# Hölder and Kurokawa meet Borwein-Dykshoorn and Adamchik 

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#### Abstract

Following our discovery of a nice identity in a recent preprint of Hu and Kim, we show a link between the Kurokawa multiple trigonometric functions and two functions introduced respectively by BorweinDykshoorn and by Adamchik. In particular several identities involving $\zeta(3), \pi$ and the Catalan constant $G$ that are proved in these three papers are related.


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## 1 Introduction

There is a wealth of special functions arising from geometry and from transcendence theory. The purpose of this paper is to provide identities relating some of them. The beginning of the story here is a recent preprint of Hu and Kim [11] that gives the nice identity

$$
\zeta(3)=\frac{4 \pi^{2}}{21} \log \left(\frac{e^{\frac{4 G}{\pi}} \mathcal{C}_{3}\left(\frac{1}{4}\right)^{16}}{\sqrt{2}}\right)
$$

where $G=\sum_{n \geq 0} \frac{(-1)^{n}}{(2 n+1)^{2}}$ is the Catalan constant, and $\mathcal{C}_{3}$ is the Kurokawa-Koyama triple cosine function (see below).

The (slightly hidden) occurrences of $7 \zeta(3) / 4 \pi^{2}$ and of $e^{G / 2 \pi}$ reminded us two (out of four) identities in a paper of Kachi and Tzermias [12], namely:

$$
\lim _{n \rightarrow \infty} \prod_{k=1}^{2 n+1} e^{-1 / 4}\left(1-\frac{1}{k+1}\right)^{\frac{k(k+1)}{2}(-1)^{k}}=\exp \left(\frac{7 \zeta(3)}{4 \pi^{2}}+\frac{1}{4}\right)
$$

and

$$
\lim _{n \rightarrow \infty} \prod_{k=1}^{2 n+1}\left(1-\frac{2}{2 k+1}\right)^{k(-1)^{k}}=\exp \left(\frac{2 G}{\pi}+\frac{1}{2}\right)
$$

In that paper Kachi and Tzermias proved four identities, and they indicated that they were not able to deduce them directly from the values of a function introduced in 1993 by Borwein and Dykshoorn [4]. We provided in [2] a proof of their identities, using the paper of Borwein and Dykshoorn and their function $D$ for two of the identities, and a paper of Adamchik [1] and his function $E$ for the remaining two. It was thus tempting to relate these functions $D$ and $E$ to the Kurokawa-Koyama triple cosine.

Furthermore, looking at the papers of Kurokawa, we found an expression of $\zeta(3)$ resembling the identity given by Hu et Kim, namely (see [14, p. 62], also see [17, Theorem 2, p. 209])

$$
\zeta(3)=\frac{8 \pi^{2}}{7} \log \left(\frac{2^{1 / 4}}{\mathcal{S}_{3}(1 / 2)}\right)
$$

where $\mathcal{S}_{3}$ is the triple sine function of Kurokawa (in particular

$$
\mathcal{S}_{3}(1 / 2)=e^{1 / 8} \prod_{n \geq 1}\left(\left(1-\frac{1}{4 n^{2}}\right)^{n^{2}} e^{\frac{1}{4}}\right)
$$

It was then even more tempting to study these multiple trigonometric functions and to (try to) find a link between all these results.

A possibly surprising fact is that the literature related to the functions of Borwein-Dykshoorn and of Adamchik appears to be essentially disjoint from the literature related to the Kurokawa multiple trigonometric functions, with the notable exception of the book [28] where several papers of Kurokawa et al. are cited in the references but do not seem to be exploited in the text. We propose to enlarge the bridge between these two branches of the theory of special functions.

In the sequel, we will recall the definitions of the multiple trigonometric functions first introduced by Kurokawa in [14, 15], then the definitions of the Borwein-Dykshoorn function $D$ given in [4], and of the Adamchik function $E$ given in [1]. We will obtain close relations between these functions. Furthermore we will show that an identity due to Holcombe [9] can also be obtained using multiple trigonometric functions. We will also mention a link between these functions and the generalized Euler constant function in [8] (also see [30] and [27]). We will end with two questions, one of which concerning an identity proved in [24], that we were not able to address with multiple trigonometric functions.

