Hölder and Kurokawa meet Borwein–Dykshoorn and Adamchik

J.-P. Allouche

CNRS, IMJ-PRG, Sorbonne 4 Place Jussieu F-75252 Paris Cedex 05, France jean-paul.allouche@imj-prg.fr

Abstract

Following our discovery of a nice identity in a recent preprint of Hu and Kim, we show a link between the Kurokawa multiple trigonometric functions and two functions introduced respectively by Borwein-Dykshoorn and by Adamchik. In particular several identities involving $\zeta(3)$, π and the Catalan constant G that are proved in these three papers are related.

2010 Mathematics Subject Classification: 11M06; 33B15; 11M35; 33E20.

Key words: Kurokawa multiple sine; Borwein-Dykshoorn function; Adamchik function; zeta values.

1 Introduction

There is a wealth of special functions arising from geometry and from transcendence theory. The purpose of this paper is to provide identities relating some of them. The beginning of the story here is a recent preprint of Hu and Kim [11] that gives the nice identity

$$\zeta(3) = \frac{4\pi^2}{21} \log\left(\frac{e^{\frac{4G}{\pi}} C_3\left(\frac{1}{4}\right)^{16}}{\sqrt{2}}\right).$$

where $G = \sum_{n\geq 0} \frac{(-1)^n}{(2n+1)^2}$ is the Catalan constant, and C_3 is the Kurokawa-Koyama triple cosine function (see below).

The (slightly hidden) occurrences of $7\zeta(3)/4\pi^2$ and of $e^{G/2\pi}$ reminded us two (out of four) identities in a paper of Kachi and Tzermias [12], namely:

$$\lim_{n \to \infty} \prod_{k=1}^{2n+1} e^{-1/4} \left(1 - \frac{1}{k+1} \right)^{\frac{k(k+1)}{2}(-1)^k} = \exp\left(\frac{7\zeta(3)}{4\pi^2} + \frac{1}{4}\right)$$

and

$$\lim_{n \to \infty} \prod_{k=1}^{2n+1} \left(1 - \frac{2}{2k+1} \right)^{k(-1)^k} = \exp\left(\frac{2G}{\pi} + \frac{1}{2}\right).$$

In that paper Kachi and Tzermias proved four identities, and they indicated that they were not able to deduce them directly from the values of a function introduced in 1993 by Borwein and Dykshoorn [4]. We provided in [2] a proof of their identities, using the paper of Borwein and Dykshoorn and their function D for two of the identities, and a paper of Adamchik [1] and his function E for the remaining two. It was thus tempting to relate these functions D and E to the Kurokawa-Koyama triple cosine.

Furthermore, looking at the papers of Kurokawa, we found an expression of $\zeta(3)$ resembling the identity given by Hu et Kim, namely (see [14, p. 62], also see [17, Theorem 2, p. 209])

$$\zeta(3) = \frac{8\pi^2}{7} \log\left(\frac{2^{1/4}}{\mathcal{S}_3(1/2)}\right)$$

where S_3 is the triple sine function of Kurokawa (in particular

$$S_3(1/2) = e^{1/8} \prod_{n \ge 1} \left(\left(1 - \frac{1}{4n^2} \right)^{n^2} e^{\frac{1}{4}} \right).$$

It was then even more tempting to study these multiple trigonometric functions and to (try to) find a link between all these results.

A possibly surprising fact is that the literature related to the functions of Borwein-Dykshoorn and of Adamchik appears to be essentially disjoint from the literature related to the Kurokawa multiple trigonometric functions, with the notable exception of the book [28] where several papers of Kurokawa et al. are cited in the references but do not seem to be exploited in the text. We propose to enlarge the bridge between these two branches of the theory of special functions.

In the sequel, we will recall the definitions of the multiple trigonometric functions first introduced by Kurokawa in [14, 15], then the definitions of the Borwein-Dykshoorn function D given in [4], and of the Adamchik function E given in [1]. We will obtain close relations between these functions. Furthermore we will show that an identity due to Holcombe [9] can also be obtained using multiple trigonometric functions. We will also mention a link between these functions and the generalized Euler constant function in [8] (also see [30] and [27]). We will end with two questions, one of which concerning an identity proved in [24], that we were not able to address with multiple trigonometric functions.

