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Ramanujan Summation for Odd and Even Numbered Triangles

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ABSTRACT

Pascal's triangle is the simplest triangle containing binomial coefficients and possessing exclusive mathematical properties in it. There are several triangles of numbers exhibiting exotic mathematical oddities. In this paper, we will introduce two number triangles and determine Ramanujan summation related to their entries. In particular, we have proved new results related to Ramanujan summation values of series of odd and even powers of numbers considered from the two number triangles.

Keywords: Ramanujan Summation, Divergent Series, Number Triangles, Centered Numbers, Bernoulli Numbers.

1. Introduction

Among several methods applied for summing a divergent series, Cesaro method is well known for its novelty and depth. One of such methods of summing divergent series was introduced by great Indian mathematician Srinivasa Ramanujan during early part of 20th century. His results related to Ramanujan summation methods created great interest and development in Analytic Number Theory. In this paper, by introducing two number triangles, we will determine Ramanujan summation for various series obtained from the entries of the triangles.

2. Definition

Let $\sum_{n=1}^{\infty} a_n$ be a divergent series of real numbers. The Ramanujan summation abbreviated as *RS* (see **[1]**) of $\sum_{n=1}^{\infty} a_n$ is defined by $$(RS)\left(\sum_{n=1}^{\infty} a_n\right) = \int_{n=-1}^{0}\left(\sum_{k=1}^{n} a_k\right) dn \quad (2.1)$$

3. Construction of Number Triangles

In this section, we will introduce two number triangles, the first consisting of odd natural numbers and second that of even natural numbers displayed in Figures 1 and 2 respectively. We notice that in both cases, for any natural number *n*, row *n* contains *n* successive odd or successive even numbers. We shall call the number triangle consisting of odd natural numbers in Figure 1 as odd numbered triangle and that of even numbers in Figure 2 as even numbered triangle. In the following sections, we will prove some properties related to these triangles and determine the Ramanujan summation using those properties. Further we define centred numbers of both triangles as numbers which are located as middle terms in the odd numbered rows. Such numbers are shown in pink color in Figures 1 and 2 respectively.

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1
3 5
7 **9** 11
13 15 17 19
21 23 **25** 27 29
31 33 35 37 39 41
43 45 47 **49** 51 53 55
. . . .

Figure 1: Odd Numbered Triangle

2
4 6
8 **10** 12
14 16 18 20
22 24 **26** 28 30
32 34 36 38 40 42
44 46 48 **50** 52 54 56
. . . .

Figure 2: Even Numbered Triangle

4. Properties of Odd Numbered Triangle

Let $u_{n,m}$ denote the *n*th row, *m*th number of the odd numbered triangle in Figure 1. We call $u_{n,m}$ as the general term of odd numbered triangle. We notice that the general term $u_{n,m}$ is given by $u_{n,m} = n(n-1)+(2m-1)$ (4.1) where $1 \le m \le n$. Using (4.1), we will prove some results in the following sections.

4.1 Theorem 1

The centered numbers of odd numbered triangle are odd perfect squares.

That is, $u_{2k-1,k} = (2k-1)^2$ (4.2)

Proof: The middle term in the odd numbered row of odd numbered triangle in Figure 1 can be considered as centered numbers of the triangle. Referring to Figure 1, such numbers will be of the form $u_{2k-1,k}$ for any natural number *k*. Now using (4.1), we have

$$u_{2k-1,k} = \left(2k-1\right)\left(2k-2\right)+\left(2k-1\right) = (2k-1)^2$$

Since $(2k-1)^2$ are odd squares for any natural number *k*, this completes the proof.

4.2 Theorem 2

The sum of first *n* centered numbers of odd numbered triangle is given by $\dfrac{n(4n^2-1)}{3}$ (4.3)

Proof: From (4.2), the required sum is given by

$$\sum_{k=1}^{n} u_{2k-1,k} = \sum_{k=1}^{n}(2k-1)^2 = 4\sum_{k=1}^{n}k^2 - 4\sum_{k=1}^{n}k + n$$
$$= \frac{2n(n+1)(2n+1)}{3} - 2n(n+1) + n = \frac{n(4n^2-1)}{3}$$

This completes the proof.

