SERIES OF RECIPROCAL POWERS OF $k-$ALMOST PRIMES

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ABSTRACT. Sums over inverse s-th powers of semiprimes and k-almost primes are transformed into sums over products of powers of ordinary prime zeta functions. Multinomial coefficients known from the cycle decomposition of permutation groups play the role of expansion coefficients. Founded on a known convergence acceleration for the ordinary prime zeta functions, the sums and first derivatives are tabulated with high precision for indices $k = 2, \ldots, 6$ and integer powers $s = 2, \ldots, 8$.

1. Overview

Series over rational polynomials evaluated at integer arguments contain subseries summing over integers classified by the count of their prime factors. The core example is the Riemann zeta function ζ which accumulates the prime zeta function P_1 plus what we shall define the almost-prime zeta functions P_k (Section 2). The central observation of this manuscript is that the almost-prime zeta functions are combinatorial sums over the prime zeta function (Section 3). Since earlier work by Cohen, Sebah and Gourdon has pointed at efficient numerical algorithms to compute P_1, series over reciprocal almost-primes—which may suffer from slow convergence in their defining format—may be computed efficiently by reference to the P_1.

In consequence, any converging series over the positive integers which has a Taylor expansion in reciprocal powers of these integers splits into k-almost prime components. Section 4 illustrates this for the most basic formats.

Number theory as such will not be advanced. The meromorphic landscape of the prime zeta functions as a function of their main variable appears to be more complicated than what is known for their host, the Riemann zeta function; so only some remarks on the calculation of first derivatives are dropped.

2. Prime Zeta Function

Definition 1. *The prime zeta function $P(s)$ is the sum over the reciprocal s-th powers of the prime numbers p* [16, 15, 27]

$$P(s) \equiv \sum_{p}^{\infty} \frac{1}{p^s}; \quad \Re s > 1. \tag{1}$$

Remark 1. *The primes are represented by sequence A000040 in the Online-Encyclopedia of Integer Sequences* [32]*, and we will adopt the nomenclature that a letter A followed by a 6-digit number points at a sequence in this database. Accurate values*

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$P(s)$ for $s \le 9$ are provided by the sequences A085548, A085541, and A085964–A085969.

Table 1 complements these by using [4, 30, 13]

$$P(s) = \sum_{p\le M} \frac{1}{p^s} + \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log P(M, sn) \tag{2}$$

for a suitably large prime M, where μ is the Möbius function [1, (24.3.1)], where ζ is the Riemann zeta function [11, 22][18, (9.5)], and

$$P(M, s) \equiv \zeta(s) \prod_{p\le M} \left(1 - p^{-s}\right) \tag{3}$$

an associated definition of a partial product.

Moments are listed in Table 2. The first line,

$$\sum_{s=1}^{\infty} \frac{1}{s} P(s) = -\sum_p \left[\log\left(1 - \frac{1}{p}\right) + \frac{1}{p}\right] = \lim_{s\to 1}[\zeta(s) - \sum_p \frac{1}{p^s}], \tag{4}$$

is A143524 [32, 13]. The total of all values in Table 2 is

$$\begin{aligned} \sum_{u=1}^{\infty}\sum_{s=2}^{\infty} \frac{1}{s^u} P(s) &= \sum_{s=2}^{\infty}\sum_p \frac{1}{(s-1)p^s} = -\sum_p \frac{\log(1-1/p)}{p} \\ &= 0.58005849381391172358283349737677118691587319037\ldots \end{aligned} \tag{5}$$

The first derivatives $dP(s)/ds$ of (1) are evaluated as the first derivatives of (2) [1, (3.3.6)],

$$P'(s) = -\sum_p \frac{\log p}{p^s} = -\sum_{p\le M} \frac{\log p}{p^s} + \sum_{n=1}^{\infty} \mu(n) \frac{P'(M, sn)}{P(M, sn)}; \tag{6}$$

$$\frac{P'(M, s)}{P(M, s)} = \frac{\zeta'(s)}{\zeta(s)} + \sum_{p\le M} \frac{\log p}{p^s(1 - p^{-s})}. \tag{7}$$

Here, primes denote derivatives with respect to the main argument, which is the second argument for the case of $P(.,.)$. Table 3 shows some of them for selected small integer s.

