

RAPIDLY CONVERGING FORMULAE FOR $\zeta(4k \pm 1)$

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ABSTRACT. We provide rapidly converging formulae for the Riemann zeta function at odd integers using the Lambert series $\mathcal{L}_q(s) = \sum_{n=1}^{\infty} n^s q^n / (1 - q^n)$, $s = -(4k \pm 1)$. Our main formula for $\zeta(4k - 1)$ converges at rate of about $e^{-\sqrt{15}\pi}$ per term, and the formula for $\zeta(4k + 1)$, at the rate of $e^{-4\pi}$ per term. For example, the first order approximation yields $\zeta(3) \approx \frac{\pi^3 \sqrt{15}}{100} + e^{-\sqrt{15}\pi} \left[\frac{9}{4} + \frac{4}{\sqrt{15}} \sinh\left(\frac{\sqrt{15}\pi}{2}\right) \right]$ which has an error only of order 10^{-10} .

1. INTRODUCTION

In a recent paper [3] we analyzed the $|q| \rightarrow 1^-$ asymptotic behavior of a Lambert series of the type

$$(1.1) \quad \mathcal{L}_q(s, x) = \sum_{n=1}^{\infty} \frac{n^s q^{nx}}{1 - q^n}, \quad s \in \mathbb{C},$$

with $|q| < 1$ and $x > 0$. At $x = 1$, the Lambert series $\mathcal{L}_q(s, x)$ is the generating function for the divisor function $\sigma_s(n)$, the sum of the s th powers of divisors of an integer n [1]:

$$(1.2) \quad \sum_{n=1}^{\infty} \sigma_s(n) q^n = \mathcal{L}_q(s, 1).$$

For odd positive integer s values, the Lambert series is related to the Eisenstein series

$$(1.3) \quad E_{2k}(q) = 1 - \frac{4k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{2k-1}(n) q^n,$$

that are modular in nature [2].

In this paper we focus on the Lambert series at negative odd integer values of $s = -(4k \pm 1)$, $k = 1, 2, 3, \dots$ and at $x = 1$. This Lambert series is related to the Riemann zeta function at the corresponding positive integer values, $\zeta(4k \pm 1)$, [4]. Our main results are stated in Sections 3, 4, and 5 where we provide a series of rapidly converging formulae for $\zeta(4k \pm 1)$. Where comparison is possible, our results agree with those obtained using powerful computational methods [5, 7].

Key words and phrases. Lambert, Eisenstein, q-series, modular, Riemann zeta, divisor sum.

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2. LAMBERT SERIES

In this section we focus on the Lambert series at negative odd integer arguments and $x = 1$. Since x is set to 1 in this paper, to avoid redundancy we use the notation $\mathcal{L}_q(s, 1) \equiv \mathcal{L}_q(s)$.

Theorem 2.1. *For odd negative integers s the Lambert series at $x = 1$ satisfies the following relations:*

(1) For $s = -1$,

$$\mathcal{L}_{e^{-2\pi t}}(-1) - \mathcal{L}_{e^{-\frac{2\pi}{t}}}(-1) = \frac{1}{2} \log t - \frac{\pi}{6} \sinh(\log t).$$

(2) For $s = -(4k - 1) = -3, -7, -11, \dots$,

$$\begin{aligned} & \frac{1}{t^{2k-1}} \mathcal{L}_{e^{-2\pi t}}(-4k+1) + t^{2k-1} \mathcal{L}_{e^{-\frac{2\pi}{t}}}(-4k+1) \\ &= (2\pi)^{4k-1} \sum_{j=0}^k \frac{(-1)^{j+1} B_{2j} B_{4k-2j} \cosh[(2k-2j) \log t]}{(2j)! (4k-2j)! (1 + \delta_{jk})} \\ & \quad - \zeta(4k-1) \cosh[(2k-1) \log t]. \end{aligned}$$

(3) For $s = -(4k + 1) = -5, -9, -13, \dots$,

$$\begin{aligned} & \frac{1}{t^{2k}} \mathcal{L}_{e^{-2\pi t}}(-4k-1) - t^{2k} \mathcal{L}_{e^{-\frac{2\pi}{t}}}(-4k-1) \\ &= (2\pi)^{4k+1} \sum_{j=0}^k \frac{(-1)^{j+1} B_{2j} B_{4k+2-2j} \sinh[(2k+1-2j) \log t]}{(2j)! (4k+2-2j)!} \\ & \quad + \zeta(4k+1) \sinh(2k \log t), \end{aligned}$$

where B_k is the k th Bernoulli number and δ_{jk} is the Kronecker delta.

Remark Although the theorem is stated for negative odd integer arguments of the Lambert series, it holds true for positive odd arguments as well. The results for positive odd arguments, obtained by using negative k values in the theorem, reproduce modular properties of the Eisenstein series (1.3).

Proof. For the $s = -1$ case, by taking the logarithm of the complete expansion of the q -Pochhammer symbol provided in the Referee Remark 3.3 of Ref. [3] we get

$$\begin{aligned} (2.1) \quad \mathcal{L}_q(-1) &= \frac{\pi^2}{6t} + \frac{1}{2} \log \frac{t}{2\pi} - \frac{t}{24} + \sum_{n=1}^{\infty} \text{Li}_1(e^{-\frac{4n\pi^2}{t}}) \\ &= \frac{\pi^2}{6t} + \frac{1}{2} \log \frac{t}{2\pi} - \frac{t}{24} + \mathcal{L}_{e^{-\frac{4\pi^2}{t}}}(-1), \end{aligned}$$

where $q = e^{-t}$ and $\text{Li}_s(q)$ is the polylogarithm function. Replacing t with $2\pi t$ and rearranging the terms completes the proof.

For $s = -4k+1$ we begin with the result in Corollary 2.4(2) of Ref. [3]. Replacing t with $2\pi t$, writing $\zeta'(2-4k)$ in terms of $\zeta(4k-1)$, and writing the zeta function

at even arguments in terms of Bernoulli numbers gives

$$\begin{aligned} & \frac{1}{t^{2k-1}} \mathcal{L}_{e^{-2\pi t}}(-4k+1) \\ & \approx (2\pi)^{4k-1} \left[\frac{(-1)^{k+1} B_{2k}^2}{(2k)!^2 2} + \sum_{j=0}^{k-1} \frac{(-1)^{j+1} B_{2j} B_{4k-2j} \cosh[(2k-2j) \log t]}{(2j)! (4k-2j)!} \right] \\ & \quad - \zeta(4k-1) \cosh[(2k-1) \log t]. \end{aligned}$$

The right hand side of the equation above is completely symmetric with respect to a $t \rightarrow 1/t$ transformation. Thus adding $t^{2k-1} \mathcal{L}_{e^{-\frac{2\pi}{t}}}(-4k+1)$ to the left hand side of the equation to make it completely symmetric yields the result in the theorem. Note that such symmetry arguments only guarantee numerical accuracy. However, a comparison with Entry 21(i) in [4] shows that the result is exact, and there are no other error terms. In the final result the B_{2k}^2 term is included in the sum over j by using δ_{jk} and extending the limit of the sum from $k-1$ to k .

