On the asymptotic prime partitions of integers

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Abstract

In this paper we discuss $\mathcal{P}(n)$, the number of ways a given integer can be written as a sum of primes. We adopt methods used in quantum statistics, where the central problem is the number of ways in which energy is distributed among particles occupying single-particle states. The partition function in statistical mechanics plays the role of the generating function of partitions. The bosonic partition function of primes is constructed, and using the saddle-point approximation the density of states is evaluated in the limit of large numbers n. This directly gives the asymptotic number of prime partitions $\mathcal{P}_{as}(n)$. We discuss corrections to the leading asymptotic expression and compare various approximations with the exact numerical values of $\mathcal{P}(n)$ up to $n \sim 10^7$.

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I. INTRODUCTION

It has long been known that the asymptotic density of states of ideal bosons in a quantum-mechanical harmonic oscillator in one dimension is identical to the number of ways of partitioning an integer n into a sum of integers $\leq n$, and is given by the famous Hardy-Ramanujan formula [1]. It turns out that the generating function of the partitions given by Hardy and Ramanujan is the canonical partition function of ideal bosons, with the number of particles $N \to \infty$ trapped in a one-dimensional harmonic oscillator potential. Alternately, it may also be regarded as the grand partition function with chemical potential zero. It is interesting to note that the quantum partition fucntion of bosons was written down by Hardy and Ramanujan almost a decade before the advent of quantum statistical mechanics, when Bose-Einstein statistics was discovered.

In an earlier collaboration by some of the authors [2], methods of statistical mechanics were used to rederive the Hardy-Ramanujan formula for partitions of integers, as well as to derive some general results regarding distinct partitions of various types. A similar technique is applied in the present paper to derive an asymptotic formula for prime partitions, i.e., the number of ways $\mathcal{P}(n)$ that a given integer n can be expressed as a sum of primes. We consider a many-body system whose energies are integers n, with degeneracies $\mathcal{P}(n)$. For $n \to \infty$, the leading asymptotic result is already available [3, 4]. Corrections to the leading asymptotic result have been derived by Vaughan [5] using the saddle-point method. While our leading-order (LO) result, including the pre-exponential factor, agrees with the one given by Vaughan [5], our next to leading-order (NLO) term in the exponent has a different coefficient. Our asymptotic result, which we denote by $\mathcal{P}_{as}(n)$, is compared numerically with the exactly computed $\mathcal{P}(n)$ and found to be superior to both the LO result [3, 4] and that given by Vaughan [5]. However, even for as large numbers as $n \sim 10^7$, all of the asymptotic expressions discussed here are still far from reaching the exact $\mathcal{P}(n)$.

In Section II, we first outline the the saddle-point method in general and then apply it to obtain the density of states of a prime gas. We rederive the LO result and also the most important NLO corrections. In Section III the asymptotic result for prime partitions is compared numerically in detail with the exact prime partitions and the other asymptotic results discussed here. We conclude the paper with a short summary in Section IV. Some details about the density of primes, relevant to our analysis, are presented in the Appendix.

II. UNRESTRICTED PARTITIONS WITH PRIMES

A. Saddle-point asymptotics for a single-particle spectrum

To set the notation and to outline the method, we begin with a discrete single-particle spectrum given by ϵ_k; $\quad k = 1, \cdots, \infty$ in an infinite trap. Consider a system with a large number of particles, such that $N \to \infty$. At a given occupancy, the energy of the system is given by

$$E = \sum_{k=1}^{\infty} n_k \epsilon_k, \tag{1}$$

where n_k is the occupancy of the level ϵ_k which may be zero or a positive integer. In general for any given single-particle spectrum, there are many ways of partitioning the energy E into summands of ϵ_k.

The N-body canonical partition function of the system is given by

$$Z_N(\beta) = \sum_k \eta_k \exp(-\beta E_k) = \int_0^{\infty} dE \; \rho_N(E) \exp(-\beta E), \tag{2}$$

where $\beta = 1/kT$ is the inverse temperature, E_k are the eigenenergies of the N-particle system each with a degeneracy η_k. The N-particle density of states is denoted by

$$\rho_N(E) = \sum_k \eta_k \delta(E - E_k) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} d\beta \; \exp(\beta E) Z_N(\beta). \tag{3}$$

The integral representation on the r.h.s. above is simply the Laplace inverse of the partition function which we will use later.

In terms of the single-particle spectrum, the partition function may be written, by taking the limit $N \to \infty$, as

$$Z(\beta) = \prod_{k=1}^{\infty} \frac{1}{1 - e^{-\beta \epsilon_k}} = \prod_{k=1}^{\infty} \frac{1}{1 - x^{\epsilon_k}}, \qquad x = \exp(-\beta) < 1. \tag{4}$$

The limit $N \to \infty$ means that the partition of the total energy is unrestricted, allowing any number of summands that is allowed by the value of the energy. The last part of Eq. (4) has the familiar form of the generating function of partitions used in number theory [1]. In order to evaluate the density through the Laplace transform in Eq. (3), we define the function

$$S(\beta) = \beta E + \ln Z(\beta)\,, \tag{5}$$

which defines the canonical entropy, and

$$\ln Z(\beta) = -\sum_{k=1}^{\infty} \ln(1 - e^{-\beta \epsilon_k}) \,. \tag{6}$$

We evaluate the inverse Laplace transform in Eq. (3) using the method of steepest descent, or saddle-point method, so that

$$\rho(E) = \frac{e^{S(\beta_0)}}{\sqrt{2\pi S^{(2)}(\beta_0)}} \left[1 + \cdots\right], \tag{7}$$

where $S(\beta_0)$ is the entropy evaluated at the saddle point β_0 of the integrand. The dots indicate so-called cumulants involving higher derivatives of the entropy, which become more important for large β (see, e.g., Ref. [6]). Since we are interested here in the limit $\beta \to 0$ relevant for the asymptotics of large N, we shall neglect them. The derivatives of $S(\beta)$ evaluated at the saddle point are denoted by

$$S^{(n)}(\beta_0) = \left. \frac{\partial^n S(\beta)}{\partial \beta^n} \right|_{\beta_0} . \tag{8}$$

The energy E is determined through the saddle-point condition, namely

$$\left. \frac{\partial S(\beta)}{\partial \beta} \right|_{\beta_0} = S^{(1)}(\beta_0) = E + \frac{Z'(\beta_0)}{Z(\beta_0)} = 0 \,, \tag{9}$$

if there exists a saddle point β_0 fulfilling this equation.