4.3 Theorem 3

The sum of first n outer diagonal numbers of odd numbered triangle are given by $\sum_{k=1}^{n} u_{k,1} = \frac{n(n^2+2)}{3}$ (4.4) and $\sum_{k=1}^{n} u_{k,k} = \frac{n(n^2+3n-1)}{3}$ (4.5)

Proof: Using (4.1), we have

$$\sum_{k=1}^{n} u_{k,1} = \sum_{k=1}^{n}\left[k(k-1)+1\right] = \sum_{k=1}^{n}k^2 - \sum_{k=1}^{n}k + n = \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + n = \frac{n(n^2+2)}{3}$$

$$\sum_{k=1}^{n} u_{k,k} = \sum_{k=1}^{n}\left[k(k-1)+(2k-1)\right] = \sum_{k=1}^{n}k^2 + \sum_{k=1}^{n}k - n = \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} - n = \frac{n(n^2+3n-1)}{3}$$ This completes the proof.

5. Properties of Even Numbered Triangle

Let $v_{n,m}$ denote the nth row, mth number of the even numbered triangle in Figure 2. We call $v_{n,m}$ as the general term of even numbered triangle. We notice that the general term $v_{n,m}$ is given by $v_{n,m} = n(n-1)+2m$ (5.1) where $1 \le m \le n$. Using (5.1), I will prove some results in the following sections.

5.1 Theorem 4

The centered numbers of even numbered triangle are one more than odd perfect squares

That is, $v_{2k-1,k} = (2k-1)^2 + 1$ (5.2)

Proof: The middle term in the odd numbered row of even numbered triangle in Figure 2 can be considered as centered numbers of the triangle. Referring to Figure 2, such numbers will be of the form $v_{2k-1,k}$ for any natural number k. Now using (5.1), we have

$$v_{2k-1,k} = (2k-1)(2k-2) + 2k = 4k^2 - 4k + 1 + 1 = (2k-1)^2 + 1$$

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Since $(2k-1)^2$ are odd squares for any natural number *k*, this completes the proof.

5.2 Theorem 5

The sum of first *n* centered numbers of even numbered triangle is given by $\frac{2n(2n^2+1)}{3}$ (5.3)

Proof: From (5.2), the required sum is given by

$$\sum_{k=1}^{n} v_{2k-1,k} = \sum_{k=1}^{n}\left[(2k-1)^2+1\right] = 4\sum_{k=1}^{n}k^2 - 4\sum_{k=1}^{n}k + 2n$$
$$= \frac{2n(n+1)(2n+1)}{3} - 2n(n+1) + 2n = \frac{2n(2n^2+1)}{3}$$

This completes the proof.

5.3 Theorem 6

The sum of first *n* outer diagonal numbers of even numbered triangle are given by $\sum_{k=1}^{n} v_{k,1} = \frac{n(n^2+5)}{3}$ (5.4) and

$$\sum_{k=1}^{n} v_{k,k} = \frac{n(n+1)(n+2)}{3} \quad (5.5)$$

Proof: Using (5.1), we have

$$\sum_{k=1}^{n} v_{k,1} = \sum_{k=1}^{n}\left[k(k-1)+2\right] = \sum_{k=1}^{n}k^2 - \sum_{k=1}^{n}k + 2n = \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + 2n = \frac{n(n^2+5)}{3}$$

$$\sum_{k=1}^{n} v_{k,k} = \sum_{k=1}^{n}\left[k(k-1)+2k\right] = \sum_{k=1}^{n}k^2 + \sum_{k=1}^{n}k = \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{3}$$

This completes the proof.

6. Ramanujan Summation

In this section, using the results obtained in sections 4 and 5. we will prove Ramanujan summation formulas for some of the divergent series from Odd and Even numbered triangles presented in Figures 1 and 2 respectively.