2 Multiple trigonometric functions

In 1991–1992 (see [14, 15]) Kurokawa introduced the multiple sine functions defined by $S_1(z) := 2\sin(\pi z)$, and for $r \ge 2$,

$$\mathcal{S}_r(z) := \exp\left(\frac{z^{r-1}}{r-1}\right) \prod_{n \ge 1} \left(P_r\left(\frac{z}{n}\right) P_r\left(-\frac{z}{n}\right)^{(-1)^{r-1}} \right)^{n^{r-1}}$$

where

$$P_r(z) = (1-z) \exp\left(z + \frac{z^2}{2} + \ldots + \frac{z^r}{r}\right).$$

Since $P_r(z) = P_{r-1}(z)e^{z^r/r}$, so that $e^{(z/n)^r} \cdot (e^{(-z/n)^r})^{(-1)^{r-1}} = 1$, we can clearly write, for $r \ge 2$,

$$S_r(z) := \exp\left(\frac{z^{r-1}}{r-1}\right) \prod_{n \ge 1} \left(P_{r-1}\left(\frac{z}{n}\right) P_{r-1}\left(-\frac{z}{n}\right)^{(-1)^{r-1}} \right)^{n^{r-1}}.$$

Note that in particular

$$S_2(z) = e^z \prod_{n \ge 1} \left(\left(\frac{1 - \frac{z}{n}}{1 + \frac{z}{n}} \right)^n e^{2z} \right)$$

is equal to the function F studied by Hölder in [10, Eq. (4), p. 515]. Note that S_r is equal to $1/\Lambda_r$ for r > 1 where Λ_r is the function defined by Rovinskii in 1991 (compare [14, (1) p. 62] and [25, (5) p. 74]). Also see [22] and [26].

We are not going to give more details in this section about multiple trigonometric functions, except for one of the motivations of Kurokawa, who writes in [14]: as an application we report the calculation of the "gamma factors" of Selberg-Gangolli-Wakayama zeta functions of rank one locally symmetric spaces, and for pointers to a limited number of references, e.g., [29] for an early approach of a related question, and the survey of Manin [20].

3 The Borwein-Dykshoorn function

The Borwein-Dykshoorn function

$$D(x) = \lim_{n \to \infty} \prod_{k=1}^{2n+1} \left(1 + \frac{x}{k} \right)^{k(-1)^{k+1}}$$

was introduced in [4] as a generalization of a result of Melzak [23] proving that

$$\lim_{n \to \infty} \prod_{k=1}^{2n+1} \left(1 + \frac{2}{k} \right)^{k(-1)^{k+1}} = \frac{\pi e}{2}.$$

This function can be extended to a meromorphic function on \mathbb{C} with poles at the negative even integers. We prove that this function is related to S_2 .

Theorem 1 The following equality holds:

$$\frac{D(x)}{D(-x)} = e^x \frac{\mathcal{S}_2(x/2)^4}{\mathcal{S}_2(x)}.$$

Proof. Let $D_n(x)$ be defined by

$$D_n(x) := \prod_{k=1}^{2n+1} \left(1 + \frac{x}{k}\right)^{k(-1)^{k+1}}.$$

We can write

$$D_n(x) = \prod_{j=1}^n \left(1 + \frac{x}{2j}\right)^{-2j} \prod_{j=0}^n \left(1 + \frac{x}{2j+1}\right)^{2j+1} = \prod_{j=1}^n \left(1 + \frac{x}{2j}\right)^{-2j} \frac{\prod_{k=1}^{2n+1} \left(1 + \frac{x}{k}\right)^k}{\prod_{k=1}^n \left(1 + \frac{x}{2k}\right)^{2k}} \cdot \frac{1}{\sum_{k=1}^n \left(1 +$$

Thus

$$D_n(x) = \frac{\prod_{k=1}^{2n+1} \left(1 + \frac{x}{k}\right)^k}{\prod_{j=1}^n \left(1 + \frac{x}{2j}\right)^{4j}}$$

which implies

$$\frac{D_n(x)}{D_n(-x)} = \prod_{k=1}^{2n+1} \left(\frac{1+\frac{x}{k}}{1-\frac{x}{k}}\right)^k \left(\prod_{k=1}^n \left(\frac{1-\frac{x/2}{k}}{1+\frac{x/2}{k}}\right)^k\right)^4.$$