B. Partition function for primes

The canonical partition function of a prime gas in the $N \to \infty$ limit is

$$Z(\beta) = \prod_p \frac{1}{[1 - e^{-\beta p}]} \,, \tag{10}$$

where the product runs over all primes p. This is also the generating function of the prime partitions. Taking the logarithm of the partition function (10) gives a sum over all primes p which we may also write as an integral

$$\ln Z(\beta) = -\sum_p \ln(1 - e^{-\beta p}) = -\int_{x_0}^{\infty} dx \, g(x) \ln(1 - e^{-\beta x}) \,, \tag{11}$$

where x_0 is any real number smaller than the lowest prime: $x_0 < p_1 = 2$, and $g(x)$ is the exact density of primes given by the sum of delta function distributions

$$g(x) = \sum_p \delta(x - p) \,. \tag{12}$$

For the study of asymptotics, we replace the exact $g(x)$ by the average prime density $g_{av}(x)$ which should be sufficient to obtain the leading contributions. By substituting this function for $g(x)$ in (11), we define the logarithm of the average partition function

$$\ln Z_{av}(\beta) = -\int_a^\infty dx\, g_{av}(x) \ln(1 - e^{-\beta x})\,, \tag{13}$$

where the constant a must be chosen carefully, as will be discussed in the following. As a specific choice, we use for $g_{av}(x)$ the asymptotic prime density that is well-known from number theory (see the Appendix):

$$g_{av}(x) = 1/\ln(x)\,. \tag{14}$$

Since we are only interested in asymptotic results, it will be sufficient to look at the limit $\beta \to 0$, i.e., the high-temperature limit of the partition function.

The integrand (14) in (13) has a pole at $x = 1$, which becomes relevant when $a < 1$. We therefore define the following principal-value integral

$$I(a,\beta) = -\lim_{\epsilon\to 0}\left[\int_a^{1-\epsilon} dx\,\frac{1}{\ln(x)}\ln(1-e^{-\beta x}) + \int_{1+\epsilon}^\infty dx\,\frac{1}{\ln(x)}\ln(1-e^{-\beta x})\right], \quad (a \neq 1) \tag{15}$$

which in the following is denoted by the symbol $⨍_a^\infty dx(\ldots)$, so that

$$\ln Z_{av}(a,\beta) = I(a,\beta) = -⨍_a^\infty dx\,\frac{1}{\ln(x)}\ln(1-e^{-\beta x})\,. \tag{16}$$

This integral exists for any $a \neq 1$ and for finite β. We now make the change of variable $y = \beta x$ to obtain

$$I(a,\beta) = \frac{1}{\beta\ln(\beta)}⨍_{a\beta}^\infty dy\,\frac{1}{\left[1-\frac{\ln(y)}{\ln(\beta)}\right]}\ln\left(1-e^{-y}\right). \tag{17}$$

In order to make the next step more clear, we define

$$\tau = 1/\beta \tag{18}$$

and rewrite (17) as

$$I(a,\tau) = -\frac{\tau}{\ln(\tau)}⨍_{a/\tau}^\infty dy\,\frac{1}{\left[1+\frac{\ln(y)}{\ln(\tau)}\right]}\ln\left(1-e^{-y}\right), \tag{19}$$

which we want to evaluate asymptotically in the limit $\tau \to \infty$. We split it into two parts, writing

$$I(a,\tau) = -\frac{\tau}{\ln(\tau)}\left[⨍_{a/\tau}^\tau dy\,\frac{1}{\left[1+\frac{\ln(y)}{\ln(\tau)}\right]}\ln(1-e^{-y}) + \int_\tau^\infty dy\,\frac{1}{\left[1+\frac{\ln(y)}{\ln(\tau)}\right]}\ln(1-e^{-y})\right]. \tag{20}$$

If we fix a to an arbitrary value in the limits $1 < a < 2$ and take $\tau > 1$, we may approximate the first integral by the first term of the binomial expansion of its denominator and write

$$I(a,\tau) \simeq -\frac{\tau}{\ln(\tau)}\left[\int_{a/\tau}^{\tau} dy \left(1 - \frac{\ln(y)}{\ln(\tau)}\right)\ln(1-e^{-y}) + \int_{\tau}^{\infty} dy \frac{1}{\left[1+\frac{\ln(y)}{\ln(\tau)}\right]}\ln(1-e^{-y})\right]. \quad (21)$$

In the limit $\tau \to \infty$, the second integral goes to zero and the first integral gives the asymptotic approximation

$$I_{as}(a,\tau) = -\frac{\tau}{\ln(\tau)}\int_{0}^{\infty} dy \left(1 - \frac{\ln(y)}{\ln(\tau)}\right)\ln(1-e^{-y}). \quad (22)$$

Using (18) and (16), we obtain the following asymptotic form for the logarithm of the partition function, which we call $\ln Z_{as}(\beta)$ and which we can evaluate analytically:

$$\ln Z_{as}(\beta) = \frac{1}{\beta\ln(\beta)}\int_{0}^{\infty} dy \left(1 + \frac{\ln(y)}{\ln(\beta)}\right)\ln(1-e^{-y}) = -\frac{f_1}{\beta\ln(\beta)} + \frac{f_2}{\beta\ln^2(\beta} \quad (23)$$

with

$$f_1 = \frac{\pi^2}{6}, \qquad f_2 = \frac{C\pi^2}{6} + \sum_k \frac{\ln(k)}{k^2} = 1.88703\,, \quad (24)$$

where $C = 0.577216$ is the Euler constant and the sum over k has been evaluated numerically (with $k_{max} \sim 10'000$). Note that the result (23) does not depend on the precise value of a that was chosen.