6.1 Theorem 7

$$(RS)(1+3+5+7+\cdots) = \frac{1}{3} \quad (6.1), \quad (RS)(2+4+6+8+\cdots) = -\frac{1}{6} \quad (6.2)$$

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$(RS)(1^2+3^2+5^2+7^2+\cdots)=-\frac{1}{6}$ (6.3), $(RS)(2^2+4^2+6^2+8^2+\cdots)=0$ (6.4)

$(RS)(1+3+7+13+21+31+\cdots)=-\frac{5}{12}$ (6.5), $(RS)(1+5+11+19+29+41+\cdots)=\frac{5}{12}$ (6.6)

$(RS)(2+4+8+14+22+32+\cdots)=-\frac{11}{12}$ (6.7), $(RS)(2+6+12+20+30+42+\cdots)=-\frac{1}{12}$ (6.8)

Proof: Using (2.1) we can prove the stated results.

Since $1+3+5+7+\cdots+(2n-1)=n^2$ we get $(RS)(1+3+5+7+\cdots)=\int_{n=-1}^{0} n^2\,dn=\frac{1}{3}$

This proves (6.1).

Since $2+4+6+8+\cdots+2n=n^2+n$ we get $(RS)(2+4+6+8+\cdots)=\int_{n=-1}^{0}\left(n^2+n\right)dn=\frac{1}{3}-\frac{1}{2}=-\frac{1}{6}$ This proves (6.2).

Now we notice that $1^2 = 1$, $3^2 = 9$, $5^2 = 25$, $7^2 = 49$, . . . are precisely the centered numbers of the odd numbered triangle. Hence by (2.1) and (4.3), we get

$(RS)(1^2+3^2+5^2+7^2+\cdots)=\int_{n=-1}^{0}\frac{n(4n^2-1)}{3}dn=-\frac{1}{3}+\frac{1}{6}=-\frac{1}{6}$. This proves (6.3).

Since $2^2+4^2+6^2+8^2+\cdots+(2n)^2=2^2(1^2+2^2+3^2+\cdots+n^2)=\frac{4}{3}n^3+2n^2+\frac{2}{3}n$ we have

$(RS)(2^2+4^2+6^2+8^2+\cdots)=\int_{n=-1}^{0}\left(\frac{4}{3}n^3+2n^2+\frac{2}{3}n\right)dn=-\frac{1}{3}+\frac{2}{3}-\frac{1}{3}=0$. This proves (6.4).

Since 1, 3, 7, 13, 21, 31, . . . forms south – west outer diagonal of odd numbered triangle of Figure 1, using (2.1) and (4.4), we have

$(RS)(1+3+7+13+21+31+43+\cdots)=\int_{n=-1}^{0}\left(\frac{1}{3}n^3+\frac{2}{3}n\right)dn=-\frac{1}{12}-\frac{1}{3}=-\frac{5}{12}$. This proves (6.5)

Since 1, 5, 11, 19, 29, 41, . . . forms south – east outer diagonal of odd numbered triangle of Figure 1, using (2.1) and (4.5), we have

$(RS)(1+5+11+19+29+41+\cdots)=\int_{n=-1}^{0}\left(\frac{1}{3}n^3+n^2-\frac{1}{3}n\right)dn=-\frac{1}{12}+\frac{1}{3}+\frac{1}{6}=\frac{5}{12}$. This proves (6.6)

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Since 2, 4, 8, 14, 22, 32, . . . forms south – west outer diagonal of even numbered triangle of Figure 2, using (2.1) and (5.4), we have

$$(RS)(2+4+8+14+22+32+\cdots)=\int_{n=-1}^{0}\left(\frac{1}{3}n^3+\frac{5}{3}n\right)dn=-\frac{1}{12}-\frac{5}{6}=-\frac{11}{12}$$. This proves (6.7)

Since 2, 6, 12, 20, 30, 42, . . . forms south – east outer diagonal of even numbered triangle of Figure 2, using (2.1) and (5.5), we have

$$(RS)(2+6+12+20+30+\cdots)=\int_{n=-1}^{0}\left(\frac{1}{3}n^3+n^2+\frac{2}{3}n\right)dn=-\frac{1}{12}+\frac{1}{3}-\frac{1}{3}=-\frac{1}{12}$$. This proves (6.8)

This completes the proof.