Remark 2. *The relation $-\zeta'(s)/\zeta(s) = \sum_p \log(p)/(p^s - 1)$ is a gateway to acceleration of series involving logarithmic numerators. Examples are growth rate coefficients of unitary square-free divisors* [33],

$$\begin{aligned} \sum_p \frac{(2p+1)\log p}{(p+1)(p^2+p-1)} &= \sum_p \log p \Big[\frac{2}{p^2-1} - \frac{3}{p^3-1} + \frac{4}{p^4-1} - \frac{10}{p^5-1} + \frac{18}{p^6-1} \\ &\quad - \frac{28}{p^7-1} + \frac{40}{p^8-1} - \frac{72}{p^9-1} + \cdots\Big] = 0.748372333429674\ldots \end{aligned} \tag{8}$$

or of unitary cube-free divisors,

$$\begin{aligned} \sum_p \frac{(4p^2-2-p)\log p}{(p^2-1)(p^2+p-1)} &= \sum_p \log p \Big[\frac{4}{p^2-1} - \frac{5}{p^3-1} + \frac{7}{p^4-1} - \frac{17}{p^5-1} + \frac{31}{p^6-1} \\ &\quad - \frac{48}{p^7-1} + \frac{69}{p^8-1} - \frac{124}{p^9-1} + \cdots\Big] = 1.647948081159756\ldots \end{aligned} \tag{9}$$

TABLE 1. The Prime Zeta Function of some integer arguments. In a style adopted from [1], optional trailing parentheses contain an additional power of 10. Example: The number 3.45×10^{-3} may appear as 0.00345 or as 3.45(−3) or .345(−2). Trailing dots indicate that more digits are chopped off, not rounded, at the rightmost places.

s	$P(s)$
10	.99360357443698021785585070014773941630187254528520332055356666(-3) ...
11	.49394726910465497569162176833439871215593970096049521818660074(-3) ...
12	.24602647003454567952664859216508092797993226794732319217414159(-3) ...
13	.12269836752786927990548879243140332391474285255776901352565628(-3) ...
14	.61244396725464478377508034299874541972821268723780135418853003(-4) ...
15	.30587302823270052567554629313712628001301145253898093307659981(-4) ...
16	.15282026219339344180801926411890559774661269877603931107880060(-4) ...
17	.76371393706458972509045560439397620175698390421626625202513445(-5) ...
18	.38172787031749966312275153163110913616249426363826141957480077(-5) ...
19	.19082090769262825721861799879697766181456161950689863811657765(-5) ...
20	.95396112410362332635288349397700579557005558858221343649929886(-6) ...
21	.47693275936842725050837266188188761060419081025437783112861007(-6) ...
22	.23845044587670192812631168520159550867873250699147067361389961(-6) ...
23	.11921991175318828561602464533833985771083041165914134967504467(-6) ...
24	.59608185498334532971130666550086201311465824801177155987249992(-7) ...
25	.29803502626438658766626594017781459495927788278311399231662997(-7) ...
26	.14901554606314570543459077394423733840265740940037170948263319(-7) ...
27	.74507117343233007801645461240936933495593461489271525171775541(-8) ...
28	.37253340109105063518339122876930717530071331761809587555440001(-8) ...
29	.18626597200435749075221451133536011728833471613165710906770067(-8) ...
30	.93132743155230192067706645896544775909511359173598450541427758(-9) ...
31	.46566290628653721880247561689245507483711109046830714608488003(-9) ...
32	.23283118331344031491367214292901343839560128393537926955495588(-9) ...
33	.11641550171345264966007162860197173019006429517590258455552278(-9) ...
34	.58207720875638873612961103292798914611355443714618707780501557(-10) ...
35	.29103850444123963340305283132124818097185438356356095826813888(-10) ...
36	.14551921890830225902161329050874685290735640454455298546815582(-10) ...
37	.72759598350045414391584848176711312860098068027996528848736652(-11) ...
38	.36379795473654162977432391725914219152604119208123529133016657(-11) ...
39	.18189896503037572246857639039058566200252330075315083216341193(-11) ...

3. ZETA FUNCTIONS OF ALMOST-PRIMES

3.1. **Nomenclature.** We generalize the notation, and define the k-almost prime zeta functions by summation over inverse powers of k-almost primes q_j,

Definition 2.

$$P_k(s) \equiv \sum_{j=1}^{\infty} \frac{1}{q_j^s}. \tag{10}$$

Table 2. Moments of the Prime Zeta function. The difference between the first two lines has been quoted by Chan [10].

u	$\sum_{s=2}^{\infty} P(s)/s^u = \sum_{s=2}^{\infty}\sum_p 1/(s^u p^s)$
1	.31571845205389007685108525147370657199059268767872439261370 ...
2	.139470639611308681803077672937394594285963216485548292900601 ...
3	.646081378884568610840336242581546273479300400893390674692107(-1) ...
4	.307990601127799353645768912808351277991590439776491610562576(-1) ...
5	.149414104078444272503589211881003770331602105485328925740449(-1) ...
6	.732763762441199457647551708167392749211019881968761942480213(-2) ...