We now want to test the quality of the approximation (23), which should become accurate in the limit $\beta \to 0$. To that purpose we first integrate the principal-value integral $\ln Z_{av}(a,\beta)$ in (16). Here we choose $a = 0$ for definiteness; we emphasize that this choice is *a priori* independent of the fact that $1 < a < 2$ was used to derive the approximation (23), and of the fact that the lower integration limit in (23) is also zero. Then we compare it to the exact function (11) and to the approximation $\ln Z_{as}(\beta)$ in (23). The results are shown in Fig. 1. We see that both approximations approach the exact values closely for small β, while $\ln Z_{av}(\beta)$ is better than $\ln Z_{as}(\beta)$ for the largest values of β. In Fig. 2, we see the same in a region of smaller values for β. The approximation $\ln Z_{as}(\beta)$ given in (23) crosses the exact curve near $\beta \sim 0.008$ and appears to stay below it for $\beta \to 0$. But it reveals itself as an excellent asymptotic approximation to the exact $\ln Z(\beta)$ in the small-β limit.

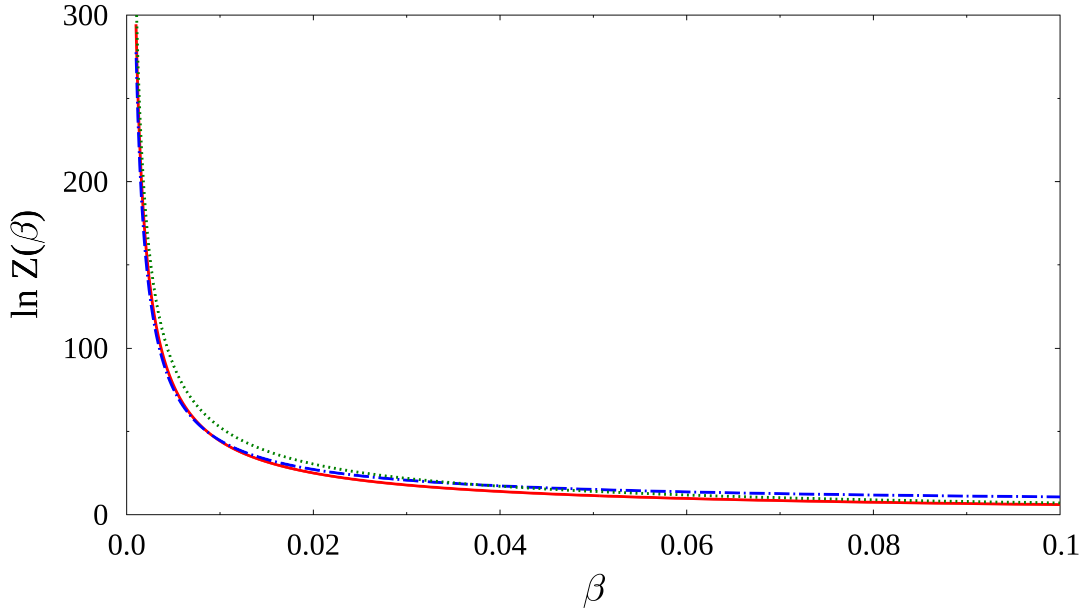

FIG. 1: Logarithm $\ln Z(\beta)$ of the partition function plotted versus β. Solid line (red): exact function (11). Dotted line (green): numerically integrated principal-value integral $\ln Z_{av}(a, \beta)$ in (16) with $a = 0$. Dash-dotted (blue) line: asymptotic approximation $\ln Z_{as}(\beta)$ in (23).

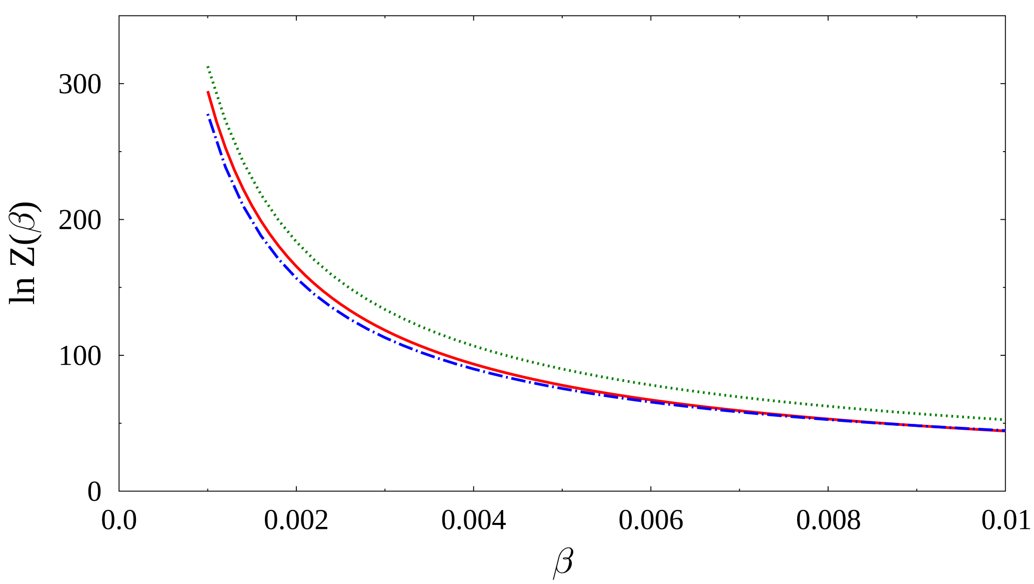

FIG. 2: Same as Fig. 1, but shown in limit of small β.

Using the analytical approximate form (23) of the partition function, the inverse Laplace transform (3) can now be evaluated in the saddle-point approximation, as outlined at the end of Sect. II.A.