Multiplying the first product by $e^{-2x(2n+1)}$ and the second product by $(e^{\frac{x}{2}}e^{nx})^4$ does not change the quantity $D_n(x)D_n(-x)$, thus

$$\frac{D_n(x)}{D_n(-x)} = \prod_{k=1}^{2n+1} \left(e^{-2x} \left(\frac{1+\frac{x}{k}}{1-\frac{x}{k}} \right)^k \right) \left(e^{\frac{x}{2}} \prod_{k=1}^n \left(e^x \left(\frac{1-\frac{x/2}{k}}{1+\frac{x/2}{k}} \right)^k \right) \right)^4$$

which gives, when n tends to infinity,

$$\frac{D(x)}{D(-x)} = e^x \frac{\mathcal{S}_2(x/2)^4}{\mathcal{S}_2(x)} \cdot \square$$

A corollary of this result gives the value of an infinite product studied in [12] which was proved again in [2] by using the Borwein–Dykshoorn function.

Corollary 1 The following identities hold

$$\lim_{n \to \infty} \prod_{k=1}^{2n} \left(1 - \frac{2}{2k+1} \right)^{k(-1)^k} = \exp\left(\frac{2G}{\pi} - \frac{1}{2}\right)$$

and

$$\lim_{n \to \infty} \prod_{k=1}^{2n+1} \left(1 - \frac{2}{2k+1} \right)^{k(-1)^k} = \exp\left(\frac{2G}{\pi} + \frac{1}{2}\right)$$

where $G = \sum_{n \ge 0} \frac{(-1)^n}{(2n+1)^2}$ is the Catalan constant.

Proof. Since $\left(1 - \frac{2}{4n+3}\right)^{-(2n+1)}$ tends to *e* (take the logarithm), it suffices to prove the second identity. We write

$$\prod_{k=1}^{2n+1} \left(1 - \frac{2}{2k+1}\right)^{k(-1)^k} = \prod_{k=1}^{2n+1} \left(\frac{2k+1}{2k-1}\right)^{k(-1)^{k+1}} = \prod_{k=1}^{2n+1} \left(1 + \frac{1}{2k}\right)^{k(-1)^{k+1}} \cdot \prod_{k=1}^{2n+1} \left(1 - \frac{1}{2k}\right)^{k(-1)^{k+1}} \cdot$$

Hence, using Theorem 1

$$\lim_{n \to \infty} \prod_{k=1}^{2n+1} \left(1 - \frac{2}{2k+1} \right)^{k(-1)^k} = \frac{D(1/2)}{D(-1/2)} = e^{1/2} \frac{S_2(1/4)^4}{S_2(1/2)}.$$

But we have $S_2(1/2) = 2^{1/2}$ and $S_2(1/4) = 2^{1/8} \exp\left(\frac{G}{2\pi}\right)$ (see [16, p. 852], hence

$$\lim_{n \to \infty} \prod_{k=1}^{2n+1} \left(1 - \frac{2}{2k+1} \right)^{k(-1)^k} = e^{1/2} \frac{\mathcal{S}_2(1/4)^4}{\mathcal{S}_2(1/2)} = \exp\left(\frac{2G}{\pi} + \frac{1}{2}\right) \cdot$$

Remark 1

• The function D can also be written (see, e.g., [2])

$$D(x) = e^{x} \prod_{k \ge 1} \left(e^{-x} \left(1 + \frac{x}{k} \right)^{k} \right)^{(-1)^{k+1}}$$

- The left hand term of the equality in Theorem 1, say f(x) := D(x)/D(-x), has the property that f(-x) = 1/f(x). Hence this is the same for the right hand term, but it is of course easy to see that the function $S_2(x)$ itself satisfies the identity $S_2(-x) = (S_2(x))^{-1}$ (see, e.g., [10, Eq.(5), p. 516]).
- Note that [4, Theorem, p. 204] gives an explicit (finite) formula in terms of the function Γ and of the generalized Gamma function Γ_1 defined in [3] and itself closely related to the Barnes function. There is also a close formula for D(x) in [1, Proposition 5, p. 284]. Also see [24, Example 20, p. 139]. Finally, an expression of D(x) in terms of the function γ_{α} and its derivative is given in [2, p. 86], where γ_{α} (see [8] and [27]; also see [30]) is defined by:

$$\gamma_{\alpha}(z) := \sum_{n \ge 1} z^{n-1} \left(\alpha - n \log \left(1 + \frac{\alpha}{n} \right) \right).$$