6.2 Bernoulli Numbers

Bernoulli Numbers are numbers which occur as coefficients of $\frac{x^n}{n!}$ in the Taylor's series expansion of $\frac{x}{e^x-1}$ about $x = 0$.

We denote the nth Bernoulli Number by B_n . Thus by definition we get $$\frac{x}{e^x-1}=\sum_{n=0}^{\infty}B_n\frac{x^n}{n!} \quad (6.9)$$.

We notice that the constant term of $\frac{x}{e^x-1}$ is 1 and so from (6.9) it follows that $B_0=1$.

The first few values of Bernoulli Numbers are given by

$$B_0=1, B_1=-\frac{1}{2}, B_2=\frac{1}{6}, B_3=0, B_4=-\frac{1}{30}, B_5=0, B_6=\frac{1}{42}, B_7=0, B_8=-\frac{1}{30}, B_9=0$$
$$B_{10}=\frac{5}{66}, B_{11}=0, B_{12}=-\frac{691}{2730}, B_{13}=0, B_{14}=\frac{7}{6}, B_{15}=0, B_{16}=-\frac{3617}{510},... \quad (6.10)$$

From the above values we observe that except for B_1, $B_n = 0$ for all odd values of n.

Srinivasa Ramanujan proved a formula connecting Riemann zeta function with Bernoulli numbers (for proof see **[2]**). The formulas called as Ramanujan Summation were given by

$$(RS)(1^{2r}+2^{2r}+3^{2r}+\cdots)=\int_{n=-1}^{0}\left(\sum_{k=1}^{n}k^{2r}\right)dn=\zeta(-2r)=0 \quad (6.11)$$

$$(RS)(1^{2r-1}+2^{2r-1}+3^{2r-1}+\cdots)=\int_{n=-1}^{0}\left(\sum_{k=1}^{n}k^{2r-1}\right)dn=\zeta(1-2r)=-\frac{B_{2r}}{2r} \quad (6.12)$$

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7. Ramanujan Summation for Odd and Even Numbered Triangles

In this section we will prove formulas related to general integer powers of entries of odd and even numbered triangles considered in Figures 1 and 2.

7.1 Theorem 8

If B_n is the *n*th Bernoulli number and if *r* is any positive integer then

$$(RS)(1^{2r}+3^{2r}+5^{2r}+7^{2r}+\cdots)=\sum_{s=1}^{r}\binom{2r}{2s-1}2^{2r-2s+1}\frac{B_{2r-2s+2}}{2r-2s+2}-\frac{1}{2} \qquad (7.1)$$

$$(RS)(1^{2r-1}+3^{2r-1}+5^{2r-1}+7^{2r-1}+\cdots)=-\sum_{s=1}^{r}\binom{2r-1}{2s-2}2^{2r-2s+1}\frac{B_{2r-2s+2}}{2r-2s+2}+\frac{1}{2} \qquad (7.2)$$

$$(RS)(2^{2r}+4^{2r}+6^{2r}+8^{2r}+\cdots)=0 \qquad (7.3)$$

$$(RS)(2^{2r-1}+4^{2r-1}+6^{2r-1}+8^{2r-1}+\cdots)=-2^{2r-1}\frac{B_{2r}}{2r} \qquad (7.4)$$

Proof: First, I will prove the results (7.3) and (7.4) corresponding to even and odd powers of entries of even numbered triangle in Figure 2.

$$2^{2r}+4^{2r}+6^{2r}+8^{2r}+\cdots=2^{2r}(1^{2r}+2^{2r}+3^{2r}+4^{2r}+\cdots).$$

Hence, using (6.11), we get

$$(RS)(2^{2r}+4^{2r}+6^{2r}+8^{2r}+\cdots)=2^{2r}(RS)(1^{2r}+2^{2r}+3^{2r}+\cdots)=2^{2r}\times 0=0.$$

This proves (7.3).