C. Saddle-point approximation

In order to find the extremum we shall isolate the most singular terms in $S(\beta)$ in the high-temperature limit. We first write entropy, using (23) above, in the form

$$S(\beta) = \beta E - \frac{f_1}{\beta \ln(\beta)} + \frac{f_2}{\beta \ln^2(\beta)}, \tag{25}$$

where we for simplicity omit the subscript "as" henceforth. Since the entropy above is given up to order $1/\ln^2(\beta)$, all further calculations will be done up to this order. To begin with we need the following derivatives of the entropy

$$S^{(1)}(\beta) = E + \frac{f_1}{\beta^2 \ln(\beta)} + \frac{f_1}{\beta^2 \ln^2(\beta)} - \frac{f_2}{\beta^2 \ln^2(\beta)} + \cdots, \tag{26}$$

$$S^{(2)}(\beta) = -\frac{2f_1}{\beta^3 \ln(\beta)} - \frac{3f_1}{\beta^3 \ln^2(\beta)} + \frac{2f_2}{\beta^3 \ln^2(\beta)} + \cdots. \tag{27}$$

The saddle-point solution β_0 is given by the following convenient form

$$\beta_0 E = -\frac{f_1}{\beta_0 \ln(\beta_0)} + \frac{f_2}{\beta_0 \ln^2(\beta_0)} - \frac{f_1}{\beta_0 \ln^2(\beta_0)} + \cdots. \tag{28}$$

This is a transcendental equation whose solution can be obtained iteratively as outlined in Sec. III.B below. However, we may use the above condition directly in $S(\beta_0)$ to obtain

$$S(\beta_0) = 2\beta_0 E + \frac{f_1}{\beta_0 \ln^2(\beta_0)} + \cdots, \tag{29}$$

and

$$S^{(2)}(\beta_0) = \frac{1}{\beta_0^2}\left[2\beta_0 E - \frac{f_1}{\beta_0 \ln^2(\beta_0)} + \cdots\right]. \tag{30}$$

Using the above solutions in terms of as yet undetermined β_0, we obtain the asymptotic density of a prime gas given by

$$\rho(E) = \frac{\exp(2\beta_0 E + \frac{f_1}{\beta_0 \ln^2(\beta_0)} + \cdots)}{\sqrt{(2\pi/\beta_0^2)\left[2\beta_0 E - \frac{f_1}{\beta_0 \ln^2(\beta_0)} + \cdots\right]}} \tag{31}$$

This asymptotic density is the same as the asymptotic formula for prime partitions denoted here by $\mathcal{P}(n) = \rho(n{=}E)$. Any further analysis requires the solution of the saddle-point condition (9) for $\beta_0(E)$. Since this is a transcendental equation, its solution is not straight forward, even with the approximations we have made. We shall outline this next.

D. Saddle-point solution

The solution of the saddle-point equation

$$\beta E = -\frac{f_1}{\beta \ln \beta}\,, \tag{32}$$

keeping only the leading-order, may be worked out successively. Let $\tau = 1/\beta$:

$$\frac{f_1}{E} = \frac{\ln(\tau)}{\tau^2}. \tag{33}$$

We start by assuming the solution to be of the form

$$\tau = a_1 E^{a_2}[\ln(E)]^{a_3}, \tag{34}$$

where a_1, a_2, and a_3 are constants to be determined using the Eq. (32). Upon substitution, assuming large E, we get

$$\frac{f_1}{E} = \frac{1}{a_1^2 E^{2a_2}(\ln E)^{2a_3}}[\ln a_1 + a_2 \ln(E) + a_3 \ln\ln(E)] \approx \frac{a_2}{a_1^2 E^{2a_2}(\ln E)^{2a_3-1}}. \tag{35}$$

First we determine the leading term, comparing powers, to find the solutions

$$a_3 = \frac{1}{2}\,, \quad a_2 = \frac{1}{2}\,, \quad a_1^2 = \frac{a_2}{f_1} = \frac{3}{\pi^2}\,. \tag{36}$$

Thus we have the leading solution given by

$$\tau = \frac{1}{\beta_0} = \sqrt{\frac{3}{\pi^2} E \ln(E)}\,. \tag{37}$$

To leading order, therefore, we have the following result for the density of the prime gas, or equivalently for unrestricted prime partitions:

$$\rho(E) = \frac{e^{S(\beta_0)}}{\sqrt{2\pi S''(\beta_0)}} = \frac{e^{2\pi\sqrt{E/[3\ln(E)]}}}{\sqrt{4E^{3/2}[3\ln(E)]^{1/2}}}\,. \tag{38}$$

Apart from the prefactor, it is well known [3, 4] that $\ln[\rho(E)] \approx 2\pi\sqrt{E/(3\ln E)}$. In the paper by Vaughan [5] the prefactor has also been given by calculating $\sqrt{2\pi S^{(2)}(\beta)}$ which agrees with the calculation given here.

Next we consider corrections to the the asymptotic result given in Eq. (38).

E. Non-leading order corrections

The results of the previous subsection may be further improved by including additional terms that were neglected in Eq. (35). This is done by assuming the solution to be of the form

$$\beta_0 = \pi \sqrt{\frac{1}{3E\ln(E)}} \left[1 + a\, \frac{\ln[\ln(E)]}{\ln(E)} + b\, \frac{1}{\ln(E)} \cdots \right], \tag{39}$$

where a, b are arbitrary coefficients to be determined using the equation above. The form of the solution is suggested by the transcendental equation (35) itself. Since the LHS of (35) is a monomial in E, the only way this can be satisfied is to have additional corrections to cancel the non-leading terms. Writing

$$\beta_0^2 \ln(\beta_0) = -\frac{\pi^2}{6E}, \tag{40}$$

we expand the unknowns on the LHS to the desired order $1/\ln(E)$ in the limit of large E.

$$\beta_0^2 = \frac{\pi^2}{3E\ln(E)} \left[1 + 2a\, \frac{\ln[\ln(E)]}{\ln(E)} + 2b\, \frac{1}{\ln(E)} + O\{1/\ln^2(E)\} \right],$$

$$\ln(\beta_0) = -\frac{1}{2}\ln(E) \left[1 + \frac{\ln[\ln(E)]}{\ln(E)} - \ln\left(\frac{\pi^2}{3}\right) \frac{1}{\ln(E)} - 2a\, \frac{\ln[\ln(E)]}{\ln^2(E)} - 2b\, \frac{1}{\ln^2(E)} \right].$$

Substituting these in the Eq. (35), we have

$$\beta_0^2 \ln(\beta_0) = -\frac{\pi^2}{6E} \left[1 + (2a+1) \frac{\ln[\ln(E)]}{\ln(E)} + \frac{2b - \ln(\pi^2/3)}{\ln(E)} + O\{1/\ln^2(E)\} \right] = -\frac{\pi^2}{6E},$$

which now determines the constants $a = -1/2, b = \ln(\pi/\sqrt{3})$ and therefore

$$\beta_0 = \pi \sqrt{\frac{1}{3E\ln(E)}} \left[1 - \frac{1}{2}\, \frac{\ln[\ln(E)]}{\ln(E)} + \frac{1}{2}\, \frac{\ln(\pi^2/3)}{\ln(E)} \cdots \right]. \tag{41}$$