Table 3. The first derivative of the Prime Zeta Function at some integer arguments s. The value in the first line differs from Cohen's value [13] after 42 digits.

s	$P'(s)$
2	-4.930911093687644621978262050564912580555881263464682907133271(-1) ...
3	-1.507575555439504221798365163653429195755011615306893318187976(-1) ...
4	-6.060763335077006339223098370971337840638287746125984399112768(-2) ...
5	-2.683860127679835742218751329245015994333014955355822812481980(-2) ...
6	-1.245908072279999152702779277468997004091135047157587587410933(-2) ...
7	-5.940689039148196142550592829016609019368189505929351075166813(-3) ...
8	-2.879524708729247391346028423857334064998983761675865841067618(-3) ...
9	-1.410491921424531291554196456308199977901657131693496192836500(-3) ...
10	-6.956784473446204802000701977708415913844863703329838954712256(-4) ...
11	-3.446864256305149016520798301347221055148509398720732052598028(-4) ...
12	-1.712993524462175657532493112138275372004981118241302276420951(-4) ...
13	-8.530310916711056635208876017215691972617326615054214472499073(-5) ...
14	-4.253630557412291035554757415368617516720893534438843631304558(-5) ...
15	-2.122979056274934599669348621302375720453112762226994727150844(-5) ...
16	-1.060211861676127903320578231686279299852887328732516230264968(-5) ...
17	-5.296802557643848074496697331902062291354582070044729659167083(-6) ...
18	-2.646982787802997352263261854182101806956865359404392741570106(-6) ...
19	-1.323018648512292735443206851957658773372595611301942028763990(-6) ...
20	-6.613517594172600210891457029052100560435779681754585968634604(-7) ...
21	-3.306233614825208657730023089591286331399889373034673500892396(-7) ...
22	-1.652941753425972669328543237067224505237754606957895309294978(-7) ...
23	-8.264125267365738127779160862943622945349018242909935277495509(-8) ...
24	-4.131868136465068742054546598016395808846430264964223880702300(-8) ...
25	-2.065869236367122379085627896761801003317031762395199472960982(-8) ...
26	-1.032913007669833840610060139473968867069681131430230354135307(-8) ...
27	-5.164493003519525183949097124602884025560927707601929381567114(-9) ...
28	-2.582222490193098373680412778703362364263231350157972835761708(-9) ...
29	-1.291103241249637065884459649285079993129747526229207669548247(-9) ...

In slight violation of the almost-*terminology, the Prime Zeta Function is incorporated as just one special case,*

$$P_1(s) \equiv P(s). \tag{11}$$

Remark 3. *The sequence q_j is given by the primes A000040 if $k=1$, by the semiprimes A001358 if $k=2$, by the 3-almost primes A014612 if $k=3$, by the 4-almost primes A014613 if $k=4$, by the 5-almost primes A014614 if $k=5$, by the 6-almost primes A046306 if $k=6$, by the 7-almost primes A046308 if $k=7$ etc. $P_2(2)$ is A117543, and $P_3(2)$ is A131653.*

Remark 4. *One step further defines prime multi-zeta functions of the form*

$$P(s_1, s_2, \ldots, s_k) \equiv \sum_{p_1, p_2, \ldots, p_k = 2,3,5\ldots} \frac{1}{p_1^{s_1} p_2^{s_2} \cdots p_k^{s_k}}. \tag{12}$$

If the exponents are restricted to a common number s, they retrieve the information of the $P_k(s)$ in slightly entangled form, for example $P(s,s) = 2P_2(s) - P(2s)$ [8].

Each integer $n > 0$ is either a member of the set $\{1\}$, or of the set of primes, or of the set of semi-primes, etc. These disjoint sets are labeled by the sum of the exponents of the prime number factorizations of their members,

$$\Omega(n) = k; \quad \Omega(1) \equiv 0. \tag{13}$$

The Riemann zeta function may be partitioned into sums over the almost-prime zeta functions,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \sum_{k=1}^{\infty} \sum_{\substack{n=1 \\ \Omega(n)=k}}^{\infty} \frac{1}{n^s} = 1 + \sum_{k=1}^{\infty} P_k(s). \tag{14}$$

Remark 5. *$\zeta(s)$ may be taken from* [1, (Table 23.3)] *for $s \le 42$, or from A013661, A002117, A013662 – A013678 while $s \le 20$, or from the link to "Recent additions of tables" in Plouffe's database* [28] *while $s \le 99$. $\Omega(n)$ is tabulated in A001222.*