4 The Adamchik function E

Adamchik defined in [1] a function E by

$$E(x) := \lim_{n \to \infty} \prod_{k=1}^{2N} \left(1 - \frac{4x^2}{k^2} \right)^{k^2(-1)^{k+1}}.$$

It is easy to see that E(x) has also the following expression:

$$E(x) = \prod_{k \ge 1} \left(e^{4x^2} \left(1 - \frac{4x^2}{k^2} \right)^{k^2} \right)^{(-1)^{k+1}}.$$

Recall that $\mathcal{S}_3(x)$ is defined by

$$S_3(x) = e^{\frac{x^2}{2}} \prod_{n \ge 1} \left(e^{x^2} \left(1 - \frac{x^2}{n^2} \right)^{n^2} \right).$$

Also recall that the function $C_3(x)$ is defined by (see, e.g., [18, p. 1])

$$C_3(x) = \prod_{n \ge 1} \left(e^{x^2} \left(1 - \frac{4x^2}{(2n-1)^2} \right)^{\frac{(2n-1)^2}{4}} \right)$$

Now we state the following result.

Theorem 2 We have the following identity:

$$E(x) = e^{2x^2} \frac{S_3(2x)}{S_3(x)^8} = e^{2x^2} \frac{C_3(x)^8}{S_3(2x)}$$

Proof. We write

$$E(x) = \prod_{k \ge 1} \left(e^{4x^2} \left(1 - \frac{4x^2}{k^2} \right)^{k^2} \right)^{(-1)^{k+1}} = \frac{\prod_{\substack{k \ge 1 \\ k \text{ odd}}} \left(e^{4x^2} \left(1 - \frac{4x^2}{k^2} \right)^{k^2} \right)}{\prod_{k \ge 1} \left(e^{4x^2} \left(1 - \frac{x^2}{k^2} \right)^{4k^2} \right)} = \frac{\prod_{k \ge 1} \left(e^{4x^2} \left(1 - \frac{4x^2}{k^2} \right)^{k^2} \right)}{\prod_{k \ge 1} \left(e^{4x^2} \left(1 - \frac{x^2}{k^2} \right)^{4k^2} \right)^2}$$

Hence

$$E(x) = \left(S_3(2x)e^{-2x^2}\right) \left(S_3(x)e^{-x^2/2}\right)^{-8} = e^{2x^2} \frac{S_3(2x)}{S_3(x)^8}$$

Now (see, e.g., [18, Theorem 1.1, p. 123]),

$$\mathcal{S}_3(2x) = \mathcal{S}_3(x)^4 \mathcal{C}_3(x)^4,$$

which gives

$$E(x) = e^{2x^2} \frac{\mathcal{C}_3(x)^8}{\mathcal{S}_3(2x)} \qquad \Box$$

Remark 2

* The first equality in Theorem 2 above and Equation (2.15) in [16, p. 850] imply Corollary 1 in the paper of Adamchik [1, p. 286] where E(x) is expressed in terms of $\operatorname{Li}_2(\pm e^{2i\pi x})$ and $\operatorname{Li}_3(\pm e^{2i\pi x})$, up to replacing *i* with -i, which does not change the result since E(x) is real (recall that $\operatorname{Li}_k(x) := \sum_{n \ge 1} \frac{\sin nx}{n^k}$). It is also interesting to compare these expressions and [31, Proposition 2 p. 618]. * Note that [1, Proposition 6, p. 286] gives a close formula for E in terms of the Barnes function and of the generalized gamma function Γ_3 .

We have as a corollary the following particular cases given in [1, p. 287].

Corollary 2 We have

$$\lim_{N \to \infty} \prod_{k=1}^{2N} \left(1 - \frac{1}{4k^2} \right)^{k^2 (-1)^{k+1}} = \exp\left(\frac{1}{8} - \frac{2G}{\pi} + \frac{7\zeta(3)}{2\pi^2} \right)$$

and

$$\lim_{N \to \infty} \prod_{k=2}^{2N} \left(1 - \frac{1}{k^2} \right)^{k^2 (-1)^{k+1}} = -\frac{\pi}{4} \exp\left(\frac{1}{2} + \frac{7\zeta(3)}{\pi^2} \right)$$

Proof.

