then as upper limit 7,854,066. And from these two, again, a more precise lower limit 7,853,885. This results in the ratio of the periphery to the diameter being less than 31,416$\frac{1}{4}$, and more than 31,415, to 10,000.

And since the difference between the upper limit 7,854,066 and the real length of the arc

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is less than 85 parts (indeed, arc AB, according to what we proved above is larger than 7,853,981), and since 85 parts are less than two seconds, which is $\frac{2}{1{,}296{,}000}$ of the circumference, because the whole circumference has more than 60,000,000 parts, it is obvious that, if we try to find the angles of a right triangle using the given sides, the same way as we did for the upper limit above, the error can never be more than two seconds; even if the sides of the right angle are equal, as they were here in triangle DAM.

But if the ratio of side DM to MA is such that the angle ADM does not exceed a quarter of a right angle, the error will not be more than a third scruple . For, taking arc AB equal to $\frac{1}{10}$ of the circumference, AM will be half of the side of the equilateral octagon inscribed in the circle, and equal to 382,683,433 parts and not more; but, AB will be the side of the sixteen-gon and then will contain 390,180,644 parts, and not one more, with the radius DB containing 1,000,000,000 parts. In this way is found a first lower limit, of the length of arc AB, of 392,679,714 parts. And the upper limit is 392,699,148. And from there again a lower limit of 392,699,010. But, what was proved above results in arc AB, $\frac{1}{10}$ of the circumference, being larger than 392,699,081 parts, which the upper limit exceeds by 67 parts. But these are less than a third scruple, which is $\frac{1}{77{,}760{,}000}$ of the whole circumference , since the circumference is larger than 6,000,000,000.

Then, out of these new limits just found, the ratio of the circumference to the diameter will come smaller than 314,593$\frac{1}{6}$ to 1,000,000 but larger than 314,592 to 1,000,000.

And if we take an arc AB equal to $\frac{1}{60}$ of the circumference, which is six parts of the total 360, AM will be half of the side of the (inscribed) 30-gon, made of 10,452,846,326,766 parts, and not one less, when the radius has 100,000,000,000,000. And AB is the side of the (inscribed) 60-gon, 10,467,191,248,588 parts and not one more. Through this data is found arc AB, according to the first lower limit, 10,471,972,889,195. Then the upper limit 10,471,975,512,584. And from there the other lower limit, 10,471,975,511,302. This results in the ratio of the periphery to the diameter being less than

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31,415,926,538 to 10,000,000,000, but larger than 31,415,926,533 to 10,000,000,000.

If we had to find these limits through adding the sides of the inscribed and circumscribed polygons, we would have to go up to 400,000 sides . Because with the 60 angles inscribed and circumscribed polygons, we only prove that the ratio of the periphery to the diameter is less than 3,145 to 1,000, and larger than 3,140. Thus, the number of exact digits through this calculation seems to be three times higher, and even more. But if someone tries it, he will see that the same always happens with the following polygons [as well]; we know why but it would take a long explanation .

On the other hand, I believe it is clear enough how, for any other inscribed polygons, it is possible to find, through the above methods, the length of the arcs subtended. Because, if they are larger than the side of the inscribed square, we will have to find the length of the remaining arc on the half circumference, the chord of which is then also given. But we must also know how to find the chords of the half-arcs, when the chord of the full arc is given. And this way, if we want to use bisections, we will be able to find without any difficulty for any chord the length of its arc, as close as we want. This is useful for examining tables of sines, and even for their composition; because, knowing the chord of a given arc, we can determine with sufficient accuracy the length of the arc that is slightly larger or smaller.

Chapter 4

Demotica Cyclometrica

4.1 Irving Kaplansky's "Song of Pi"

The distinguished mathematician Irving Kaplansky (1917–) is also a fine musician, whose daughter Lucy is an accomplished folk singer. In February of 1973, Kaplansky composed a popular "Song about pi" of *Type 2* in the sense he describes in more detail in *More Mathematical People* [1, pp.121-122]. He found that out of 100 popular songs he surveyed, 70 were of Type 1 (a simple 'AABA' refrain), 20 were of the more complex Type 2, and ten were irregular. Half of songs in Woody Allen films are of Type 2.