3.2. **Numerical Scheme.** Table 4 of $P_k(s)$ is deduced from

Theorem 1. [21]

$$P_2(s) = \frac{P(2s) + P^2(s)}{2!}; \tag{15}$$

$$P_3(s) = \frac{2P(3s) + 3P(2s)P(s) + P^3(s)}{3!}; \tag{16}$$

$$P_4(s) = \frac{6P(4s) + 8P(3s)P(s) + 3P^2(2s) + 6P(2s)P^2(s) + P^4(s)}{4!}; \tag{17}$$

$$P_k(s) = \frac{1}{k!} \sum_{\substack{k_1 + 2k_2 + 3k_3 + \cdots + kk_k = k \\ k_k \ge 0}} (k; k_1 k_2 \ldots k_k)^* P^{k_1}(s) P^{k_2}(2s) \cdots P^{k_k}(ks), \tag{18}$$

utilizing values of the prime zeta function as discussed above and summing over the partitions $\pi(k)$ of k with weight coefficients

$$(k; k_1 k_2 \ldots k_k)^* = k! / \prod_{m=1}^{k} (m^{k_m} k_m!) \tag{19}$$

of Table 24.2 in [1] *(multinomials M_2) and A036039 or A102189.*

Proof. (18) is the main result of the paper. For small k, explicit verification can be done along Price's [29] construction, where the k-almost primes fill triangular, ($k = 2$), tetrahedral ($k = 3$) etc. sections of a k-dimensional Euclidean lattice labeled by the prime numbers along its Cartesian axes [5]. The case $P_1(s) = P(s)$ just repeats the definition (10). The case (15) accumulates in $P(2s)$ the sum over the squares, and in $P^2(s)$—with the binomial expansion— again the sum over the squares and twice the sum over products of distinct primes. After division through 2!, each semiprime is effectively represented once.

The generic proof follows through induction: the terms of the right hand side of (18) contain the factor $(k; k_1k_2, \dots k_k)^*$, which is the number of distinct permutations with k_m cycles of length m for $m = 1, 2 \dots k$ [6, (p. 123)][1, (24.1.2)]. The right hand side is the cycle index

$$Z(S_k) \equiv \frac{1}{k!} \sum_{\substack{k_1+2k_2+3k_3+\cdots+kk_k=k \\ k_k \ge 0}} (k; k_1k_2 \dots k_k)^* P^{k_1}(s) P^{k_2}(2s) \cdots P^{k_k}(ks) \tag{20}$$

of the symmetric group S_k [20, (2.2.5)] with $P(ms)$ substituted for the indeterminates of cycle length m. Skipping any interpretation within a Redfield-Pólya symmetry, its recurrence

$$Z(S_k) = \frac{1}{k} \sum_{j=1}^{k} P(js) Z(S_{k-j}) \tag{21}$$

is already established [20, (2.2.9)]. This matches precisely the recurrence on the left hand side which generates P_k by a combination of products of lower-indexed almost-primes,

$$P_k(s) = \frac{1}{k} \sum_{j=1}^{k} P(js) P_{k-j}(s). \tag{22}$$

This recurrence is valid because each k-almost prime which appears on the left hand side of this equation can be generated in k ways by a product of the form $P(js)P_{k-j}(s)$: in $\omega(k)$ ways by splitting off a prime number and multiplication with a number of the sum in $P_{k-1}(s)$, for each divisor of the k-almost prime which is a square of some prime in addition by multiplication of the square with a term in the sum in $P_{k-2}(s)$, and so on for divisors that are cubes of some prime etc. □

The simple pole of $\zeta(s)$ at $s-1$ with Stieltjes constants γ_j [12, 3, 23],

$$\zeta(s) = \frac{1}{s-1} + \gamma + \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} \gamma_j (s-1)^j, \tag{23}$$

is associated with a logarithmic singularity of $P(s)$ at $s = 1$ and singularities of $P(s)$ on the real line between $s = 0$ and $s = 1$ where s is the inverse of a square-free integer [16]. If $k > 1$, the k-almost zeta functions inherit these and add more by the mechanism evident from the multipliers in Theorem 1. $P_2(s)$ in (15), for example, inherits singularities at 1/2, 1/3, 1/5, 1/6, 1/7, 1/10 etc. from the term $P^2(s)$, and singularities at 1/2, 1/4, 1/6, 1/10, 1/12 etc. from the term $P(2s)$, illustrated in Fig. 1.