The density of prime partitions is then obtained by substituting the above solution into

$$\rho(E) = \frac{\exp\left[2\beta_0 E \left(1 + \frac{(\beta_0 E)^2}{2f_1 E} \right) + \cdots \right]}{\sqrt{2\pi(2\beta_0 E/\beta_0^2) \left[1 - \frac{(\beta_0 E)^2}{2f_1 E} + \cdots \right]}}, \tag{42}$$

where we have kept the NLO term in the density consistent with the order to which the solution has been obtained. Substituting for $\beta_0 E$ from Eq. (41) we finally obtain

$$\rho(E) = \frac{\exp\left\{ 2\pi \sqrt{\frac{E}{3\ln(E)}} \left[1 - \frac{1}{2} \frac{\ln[\ln(E)]}{\ln(E)} + \frac{1 + \ln(\pi/\sqrt{3})}{\ln(E)} + \cdots \right] \right\}}{\sqrt{\{4[3\ln(E)]^{1/2} E^{3/2} + \cdots\}}}. \tag{43}$$

Identifying $\rho(E)$ with $\mathcal{P}(n{=}E)$, the above equation gives the asymptotic prime partitions of an integer n. The first correction to the exponent given above, proportional to $\ln[\ln(E)]/\ln(E)$, is similar to that given by Vaughan [5] except that its coefficient here is $-\frac{1}{2}$ intead of $+1$. In the following section we shall test the approximation obtained by ignoring all higher-order terms indicated by the dots above, thus defining

$$\mathcal{P}_{as}(n) = \frac{1}{2[3\ln(E)]^{1/4}E^{3/4}} \exp\left\{2\pi\sqrt{\frac{n}{3\ln(n)}}\left[1 - \frac{1}{2}\frac{\ln[\ln(n)]}{\ln(n)} + \frac{1+\ln(\pi/\sqrt{3})}{\ln(n)}\right]\right\}. \quad (44)$$

III. NUMERICAL STUDIES OF THE ASYMPTOTIC PRIME PARTITION

A. Evaluation of data base for $\mathcal{P}(n)$

We evaluate the prime partition $\mathcal{P}(n)$ using a standard method. Given an integer n, find the distinct primes that divides n. The sum of distinct prime factors that decompose n is denoted by $\mathscr{S}(n)$ [7]. For example, $\mathscr{S}(4) = 2$ since $4 = 2 \cdot 2$ has only one distinct prime that divides it; $\mathscr{S}(6) = 5$ since $6 = 2 \cdot 3$, or $\mathscr{S}(52) = 15$ since $52 = 2 \cdot 2 \cdot 13$ (Note: if a prime factor occurs several times, it should only be counted once.) Once the sum of prime factors $\mathscr{S}(n)$ is generated in a table, the following recursion relation [8] is used to compute the prime partitions (without any restriction)

$$\mathcal{P}(n) = \frac{1}{n}\left[\mathscr{S}(n) + \sum_{k=1}^{n-1}\mathscr{S}(k)\cdot\mathcal{P}(n-k)\right]. \quad (45)$$

which involves all prime partitions of integers less than n. This procedure is very time consuming for large n. We have been able to compute $\mathcal{P}(n)$ for n up to $8'654'775$. But, as we shall see, even that large number is not sufficient to reach the asymptotics of $\mathcal{P}(n)$.

B. Numerical study of $\mathcal{P}_{as}(n)$

Using the above derived data base for the exact $\mathcal{P}(n)$, we now test various approximations for their asymptotic behavior. Rather than calculating the exponentially growing full function $\mathcal{P}(n)$, we look at its logarithm. We compare numerically the logarithm of the exact $\mathcal{P}(n)$ with that of the following approximations:

- To lowest order (LO), we set the prefactor of the exponent in (44) to unity, ignoring its denominator, and just keep the leading exponential term

$$\mathcal{P}_0(n) = \exp\left\{2\pi\sqrt{\frac{n}{3\ln(n)}}\right\}, \tag{46}$$

an asymptotic result that has been known for a long time [3, 4].

- The next approximation is that of Vaughan [5]:

$$\mathcal{P}_V(n) = \frac{1}{2[3\ln(E)]^{1/4}E^{3/4}} \exp\left\{2\pi\sqrt{\frac{n}{3\ln(n)}}\left[1+\frac{\ln[\ln(n)]}{\ln(n)}\right]\right\}. \tag{47}$$

Note that the correction in the exponent here has a different coefficient from that in our result 44).

- The third approximation we investigate is our asymptotic result (44) derived in the previous section.

The numerical comparison of the above three expressions with the exact prime partitions is now discussed in several steps.

We first plot $\ln\mathcal{P}(n)$ versus n for the various approximations in Fig. 3. The solid (black) curve gives the exact values $\ln\mathcal{P}(n)$. Our present approximation (44), shown by the dashed (red) line, comes closest to it, improving somewhat over the lowest-order approximation $\ln\mathcal{P}_0(n)$ (46) shown by the dash-dotted (blue) line. The approximation (47) of Vaughan, shown by the dotted (green) curve, overshoots the exact values substiantially.

From this figure we can, however, not assess the way in which the various approximations approach the correct asymptotics. To this purpose we next show in Fig. 4 the relative differences of the approximated logarithms, $[\ln\mathcal{P}_{app}(n) - \ln\mathcal{P}(n)]/\ln\mathcal{P}_0(n)$, and plot them versus $1/n$ so that they should tend to zero for $n \to \infty$ (i.e., towards the left vertical axis in the figure). Shown are, with the same symbols (and colors) as above, our present approximation (44), the leading term (46), and that of Vaughan (47).

Some weak oscillations can be seen for values of n less than ~ 20 (i.e., $1/n$ larger than ~ 0.05). For the rest of the region, the oscillations damp out and all curves appear very smooth. On this scale we cannot see, however, what goes on for larger n. To this purpose we show in Fig. 5 the same results in the region $0 \le 1/n \le 0.001$.