Similarly, for $s = -4k - 1$ case from Corollary 2.4(2) of Ref. [3] we obtain

$$\begin{aligned} & \frac{1}{t^{2k}} \mathcal{L}_{e^{-2\pi t}}(-4k-1) \\ & \approx (2\pi)^{4k+1} \sum_{j=0}^k \frac{(-1)^{j+1} B_{2j} B_{4k+2-2j} \sinh[(2k+1-2j) \log t]}{(2j)! (4k+2-2j)!} \\ & \quad + \zeta(4k+1) \sinh(2k \log t). \end{aligned}$$

In this case the right hand side of the equation is antisymmetric with respect to a $t \rightarrow 1/t$ transformation. Thus, we add $t^{2k} \mathcal{L}_{e^{-\frac{2\pi}{t}}}(-4k-1)$ to the left hand side of the equation to make the equation completely antisymmetric. As in the case above, a comparison with Entry 21(i) in [4] shows that the result is exact, which yields the result in the theorem. \square

Corollary 2.2. *For positive odd integers $(4k-1) = 3, 7, 11, \dots$ we have*

$$\zeta(4k-1) = (2\pi)^{4k-1} \sum_{j=0}^k \frac{(-1)^{j+1} B_{2j} B_{4k-2j}}{(2j)! (4k-2j)! (1+\delta_{jk})} - 2\mathcal{L}_{e^{-2\pi}}(-4k+1).$$

Proof. Substituting $t = 1$ in Theorem 2.1(2) and solving for $\zeta(4k-1)$ completes the proof. \square

Corollary 2.3. *For positive odd integers $(4k+1) = 5, 9, 13, \dots$ we have*

$$\begin{aligned} \zeta(4k+1) &= (2\pi)^{4k+1} \sum_{j=0}^k \frac{(-1)^j (2k+1-2j) B_{2j} B_{4k+2-2j}}{2k (2j)! (4k+2-2j)!} \\ & \quad - \frac{2\pi e^{-2\pi} \mathcal{L}'_{e^{-2\pi}}(-4k-1)}{k} - 2\mathcal{L}_{e^{-2\pi}}(-4k-1), \end{aligned}$$

where $\mathcal{L}' = d\mathcal{L}/dq$.

Proof. Differentiating Theorem 2.1(3) with respect to t at $t = 1$ and solving for $\zeta(4k+1)$ completes the proof. \square

Remark The results in Corollaries 2.2 and 2.3 are well known [4–6] but have been reproduced here for the sake of completeness. The results hold true for negative values of k as well.

3. FORMULAE FOR $\zeta(4k+1)$

In Corollary 2.3 we established a formula for $\zeta(4k+1)$ that involved the Lambert series and its derivative with respect to q at $q = e^{-2\pi}$. In this section we establish three new formulae for $\zeta(4k+1)$ that are in terms of the Lambert series only, and not its derivative.

First, we prove a general identity for the Lambert series that is true for all s values in the form of the following lemma.

Lemma 3.1. *For any positive prime integer p and $s \in \mathbb{C}$ the Lambert series at $x = 1$ satisfies:*

$$\sum_{n=0}^{p-1} \mathcal{L}_{q^{\frac{1}{p}} e^{i\frac{2\pi n}{p}}}(s) = (p^{s+1} + p)\mathcal{L}_q(s) - p^{s+1}\mathcal{L}_{q^p}(s).$$

Proof. Writing the Lambert series in terms of the divisor function using (1.2) gives

$$\begin{aligned} \frac{1}{p} \sum_{n=0}^{p-1} \mathcal{L}_{q^{\frac{1}{p}} e^{i\frac{2\pi n}{p}}}(s) &= \frac{1}{p} \sum_{n=0}^{p-1} \sum_{k=1}^{\infty} \sigma_s(k) q^{\frac{k}{p}} e^{i\frac{2\pi nk}{p}} \\ &= \frac{1}{p} \sum_{k=1}^{\infty} \sigma_s(k) q^{\frac{k}{p}} \sum_{n=0}^{p-1} e^{i\frac{2\pi nk}{p}} \\ &= \sum_{l=1}^{\infty} \sigma_s(lp) q^l \\ &= (p^s + 1) \sum_{l=1}^{\infty} \sigma_s(l) q^l - p^s \sum_{\frac{p}{p}|l} \sigma_s\left(\frac{l}{p}\right) q^l \\ &= (p^s + 1) \sum_{l=1}^{\infty} \sigma_s(l) q^l - p^s \sum_{m=1}^{\infty} \sigma_s(m) q^{mp}, \end{aligned}$$

where $p|l$ indicates sum only over integers l that are divisible by p . Rewriting the two infinite sums in terms of Lambert series using (1.2) and multiplying by p on both sides proves the lemma. In the proof above we used the following property of the divisor function [1]

$$\sigma_s(lp) = (p^s + 1)\sigma_s(l) - p^s \sigma_s\left(\frac{l}{p}\right),$$

where $\sigma_s\left(\frac{l}{p}\right)$ is non-zero only when p divides l . □

Remark For $p \neq 2$ the limits of the sum over n in the lemma can be replaced with the symmetric limits ranging from $-(p-1)/2$ to $(p-1)/2$. It is this symmetric range that we use in later proofs when $p = 3$ and $p = 5$.

Theorem 3.2. *The Riemann zeta function at odd integers $4k+1$ other than unity can be written in terms of Lambert series at $q = e^{-2\pi}$ and $e^{-4\pi}$ as:*

$$\begin{aligned} \zeta(4k+1) &= (2\pi)^{4k+1} \sum_{j=0}^k \frac{(-1)^{j+1} b_{jk} B_{2j} B_{4k+2-2j}}{(2j)!(4k+2-2j)!} \\ &\quad - \frac{2a_k + 4}{a_k} \mathcal{L}_{e^{-2\pi}}(-4k-1) + \frac{4}{a_k} \mathcal{L}_{e^{-4\pi}}(-4k-1), \end{aligned}$$

where

$$\begin{aligned} a_k &= 2^{4k+1} - (-1)^k 2^{2k} - 1, \\ b_{jk} &= \frac{2^{2j-1} [1 + (1+i)^{4k+1-2j}] - 2^{4k+1-2j} [1 + (1+i)^{2j-1}]}{a_k}. \end{aligned}$$

Proof. Applying Theorem 2.1(3) at $t = \frac{1}{2}$ gives

$$\begin{aligned} &2^{2k} \mathcal{L}_{e^{-\pi}}(-4k-1) - \frac{1}{2^{2k}} \mathcal{L}_{e^{-4\pi}}(-4k-1) \\ (3.1) \quad &= - (2\pi)^{4k+1} \sum_{j=0}^k \frac{(-1)^{j+1} B_{2j} B_{4k+2-2j} \sinh(\log 2^{2k+1-2j})}{(2j)!(4k+2-2j)!} \\ &\quad - \zeta(4k+1) \sinh(\log 2^{2k}). \end{aligned}$$

Applying Theorem 2.1(3) at $t = \frac{1}{2} - \frac{i}{2} = 1/(1+i)$ gives

$$\begin{aligned} &(1+i)^{2k} \mathcal{L}_{e^{-\pi}}(-4k-1, 1) - \frac{1}{(1+i)^{2k}} \mathcal{L}_{e^{-2\pi}}(-4k-1) \\ (3.2) \quad &= - (2\pi)^{4k+1} \sum_{j=0}^k \frac{(-1)^{j+1} B_{2j} B_{4k+2-2j} \sinh[\log(1+i)^{2k+1-2j}]}{(2j)!(4k+2-2j)!} \\ &\quad - \zeta(4k+1) \sinh[\log(1+i)^{2k}]. \end{aligned}$$

Applying Lemma 3.1 at $q = e^{-2\pi}$ with $p = 2$ gives

$$\begin{aligned} &\mathcal{L}_{e^{-\pi}}(-4k-1) + \mathcal{L}_{e^{-\pi}}(-4k-1) \\ (3.3) \quad &= (2^{-4k} + 2) \mathcal{L}_{e^{-2\pi}}(-4k-1) - 2^{-4k} \mathcal{L}_{e^{-4\pi}}(-4k-1). \end{aligned}$$