## 2 Multiple trigonometric functions

In 1991-1992 (see [14, 15]) Kurokawa introduced the multiple sine functions defined by $\mathcal{S}_{1}(z):=2 \sin (\pi z)$, and for $r \geq 2$,

$$
\mathcal{S}_{r}(z):=\exp \left(\frac{z^{r-1}}{r-1}\right) \prod_{n \geq 1}\left(P_{r}\left(\frac{z}{n}\right) P_{r}\left(-\frac{z}{n}\right)^{(-1)^{r-1}}\right)^{n^{r-1}}
$$

where

$$
P_{r}(z)=(1-z) \exp \left(z+\frac{z^{2}}{2}+\ldots+\frac{z^{r}}{r}\right)
$$

Since $P_{r}(z)=P_{r-1}(z) e^{z^{r} / r}$, so that $e^{(z / n)^{r}} \cdot\left(e^{(-z / n)^{r}}\right)^{(-1)^{r-1}}=1$, we can clearly write, for $r \geq 2$,

$$
\mathcal{S}_{r}(z):=\exp \left(\frac{z^{r-1}}{r-1}\right) \prod_{n \geq 1}\left(P_{r-1}\left(\frac{z}{n}\right) P_{r-1}\left(-\frac{z}{n}\right)^{(-1)^{r-1}}\right)^{n^{r-1}}
$$

Note that in particular

$$
\mathcal{S}_{2}(z)=e^{z} \prod_{n \geq 1}\left(\left(\frac{1-\frac{z}{n}}{1+\frac{z}{n}}\right)^{n} e^{2 z}\right)
$$

is equal to the function $F$ studied by Hölder in [10, Eq. (4), p. 515]. Note that $\mathcal{S}_{r}$ is equal to $1 / \Lambda_{r}$ for $r>1$ where $\Lambda_{r}$ is the function defined by Rovinskii in 1991 (compare [14, (1) p. 62] and [25, (5) p. 74]). Also see [22] and 26].

We are not going to give more details in this section about multiple trigonometric functions, except for one of the motivations of Kurokawa, who writes in [14]: as an application we report the calculation of the "gamma factors" of Selberg-Gangolli-Wakayama zeta functions of rank one locally symmetric spaces, and for pointers to a limited number of references, e.g., [29] for an early approach of a related question, and the survey of Manin [20].

## 3 The Borwein-Dykshoorn function

The Borwein-Dykshoorn function

$$
D(x)=\lim _{n \rightarrow \infty} \prod_{k=1}^{2 n+1}\left(1+\frac{x}{k}\right)^{k(-1)^{k+1}}
$$

was introduced in 4] as a generalization of a result of Melzak [23] proving that

$$
\lim _{n \rightarrow \infty} \prod_{k=1}^{2 n+1}\left(1+\frac{2}{k}\right)^{k(-1)^{k+1}}=\frac{\pi e}{2}
$$

This function can be extended to a meromorphic function on $\mathbb{C}$ with poles at the negative even integers. We prove that this function is related to $\mathcal{S}_{2}$.

Theorem 1 The following equality holds:

$$
\frac{D(x)}{D(-x)}=e^{x} \frac{\mathcal{S}_{2}(x / 2)^{4}}{\mathcal{S}_{2}(x)}
$$

Proof. Let $D_{n}(x)$ be defined by

$$
D_{n}(x):=\prod_{k=1}^{2 n+1}\left(1+\frac{x}{k}\right)^{k(-1)^{k+1}}
$$

We can write

$$
D_{n}(x)=\prod_{j=1}^{n}\left(1+\frac{x}{2 j}\right)^{-2 j} \prod_{j=0}^{n}\left(1+\frac{x}{2 j+1}\right)^{2 j+1}=\prod_{j=1}^{n}\left(1+\frac{x}{2 j}\right)^{-2 j} \frac{\prod_{k=1}^{2 n+1}\left(1+\frac{x}{k}\right)^{k}}{\prod_{k=1}^{n}\left(1+\frac{x}{2 k}\right)^{2 k}}
$$