Here we see that sign changes occur in the two lowest curves: at $n \sim 5'800$ for (44), and at $n \sim 13'000$ for (46). They therefore approach zero for $n \to \infty$ from below, while

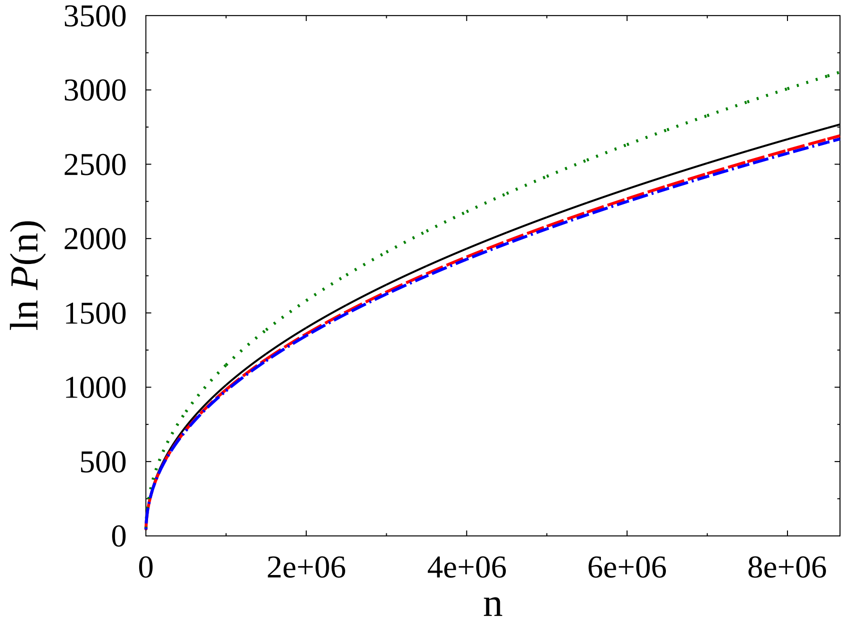

FIG. 3: Logarithms $\ln \mathcal{P}(n)$ in various approximations. Solid line (black): exact numerical values. Dashed (red): $\ln \mathcal{P}_{as}(n)$ (44), dash-dotted (blue): LO $\ln \mathcal{P}_0(n)$ (46), dotted(green): Vaughan (47).

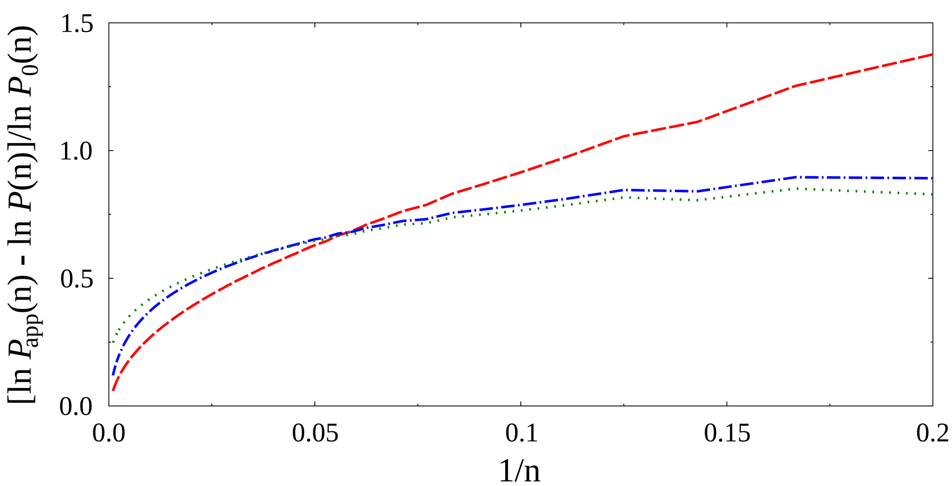

FIG. 4: Relative differences $[\ln \mathcal{P}_{app}(n) - \ln \mathcal{P}(n)]/\ln \mathcal{P}_0(n)$ plotted versus $1/n$. Dashed (red): present (44), dash-dotted (blue): LO term (46), dotted (green): Vaughan (47).

the curve of Vaughan (47) stays on the positive side. We note that our result (44) brings a considerable improvement for the asymptotics over Vaughan's result.

In order to see how (or if) the two lower curves approach the asymptotic result 0, we focus on the largest region of n available in our computation and further reduce the scale to $1/n \leq 10^{-5}$, as seen in Fig. 6; Vaughan's curve stays outside this picture.

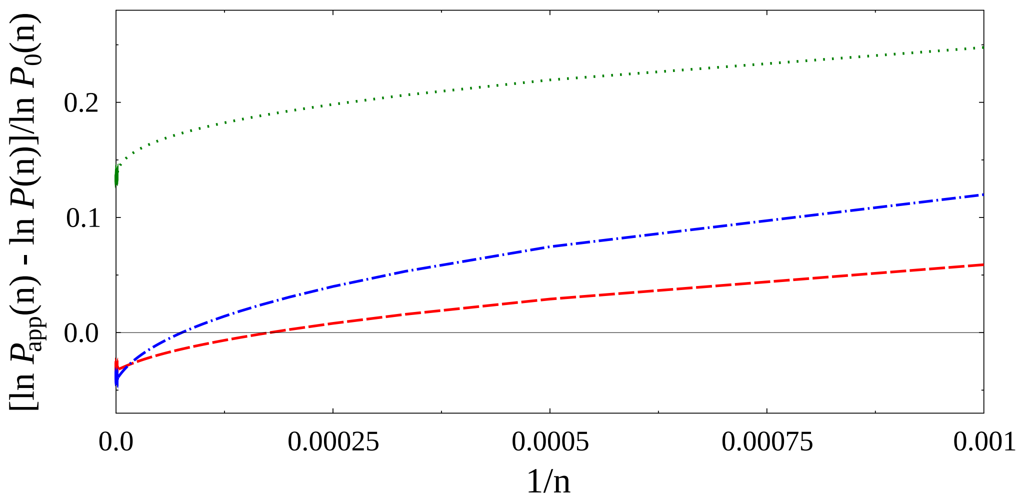

FIG. 5: Same as in Fig. 4 in a different region of $1/n$.