Eliminating $\mathcal{L}_{e^{-\pi}}(-4k-1)$ and $\mathcal{L}_{e^{-\pi}}(-4k-1)$ from Equations (3.1), (3.2), and (3.3) and solving for $\zeta(4k+1)$ in terms of $\mathcal{L}_{e^{-2\pi}}(-4k-1)$ and $\mathcal{L}_{e^{-4\pi}}(-4k-1)$ completes the proof of the theorem. \square

Theorem 3.3. *The Riemann zeta function at odd integers $4k+1$ other than unity can be written in terms of Lambert series at $q = -e^{-3\pi}$, $e^{-4\pi}$, and $e^{-6\pi}$ as:*

$$\begin{aligned} \zeta(4k+1) &= \frac{(2\pi)^{4k+1}}{2b_k} \sum_{j=0}^k \frac{(-1)^{j+1} c_{jk} B_{2j} B_{4k+2-2j}}{(2j)!(4k+2-2j)!} + \frac{(-1)^k 2^{2k+1}}{b_k} \mathcal{L}_{e^{-3\pi}}(-4k-1) \\ &\quad - \frac{a_k}{2^{4k-1} b_k} \mathcal{L}_{e^{-4\pi}}(-4k-1) + \frac{2^{2k}}{2^{2k-1} b_k} \mathcal{L}_{e^{-6\pi}}(-4k-1) \end{aligned}$$

where

$$\begin{aligned}
a_k &= \frac{2^{4k} (3^{4k+1} + 1)}{2^{4k+1} - (-1)^k 2^{2k} + 1}, \\
b_k &= \frac{3^{4k+1} - 1}{2} - (-1)^k 2^{2k} - a_k \left[\frac{2^{4k+1} - (-1)^k 2^{2k} - 1}{2^{4k+1}} \right], \\
c_{jk} &= 3^{2j-1} \sum_{n=-1}^1 (1+in)^{4k+1-2j} - 3^{4k+1-2j} \sum_{n=-1}^1 (1+in)^{2j-1} \\
&\quad - a_k \left[\frac{1 + (1+i)^{4k+1-2j}}{2^{4k+1-2j}} - \frac{1 + (1+i)^{2j-1}}{2^{2j-1}} \right].
\end{aligned}$$

Remark This formula converges at about 4.09 digits for each additional term in the Lambert series. The first few examples of this theorem are listed in the Appendix. If needed, the Lambert series at $q = -e^{-3\pi}$ can be replaced by Lambert series at positive only q values using Lemma 3.1 at $q = e^{-6\pi}$ with $p = 2$,

$$\begin{aligned}
\mathcal{L}_{-e^{-3\pi}}(-4k-1) &= -\mathcal{L}_{e^{-3\pi}}(-4k-1) + (2^{-4k} + 2)\mathcal{L}_{e^{-6\pi}}(-4k-1) \\
&\quad - 2^{-4k}\mathcal{L}_{e^{-12\pi}}(-4k-1).
\end{aligned}$$

However, the result as stated in the theorem is more compact and has the same rate of convergence as the one with positive q values. Similar replacement for Lambert series at negative q values can be made in Theorems 3.4 and 4.3 as well.

Proof. The proof is a generalization of the proof of Theorem 3.2 to the $p = 3$ case. Applying Theorem 2.1(3) at $t = \frac{1}{3}$ and $t = \frac{1}{3} \pm \frac{i}{3}$ yields three equations involving $\mathcal{L}_{e^{-\frac{2\pi}{3}}}(-4k-1)$, $\mathcal{L}_{e^{-\frac{2\pi}{3} \pm \frac{2\pi i}{3}}}(-4k-1)$, and $\zeta(4k+1)$ in terms of $\mathcal{L}_{-e^{-3\pi}}(-4k-1)$, and $\mathcal{L}_{e^{-6\pi}}(-4k-1)$. Applying Lemma 3.1 at $q = e^{-2\pi}$ with $p = 3$ gives

$$\sum_{n=-1}^1 \mathcal{L}_{e^{-\frac{2\pi}{3} + i\frac{2\pi n}{3}}}(-4k-1) = (3^{-4k} + 3)\mathcal{L}_{e^{-2\pi}}(-4k-1) - 3^{-4k}\mathcal{L}_{e^{-6\pi}}(-4k-1).$$

Theorem 3.2 supplies the fifth equation that relates $\mathcal{L}_{e^{-2\pi}}(-4k-1)$ to $\mathcal{L}_{e^{-4\pi}}(-4k-1)$ and $\zeta(4k+1)$.

Using these five equations to eliminate $\mathcal{L}_{e^{-\frac{2\pi}{3}}}(-4k-1)$, $\mathcal{L}_{e^{-\frac{2\pi}{3} \pm \frac{2\pi i}{3}}}(-4k-1)$, and $\mathcal{L}_{e^{-2\pi}}(-4k-1)$ and solving for $\zeta(4k+1)$ in terms of $\mathcal{L}_{-e^{-3\pi}}(-4k-1)$, $\mathcal{L}_{e^{-4\pi}}(-4k-1)$, and $\mathcal{L}_{e^{-6\pi}}(-4k-1)$ completes the proof of the theorem. \square

Theorem 3.4. *The Riemann zeta function at odd integers $4k+1$ other than unity can be written in terms of Lambert series at $q = e^{-4\pi}$, $e^{-5\pi}$, and $e^{-10\pi}$ as:*

$$\begin{aligned}
\zeta(4k+1) &= \frac{(2\pi)^{4k+1}}{2b_k} \sum_{j=0}^k \frac{(-1)^{j+1} c_{jk} B_{2j} B_{4k+2-2j}}{(2j)!(4k+2-2j)!} - \frac{1}{2^{4k-1} b_k} \mathcal{L}_{e^{-4\pi}}(-4k-1) \\
&\quad + \frac{(-1)^k 2^{2k+1} a_k}{b_k} \mathcal{L}_{-e^{-5\pi}}(-4k-1) + \frac{2a_k}{b_k} \mathcal{L}_{e^{-10\pi}}(-4k-1),
\end{aligned}$$

where

$$\begin{aligned}
 a_k &= \frac{2^{4k+1} - (-1)^k 2^{2k} + 1}{2^{4k} [5^{4k+1} - 2 \times 5^{2k} \cos(4k \tan^{-1} 2) + 1]}, \\
 b_k &= \frac{a_k}{2} \left[5^{4k+1} - \sum_{n=-2}^2 (1+in)^{4k} \right] - \frac{2^{4k+1} - (-1)^k 2^{2k} - 1}{2^{4k+1}}, \\
 c_{jk} &= a_k \left[5^{2j-1} \sum_{n=-2}^2 (1+in)^{4k+1-2j} - 5^{4k+1-2j} \sum_{n=-2}^2 (1+in)^{2j-1} \right] \\
 &\quad - \left[\frac{1 + (1+i)^{4k+1-2j}}{2^{4k+1-2j}} - \frac{1 + (1+i)^{2j-1}}{2^{2j-1}} \right].
 \end{aligned}$$

Remark This formula has a convergence rate of about 5.45 digits per term and is our best result for $\zeta(4k+1)$. The first few examples of this theorem are listed in the Appendix.

Proof. The proof of this theorem is very similar to that of Theorem 3.3. Theorem 2.1(3) is now applied at $t = \frac{1}{5}, \frac{1}{5} \pm \frac{2}{5},$ and $\frac{1}{5} \pm \frac{2i}{5}$ and Lemma 3.1 is applied at $q = e^{-2\pi}$ with $p = 5$ instead of $p = 3$. Rest of the logic remains the same. \square

4. ADDITIONAL FORMULAE FOR $\zeta(4k+1)$

In the previous section we provided formulae for $\zeta(4k+1)$ in terms of Lambert series at q values of the form $e^{-2^n p \pi}$ where p was a prime equal to 2, 3, or 5 and n was 0, 1, or 2. In this section we provide an additional class of formulae for $\zeta(4k+1)$ using Lambert series with q values of the form $e^{-2^n \sqrt{m} \pi}$ where m is equal to 3, 7, or 15 and n is 0, 1, or 2.