TABLE 4. Almost-prime zeta functions $P_k(s)$ at small integer arguments s.

k	s	$P_k(s)$
2	2	1.407604343490233882227509254138772537749192760048802639241489(-1) . . .
2	3	2.380603347277195967869595585283620062893217848034845684562765(-2) . . .
2	4	4.994674468637339635276874049579289322502057848230867728509096(-3) . . .
2	5	1.136012424856354766515556190735772665693748056026108556151424(-3) . . .
2	6	2.687071675614096324217387396140875535798787447125719642936101(-4) . . .
2	7	6.493314175691145578854061507836714519989975167152833237216508(-5) . . .
2	8	1.588851988525958888572372095351879234527858971327233300748108(-5) . . .
3	2	3.851619298269464091283792262806039543890016747838157193719155(-2) . . .
3	3	3.049362082334312946748098847079302999848694548619577993637287(-3) . . .
3	4	3.144274968329417421821246641907192073071706953574340102524412(-4) . . .
3	5	3.557725337068269111888017622799305930206716602282958084700573(-5) . . .
3	6	4.201275533960671214387834295923202794959720951879928447823823(-6) . . .
3	7	5.073887994515979227127878654920650441797124899213497891489656(-7) . . .
3	8	6.206813624161469945551964458392656691354524774013736254471908(-8) . . .
4	2	1.000943620148325082041084351808525466652473851036634849174401(-2) . . .
4	3	3.839045346157269074628008425162843300890790106333110559279434(-4) . . .
4	4	1.967963362818191467940855961573410955099879950428233958199339(-5) . . .
4	5	1.112105498394147042065416843932409614810339288829649464410277(-6) . . .
4	6	6.564866966272364593992630942917565336419279606893244510336952(-8) . . .
4	7	3.964020093813893558567748245375642870705531500802782383854435(-9) . . .
4	8	2.424542067198129719213221460827573885198415530509018971808441(-10) . . .
5	2	2.545076168069302058221776985605516223099431333404435645812102(-3) . . .
5	3	4.808940110832567973019045453666287670709263774628437825310148(-5) . . .
5	4	1.230321747728495443208363890849979176133316153001252000782537(-6) . . .
5	5	3.475459860092756789327837058184938607371038782365590655202548(-8) . . .
5	6	1.025765593034930528602801778254441805529589443170682127946500(-9) . . .
5	7	3.096892760390829520074774635913281487435638340106655991757766(-11) . . .
5	8	9.470868287557099531707292414885674011014480791459591532172493(-13) . . .
6	2	6.410338528642807128627067320846767912178898525413482485882042(-4) . . .
6	3	6.014928780179108948186295382866155223857573977633433216567001(-6) . . .
6	4	7.689936414615761724089452218863183766130186637100554867671046(-8) . . .
6	5	1.086086563383684175516301346294503063069865950783005109331817(-9) . . .
6	6	1.602759442759730816486790711011122437362717548888087849039669(-11) . . .
6	7	2.419447563344257658856721989501609732074582033968956770368419(-13) . . .
6	8	3.699557937592796079194223671580493130944594965925089150589532(-15) . . .

Remark 6. *Sums over odd indices are* [31, 2]*:*

$$\sum_{k=1}^{\infty} P_{2k-1}(2) = \frac{\pi^2}{20}; \qquad \sum_{k=1}^{\infty} P_{2k-1}(2s) = \frac{\zeta^2(2s)-\zeta(4s)}{2\zeta(2s)}. \tag{24}$$

Sums over even indices follow from there as the complement in (14).

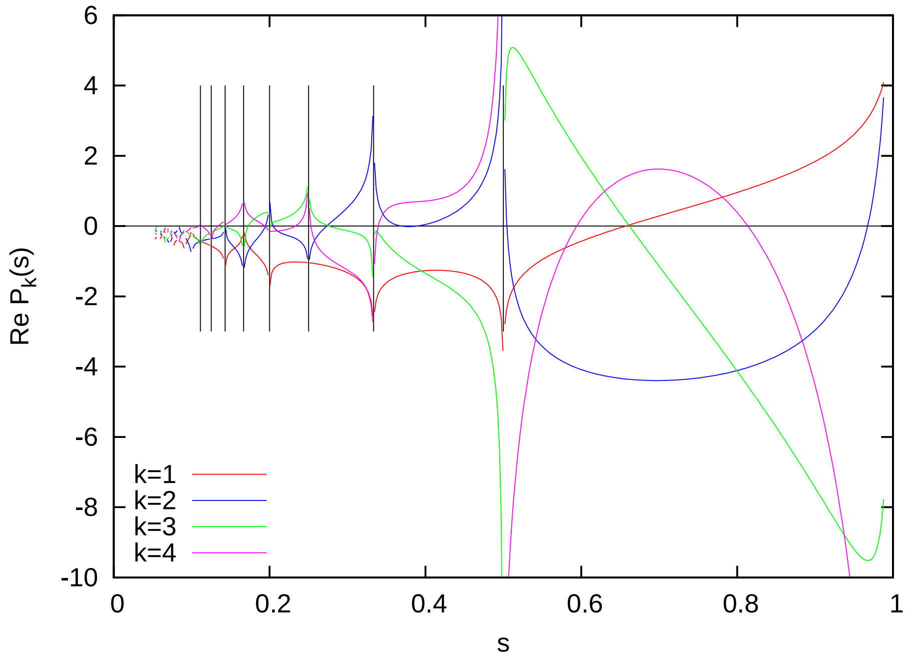

FIGURE 1. Structure of poles and logarithmic singularities: $\Re P_k(s)$ on the real line between 0 and 1.