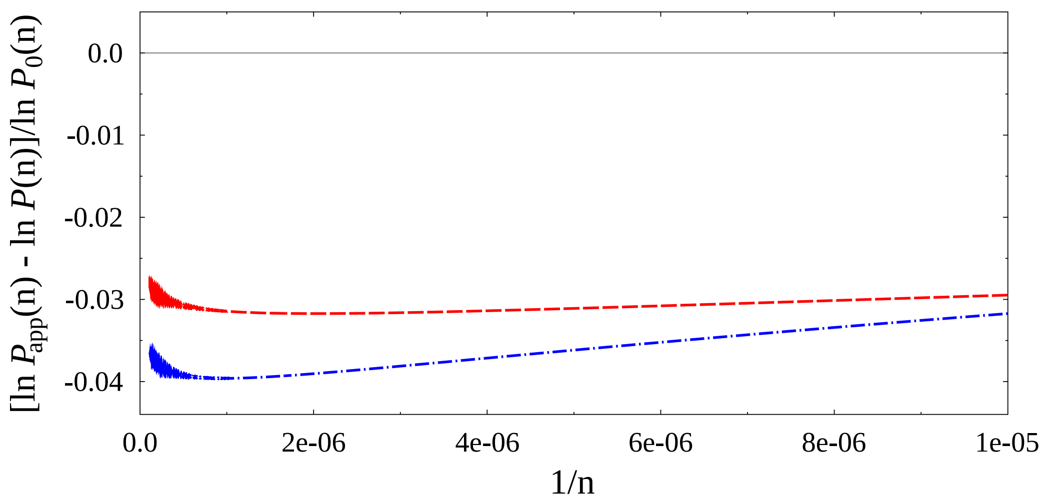

FIG. 6: Same as in Fig. 5 in the lowest region $1/n \leq 10^{-5}$; Vaughan curve not seen at this scale.

Clearly, the differences are still quite far from reaching zero, even for our largest value $n_{max} = 8'654'775$. We must therefore ask how far one has to go for the logarithm of our theoretically well-founded asymptotics (44) to go over into the exact $\ln \mathcal{P}(n)$. Although the two curves in Fig. 6 do bend up towards zero for $1/n \to 0$, the slopes at their ends (corresponding to n_{max}) are still rather small, so that there may be a very long way to go – too long perhaps to be covered by any numerical computation of the exact $\mathcal{P}(n)$.

We conclude that our result $\mathcal{P}_{as}(n)$ in (44) appears to have the correct asymptotic behaviour, but that even the included corrections beyond the LO are not sufficient to reach the exact partitions in our numerically accessible region.

IV. SUMMARY

In this paper we have discussed $\mathcal{P}(n)$, the number of ways a given integer may be written as a sum of primes – a central theme in number theory. We have adopted methods used in quantum statistics where the central problem is the number of ways in which energy is distributed among particles occupying single-particle states. The partition function in statistical mechanics plays the role of the generating function of partitions. We have discussed the method in detail and then applied the same to the problem of prime partitions of an integer. The dominant integral is evaluated using the saddle-point method.

The main results of the paper may be summarised as follows:

- While the main asymptotic form Eq. (46) has been known for some time, we derive non-leading order (NLO) corrections to the exponent. There has not been much discussion in the literature on the prefactor to the exponential form (46), needed to calculate the absolute value of the prime partitions. An exception is Vaughan [5] who derived the prefactor and also a correction to the exponent in (46), leading to the expression given in (47). We obtain the same prefactor but a different NLO contribution to the exponent of our result (44), which brings a considerable improvement for the asymptotics compared to that of Vaughan.

- We use a well-known algorithm to compute the exact prime partitions, in order to compare analytical expressions for asymptotic prime partitions numerically. We have been able to do this up to more than 8 million in n. To our knowledge, a numerical comparison of the exact results with asymptotic expressions has not been done before up to this range.

- It has been known from earlier works, see for example Ref. [2], that for partitions of ordinary integer numbers into integers (or their powers), the asymptotic expressions become almost exact very soon – for n of the order of 100 or more. However, in the case of prime partitions it is surprising that even for n around 8 million or more, the asymptotic form $\mathcal{P}_{as}(n)$ given in Eq. (44) has by no means reached the exact $\mathcal{P}(n)$. The relative error here remains much larger than in the case of other known integer partitions.

- Although both the exact $\mathcal{P}(n)$ and the asymptotic form $\mathcal{P}_{as}(n)$ given in (44) are monotonously increasing, there difference is not monotonic. In fact, we found that $\mathcal{P}_{as}(n)$ crosses $\mathcal{P}(n)$ around $n \sim 5'800$ and approaches it from below for $n \to \infty$ (within the limits of our data).

- Our main conclusion is that our result $\mathcal{P}_{as}(n)$ given in (44) appears to have the correct asymptotic behaviour, but that even the corrections included beyond the LO expression $\mathcal{P}_0(n)$ in (46) are not sufficient to reach the exact $\mathcal{P}(n)$ in the numerically accessible region.

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Appendix: Some details about the density of primes

In this section, we discuss two approximations to the density of primes $g(x)$ defined in (12), which is related to the function $\pi(x)$ that counts the number of primes $p \leq x$ by a differentiation:

$$g(x) = \frac{d\pi(x)}{dx}\,. \tag{A.1}$$

Both $\pi(x)$ and $g(x)$ have been the object of a lot of research in number theory. $\pi(x)$ is a stair-case function whose average part is given by the asymptotic form

$$\pi(x) \sim \frac{x}{\ln(x)}\,, \tag{A.2}$$

which is a consequence of the prime number theorem. A more refined asymptotic form is (see, e.g., [9]):

$$\pi(x) \sim \frac{x}{\ln(x)} + \frac{x}{[\ln(x)]^2} + \cdots + (n-1)!\,\frac{x}{[\ln(x)]^n}\,. \tag{A.3}$$

Differentiating it yields the asymptotic expression for the density of primes

$$g(x) \sim 1/\ln(x)\,, \tag{A.4}$$

whereby all higher-order terms coming from (A.3) have cancelled successively. In Sec. III we have used the above asymptotic form for the average prime density $g_{av}(x)$.