First we state the following two lemmas on which the theorems in this section are based. Both lemmas combine two Lambert series to produce a series with a stronger convergence rate per term.

Lemma 4.1. *For any $s \in \mathbb{C}$ the Lambert series at $x = 1$ satisfies:*

$$\begin{aligned}
 \mathcal{L}_{i\sqrt{q}}(s) + \mathcal{L}_{-i\sqrt{q}}(s) &= -(2^{s+1} + 2) \mathcal{L}_q(s) + (2^{2s+2} + 3 \times 2^{s+1} + 4) \mathcal{L}_{q^2}(s) \\
 &\quad - (2^{2s+2} + 2^{s+2}) \mathcal{L}_{q^4}(s).
 \end{aligned}$$

Proof. Applying Lemma 3.1 at $p = 2$ and replacing q with $-q$ gives

$$(4.1) \quad \mathcal{L}_{i\sqrt{q}}(s) + \mathcal{L}_{-i\sqrt{q}}(s) = (2^{s+1} + 2) \mathcal{L}_{-q}(s) - 2^{s+1} \mathcal{L}_{q^2}(s).$$

Now using Lemma 3.1 at $p = 2$ and replacing q with q^2 gives

$$(4.2) \quad \mathcal{L}_q(s) + \mathcal{L}_{-q}(s) = (2^{s+1} + 2) \mathcal{L}_{q^2}(s) - 2^{s+1} \mathcal{L}_{q^4}(s).$$

Solving for $\mathcal{L}_{-q}(s)$ in the second equation and replacing it in the first equation above completes the proof. \square

Lemma 4.2. *For any $s \in \mathbb{C}$ the Lambert series at $x = 1$ satisfies:*

$$\mathcal{S}_q(s) \equiv i [\mathcal{L}_{i\sqrt{q}}(s) - \mathcal{L}_{-i\sqrt{q}}(s)] = \sum_{n=0}^{\infty} (-1)^{n+1} (2n+1)^s \operatorname{sech} \left[\left(n + \frac{1}{2} \right) \log q \right].$$

Proof.

$$\begin{aligned}
\mathcal{L}_{iq}(s) - \mathcal{L}_{-iq}(s) &= \sum_{n=1}^{\infty} \frac{n^s (iq)^n}{1 - (iq)^n} - \sum_{n=1}^{\infty} \frac{n^s (-iq)^n}{1 - (-iq)^n} \\
&= \sum_{m=0}^{\infty} \frac{(2m+1)^s (iq)^{2m+1}}{1 - (iq)^{2m+1}} - \sum_{m=0}^{\infty} \frac{(2m+1)^s (-iq)^{2m+1}}{1 - (-iq)^{2m+1}} \\
&= \sum_{m=0}^{\infty} \frac{2(2m+1)^s (iq)^{2m+1}}{1 - (iq)^{4m+2}} \\
&= -2i \sum_{m=0}^{\infty} \frac{(-1)^{m+1} (2m+1)^s q^{2m+1}}{1 + q^{4m+2}} \\
&= -i \sum_{m=0}^{\infty} (-1)^{m+1} (2m+1)^s \operatorname{sech}[(2m+1) \log q].
\end{aligned}$$

Replacing q with \sqrt{q} and multiplying by i on both sides completes the proof. \square

Now we state the three formulae for $\zeta(4k+1)$ in increasing order of their convergence rates in the form of the following theorems.

Theorem 4.3. *The Riemann zeta function at odd integers $4k+1$ other than unity can be written in terms of the Lambert series at $q = -e^{-\sqrt{3}\pi}$ as:*

$$\zeta(4k+1) = (2\pi)^{4k+1} \sum_{j=0}^k \frac{(-1)^j b_{jk} B_{2j} B_{4k+2-2j}}{(2j)!(4k+2-2j)!} - a_k \mathcal{L}_{-e^{-\sqrt{3}\pi}}(-4k-1)$$

where

$$\begin{aligned}
a_k &= \frac{2^{4k+1} + 1}{2^{4k} - \cos\left(\frac{2\pi k}{3}\right)}, \\
b_{jk} &= \frac{2^{2j-1} \sin\left[\frac{(2k-1-j)\pi}{3}\right] + 2^{4k+1-2j} \sin\left[\frac{(j+1)\pi}{3}\right]}{2^{4k} - \cos\left(\frac{2\pi k}{3}\right)}.
\end{aligned}$$

Proof. Substituting $t = (\sqrt{3} \pm i)/2$ in Theorem 2.1(3) and adding gives one equation in $\mathcal{L}_{-e^{-\sqrt{3}\pi}}(-4k-1)$ and $\zeta(4k+1)$. Solving for $\zeta(4k+1)$ yields the final result. \square

Theorem 4.4. *The Riemann zeta function at odd integers $4k+1$ other than unity can be written in terms of the Lambert series at $q = e^{-\sqrt{7}\pi}$, $e^{-2\sqrt{7}\pi}$, and $e^{-4\sqrt{7}\pi}$ as:*

$$\begin{aligned}
\zeta(4k+1) &= (2\pi)^{4k+1} \sum_{j=0}^k \frac{(-1)^j c_{jk} B_{2j} B_{4k+2-2j}}{(2j)!(4k+2-2j)!} + a_k \mathcal{L}_{e^{-\sqrt{7}\pi}}(-4k-1) \\
&\quad + b_k \mathcal{L}_{e^{-2\sqrt{7}\pi}}(-4k-1) + 2^{-4k} a_k \mathcal{L}_{e^{-4\sqrt{7}\pi}}(-4k-1),
\end{aligned}$$

where

$$\begin{aligned}
a_k &= \frac{2 + 2^{-4k} - 2^{-2k+1} \cos(4k\theta)}{1 - 2^{-2k} \cos(4k\theta)}, \\
b_k &= -\frac{4 + 3 \times 2^{-4k} + 2^{-8k} - 2^{-2k+2} \cos(4k\theta) - 2^{-6k+1} \cos(4k\theta)}{1 - 2^{-2k} \cos(4k\theta)}, \\
c_{jk} &= \frac{2^{2k+\frac{1}{2}-j} \cos[(2j-1)\theta] - 2^{j-\frac{1}{2}} \cos[(4k+1-2j)\theta]}{2^{2k} - \cos(4k\theta)},
\end{aligned}$$

and $\theta = \cot^{-1} \sqrt{7}$.