Corollary 1. *The product rule* [1, (3.3.3)] *applied to (18) yields the first derivative*

$$P_k'(s) = \frac{1}{k!} \sum_{\substack{k_1+2k_2+3k_3+\cdots+kk_k=k \\ k_k \geq 0}} (k; k_1k_2\ldots k_k)^* P^{k_1}(s)P^{k_2}(2s)\cdots P^{k_k}(ks) \times \left[\frac{k_1P'(s)}{P(s)} + \frac{2k_2P'(2s)}{P(2s)} + \cdots + \frac{kk_kP'(ks)}{P(ks)}\right]. \tag{25}$$

Numerical evaluation yields Table 5. The underivative $\int_x^\infty P_k(s)ds = \sum_i 1/[p_i^x \log p_i]$ has been evaluated numerically for $k = 1$ by Cohen [13, 25].

3.3. **Möbius (Square-free) Variant.** Reduction of the summation to k-almost primes with k distinct prime factors defines a signed variant of the prime zeta functions:

Definition 3.

$$P_k^{(\mu)}(s) \equiv \sum_{\substack{n=1 \\ \Omega(n)=k}}^{\infty} \frac{\mu(n)}{n^s} = (-1)^k \sum_{\substack{n=1 \\ \Omega(n)=\omega(n)=k}}^{\infty} \frac{1}{n^s}; \quad P_1^{(\mu)}(s) = -P(s), \tag{26}$$

where $\omega(.)$ is the number of distinct prime factors of its argument.

Remark 7. *The criterion $\Omega(n) = \omega(n) = k$ selects the prime number products (square-free k-almost primes) of A000040 ($k = 1$), A006881 ($k = 2$), A007304 ($k = 3$), A046386 ($k = 4$), A046387 ($k = 5$), A067885 ($k = 6$), A123321 ($k = 7$), A123322 ($k = 8$), A115343 ($k = 9$), etc.*

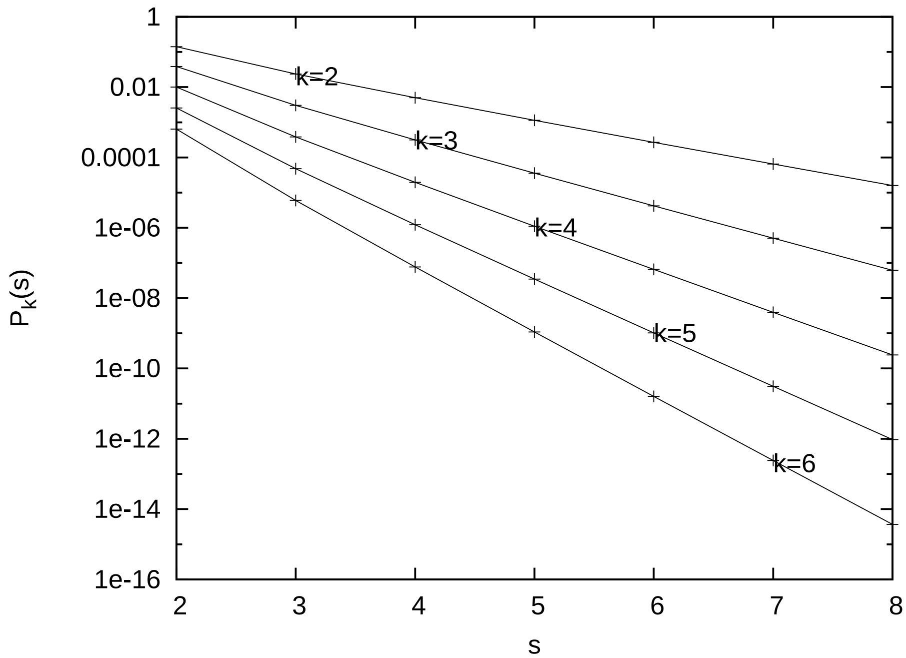

FIGURE 2. A synopsis of table 4 on a semi-logarithmic scale, indicating that the $P_k(s)$ fall off approximately exponentially $P_k(s) \sim 2^{-ks}$ as $s \to \infty$ along the real s-axis.