Thus

$$
D_{n}(x)=\frac{\prod_{k=1}^{2 n+1}\left(1+\frac{x}{k}\right)^{k}}{\prod_{j=1}^{n}\left(1+\frac{x}{2 j}\right)^{4 j}}
$$

which implies

$$
\frac{D_{n}(x)}{D_{n}(-x)}=\prod_{k=1}^{2 n+1}\left(\frac{1+\frac{x}{k}}{1-\frac{x}{k}}\right)^{k}\left(\prod_{k=1}^{n}\left(\frac{1-\frac{x / 2}{k}}{1+\frac{x / 2}{k}}\right)^{k}\right)^{4}
$$

Multiplying the first product by $e^{-2 x(2 n+1)}$ and the second product by $\left(e^{\frac{x}{2}} e^{n x}\right)^{4}$ does not change the quantity $D_{n}(x) D_{n}(-x)$, thus

$$
\frac{D_{n}(x)}{D_{n}(-x)}=\prod_{k=1}^{2 n+1}\left(e^{-2 x}\left(\frac{1+\frac{x}{k}}{1-\frac{x}{k}}\right)^{k}\right)\left(e^{\frac{x}{2}} \prod_{k=1}^{n}\left(e^{x}\left(\frac{1-\frac{x / 2}{k}}{1+\frac{x / 2}{k}}\right)^{k}\right)\right)^{4}
$$

which gives, when $n$ tends to infinity,

$$
\frac{D(x)}{D(-x)}=e^{x} \frac{\mathcal{S}_{2}(x / 2)^{4}}{\mathcal{S}_{2}(x)}
$$

A corollary of this result gives the value of an infinite product studied in [12] which was proved again in [2] by using the Borwein-Dykshoorn function.

Corollary 1 The following identities hold

$$
\lim _{n \rightarrow \infty} \prod_{k=1}^{2 n}\left(1-\frac{2}{2 k+1}\right)^{k(-1)^{k}}=\exp \left(\frac{2 G}{\pi}-\frac{1}{2}\right)
$$

and

$$
\lim _{n \rightarrow \infty} \prod_{k=1}^{2 n+1}\left(1-\frac{2}{2 k+1}\right)^{k(-1)^{k}}=\exp \left(\frac{2 G}{\pi}+\frac{1}{2}\right)
$$

where $G=\sum_{n \geq 0} \frac{(-1)^{n}}{(2 n+1)^{2}}$ is the Catalan constant.
Proof. Since $\left(1-\frac{2}{4 n+3}\right)^{-(2 n+1)}$ tends to $e$ (take the logarithm), it suffices to prove the second identity. We write

$$
\prod_{k=1}^{2 n+1}\left(1-\frac{2}{2 k+1}\right)^{k(-1)^{k}}=\prod_{k=1}^{2 n+1}\left(\frac{2 k+1}{2 k-1}\right)^{k(-1)^{k+1}}=\frac{\prod_{k=1}^{2 n+1}\left(1+\frac{1}{2 k}\right)^{k(-1)^{k+1}}}{\prod_{k=1}^{2 n+1}\left(1-\frac{1}{2 k}\right)^{k(-1)^{k+1}}}
$$

Hence, using Theorem 1

$$
\lim _{n \rightarrow \infty} \prod_{k=1}^{2 n+1}\left(1-\frac{2}{2 k+1}\right)^{k(-1)^{k}}=\frac{D(1 / 2)}{D(-1 / 2)}=e^{1 / 2} \frac{\mathcal{S}_{2}(1 / 4)^{4}}{\mathcal{S}_{2}(1 / 2)}
$$