Kaplansky wrote the Pi song to illustrate the superiority of type 2: "the idea being that you could take such unpromising material as the first fourteen digits of pi and make a passable song out of it if you used type 2." We reproduce his score here in Figures 4.1 to 4.5.

4.2 Ludolph van Ceulen's Tombstone

Ludolph van Ceulen (1540-1610) was the last to compute π seriously using Archimedes' method. He computed 39 digits with 35 correct in 1610 (published posthumously in 1615). He was sufficiently proud of this accomplishment that he had the number inscribed on his tombstone in Leiden. The tombstone vanished long ago though the number was and is still called Ludolph's number in parts of Europe. The tombstone was redesigned and rebuilt from surviving descriptions and sculpted by Cornelia Bakkum as reconsecrated July 5, 2000. (See also [11] for a mathematical paper written for the occasion.)

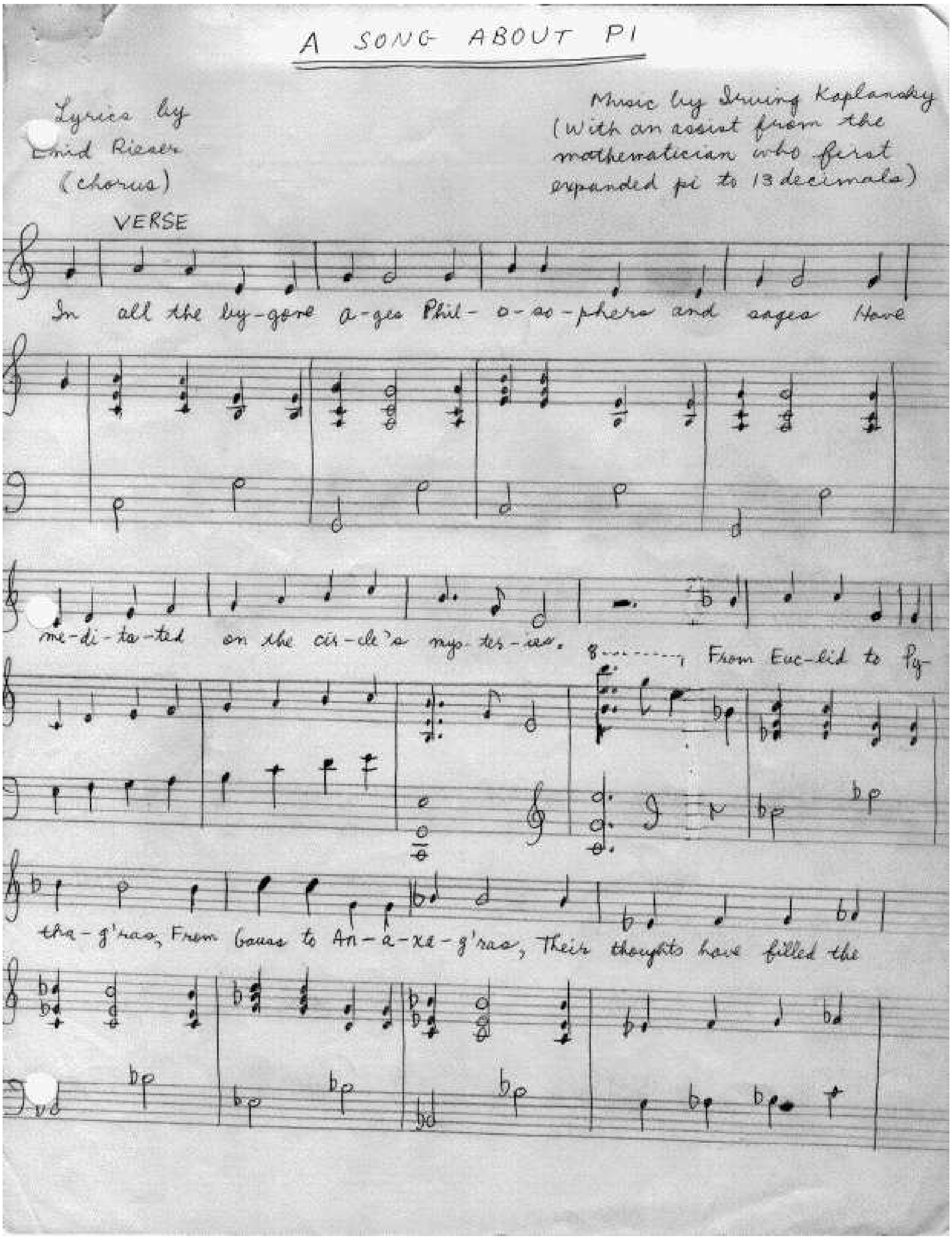

Figure 4.1: Kaplansky's "Song of Pi", page one

Figure 4.2: "Song of Pi", page two

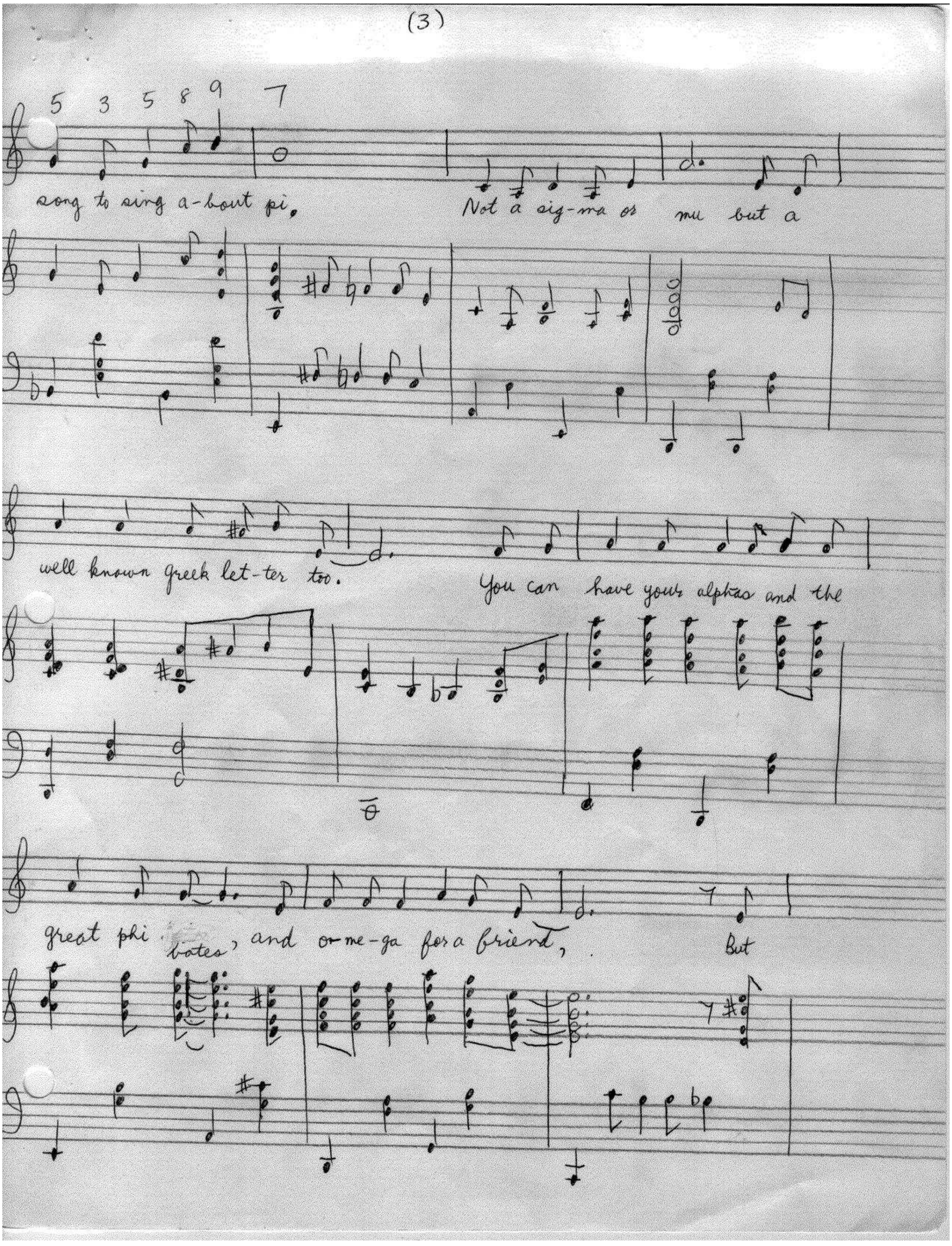

Figure 4.3: “Song of Pi”, page three

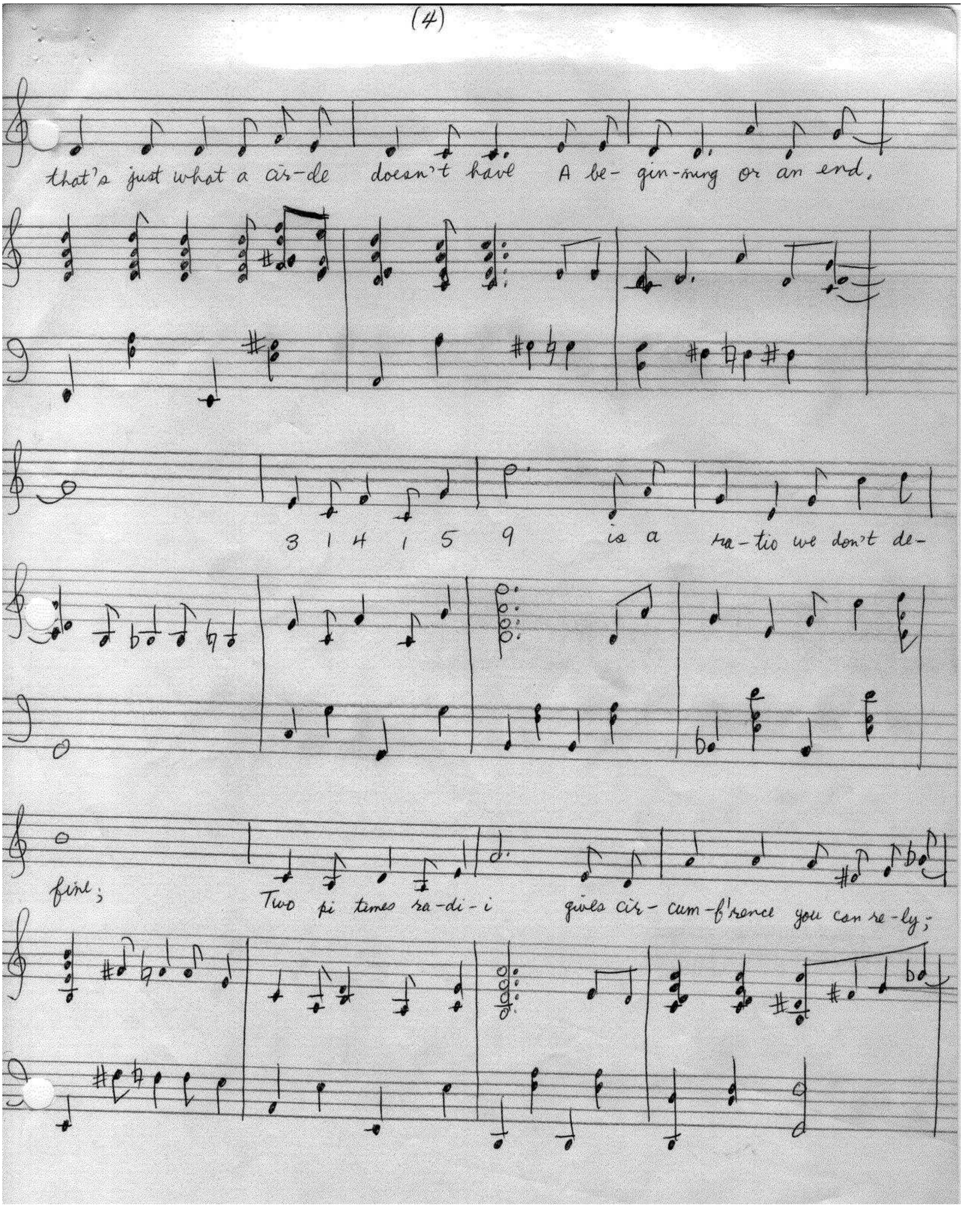

Figure 4.4: "Song of Pi", page four

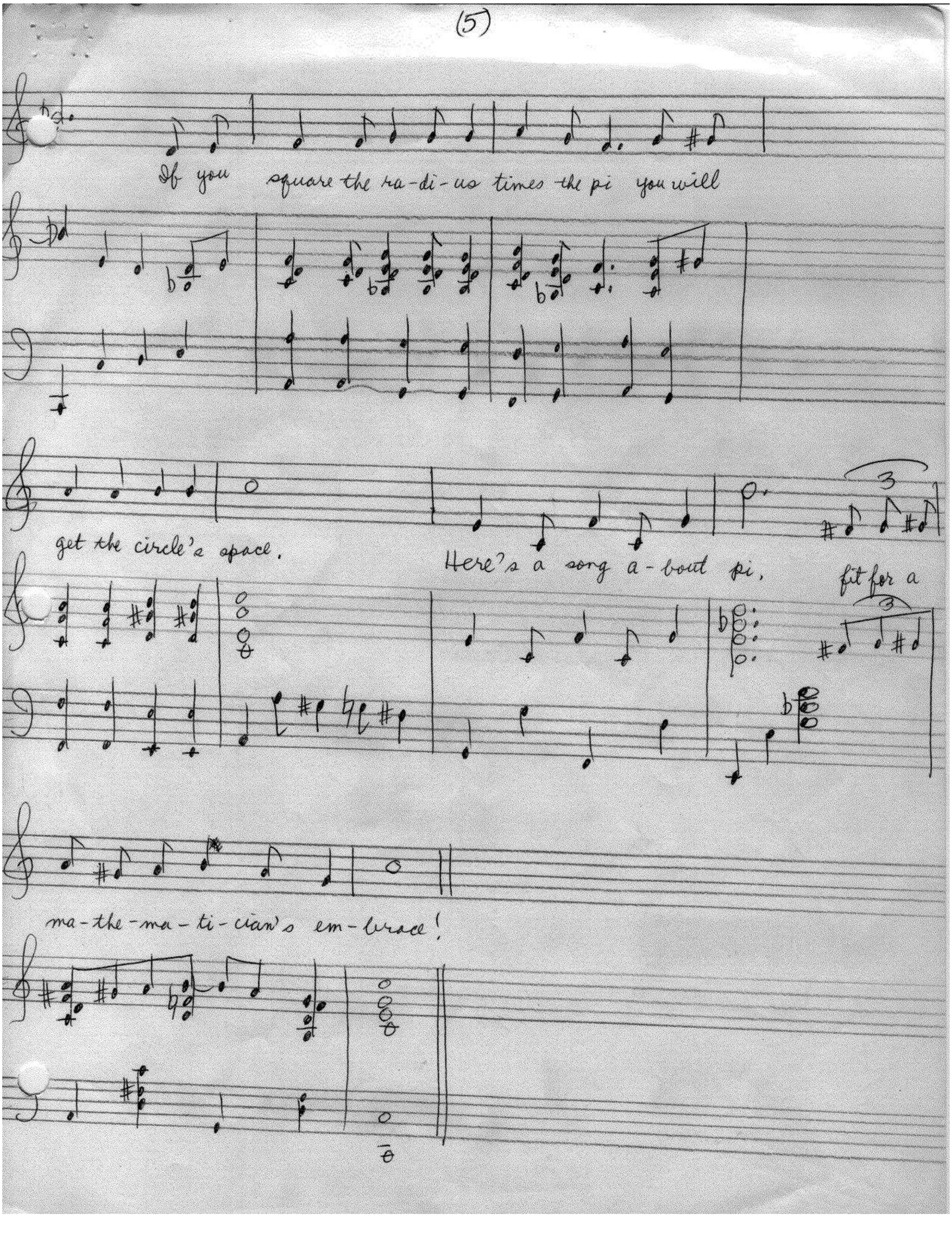

Figure 4.5: Kaplansky's "Song of Pi", page five

Figure 4.6: Ludolph's rebuilt tombstone

Bibliography

[1] Donald J. Albers, Gerald L. Alexanderson, and Constance Reid (Editors). *More Mathematical People: contemporary Conversations.* Academic Press, New York, 1997.

[2] Jorg Arndt and Christoph Haenel. *Pi Unleashed.* Springer, New York, 2001.

[3] David H. Bailey. The computation of pi to 29,360,000 decimal digits using Borweins' quartically convergent algorithm. *Mathematics of Computation*, 150:283–296, 1988.