Proof. Substituting $t = (\sqrt{7} \pm i)/4$ in Theorem 2.1(3) and adding gives one equation in $\mathcal{L}_{\pm i e^{-\sqrt{7}\pi/2}}(-4k-1)$, $\mathcal{L}_{-e^{-\sqrt{7}\pi}}(-4k-1)$, and $\zeta(4k+1)$. Eliminating the sum of $\mathcal{L}_{\pm i e^{-\sqrt{7}\pi/2}}(-4k-1)$ using Lemma 4.1 at $q = e^{-\sqrt{7}\pi}$, and eliminating $\mathcal{L}_{-e^{-\sqrt{7}\pi}}(-4k-1)$ using Lemma 3.1 for $p = 2$ at $q = e^{-2\sqrt{7}\pi}$ yields the final result. \square

Theorem 4.5. *The Riemann zeta function at odd integers $4k+1$ other than unity can be written in terms of the Lambert series at $q = e^{-\sqrt{15}\pi}$, $e^{-2\sqrt{15}\pi}$, and $e^{-4\sqrt{15}\pi}$, and the series $\mathcal{S}_{e^{-\sqrt{15}\pi}}(-4k-1)$ as:*

$$\begin{aligned} \zeta(4k+1) &= (2\pi)^{4k+1} \sum_{j=0}^k \frac{(-1)^j c_{jk} B_{2j} B_{4k+2-2j}}{(2j)!(4k+2-2j)!} + \cot(2k\theta) \mathcal{S}_{e^{-\sqrt{15}\pi}}(-4k-1) \\ &\quad + (2^{-4k} + 2) \mathcal{L}_{e^{-\sqrt{15}\pi}}(-4k-1) \\ &\quad - (2^{-8k} + 3 \times 2^{-4k} + 4) \mathcal{L}_{e^{-2\sqrt{15}\pi}}(-4k-1) \\ &\quad + (2^{-8k} + 2^{-4k+1}) \mathcal{L}_{e^{-4\sqrt{15}\pi}}(-4k-1), \end{aligned}$$

where

$$c_{jk} = \csc(2k\theta) \sin[(2k+1-2j)\theta],$$

and $\theta = \cot^{-1} \sqrt{15}$.

Remark This result converges almost as rapidly as Theorem 3.4 at 5.28 decimal digits per term. Some examples of this theorem are listed in the Appendix.

Proof. Substituting $t = (\sqrt{15} \pm i)/4$ in Theorem 2.1(3) and adding gives one equation containing both the sum and difference of $\mathcal{L}_{\pm i e^{-\sqrt{15}\pi/2}}(-4k-1)$, and $\zeta(4k+1)$. Eliminating the sum of $\mathcal{L}_{\pm i e^{-\sqrt{15}\pi/2}}(-4k-1)$ using Lemma 4.1 at $q = e^{-\sqrt{15}\pi}$, and eliminating the difference of $\mathcal{L}_{\pm i e^{-\sqrt{15}\pi/2}}(-4k-1)$ using Lemma 4.2 at $q = e^{-\sqrt{15}\pi}$ yields the final result. \square

5. FORMULAE FOR $\zeta(4k-1)$

The methods of Section 4 are mirrored with minor variations to produce similar results for $\zeta(4k-1)$ stated in terms of the three theorems below.

Theorem 5.1. *The Riemann zeta function at odd integers $4k-1$ can be written in terms of Lambert series at $q = e^{-\sqrt{3}\pi}$, $e^{-2\sqrt{3}\pi}$, and $e^{-4\sqrt{3}\pi}$ as:*

$$\begin{aligned} \zeta(4k-1) &= (2\pi)^{4k-1} \sum_{j=0}^k \frac{(-1)^{j+1} b_{jk} B_{2j} B_{4k-2j}}{(2j)!(4k-2j)!} - \frac{2^{4k+1} + 4}{a_k} \mathcal{L}_{e^{-\sqrt{3}\pi}}(-4k+1) \\ &\quad - \frac{2^{4k+2} + 2^{-4k+4} + 12 + 8 \cos\left[\frac{(2k-1)\pi}{3}\right]}{a_k} \mathcal{L}_{e^{-2\sqrt{3}\pi}}(-4k+1) \\ &\quad + \frac{2^{-4k+4} + 8}{a_k} \mathcal{L}_{e^{-4\sqrt{3}\pi}}(-4k+1), \end{aligned}$$

where

$$a_k = 2^{4k} + 4 \sin\left[\frac{(4k+1)\pi}{6}\right],$$

$$b_{jk} = \frac{2^{4k+1-2j} \cos\left[\frac{(2j-1)\pi}{6}\right] + 2^{2j+1} \cos\left[\frac{(4k-1-2j)\pi}{6}\right]}{a_k(1 + \delta_{jk})}.$$

Proof. Substituting $t = (\sqrt{3} \pm i)/4$ in Theorem 2.1(2) and adding the results gives one equation in $\mathcal{L}_{\pm ie^{-\sqrt{3}\pi/2}}(-4k+1)$, $\mathcal{L}_{e^{-2\sqrt{3}\pi}}(-4k+1)$, and $\zeta(4k-1)$. Eliminating $\mathcal{L}_{\pm ie^{-\sqrt{3}\pi/2}}(-4k+1)$ in favor of $\mathcal{L}_{e^{-2\sqrt{3}\pi}}(-4k+1)$, $\mathcal{L}_{e^{-2\sqrt{3}\pi}}(-4k+1)$, and $\mathcal{L}_{e^{-4\sqrt{3}\pi}}(-4k+1)$ using Lemma 3.1 for $p = 2$ at $q = e^{-2\sqrt{3}\pi}$ yields the final result. \square

Theorem 5.2. *The Riemann zeta function at odd integers $4k - 1$ can be written in terms of Lambert series at $q = e^{-\sqrt{7}\pi}$, $e^{-2\sqrt{7}\pi}$, and $e^{-4\sqrt{7}\pi}$ as:*

$$\begin{aligned} \zeta(4k-1) &= (2\pi)^{4k-1} \sum_{j=0}^k \frac{(-1)^{j+1} c_{jk} B_{2j} B_{4k-2j}}{(2j)!(4k-2j)!} + \frac{b_k}{a_k} \mathcal{L}_{e^{-\sqrt{7}\pi}}(-4k+1) \\ &\quad - \frac{(2^{-4k+2} + 2) b_k - 2^{-2k+2}}{a_k} \mathcal{L}_{e^{-2\sqrt{7}\pi}}(-4k+1) \\ &\quad + \frac{2^{-4k+2} b_k}{a_k} \mathcal{L}_{e^{-4\sqrt{7}\pi}}(-4k+1), \end{aligned}$$

where

$$\begin{aligned} a_k &= 2^{2k} + 2 \cos[(4k-2)\theta], \\ b_k &= 2^{2k+1} + 2^{-2k+2} + 2^{\frac{3}{2}} \sin[(4k+1)\theta] + 2 \cos(4k\theta), \\ c_{jk} &= \frac{2^{2k+\frac{1}{2}-j} \cos[(2j-1)\theta] + 2^{j+\frac{1}{2}} \cos[(4k-1-2j)\theta]}{a_k(1 + \delta_{jk})}, \end{aligned}$$

and $\theta = \cot^{-1} \sqrt{7}$.

Remark This formula for $\zeta(4k-1)$ has a convergence rate of about 3.6 decimal digits per term which is about 0.9 decimal digits per term faster than that of Corollary 2.2. Some examples of this theorem are listed in the Appendix.