The left hand side of first equality is equal to E(1/4) thus, using Theorem 2, to $e^{1/8}C_3(1/4)^8S_3(1/2)^{-1}$. But $S_3(1/2)$ is given in [17, p. 206], where we find

$$\zeta(3) = \frac{8\pi^2}{7} \log(2^{1/4} A^{-1}), \text{ where } A := e^{1/8} \prod_{n \ge 1} \left(e^{1/4} \left(1 - \frac{1}{4n^2} \right)^{n^2} \right)$$

which can be written

$$S_3(1/2) = 2^{1/4} \exp\left(-7\frac{\zeta(3)}{8\pi^2}\right)$$

The value of $C_3(1/4)$ is given in [18, p. 1], where we find

$$C_3(1/4) = 2^{1/32} \exp\left(\frac{21\zeta(3)}{64\pi^2} - \frac{G}{4\pi}\right)$$

Here G is the Catalan constant. Note that there seems to be a misprint in [18], namely $G/4\pi$ there should be replaced with $-G/4\pi$ as above (also see [11]). So we finally have

$$E(1/4) = \exp\left(\frac{1}{8} - \frac{2G}{\pi} + \frac{7\zeta(3)}{2\pi^2}\right).$$

5 Holcombe's infinite product

In a 2013 paper [9] Holcombe proved that

$$\pi = e^{3/2} \prod_{n \ge 2} \left(e \left(1 - \frac{1}{n^2} \right)^{n^2} \right).$$

This result can be deduced from the value of the derivative at 1 of the triple trigonometric function S_3 . Namely

$$S_3(x) = e^{\frac{x^2}{2}} \prod_{n \ge 1} \left(e^{x^2} \left(1 - \frac{x^2}{n^2} \right)^{n^2} \right)$$

so that $\mathcal{S}_3(1) = 0$ and

$$\frac{\mathcal{S}_3(x) - \mathcal{S}_3(1)}{x - 1} = -e^{\frac{3}{2}}(1 + x) \prod_{n \ge 2} \left(e\left(1 - \frac{1}{n^2}\right)^{n^2} \right).$$

Letting x tend to 1, we thus have

$$S'_{3}(1) = -2e^{\frac{3}{2}} \prod_{n \ge 2} \left(e\left(1 - \frac{1}{n^{2}}\right)^{n^{2}} \right).$$

Using the value $S'_3(1) = -2\pi$ (see [18, Prop. 4.1, p. 125]) gives Holcombe's result.

6 Miscellanea

Further links between the above mentioned functions and other special functions can be found in the literature. We give only quick remarks in this section.

Recall that the dilogarithm function, already alluded to in Remark 2, and the Clausen function are respectively defined by

$$\operatorname{Li}_{2}(z) := \sum_{n \ge 1} \frac{z^{n}}{n^{2}} \quad \text{(where } |z| \le 1\text{), and} \quad \operatorname{Cl}_{2}(\theta) = \sum_{n \ge 1} \frac{\sin n\theta}{n^{2}} \cdot$$

The Clausen function Cl_2 is related to the Barnes function G (see, e.g., [28, Equation (2) p. 175]):

$$\operatorname{Cl}_2(x) = x \log \pi - x \log \left(\sin \left(\frac{x}{2} \right) \right) + 2\pi \log \frac{G(1 - \frac{x}{2\pi})}{G(1 + \frac{x}{2\pi})}$$

But we can find in [16] the relation

$$\mathcal{S}_2(z) = (2\sin\pi z)^z \exp\left(\frac{1}{2\pi} \sum_{n\geq 1} \frac{\sin(2\pi nz)}{n^2}\right)$$

hence

$$\mathcal{S}_2(z) = (2\sin\pi z)^z \exp\left(\frac{1}{2\pi} \operatorname{Cl}_2(2\pi z)\right).$$

Thus

$$\log \frac{G(1+t)}{G(1-t)} = t \log(2\pi) - \log S_2(t).$$
 (*)

This identity can also be obtained directly from a relation for $S_2(x)$ given in [16, Theorem 2.5, p. 847] and a relation given in [5, (2.1), p. 94], namely

$$\mathcal{S}_2(x) = \exp\left(\int_0^x \pi t \cot(\pi t) \, dt\right) \quad \text{and} \quad \int_0^x \pi t \cot(\pi t) dt = x \log(2\pi) + \log \frac{G(1-x)}{G(1+x)}$$

Relation (*) can be double checked with the values of $S_2(1/2) = \sqrt{2}$ and $S_2((1/4) = 2^{1/8}e^{G/2\pi}$ given in [16, Examples 2.9, p. 852] and the values of $G(3/2)/G(1/2) = \Gamma(1/2) = \sqrt{\pi}$ and $G(5/4)/G(3/4) = 2^{1/8}\pi^{1/4}e^{-G/2\pi}$ given in [21, p. 271]. This relation can also be used to give a closed form of values of $S_2(t)$ using known values of G(1+t)/G(1-t), e.g., for t = 1/8 or t = 1/12 (see [6, Section 5] and the corrections and results in [21, p. 272–273]).