An expression for $\pi(x)$ which has been conjectured by Riemann in 1859 and proved by Mangold in 1959 is (see [9])

$$\pi(x) = \sum_{m=1}^{\infty} \frac{\mu(m)}{m}\, J(x^{1/m})\,, \tag{A.5}$$

where $\mu(m)$ is the Moebius function [$\mu(1) = 1$], the function $J(x)$ is given [9] by

$$J(x) = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{p} \Theta(x - p^n)\,, \qquad (x > 0) \tag{A.6}$$

and $\Theta(x)$ is the standard step function: $\Theta(x) = 1$ for $x \geq 0$ and $\Theta(x) = 0$ for $x < 0$. In (A.6), p again runs over all primes and n over all integers. Using an expansion of $J(x)$ derived by Riemann, one obtains the following expression for the density of primes (cf. [10])

$$g_{sc}(x) = \frac{1}{x \ln x} \sum_{m=1}^{\infty} \frac{\mu(m)}{m} \left[x^{1/m} - \frac{1}{(x^{2/m} - 1)} - 2\, x^{1/2m} \sum_{\alpha} \cos\left(\frac{\alpha}{m} \ln x\right) \right]. \tag{A.7}$$

Here $\alpha > 0$ are the zeros of the Riemann zeta function along the positive half-line, and the validity of the Riemann hypothesis has been assumed. This expression, which does not

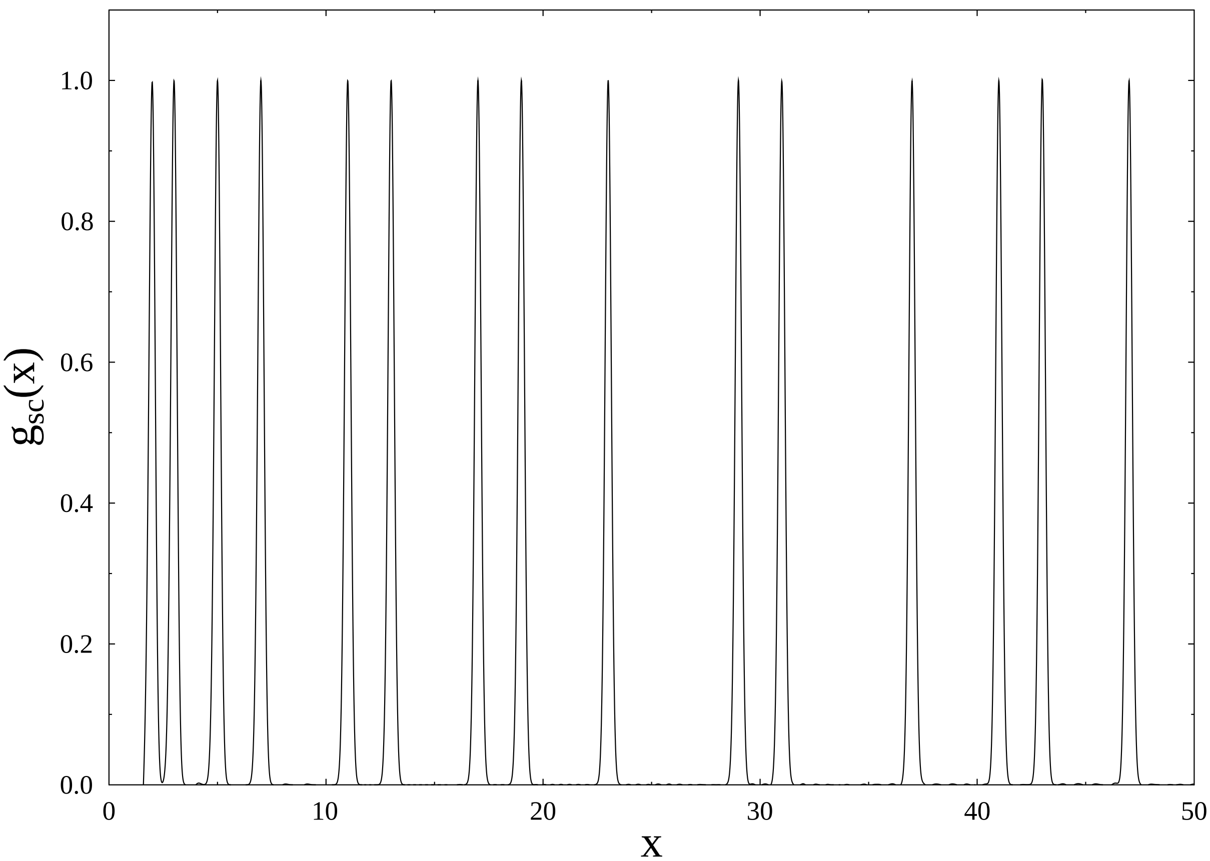

FIG. 7: Density of primes $g(x)$ obtained by the semiclassical expression $g_{sc}(x)$ in (A.7), using the lowest 3000 Riemann zeros α and $m_{max} = 14$, coarse-grained with a Gaussian width $\gamma_{sh} = 0.1$.

appear to be widely known, has the form of a semiclassical "trace formula" [11, 12] and we have therefore denoted it with the subscript "sc" for "semiclassical". Ideally, $g_{sc}(x)$ should yield the exact prime density $g(x)$ in (12) if the sum over α is not truncated, and if the Riemann hypothesis is true.

We have tested Eq. (A.7) numerically in order to convince ourselves of its validity. For practical purposes, we have coarse-grained it, replacing the delta functions in (12) by normalized Gaussians with a width γ, and correspondingly coarse-grained Eq. (A.7) as described in Sec. 5.5 of [12].

Fig. 7 shows the results, obtained using the lowest 3000 Riemann zeros α. We see that the coarse-grained trace formula indeed reproduces the Gaussian-smoothed density of primes, replacing the delta functions in (12) by Gaussians centered exactly at the primes p. (Note that the sum over m becomes finite for any finite value of x; in the case described here, $m_{max} = 14$ was sufficient.)