But we have $\mathcal{S}_{2}(1 / 2)=2^{1 / 2}$ and $\mathcal{S}_{2}(1 / 4)=2^{1 / 8} \exp \left(\frac{G}{2 \pi}\right)$ (see [16, p. 852], hence

$$
\lim _{n \rightarrow \infty} \prod_{k=1}^{2 n+1}\left(1-\frac{2}{2 k+1}\right)^{k(-1)^{k}}=e^{1 / 2} \frac{\mathcal{S}_{2}(1 / 4)^{4}}{\mathcal{S}_{2}(1 / 2)}=\exp \left(\frac{2 G}{\pi}+\frac{1}{2}\right)
$$

## Remark 1

- The function $D$ can also be written (see, e.g., [2])

$$
D(x)=e^{x} \prod_{k \geq 1}\left(e^{-x}\left(1+\frac{x}{k}\right)^{k}\right)^{(-1)^{k+1}}
$$

- The left hand term of the equality in Theorem 1 say $f(x):=D(x) / D(-x)$, has the property that $f(-x)=1 / f(x)$. Hence this is the same for the right hand term, but it is of course easy to see that the function $\mathcal{S}_{2}(x)$ itself satisfies the identity $\mathcal{S}_{2}(-x)=\left(\mathcal{S}_{2}(x)\right)^{-1}$ (see, e.g., [10, Eq.(5), p. 516]).
- Note that [4, Theorem, p. 204] gives an explicit (finite) formula in terms of the function $\Gamma$ and of the generalized Gamma function $\Gamma_{1}$ defined in [3] and itself closely related to the Barnes function. There is also a close formula for $D(x)$ in [1, Proposition 5, p. 284]. Also see [24, Example 20, p. 139]. Finally, an expression of $D(x)$ in terms of the function $\gamma_{\alpha}$ and its derivative is given in [2, p. 86], where $\gamma_{\alpha}$ (see [8] and [27]; also see (30]) is defined by:

$$
\gamma_{\alpha}(z):=\sum_{n \geq 1} z^{n-1}\left(\alpha-n \log \left(1+\frac{\alpha}{n}\right)\right) .
$$

## 4 The Adamchik function $E$

Adamchik defined in [1] a function $E$ by

$$
E(x):=\lim _{n \rightarrow \infty} \prod_{k=1}^{2 N}\left(1-\frac{4 x^{2}}{k^{2}}\right)^{k^{2}(-1)^{k+1}}
$$

It is easy to see that $E(x)$ has also the following expression:

$$
E(x)=\prod_{k \geq 1}\left(e^{4 x^{2}}\left(1-\frac{4 x^{2}}{k^{2}}\right)^{k^{2}}\right)^{(-1)^{k+1}}
$$

Recall that $\mathcal{S}_{3}(x)$ is defined by

$$
\mathcal{S}_{3}(x)=e^{\frac{x^{2}}{2}} \prod_{n \geq 1}\left(e^{x^{2}}\left(1-\frac{x^{2}}{n^{2}}\right)^{n^{2}}\right) .
$$

Also recall that the function $\mathcal{C}_{3}(x)$ is defined by (see, e.g., [18, p. 1])

$$
\mathcal{C}_{3}(x)=\prod_{n \geq 1}\left(e^{x^{2}}\left(1-\frac{4 x^{2}}{(2 n-1)^{2}}\right)^{\frac{(2 n-1)^{2}}{4}}\right)
$$

Now we state the following result.

Theorem 2 We have the following identity:

$$
E(x)=e^{2 x^{2}} \frac{\mathcal{S}_{3}(2 x)}{\mathcal{S}_{3}(x)^{8}}=e^{2 x^{2}} \frac{\mathcal{C}_{3}(x)^{8}}{\mathcal{S}_{3}(2 x)}
$$