Proof. Substituting $t = (\sqrt{7} \pm i)/4$ in Theorem 2.1(2) and adding the results gives one equation in $\mathcal{L}_{\pm ie^{-\sqrt{7}\pi/2}}(-4k+1)$, $\mathcal{L}_{e^{-\sqrt{7}\pi}}(-4k+1)$, and $\zeta(4k-1)$. Eliminating the sum of $\mathcal{L}_{\pm ie^{-\sqrt{7}\pi/2}}(-4k+1)$ using Lemma 4.1 at $q = e^{-\sqrt{7}\pi}$, and eliminating $\mathcal{L}_{e^{-\sqrt{7}\pi}}(-4k+1)$ using Lemma 3.1 for $p = 2$ at $q = e^{-2\sqrt{7}\pi}$ yields the final result. \square

Theorem 5.3. *The Riemann zeta function at odd integers $4k - 1$ can be written in terms of Lambert series at $q = e^{-\sqrt{15}\pi}$, $e^{-2\sqrt{15}\pi}$, and $e^{-4\sqrt{15}\pi}$, and the series $\mathcal{S}_{e^{-\sqrt{15}\pi}}(-4k+1)$ as:*

$$\begin{aligned} \zeta(4k-1) &= (2\pi)^{4k-1} \sum_{j=0}^k \frac{(-1)^{j+1} c_{jk} B_{2j} B_{4k-2j}}{(2j)!(4k-2j)!} + a_k \mathcal{S}_{e^{-\sqrt{15}\pi}}(-4k+1) \\ &\quad + (2^{-4k+2} + 2) b_k \mathcal{L}_{e^{-\sqrt{15}\pi}}(-4k+1) \\ &\quad - (2^{-8k+4} + 3 \times 2^{-4k+2} + 4) b_k \mathcal{L}_{e^{-2\sqrt{15}\pi}}(-4k+1) \\ &\quad + (2^{-8k+4} + 2^{-4k+3}) b_k \mathcal{L}_{e^{-4\sqrt{15}\pi}}(-4k+1), \end{aligned}$$

where

$$\begin{aligned} a_k &= \frac{\cos[(4k-1)\theta] - 2\sin(4k\theta)}{4\cos^2[(2k-1)\theta]}, \\ b_k &= \frac{2 + 2\cos(4k\theta) + \sin[(4k-1)\theta]}{4\cos^2[(2k-1)\theta]}, \\ c_{jk} &= \frac{\cos[(2k-2j)\theta]}{(1 + \delta_{jk})\cos[(2k-1)\theta]}, \end{aligned}$$

and $\theta = \cot^{-1}\sqrt{15}$.

Remark This formula has a convergence rate of about 5.28 decimal digits per term which is the best of our results for $\zeta(4k-1)$. Some examples of this theorem are listed in the Appendix.

Proof. Substituting $t = (\sqrt{15} \pm i)/4$ in Theorem 2.1(2) and adding the results gives one equation containing both the sum and difference of $\mathcal{L}_{\pm ie^{-\sqrt{15}\pi/2}}(-4k+1)$, and $\zeta(4k-1)$. Eliminating the sum of $\mathcal{L}_{\pm ie^{-\sqrt{15}\pi/2}}(-4k+1)$ using Lemma 4.1 at $q = e^{-\sqrt{15}\pi}$, and eliminating the difference of $\mathcal{L}_{\pm ie^{-\sqrt{15}\pi/2}}(-4k+1)$ using Lemma 4.2 at $q = e^{-\sqrt{15}\pi}$ yields the final result. \square

6. ADDITIONAL RESULTS FOR THE LAMBERT SERIES

Some other interesting results can be arrived at through the manipulation of results in the earlier sections. For example, from Theorem 2.1 we get

Theorem 6.1. *For any constant a with $\operatorname{Re}(a) > 0$, the Lambert series $\mathcal{L}_q(s)$ at negative odd integers s satisfies the following:*

- (1) For $s = -(4k+1) = -1, -5, -9, \dots$,

$$\begin{aligned} & \frac{1}{t^{2k}} \left[a^{2k} \mathcal{L}_{e^{-\frac{2\pi t}{a}}}(-4k-1) - (a^{2k} + a^{-2k}) \mathcal{L}_{e^{-2\pi t}}(-4k-1) \right. \\ & \quad \left. + a^{-2k} \mathcal{L}_{e^{-2\pi at}}(-4k-1) \right] \\ & - t^{2k} \left[a^{2k} \mathcal{L}_{e^{-\frac{2\pi}{at}}}(-4k-1) - (a^{2k} + a^{-2k}) \mathcal{L}_{e^{-\frac{2\pi}{t}}}(-4k-1) \right. \\ & \quad \left. + a^{-2k} \mathcal{L}_{e^{-\frac{2\pi a}{t}}}(-4k-1) \right] \\ & = (2\pi)^{4k+1} \sum_{j=0}^k \frac{(-1)^{j+1} b_{jk}(a) B_{2j} B_{4k+2-2j} \sinh[(2k+1-2j) \log t]}{(2j)! (4k+2-2j)!}. \end{aligned}$$

- (2) For $s = -(4k-1) = -3, -7, -11, \dots$,

$$\begin{aligned} & \frac{1}{t^{2k-1}} \left[a^{2k-1} \mathcal{L}_{e^{-\frac{2\pi t}{a}}}(-4k+1) - (a^{2k-1} + a^{-2k+1}) \mathcal{L}_{e^{-2\pi t}}(-4k+1) \right. \\ & \quad \left. + a^{-2k+1} \mathcal{L}_{e^{-2\pi at}}(-4k+1) \right] \\ & + t^{2k-1} \left[a^{2k-1} \mathcal{L}_{e^{-\frac{2\pi}{at}}}(-4k+1) - (a^{2k-1} + a^{-2k+1}) \mathcal{L}_{e^{-\frac{2\pi}{t}}}(-4k+1) \right. \\ & \quad \left. + a^{-2k+1} \mathcal{L}_{e^{-\frac{2\pi a}{t}}}(-4k+1) \right] \\ & = (2\pi)^{4k-1} \sum_{j=0}^k \frac{(-1)^j c_{jk}(a) B_{2j} B_{4k-2j} \cosh[(2k-2j) \log t]}{(2j)! (4k-2j)!}, \end{aligned}$$

where

$$b_{jk}(a) = a^{2k} + a^{-2k} - a^{2k+1-2j} - a^{-2k-1+2j},$$

and

$$c_{jk}(a) = \frac{a^{2k-1} + a^{-2k+1} - a^{2k-2j} - a^{-2k+2j}}{(1 + \delta_{jk})}.$$

Proof. The proof follows from making the substitutions $t \rightarrow at$ and $t \rightarrow t/a$ in Theorem 2.1 and then combining them with the original result in Theorem 2.1 in the manner stated in the theorem above. The term containing the zeta function at odd arguments, $\zeta(4k \pm 1)$, is eliminated. The special case of $s = -1$ is included as the $k = 0$ case in $s = -(4k + 1)$. The $\log t$ term is eliminated in this case. \square

As examples of Theorem 6.1, odd powers of π can be calculated in terms of the Lambert series at desired q values as stated below.

Example By choosing $t = \frac{1}{2} \pm \frac{i}{2}$ and $a = \frac{1}{2}$ in Theorem 6.1(1), eliminating $\mathcal{L}_{-e^{-\pi}}(-4k - 1)$ using Lemma 3.1 at $q = e^{-\pi}$ and $p = 2$, and eliminating the sum $\mathcal{L}_{ie^{-\frac{\pi}{2}}}(-4k - 1) + \mathcal{L}_{-ie^{-\frac{\pi}{2}}}(-4k - 1)$ using Lemma 4.1 we get

$$\begin{aligned} \pi &= 72 \mathcal{L}_{e^{-\pi}}(-1) - 96 \mathcal{L}_{e^{-2\pi}}(-1) + 24 \mathcal{L}_{e^{-4\pi}}(-1), \\ \pi^5 &= 7056 \mathcal{L}_{e^{-\pi}}(-5) - 6993 \mathcal{L}_{e^{-2\pi}}(-5) - 63 \mathcal{L}_{e^{-4\pi}}(-5), \\ \pi^9 &= \frac{28226880}{41} \mathcal{L}_{e^{-\pi}}(-9) - \frac{112920885}{164} \mathcal{L}_{e^{-2\pi}}(-9) + \frac{13365}{164} \mathcal{L}_{e^{-4\pi}}(-9). \end{aligned}$$

Example By choosing $t = 1$ and $a = \frac{1}{2}$ in Theorem 6.1(2) we get

$$\begin{aligned} \pi^3 &= 720 \mathcal{L}_{e^{-\pi}}(-3) - 900 \mathcal{L}_{e^{-2\pi}}(-3) + 180 \mathcal{L}_{e^{-4\pi}}(-3), \\ \pi^7 &= \frac{907200}{13} \mathcal{L}_{e^{-\pi}}(-7) - 70875 \mathcal{L}_{e^{-2\pi}}(-7) + \frac{14175}{13} \mathcal{L}_{e^{-4\pi}}(-7), \\ \pi^{11} &= \frac{27243216000}{4009} \mathcal{L}_{e^{-\pi}}(-11) - \frac{218158565625}{32072} \mathcal{L}_{e^{-2\pi}}(-11) + \frac{212837625}{32072} \mathcal{L}_{e^{-4\pi}}(-11). \end{aligned}$$

Remark The results for the odd powers of π in the two examples above agree with those listed by Plouffe [7].