[4] David H. Bailey. A Fortran-90 based multiprecision system. *ACM Transactions on Mathematical Software*, 21:379–387, 1995.

[5] David H. Bailey, Peter B. Borwein, and Simon Plouffe. On the rapid computation of various polylogarithmic constants. *Mathematics of Computation*, 66:903–913, 1997.

[6] David H. Bailey and Richard E. Crandall. On the random character of fundamental constant expansions. *Experimental Mathematics*, 10:175–190, 2001.

[7] David H. Bailey and Richard E. Crandall. Random generators and normal numbers. *Experimental Mathematics*, 11:527–546, 2003.

[8] Petr Beckmann. *A History of Pi.* St. Martin's Press, New York, 1971.

[9] L. Berggren, J.M. Borwein, and P.B. Borwein. *Pi: a Source Book (second edition).* Springer-Verlag, New York, 2000.

[10] Bruce C. Berndt and Robert A. Rankin (eds.). *Ramanujan: Essays and Surveys*, volume 22. AMS-LMS History of Mathematics Series, 2001.

[11] Frits Beukers. A rational approach to pi. *Nieuw Archief voor Wiskunde*, 5:372–379, 2000.

[12] Patrick Billingsley. *Probability and Measure.* John Wiley, New York, 1986.

[13] David Blatner. *The Joy of Pi.* Walker and Co., New York, 1997.

[14] Jonathan M. Borwein and David H. Bailey. *Experimentation in Mathematics: Computational paths to discovery.* A.K. Peters Ltd, 2003.

[15] Jonathan M. Borwein and David H. Bailey. *Mathematics by Experiment: Plausible reasoning in the 21st century.* A.K. Peters Ltd, 2003.

[16] Jonathan M. Borwein, David Borwein, and William F. Galway. Finding and excluding b-ary Machin-type BBP formulae. `http://www.cecm.sfu.ca/preprints/2002pp.html`, 2002.

[17] Jonathan M. Borwein and Peter B. Borwein. *Pi and the AGM: A study in analytic number theory and computational complexity.* CMS Series of Mongraphs and Advanced books in Mathematics, John Wiley & Sons, 1987.

[18] Jonathan M. Borwein, Peter B. Borwein, and David H. Bailey. Ramanujan, modular equations and pi or how to compute a billion digits of pi. *American Mathematical Monthly*, 96:201–219, 1989.

[19] Richard P. Brent. Fast multiple-precision evaluation of elementary functions. *Journal of the ACM*, 23:242–251, 1976.

[20] C. S. Calude, M. J. Dineen, and C.-K. Shu. Computing a glimpse of randomness. *Experimental Mathematics*, to appear, 2003.

[21] Helaman R. P. Ferguson, David H. Bailey, and Stephen Arno. Analysis of PSLQ, an integer relation finding algorithm. *Mathematics of Computation*, 68:351–369, 1999.

[22] Stephen Finch. *Mathematical Constants.* Cambridge University Press, Cambridge, 2003.

[23] Godfrey H. Hardy. *Ramanujan.* Chelsea, New York, 1978.

[24] Arend Heyting. *Intuitionism: An Introduction.* North-Holland, Amsterdam, 1956.

[25] Y. Kanada, Y. Ushiro, and H. Kuroda. The computation of π to 1,241,100,000,000 decimal digits. *to appear*, 2003.

[26] Donald E. Knuth. *The Art of Computer Programming*, volume 2. Addison-Wesley, New York, 1998.

[27] Joel L. Kraemer(ed.). *Perspectives on Maimonides: Philosophical and Historical Studies.* Oxford University Press, New York, 1991.

[28] L. Kuipers and H. Niederreiter. *Uniform Distribution of Sequences.* Wiley-Interscience, New York, 1974.

[29] Brian.M . Metzger and Roland E. Murphy. *The New Oxford Annotated Bible (New Revised Standard Edition).* Oxford University Press, New York, 1970.

[30] Roger Penrose. *The Emperor's New Mind*. American Philological Association, New York, 1989.

[31] S. D. Rabinowitz and S. Wagon. A spigot algorithm for pi. *American Mathematical Monthly*, 102:195–203, 1995.

[32] K. Roth. Rational approximations to algebraic numbers. *Mathematika*, 2:1–20, 1955.

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