Proof. We write

$$
E(x)=\prod_{k \geq 1}\left(e^{4 x^{2}}\left(1-\frac{4 x^{2}}{k^{2}}\right)^{k^{2}}\right)^{(-1)^{k+1}}=\frac{\prod_{\substack{k \geq 1 \\ k \text { odd }}}\left(e^{4 x^{2}}\left(1-\frac{4 x^{2}}{k^{2}}\right)^{k^{2}}\right)}{\prod_{k \geq 1}\left(e^{4 x^{2}}\left(1-\frac{x^{2}}{k^{2}}\right)^{4 k^{2}}\right)}=\frac{\prod_{k \geq 1}\left(e^{4 x^{2}}\left(1-\frac{4 x^{2}}{k^{2}}\right)^{k^{2}}\right)}{\prod_{k \geq 1}\left(e^{4 x^{2}}\left(1-\frac{x^{2}}{k^{2}}\right)^{4 k^{2}}\right)^{2}} .
$$

Hence

$$
E(x)=\left(\mathcal{S}_{3}(2 x) e^{-2 x^{2}}\right)\left(\mathcal{S}_{3}(x) e^{-x^{2} / 2}\right)^{-8}=e^{2 x^{2}} \frac{\mathcal{S}_{3}(2 x)}{\mathcal{S}_{3}(x)^{8}}
$$

Now (see, e.g., [18, Theorem 1.1, p. 123]),

$$
\mathcal{S}_{3}(2 x)=\mathcal{S}_{3}(x)^{4} \mathcal{C}_{3}(x)^{4},
$$

which gives

$$
E(x)=e^{2 x^{2}} \frac{\mathcal{C}_{3}(x)^{8}}{\mathcal{S}_{3}(2 x)}
$$

## Remark 2

* The first equality in Theorem 2 above and Equation (2.15) in [16, p. 850] imply Corollary 1 in the paper of Adamchik [1, p. 286] where $E(x)$ is expressed in terms of $L i_{2}\left( \pm e^{2 i \pi x}\right)$ and $L i_{3}\left( \pm e^{2 i \pi x}\right)$, up to replacing $i$ with $-i$, which does not change the result since $E(x)$ is real (recall that $\left.\operatorname{Li}_{k}(x):=\sum_{n \geq 1} \frac{\sin n x}{n^{k}}\right)$. It is also interesting to compare these expressions and [31, Proposition 2 p. 618].
* Note that [1, Proposition 6, p. 286] gives a close formula for $E$ in terms of the Barnes function and of the generalized gamma function $\Gamma_{3}$.
We have as a corollary the following particular cases given in [1] p. 287].
Corollary 2 We have

$$
\lim _{N \rightarrow \infty} \prod_{k=1}^{2 N}\left(1-\frac{1}{4 k^{2}}\right)^{k^{2}(-1)^{k+1}}=\exp \left(\frac{1}{8}-\frac{2 G}{\pi}+\frac{7 \zeta(3)}{2 \pi^{2}}\right)
$$

and

$$
\lim _{N \rightarrow \infty} \prod_{k=2}^{2 N}\left(1-\frac{1}{k^{2}}\right)^{k^{2}(-1)^{k+1}}=\frac{\pi}{4} \exp \left(\frac{1}{2}+\frac{7 \zeta(3)}{\pi^{2}}\right)
$$

Proof.
The left hand side of first equality is equal to $E(1 / 4)$ thus, using Theorem [2 to $e^{1 / 8} \mathcal{C}_{3}(1 / 4)^{8} \mathcal{S}_{3}(1 / 2)^{-1}$. But $\mathcal{S}_{3}(1 / 2)$ is given in [17, p. 206], where we find

$$
\zeta(3)=\frac{8 \pi^{2}}{7} \log \left(2^{1 / 4} A^{-1}\right), \text { where } A:=e^{1 / 8} \prod_{n \geq 1}\left(e^{1 / 4}\left(1-\frac{1}{4 n^{2}}\right)^{n^{2}}\right)
$$

which can be written

$$
\mathcal{S}_{3}(1 / 2)=2^{1 / 4} \exp \left(-7 \frac{\zeta(3)}{8 \pi^{2}}\right) .
$$