Proposition 6.2. *The results of Theorems 3.3 and 3.4 can be combined to produce faster converging series for π^{4k+1} (than in the example above) in terms of the Lambert series at $q = -e^{-3\pi}$, $e^{-4\pi}$, $-e^{-5\pi}$, $e^{-6\pi}$, and $e^{-10\pi}$. For example,*

$$\begin{aligned} \pi^5 &= -3686634 \mathcal{L}_{-e^{-3\pi}}(-5) + 2463048 \mathcal{L}_{e^{-4\pi}}(-5) + 402570 \mathcal{L}_{-e^{-5\pi}}(-5) \\ &\quad + \frac{1843317}{2} \mathcal{L}_{e^{-6\pi}}(-5) - \frac{201285}{2} \mathcal{L}_{e^{-10\pi}}(-5). \end{aligned}$$

Remark Combining Theorems 3.4 and 4.5 gives the fastest converging, but not the cleanest, result for π^{4k+1} .

Similarly,

Proposition 6.3. *The results of Corollary 2.2 and Theorem 5.2 can be combined to produce a faster converging series for π^{4k-1} (than in the example above) in terms*

of the Lambert series at $q = e^{-2\pi}$, $e^{-\sqrt{7}\pi}$, $e^{-2\sqrt{7}\pi}$, and $e^{-4\sqrt{7}\pi}$. For example,

$$\begin{aligned} \pi^3 = & \frac{3960}{77 - 29\sqrt{7}} \mathcal{L}_{e^{-2\pi}}(-3) + \frac{4320}{77 - 29\sqrt{7}} \mathcal{L}_{e^{-\sqrt{7}\pi}}(-3) \\ & - \frac{9360}{77 - 29\sqrt{7}} \mathcal{L}_{e^{-2\sqrt{7}\pi}}(-3) + \frac{1080}{77 - 29\sqrt{7}} \mathcal{L}_{e^{-4\sqrt{7}\pi}}(-3). \end{aligned}$$

Remark Combining the results of Theorems 5.2 and 5.3 gives the fastest converging result for π^{4k-1} .

Proposition 6.4. *By using the methods outlined in Section 3 but with $s = -1$ we can calculate the logarithm of the first three primes in terms of the Lambert series:*

$$\begin{aligned} \log 2 &= \frac{2\pi}{9} - \frac{8}{3} \mathcal{L}_{e^{-2\pi}}(-1) + \frac{8}{3} \mathcal{L}_{e^{-4\pi}}(-1), \\ \log 3 &= \frac{19\pi}{54} - \frac{32}{9} \mathcal{L}_{e^{-2\pi}}(-1) + \frac{4}{3} \mathcal{L}_{e^{-3\pi}}(-1) + \frac{8}{9} \mathcal{L}_{e^{-4\pi}}(-1) + \frac{16}{3} \mathcal{L}_{e^{-6\pi}}(-1), \\ \log 5 &= \frac{37\pi}{72} - \frac{8}{3} \mathcal{L}_{e^{-2\pi}}(-1) + \frac{2}{3} \mathcal{L}_{e^{-4\pi}}(-1) + \mathcal{L}_{e^{-5\pi}}(-1) + \mathcal{L}_{e^{-10\pi}}(-1). \end{aligned}$$

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Appendix

We list examples of the first few $\zeta(4k+1)$ generated by Theorem 3.3. For compactness only the coefficients are listed respectively for π^{4k+1} , $\mathcal{L}_{-e^{-3\pi}}(-4k-1)$, $\mathcal{L}_{e^{-4\pi}}(-4k-1)$, and $\mathcal{L}_{e^{-6\pi}}(-4k-1)$.

$$\begin{aligned} \zeta(5) &= \left[\frac{682}{201285}, -\frac{296}{355}, -\frac{488}{355}, \frac{74}{355} \right] \\ \zeta(9) &= \left[\frac{5048}{150155775}, \frac{2272}{1605}, -\frac{5624}{1605}, \frac{142}{1605} \right] \\ \zeta(13) &= \left[\frac{21462388}{62314387009875}, -\frac{1056896}{2114515}, -\frac{3188648}{2114515}, \frac{16514}{2114515} \right] \\ \zeta(17) &= \left[\frac{12292037116}{3476479836810605625}, \frac{66978304}{95520195}, -\frac{258280328}{95520195}, \frac{261634}{95520195} \right] \\ \zeta(21) &= \left[\frac{203055579851796692}{5594631411704844933908859375}, -\frac{4297066496}{12606788275}, -\frac{20920706408}{12606788275}, \right. \\ &\quad \left. \frac{4196354}{12606788275} \right] \\ \zeta(25) &= \left[\frac{91295430825021344}{245007095801727658882798940625}, \frac{274844360704}{709832878755}, -\frac{1694577218888}{709832878755}, \right. \\ &\quad \left. \frac{67100674}{709832878755} \right] \end{aligned}$$

Examples of the first few $\zeta(4k+1)$ generated by Theorem 3.4. The coefficients are listed respectively for π^{4k+1} , $\mathcal{L}_{e^{-4\pi}}(-4k-1)$, $\mathcal{L}_{-e^{-5\pi}}(-4k-1)$, and $\mathcal{L}_{e^{-10\pi}}(-4k-1)$. The result for $\zeta(5)$ matches that of Plouffe [7] when converted to positive q values.

$$\begin{aligned} \zeta(5) &= \left[\frac{694}{204813}, -\frac{6280}{3251}, -\frac{296}{3251}, \frac{74}{3251} \right] \\ \zeta(9) &= \left[\frac{6118928}{182032863705}, -\frac{3908360}{1945731}, \frac{15904}{1945731}, \frac{994}{1945731} \right] \\ \zeta(13) &= \left[\frac{4131911428}{11996181573401025}, -\frac{2441359240}{1221199811}, -\frac{1056896}{1221199811}, \frac{16514}{1221199811} \right] \\ \zeta(17) &= \left[\frac{687182059214356}{194362869568557017703375}, -\frac{1525878246920}{762905503491}, \frac{66978304}{762905503491}, \right. \\ &\quad \left. \frac{261634}{762905503491} \right] \\ \zeta(21) &= \left[\frac{2560199089127112465412}{70537137132904905751999929343125}, -\frac{953674355019400}{476839323944771}, \right. \\ &\quad \left. -\frac{4297066496}{476839323944771}, \frac{4196354}{476839323944771} \right] \\ \zeta(25) &= \left[\frac{114987316346581920808496}{308598430935986470640664020644801875}, -\frac{596046447625404680}{298023086356971651}, \right. \\ &\quad \left. \frac{274844360704}{298023086356971651}, \frac{67100674}{298023086356971651} \right] \end{aligned}$$