The sum

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \sum_{k=1}^{\infty} P_k^{(\mu)}(s) \tag{27}$$

converges for $s > \frac{1}{2}$ if the Riemann hypothesis holds [14]. Application of the multinomial expansion [1, (24.1.2)] to the powers $P^k(s)$ leads to the recurrences

Theorem 2.

$$P_2^{(\mu)}(s) = \frac{P^2(s) - P(2s)}{2!}; \tag{28}$$

$$-P_3^{(\mu)}(s) = \frac{P^3(s) - 3P(s)P(2s) + 2P(3s)}{3!}; \tag{29}$$

$$P_4^{(\mu)}(s) = \frac{P^4(s) - 6P^2(s)P(2s) + 3P^2(2s) + 8P(s)P(3s) - 6P(4s)}{4!}; \tag{30}$$

$$P_k^{(\mu)}(s) = \frac{1}{k!} \sum_{\substack{k_1+2k_2+3k_3+\cdots+kk_k=k \\ k_k \geq 0}} (-1)^m (k; k_1 k_2 \ldots k_k)^* \times P^{k_1}(s) P^{k_2}(2s) \cdots P^{k_k}(ks), \tag{31}$$

where $m \equiv k_1 + k_2 + k_3 + \cdots + k_k$.

TABLE 5. First derivatives $P'_k(s)$. $P'_1(s)$ is Table 3.

k	s	$P'_k(s)$
2	2	-2.8360681540798065222425822254827833607935057823781401341111 18(-1) ...
2	3	-3.8805869023993223366924601826587311896741264978521354099521 56(-2) ...
2	4	-7.5458966940853152069079706671963501930216394163596779085014 48(-3) ...
2	5	-1.6552931052406176407610132208680975146293969658668182800471 20(-3) ...
2	6	-3.8397694246350456257553718001194561392116258399832252378936 84(-4) ...
2	7	-9.1747980624691439523464626025702318842641288104202825947601 47(-5) ...
2	8	-2.2297035721814937323522403130216729449868586755076930362229 76(-5) ...
3	2	-1.0927644526886967182339570444603722318742776024289014891094 38(-1) ...
3	3	-7.1768131653384381438718965718681375688595376202629921788618 87(-3) ...
3	4	-6.9571839970163486779987546159176116739083746113625242788174 41(-4) ...
3	5	-7.6592770126954093063747431108080791018988690026393975133120 88(-5) ...
3	6	-8.9189219029602713702850968594500985047258262864582312485260 33(-6) ...
3	7	-1.0687346106887186358836734941326336690942480211039076243534 75(-6) ...
3	8	-1.3012956840596451752212290184488506673670136950785104849043 30(-7) ...
4	2	-3.6037260943517988485066266561811112411308366647969559629328 55(-2) ...
4	3	-1.1741163095729879469778160106188722046408164418223356286444 63(-3) ...
4	4	-5.7229988589129580060173046009023694012892508748977896764203 92(-5) ...
4	5	-3.1655663697960624496652503472309149624354749937765938198758 06(-6) ...
4	6	-1.8487630226185520006114707971908610805356606854134589449442 26(-7) ...
4	7	-1.1097306194195834323077938559497532514943078512458519616900 94(-8) ...
4	8	-6.7637711570592290998115786756788731941641578169651061064895 74(-10) ...
5	2	-1.1021620985050701831041319200537349216582999165216792109774 74(-2) ...
5	3	-1.8061349293879631179892169717077232970411869865884833166215 59(-4) ...
5	4	-4.4313644276805938999208969023370959468110756757017012221370 28(-6) ...
5	5	-1.2302032637587919426962928843124592272277404342776962115949 02(-7) ...
5	6	-3.5997235327081917841845388375229822831208082465108365321617 42(-9) ...
5	7	-1.0816382882901610115099617683225676521588992976051374954335 86(-10) ...
5	8	-3.2985690761637682632747201215210484931167657438350812925303 52(-12) ...
6	2	-3.2327203121505231183040982439695413035428417413560629149531 86(-3) ...
6	3	-2.6769153864443168037448713352769404239823084026861339691441 41(-5) ...
6	4	-3.3028846825906067816988510833674776855023287298323080448213 81(-7) ...
6	5	-4.5972349538834578195270506749575928107611293105024514133272 54(-9) ...
6	6	-6.7355203810850295867808924006008762810997432009159519905140 23(-11) ...
6	7	-1.0127332942097214153313920736275417855604770014344108735085 89(-12) ...
6	8	-1.5449373682949182756961590958029439430665543916623665633885 40(-14) ...