Similarly, since $2^{2r-1}+4^{2r-1}+6^{2r-1}+8^{2r-1}+\cdots=2^{2r-1}(1^{2r-1}+2^{2r-1}+3^{2r-1}+4^{2r-1}+\cdots)$ using (6.12), we have

$$(RS)(2^{2r-1}+4^{2r-1}+6^{2r-1}+8^{2r-1}+\cdots)=2^{2r-1}(RS)(1^{2r-1}+2^{2r-1}+3^{2r-1}+4^{2r-1}+\cdots)$$

$$=2^{2r-1}\times-\frac{B_{2r}}{2r}=-2^{2r-1}\frac{B_{2r}}{2r}$$

This proves (7.4).

For any natural number *r*, from (4.2), we have

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$$\sum_{k=1}^{n}(u_{2k-1,k})^{r}=\sum_{k=1}^{n}(2k-1)^{2r}$$
$$=\sum_{k=1}^{n}\left[(2k)^{2r}+\binom{2r}{1}(2k)^{2r-1}(-1)^{1}+\binom{2r}{2}(2k)^{2r-2}(-1)^{2}+\cdots+\binom{2r}{2r-1}(2k)^{1}(-1)^{2r-1}+(-1)^{2r}\right]$$
$$=2^{2r}\sum_{k=1}^{n}k^{2r}-2^{2r-1}\binom{2r}{1}\sum_{k=1}^{n}k^{2r-1}+2^{2r-2}\binom{2r}{2}\sum_{k=1}^{n}k^{2r-2}-\cdots-2\binom{2r}{2r-1}\sum_{k=1}^{n}k+n$$

Hence using (2.1), (6.11) and (6.12), we get

$$(RS)(1^{2r}+3^{2r}+5^{2r}+\cdots)=\int_{n=-1}^{0}\left[\sum_{k=1}^{n}(u_{2k-1,k})^{r}\right]dn=2^{2r}\int_{n=-1}^{0}\left(\sum_{k=1}^{n}k^{2r}\right)dn-2^{2r-1}\binom{2r}{1}\int_{n=-1}^{0}\left(\sum_{k=1}^{n}k^{2r-1}\right)dn$$
$$+2^{2r-2}\binom{2r}{2}\int_{n=-1}^{0}\left(\sum_{k=1}^{n}k^{2r-2}\right)dn-\cdots-2\binom{2r}{2r-1}\int_{n=-1}^{0}\left(\sum_{k=1}^{n}k\right)dn+\int_{n=-1}^{0}n\,dn$$
$$=2^{2r}\zeta(-2r)-2^{2r-1}\binom{2r}{1}\zeta(1-2r)+2^{2r-2}\binom{2r}{2}\zeta(2-2r)$$
$$-2^{2r-3}\binom{2r}{3}\zeta(3-2r)+\cdots-2\binom{2r}{2r-1}\zeta(-1)-\frac{1}{2}$$
$$=0-2^{2r-1}\binom{2r}{1}\left(-\frac{B_{2r}}{2r}\right)+0-2^{2r-3}\binom{2r}{3}\left(-\frac{B_{2r-2}}{2r-2}\right)+0-2^{2r-5}\binom{2r}{5}\left(-\frac{B_{2r-4}}{2r-4}\right)+$$
$$\cdots-2\binom{2r}{2r-1}\left(-\frac{B_{2}}{2}\right)-\frac{1}{2}=\sum_{s=1}^{r}\binom{2r}{2s-1}2^{2r-2s+1}\frac{B_{2r-2s+2}}{2r-2s+2}-\frac{1}{2}$$

This proves (7.1).