Remark 3 In [5, p. 94] the authors write that the relation

$$\int_0^x \pi t \cot(\pi t) dt = x \log(2\pi) + \log \frac{G(1-x)}{G(1+x)}$$

is originally due to Kinkelin. Actually Kinkelin proves a similar result —see the first equality after (26.) p. 135 in [13]— for a function which is actually equal to $\Gamma(x)^{x-1}/G(x)$: as written in [7, p. 136] this function has been comparatively neglected by researchers in favor of G.

This discussion opens the way to other links with several other special functions (e.g., the dilogarithm already mentioned, the inverse tangent integral $\text{Ti}_2(z) := \sum_{j \ge 1} (-1)^{j+1} y^j / j^2$ and the Legendre chi-function $\chi_2(z) := \sum_{k\ge 0} y^{2k+1} / (2k+1)^2$, both defined for $|z| \le 1$) but we will not go further in this direction: the reader can consult in particular [6, 21] and the references therein, with a special mention for the book of Lewin [19].

7 Conclusion

We end this paper with two questions.

First, we note that combining the expression of D(x)/D(-x) given above in terms of $S_2(x/2)$ and $S_2(x)$ and the expression of D(x) in terms of the Hessami Pilehrood-Sondow-Hadjicostas function given in [2], one can obtain a relation between this function and $S_2(x)$. Is it possible to obtain a relation with the higher multiple trigonometric functions $S_r(x)$?

Second, an interesting identity is given in [24, p. 125] where it is proved by unusual methods:

$$\lim_{n \to \infty} \left[e^{\frac{n}{4}(4n+1)} n^{-\frac{1}{8}-n(n+1)} (2\pi)^{-\frac{n}{2}} \prod_{k=1}^{2n} \Gamma\left(1+\frac{k}{2}\right)^{k(-1)^k} \right] = 2^{\frac{1}{12}} \exp\left(\frac{5}{24} - \frac{3}{2}\zeta'(-1) - \frac{7\zeta(3)}{16\pi^2}\right)$$

where the right hand constant can also be written

$$(2e)^{1/12}A^{3/2}\exp\left(-\frac{7\zeta(3)}{16\pi^2}\right)$$

with $A := \exp(\frac{1}{12} - \zeta'(-1))$ is the Glaisher-Kinkelin constant. We did not succeed in finding a relation between this identity with multiple trigonometric functions. Of course the quantity $7\zeta(3)/4\pi^2$ occurs, but, after all, this is nothing but $-7\zeta'(-2)$ which looks somehow more mundane. Is it possible to find a proof of this identity that uses multiple trigonometric functions?

Acknowledgments We would like to thank S. Hu and M.-S. Kim for discussions about their preprint [11]. We would also like to thank Olivier Ramaré and Sanoli Gun, in particular for their help in obtaining some of the references.

References

- V. S. Adamchik, The multiple gamma function and its application to computation of series, *Ramanujan J.* 9 (2005) 271–288.
- [2] J.-P. Allouche, A note on products involving $\zeta(3)$ and Catalan's constant, Ramanujan J. **37** (2015), 79–88.
- [3] L. Bendersky, Sur la fonction gamma généralisée, Acta Math. 61 (1933), 263–322.
- [4] P. Borwein, W. Dykshoorn, An interesting infinite product, J. Math. Anal. Appl. 179 (1993) 203–207.
- [5] J. Choi, H. M. Srivastava, Certain classes of series involving the zeta function, J. Math. Anal. Appl. 231 (1999), 91–117.
- [6] J. Choi, Junesang, H. M. Srivastava, V. S. Adamchik, Multiple gamma and related functions, Appl. Math. Comput. 134 (2003), 515–533.
- [7] S. R. Finch, *Mathematical Constants*, Encyclopedia of Mathematics and its Applications 94, Cambridge University Press, Cambridge, 2003.
- [8] K. Hessami Pilehrood, T. Hessami Pilehrood, Vacca-type series for values of the generalized Euler constant function and its derivative, J. Int. Seq. 13 (2010) Article 10.7.3.
- [9] S. R. Holcombe, A product representation for π, Amer. Math. Monthly 120 (2013) 705. Longer version available at http://arxiv.org/abs/1204.2451.
- [10] O. Hölder, Ueber eine transcendente Function, Gött. Nachr. (1886), 514–522.