The value of $\mathcal{C}_{3}(1 / 4)$ is given in [18, p. 1], where we find

$$
\mathcal{C}_{3}(1 / 4)=2^{1 / 32} \exp \left(\frac{21 \zeta(3)}{64 \pi^{2}}-\frac{G}{4 \pi}\right) .
$$

Here $G$ is the Catalan constant. Note that there seems to be a misprint in [18, namely $G / 4 \pi$ there should be replaced with $-G / 4 \pi$ as above (also see [11). So we finally have

$$
E(1 / 4)=\exp \left(\frac{1}{8}-\frac{2 G}{\pi}+\frac{7 \zeta(3)}{2 \pi^{2}}\right) .
$$

## 5 Holcombe's infinite product

In a 2013 paper [9] Holcombe proved that

$$
\pi=e^{3 / 2} \prod_{n \geq 2}\left(e\left(1-\frac{1}{n^{2}}\right)^{n^{2}}\right) .
$$

This result can be deduced from the value of the derivative at 1 of the triple trigonometric function $\mathcal{S}_{3}$. Namely

$$
\mathcal{S}_{3}(x)=e^{\frac{x^{2}}{2}} \prod_{n \geq 1}\left(e^{x^{2}}\left(1-\frac{x^{2}}{n^{2}}\right)^{n^{2}}\right)
$$

so that $\mathcal{S}_{3}(1)=0$ and

$$
\frac{\mathcal{S}_{3}(x)-\mathcal{S}_{3}(1)}{x-1}=-e^{\frac{3}{2}}(1+x) \prod_{n \geq 2}\left(e\left(1-\frac{1}{n^{2}}\right)^{n^{2}}\right) .
$$

Letting $x$ tend to 1 , we thus have

$$
\mathcal{S}_{3}^{\prime}(1)=-2 e^{\frac{3}{2}} \prod_{n \geq 2}\left(e\left(1-\frac{1}{n^{2}}\right)^{n^{2}}\right) .
$$

Using the value $\mathcal{S}_{3}^{\prime}(1)=-2 \pi$ (see [18, Prop. 4.1, p. 125]) gives Holcombe's result.

## 6 Miscellanea

Further links between the above mentioned functions and other special functions can be found in the literature. We give only quick remarks in this section.

Recall that the dilogarithm function, already alluded to in Remark 2, and the Clausen function are respectively defined by

$$
\mathrm{Li}_{2}(z):=\sum_{n \geq 1} \frac{z^{n}}{n^{2}}(\text { where }|z| \leq 1), \text { and } \mathrm{Cl}_{2}(\theta)=\sum_{n \geq 1} \frac{\sin n \theta}{n^{2}}
$$

The Clausen function $\mathrm{Cl}_{2}$ is related to the Barnes function $G$ (see, e.g., [28, Equation (2) p. 175]):

$$
\mathrm{Cl}_{2}(x)=x \log \pi-x \log \left(\sin \left(\frac{x}{2}\right)\right)+2 \pi \log \frac{G\left(1-\frac{x}{2 \pi}\right)}{G\left(1+\frac{x}{2 \pi}\right)}
$$

But we can find in [16] the relation

$$
\mathcal{S}_{2}(z)=(2 \sin \pi z)^{z} \exp \left(\frac{1}{2 \pi} \sum_{n \geq 1} \frac{\sin (2 \pi n z)}{n^{2}}\right)
$$

hence

$$
\mathcal{S}_{2}(z)=(2 \sin \pi z)^{z} \exp \left(\frac{1}{2 \pi} \mathrm{Cl}_{2}(2 \pi z)\right)
$$

Thus

$$
\begin{equation*}
\log \frac{G(1+t)}{G(1-t)}=t \log (2 \pi)-\log \mathcal{S}_{2}(t) \tag{*}
\end{equation*}
$$

This identity can also be obtained directly from a relation for $\mathcal{S}_{2}(x)$ given in [16, Theorem 2.5, p. 847] and a relation given in [5, (2.1), p. 94], namely

$$
\mathcal{S}_{2}(x)=\exp \left(\int_{0}^{x} \pi t \cot (\pi t) d t\right) \quad \text { and } \quad \int_{0}^{x} \pi t \cot (\pi t) d t=x \log (2 \pi)+\log \frac{G(1-x)}{G(1+x)}
$$