Similarly, for any natural number r, from (4.2), we have

$$\sum_{k=1}^{n}(u_{2k-1,k})^{(2r-1)/2}=\sum_{k=1}^{n}(2k-1)^{2r-1}$$
$$=\sum_{k=1}^{n}\left[(2k)^{2r-1}+\binom{2r-1}{1}(2k)^{2r-2}(-1)^{1}+\binom{2r-1}{2}(2k)^{2r-3}(-1)^{2}+\cdots+\binom{2r-1}{2r-2}(2k)^{1}(-1)^{2r-2}+(-1)^{2r-1}\right]$$
$$=2^{2r-1}\sum_{k=1}^{n}k^{2r-1}-2^{2r-2}\binom{2r-1}{1}\sum_{k=1}^{n}k^{2r-2}+2^{2r-3}\binom{2r-1}{2}\sum_{k=1}^{n}k^{2r-3}-\cdots+2\binom{2r-1}{2r-2}\sum_{k=1}^{n}k-n$$

Hence using (2.1), (6.11) and (6.12), we get

$$(RS)(1^{2r-1}+3^{2r-1}+5^{2r-1}+\cdots)=\int_{n=-1}^{0}\left[\sum_{k=1}^{n}(u_{2k-1,k})^{(2r-1)/2}\right]dn=2^{2r-1}\int_{n=-1}^{0}\left(\sum_{k=1}^{n}k^{2r-1}\right)dn$$
$$-2^{2r-2}\binom{2r-1}{1}\int_{n=-1}^{0}\left(\sum_{k=1}^{n}k^{2r-2}\right)dn+2^{2r-3}\binom{2r-1}{2}\int_{n=-1}^{0}\left(\sum_{k=1}^{n}k^{2r-3}\right)dn-\cdots+2\binom{2r-1}{2r-2}\int_{n=-1}^{0}\left(\sum_{k=1}^{n}k\right)dn-\int_{n=-1}^{0}n\,dn$$

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$$= 2^{2r-1}\zeta(1-2r) - 2^{2r-2}\binom{2r-1}{1}\zeta(2-2r) + 2^{2r-3}\binom{2r-1}{2}\zeta(3-2r) - 2^{2r-4}\binom{2r-1}{3}\zeta(4-2r) +$$

$$\cdots + 2\binom{2r-1}{2r-2}\zeta(-1) + \frac{1}{2}$$

$$= 2^{2r-1}\binom{2r-1}{0}\left(-\frac{B_{2r}}{2r}\right) - 0 + 2^{2r-3}\binom{2r-1}{2}\left(-\frac{B_{2r-2}}{2r-2}\right) - 0 + 2^{2r-5}\binom{2r-1}{4}\left(-\frac{B_{2r-4}}{2r-4}\right) -$$

$$\cdots + 2\binom{2r-1}{2r-2}\left(-\frac{B_2}{2}\right) + \frac{1}{2} = -\sum_{s=1}^{r}\binom{2r-1}{2s-2}2^{2r-2s+1}\frac{B_{2r-2s+2}}{2r-2s+2} + \frac{1}{2}$$

This proves (7.2) and completes the proof.

8. Corollary

$$(RS)(1^3+3^3+5^3+7^3+\cdots) = \frac{1}{15} \quad (8.1), \quad (RS)(1^4+3^4+5^4+7^4+\cdots) = -\frac{1}{10} \quad (8.2)$$

$$(RS)(1^5+3^5+5^5+7^5+\cdots) = \frac{13}{63} \quad (8.3), \quad (RS)(1^6+3^6+5^6+7^6+\cdots) = -\frac{1}{14} \quad (8.4)$$

$$(RS)(1^7+3^7+5^7+7^7+\cdots) = -\frac{7}{15} \quad (8.5), \quad (RS)(1^8+3^8+5^8+7^8+\cdots) = -\frac{1}{18} \quad (8.6)$$

Proof: Substituting $r = 2$ in (7.2), we get

$$(RS)(1^3+3^3+5^3+7^3+\cdots) = -\sum_{s=1}^{2}\binom{3}{2s-2}2^{5-2s}\frac{B_{6-2s}}{6-2s} + \frac{1}{2} = -\left[\binom{3}{0}2^3\frac{B_4}{4} + \binom{3}{2}2^1\frac{B_2}{2}\right] + \frac{1}{2} = \frac{1}{15}$$