Examples of the first few $\zeta(4k+1)$ generated by Theorem 4.4. The coefficients are listed respectively for π^{4k+1} , $\mathcal{L}_{e^{-\sqrt{7}\pi}}(-4k-1)$, $\mathcal{L}_{e^{-2\sqrt{7}\pi}}(-4k-1)$, $\mathcal{L}_{e^{-4\sqrt{7}\pi}}(-4k-1)$.

$$\begin{aligned} \zeta(5) &= \left[\frac{5}{558\sqrt{7}}, \frac{64}{31}, -\frac{130}{31}, \frac{4}{31} \right] \\ \zeta(9) &= \left[\frac{6451}{72571950\sqrt{7}}, \frac{1088}{543}, -\frac{8713}{2172}, \frac{17}{2172} \right] \\ \zeta(13) &= \left[\frac{684521}{751529653875\sqrt{7}}, \frac{16480}{8239}, -\frac{4219139}{1054592}, \frac{515}{1054592} \right] \\ \zeta(17) &= \left[\frac{7556214529}{808189287857201250\sqrt{7}}, \frac{261248}{130623}, -\frac{267518969}{66878976}, \frac{2041}{66878976} \right] \\ \zeta(21) &= \left[\frac{11042228011}{115045098786113871375\sqrt{7}}, \frac{4191904}{2095951}, -\frac{274720686005}{68680122368}, \frac{130997}{68680122368} \right] \\ \zeta(25) &= \left[\frac{93518263081637}{94909028455546692340078125\sqrt{7}}, \frac{67120832}{33560415}, -\frac{35190647292091}{8797661429760}, \right. \\ &\quad \left. \frac{1048763}{8797661429760} \right] \end{aligned}$$

Examples of the first few $\zeta(4k+1)$ generated by Theorem 4.5. The coefficients are listed respectively for π^{4k+1} , $\mathcal{S}_{e^{-\sqrt{15}\pi}}(-4k-1)$, $\mathcal{L}_{e^{-\sqrt{15}\pi}}(-4k-1)$, $\mathcal{L}_{e^{-2\sqrt{15}\pi}}(-4k-1)$, $\mathcal{L}_{e^{-4\sqrt{15}\pi}}(-4k-1)$.

$$\begin{aligned} \zeta(5) &= \left[\frac{5}{378\sqrt{15}}, \frac{7}{\sqrt{15}}, \frac{33}{16}, -\frac{1073}{256}, \frac{33}{256} \right] \\ \zeta(9) &= \left[\frac{19}{145530\sqrt{15}}, \frac{17}{7\sqrt{15}}, \frac{513}{256}, -\frac{262913}{65536}, \frac{513}{65536} \right] \\ \zeta(13) &= \left[\frac{5623}{4214184975\sqrt{15}}, \frac{7}{33\sqrt{15}}, \frac{8193}{4096}, -\frac{67121153}{16777216}, \frac{8193}{16777216} \right] \\ \zeta(17) &= \left[\frac{152161}{11136941565750\sqrt{15}}, -\frac{223}{119\sqrt{15}}, \frac{131073}{65536}, -\frac{17180065793}{4294967296}, \frac{131073}{4294967296} \right] \\ \zeta(21) &= \left[\frac{2100413011}{15039186678619228125\sqrt{15}}, -\frac{1673}{305\sqrt{15}}, \frac{2097153}{1048576}, -\frac{4398049656833}{1099511627776}, \right. \\ &\quad \left. \frac{2097153}{1099511627776} \right] \\ \zeta(25) &= \left[\frac{368670553}{266533834992158608875\sqrt{15}}, -\frac{8143}{231\sqrt{15}}, \frac{33554433}{16777216}, \right. \\ &\quad \left. -\frac{1125899957174273}{281474976710656}, \frac{33554433}{281474976710656} \right] \\ \zeta(29) &= \left[\frac{276635171660523838}{18471447539635216765490460984375\sqrt{15}}, \frac{30233}{3263\sqrt{15}}, \frac{536870913}{268435456}, \right. \\ &\quad \left. -\frac{288230376957018113}{72057594037927936}, \frac{536870913}{72057594037927936} \right] \end{aligned}$$

Examples of the first few $\zeta(4k-1)$ generated by Theorem 5.2. The coefficients are listed respectively for π^{4k-1} , $\mathcal{L}_{e^{-\sqrt{7}\pi}}(-4k+1)$, $\mathcal{L}_{e^{-2\sqrt{7}\pi}}(-4k+1)$, $\mathcal{L}_{e^{-4\sqrt{7}\pi}}(-4k+1)$.

$$\begin{aligned}\zeta(3) &= \left[\frac{29\sqrt{7}}{1980}, \frac{24}{11}, -\frac{52}{11}, \frac{6}{11} \right] \\ \zeta(7) &= \left[\frac{851}{963900\sqrt{7}}, \frac{240}{119}, -\frac{1927}{476}, \frac{15}{476} \right] \\ \zeta(11) &= \left[\frac{98983}{11006745750\sqrt{7}}, \frac{3984}{1991}, -\frac{510073}{127424}, \frac{249}{127424} \right] \\ \zeta(15) &= \left[\frac{120891949}{1310075958262500\sqrt{7}}, \frac{65712}{32855}, -\frac{26916047}{6728704}, \frac{4107}{33643520} \right] \\ \zeta(19) &= \left[\frac{304799492533}{321754984333646613750\sqrt{7}}, \frac{1050576}{525287}, -\frac{34425307261}{8606302208}, \frac{65661}{8606302208} \right] \\ \zeta(23) &= \left[\frac{3069248396337203}{315604617827322095616093750\sqrt{7}}, \frac{16776432}{8388215}, -\frac{1759136500931}{439784046592}, \right. \\ &\quad \left. \frac{1048527}{2198920232960} \right]\end{aligned}$$

Examples of the first few $\zeta(4k-1)$ generated by Theorem 5.3. The coefficients are listed respectively for π^{4k-1} , $\mathcal{S}_{e^{-\sqrt{15}\pi}}(-4k+1)$, $\mathcal{L}_{e^{-\sqrt{15}\pi}}(-4k+1)$, $\mathcal{L}_{e^{-2\sqrt{15}\pi}}(-4k+1)$, $\mathcal{L}_{e^{-4\sqrt{15}\pi}}(-4k+1)$.

$$\begin{aligned}\zeta(3) &= \left[\frac{\sqrt{15}}{100}, -\frac{1}{\sqrt{15}}, \frac{9}{4}, -\frac{77}{16}, \frac{9}{16} \right] \\ \zeta(7) &= \left[\frac{73}{56700\sqrt{15}}, -\frac{11}{3\sqrt{15}}, \frac{129}{64}, -\frac{16577}{4096}, \frac{129}{4096} \right] \\ \zeta(11) &= \left[\frac{82889}{6385128750\sqrt{15}}, -\frac{61}{5\sqrt{15}}, \frac{2049}{1024}, -\frac{4197377}{1048576}, \frac{2049}{1048576} \right] \\ \zeta(15) &= \left[\frac{3103}{17239847625\sqrt{15}}, -\frac{11}{3\sqrt{15}}, \frac{32769}{16384}, -\frac{1073790977}{268435456}, \frac{32769}{268435456} \right] \\ \zeta(19) &= \left[\frac{269130227}{192947512626618750\sqrt{15}}, \frac{781}{171\sqrt{15}}, \frac{524289}{262144}, -\frac{274878693377}{68719476736}, \right. \\ &\quad \left. \frac{524289}{68719476736} \right] \\ \zeta(23) &= \left[\frac{247753871371}{17365083188883060881250\sqrt{15}}, \frac{1451}{989\sqrt{15}}, \frac{8388609}{4194304}, -\frac{70368756760577}{17592186044416}, \right. \\ &\quad \left. \frac{8388609}{17592186044416} \right]\end{aligned}$$

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