Relation (*) can be double checked with the values of $\mathcal{S}_{2}(1 / 2)=\sqrt{2}$ and $\mathcal{S}_{2}\left((1 / 4)=2^{1 / 8} e^{G / 2 \pi}\right.$ given in [16, Examples 2.9, p. 852] and the values of $G(3 / 2) / G(1 / 2)=\Gamma(1 / 2)=\sqrt{\pi}$ and $G(5 / 4) / G(3 / 4)=2^{1 / 8} \pi^{1 / 4} e^{-G / 2 \pi}$ given in [21, p. 271]. This relation can also be used to give a closed form of values of $\mathcal{S}_{2}(t)$ using known values of $G(1+t) / G(1-t)$, e.g., for $t=1 / 8$ or $t=1 / 12$ (see [6, Section 5] and the corrections and results in [21, p. 272-273]).

Remark 3 In [5, p. 94] the authors write that the relation

$$
\int_{0}^{x} \pi t \cot (\pi t) d t=x \log (2 \pi)+\log \frac{G(1-x)}{G(1+x)}
$$

is originally due to Kinkelin. Actually Kinkelin proves a similar result - see the first equality after (26.) p. 135 in [13] - for a function which is actually equal to $\Gamma(x)^{x-1} / G(x)$ : as written in [7] p. 136] this function has been comparatively neglected by researchers in favor of $G$.

This discussion opens the way to other links with several other special functions (e.g., the dilogarithm already mentioned, the inverse tangent integral $\mathrm{Ti}_{2}(z):=\sum_{j \geq 1}(-1)^{j+1} y^{j} / j^{2}$ and the Legendre chi-function $\chi_{2}(z):=\sum_{k \geq 0} y^{2 k+1} /(2 k+1)^{2}$, both defined for $\left.|z| \leq 1\right)$ but we will not go further in this direction: the reader can consult in particular [6, 21] and the references therein, with a special mention for the book of Lewin [19].

## 7 Conclusion

We end this paper with two questions.
First, we note that combining the expression of $D(x) / D(-x)$ given above in terms of $\mathcal{S}_{2}(x / 2)$ and $\mathcal{S}_{2}(x)$ and the expression of $D(x)$ in terms of the Hessami Pilehrood-Sondow-Hadjicostas function given in [2], one can obtain a relation between this function and $\mathcal{S}_{2}(x)$. Is it possible to obtain a relation with the higher multiple trigonometric functions $\mathcal{S}_{r}(x)$ ?

Second, an interesting identity is given in [24, p. 125] where it is proved by unusual methods:

$$
\lim _{n \rightarrow \infty}\left[e^{\frac{n}{4}(4 n+1)} n^{-\frac{1}{8}-n(n+1)}(2 \pi)^{-\frac{n}{2}} \prod_{k=1}^{2 n} \Gamma\left(1+\frac{k}{2}\right)^{k(-1)^{k}}\right]=2^{\frac{1}{12}} \exp \left(\frac{5}{24}-\frac{3}{2} \zeta^{\prime}(-1)-\frac{7 \zeta(3)}{16 \pi^{2}}\right)
$$

where the right hand constant can also be written

$$
(2 e)^{1 / 12} A^{3 / 2} \exp \left(-\frac{7 \zeta(3)}{16 \pi^{2}}\right)
$$

with $A:=\exp \left(\frac{1}{12}-\zeta^{\prime}(-1)\right)$ is the Glaisher-Kinkelin constant. We did not succeed in finding a relation between this identity with multiple trigonometric functions. Of course the quantity $7 \zeta(3) / 4 \pi^{2}$ occurs, but, after all, this is nothing but $-7 \zeta^{\prime}(-2)$ which looks somehow more mundane. Is it possible to find a proof of this identity that uses multiple trigonometric functions?

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