Similarly, taking $r = 3$ and 4 in (7.2), we get

$$(RS)(1^5+3^5+5^5+7^5+\cdots) = -\sum_{s=1}^{3}\binom{5}{2s-2}2^{7-2s}\frac{B_{8-2s}}{8-2s} + \frac{1}{2}$$

$$= -\left[\binom{5}{0}2^5\frac{B_6}{6} + \binom{5}{2}2^3\frac{B_4}{4} + \binom{5}{4}2^1\frac{B_2}{2}\right] + \frac{1}{2} = \frac{13}{63}$$

$$(RS)(1^7+3^7+5^7+7^7+\cdots) = -\sum_{s=1}^{4}\binom{7}{2s-2}2^{9-2s}\frac{B_{10-2s}}{10-2s} + \frac{1}{2}$$

$$= -\left[\binom{7}{0}2^7\frac{B_8}{8} + \binom{7}{2}2^5\frac{B_6}{6} + \binom{7}{4}2^3\frac{B_4}{4} + \binom{7}{6}2^1\frac{B_2}{2}\right] + \frac{1}{2} = -\frac{7}{15}$$

Now substituting $r = 2$, 3 and 4 in (7.1), we get

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$$(RS)(1^4+3^4+5^4+7^4+\cdots)=\sum_{s=1}^{2}\binom{4}{2s-1}2^{5-2s}\,\frac{B_{6-2s}}{6-2s}-\frac{1}{2}=\left[\binom{4}{1}2^3\,\frac{B_4}{4}+\binom{4}{3}2^1\,\frac{B_2}{2}\right]-\frac{1}{2}=-\frac{1}{10}$$

$$(RS)(1^6+3^6+5^6+7^6+\cdots)=\sum_{s=1}^{3}\binom{6}{2s-1}2^{7-2s}\,\frac{B_{8-2s}}{8-2s}-\frac{1}{2}$$
$$=\left[\binom{6}{1}2^5\,\frac{B_6}{6}+\binom{6}{3}2^3\,\frac{B_4}{4}+\binom{6}{5}2^1\,\frac{B_2}{2}\right]-\frac{1}{2}=-\frac{1}{14}$$

$$(RS)(1^8+3^8+5^8+7^8+\cdots)=\sum_{s=1}^{4}\binom{8}{2s-1}2^{9-2s}\,\frac{B_{10-2s}}{10-2s}-\frac{1}{2}$$
$$=\left[\binom{8}{1}2^7\,\frac{B_8}{8}+\binom{8}{3}2^5\,\frac{B_6}{6}+\binom{8}{5}2^3\,\frac{B_4}{4}+\binom{8}{7}2^1\,\frac{B_2}{2}\right]-\frac{1}{2}=-\frac{1}{18}$$

This completes the proof.

9. Conjecture

In view of equations (6.3), (8.2), (8.4) and (8.6), we make the following conjecture which is an alternate form of (7.1).

$$(RS)(1^{2r}+3^{2r}+5^{2r}+7^{2r}+\cdots)=-\frac{1}{4r+2} \quad (9.1)$$

10. Conclusion

In this paper, considering two number triangles as in Figures 1 and 2 comprising of odd and even natural numbers respectively, we had proved interesting properties in first six theorems. The Ramanaujan summation values for eight divergent series obtained in theorem 7 are new and done using the properties obtained in earlier theorems. After introducing Bernoulli numbers and using its connection with Ramanujan summation we had proved four very interesting and new results in theorem 8. These results helps us to determine Ramanujan summation for odd and even powers of the entries of odd and even numbered triangles presented in Figures 1 and 2.

Using the new results obtained in theorem 8, we had determined Ramanujan summation values of third, fourth, fifth, sixth, seventh and eighth powers of odd numbers through equations (8.1) to (8.6) respectively through a corollary. Further by observing a pattern using equations (6.3), (8.2), (8.4) and (8.6), we had proposed a conjecture for Ramanujan summation value of even powers of odd numbers in (9.1). Proving this equation would be an interesting task for anyone who stretches their minds further. Thus, in this paper, by considering simple number triangles, we had obtained more than dozen new results regarding determining Ramanujan summation values. These equations and the proposed conjecture would certainly add more value to the methods of summability theory.

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