- [11] S. Hu, M.-S. Kim, Euler's integral, multiple cosine function and zeta values, Preprint (2022), https://arxiv.org/abs/2201.01124.
- [12] Y. Kachi, P. Tzermias, Infinite products involving $\zeta(3)$ and Catalan's constant, J. Int. Seq. 15 (2012) Article 12.9.4.
- [13] H. Kinkelin, Ueber eine mit der Gammafunction verwandte Transcendente und deren Anwendung auf die Integralrechnung, J. Reine Angew. Math. 57 (1860), 122–138.
- [14] N. Kurokawa, Multiple sine functions and Selberg zeta functions, Proc. Japan Acad. Ser. A Math. Sci. 67 (1991), 61–64.
- [15] N. Kurokawa, Multiple zeta functions: an example, in Zeta functions in geometry (Tokyo, 1990), 219– 226, Adv. Stud. Pure Math., 21, Kinokuniya, Tokyo, 1992.
- [16] N. Kurokawa, S.-y. Koyama, Multiple sine functions, Forum Math. 15 (2003), 839–876.
- [17] N. Kurokawa, M. Wakayama, On $\zeta(3)$, J. Ramanujan Math. Soc. 16 (2001), 205–214.
- [18] N. Kurokawa, M. Wakayama, Duplication formulas in triple trigonometry, Proc. Japan Acad. Ser. A Math. Sci. 79 (2003), 123–127.
- [19] L. Lewin, Polylogarithms and associated functions. With a foreword by A. J. Van der Poorten, North-Holland Publishing Co., New York-Amsterdam, 1981.
- [20] Y. Manin, Lectures on zeta functions and motives (according to Deninger and Kurokawa), Columbia University Number Theory Seminar (New York, 1992), Astérisque 228 (1995), 121–163.
- [21] L. Markov, Several Results Concerning the Barnes G-function, a Cosecant Integral, and Some Other Special Functions, in Advanced computing in industrial mathematics. 13th annual meeting of the Bulgarian Section of SIAM, December 18?20, 2018, Sofia, Bulgaria. Revised selected papers, I. Georgiev, H. Kostadinov, E. Lilkova (eds.), Studies in Computational Intelligence 961, Cham: Springer, p. 268–277.
- [22] Hj. Mellin, Eine Formel f
 ür den Logarithmus transcendenter Functionen von endlichem Geschlecht, Acta Math. 25 (1902), 165–183.
- [23] Z. A. Melzak, Infinite products for πe and π/e , Amer. Math. Monthly 68 (1961) 39–41.
- [24] M. Müller, D. Schleicher, Fractional sums and Euler-like identities, Ramanujan J. 21 (2010), 123–143.
- [25] M. Z. Rovinskii, Meromorphic functions connected with polylogarithms (in Russian), Funktsional. Anal. i Prilozhen. 25 (1991), 88–91; translation in Funct. Anal. Appl. 25 (1991), 74–77.
- [26] M. Rovinsky, Multiple gamma functions and L-functions, Math. Res. Lett. 3 (1996), 703–721.
- [27] J. Sondow, P. Hadjicostas, The generalized-Euler-constant function $\gamma(z)$ and a generalization of Somos's quadratic recurrence constant, J. Math. Anal. Appl. **332** (2007) 292–314.
- [28] H. M. Srivastava, J. Choi, Zeta and q-Zeta functions and associated series and integrals, Elsevier, Inc., Amsterdam, 2012.
- [29] M.-F. Vignéras, L'équation fonctionnelle de la fonction zêta de Selberg du groupe modulaire PSL(2, ℤ), Journées Arithmétiques de Luminy (1978), Astérisque 61, (1979), p. 235–249.
- [30] L.-M. Xia, The parameterized-Euler-constant function $\gamma_{\alpha}(z)$, J. Number Theory 133 (2013) 1–11.
- [31] D. Zagier, The Bloch-Wigner-Ramakrishnan polylogarithm function, Math. Ann. 286 (1990), 613–624.