Redistributing the sign with

$$(-1)^m P^{k_1}(s)P^{k_2}(2s)\cdots P^{k_k}(ks) = P_1^{(\mu)k_1}(s)P_1^{(\mu)k_2}(2s)\cdots P_1^{(\mu)k_k}(ks), \tag{32}$$

shows that a recurrence equivalent to (21) is applicable.

Remark 8. *Sums over odd indices are* [31]*:*

$$\sum_{k=1}^{\infty} P_{2k-1}^{(\mu)}(2) = -\frac{9}{2\pi^2}; \quad \sum_{k=1}^{\infty} P_{2k-1}^{(\mu)}(2s) = -\frac{\zeta^2(2s)-\zeta(4s)}{2\zeta(2s)\zeta(4s)}. \tag{33}$$

TABLE 6. Almost-prime zeta functions $P_k^{(\mu)}(s)$ at small integer arguments s computed from (31).

k	s	$P_k^{(\mu)}(s)$
2	2	6.37672945847765432801316294807193836128782162900370736592109(-2) ...
2	3	6.73594662213544672456228258677680141934623660580421211246428(-3) ...
2	4	9.33269102119509074753434906896208799524913336159126727476407(-4) ...
2	5	1.42408850419374548659705490588033249391875510740905235597858(-4) ...
2	6	2.26806975268639528950901474490066255999464767652487721194642(-5) ...
2	7	3.68874503144697741103258077849260322707848294774819695331204(-6) ...
2	8	6.06493665920244704921794541628232570617319835668401899600473(-7) ...
3	2	-3.69624416345283537839553463239466811559153971303042724974 72(-3) ...
3	3	-6.61486512463499395217298296391111156410218947274041060698 29(-5) ...
3	4	-1.72714580937223046302125882710417325726718418086639787694 00(-6) ...
3	5	-5.09401945988263568520051081132377780726644778684041068669 61(-8) ...
3	6	-1.58232291545492933890762392501476827895728535727844147235 97(-9) ...
3	7	-5.04536031146706475819392405326742480658125619492149926143 00(-11) ...
3	8	-1.63294312369382159544034164837385014572919532540256032655 54(-12) ...
4	2	1.05117508492309807485233009466098526324680558243958672947068(-4) ...
4	3	2.14173193213549705893739943930728906255490278218470044772058(-7) ...
4	4	7.29603168874401925790854604647164607340676506339572130564981(-10) ...
4	5	2.96196721173369084753821609237130107261806114215416577935264(-12) ...
4	6	1.29842711892568424473824206373082938247800573758362639862211(-14) ...
4	7	5.91005736941452577777835874048660952776699696196276661514317(-17) ...
4	8	2.74348813375914336305677937087519697186480294957450635783864(-19) ...
5	2	-1.66208220357968128224711924270382469647592506641225447100 20(-6) ...
5	3	-2.64084788254604775905672848360430516346611479985736382547 64(-10) ...
5	4	-7.62967455131527978373199543309176167710263206807889251545 80(-14) ...
5	5	-2.66748559048903027238289839616673306219358953992541696805 14(-17) ...
5	6	-1.01247718783094551198438942851637270484007795115058296019 25(-20) ...
5	7	-4.00891715824643245800165125859644329098831444088074023304 30(-24) ...
5	8	-1.62678722021987767128600247348669691960143352436631214254 23(-27) ...
6	2	1.61508116616705485066326829316589367443121196123113678467541(-8) ...
6	3	1.53530343588928080283456861104632555466175700586234793665716(-13) ...
6	4	2.99324100743804909283239785893475963049642868205678987129596(-18) ...
6	5	7.41916124138504344746395187399890095182210117294534971585873(-23) ...
6	6	2.05602063767548079577453001329402669576701618478791822725187(-27) ...
6	7	6.06313176438106233958227084211400755691448415132455986317069(-32) ...
6	8	1.85800289034470997723167773802340218486625624481826862206504(-36) ...

4. Applications

4.1. Geometric Series. By integrating [18, (1.511)]

$$\log(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} \tag{34}$$

TABLE 7. Series of the form (39).

s	$\sum_{n=2}^{\infty} 1/[n^s(n-1)]$
1	1.
2	3.550659331517735635275848333539748107810500987932015622(-1) ...
3	1.530090299921792781278466718425248200160638064527026804(-1) ...
4	7.068579628104108661184297530135691724131285453397577278(-2) ...
5	3.375804113767116028047748884432274918423193503206296081(-2) ...
6	1.641497915322202056595955905340222128241444499920939897(-2) ...
7	8.065701771299193726162009203605461682550884433970692555(-3) ...
8	3.988345573354854347476770694952996423590093784120672226(-3) ...
9	1.979952747272639929624001462540935937984242389231915677(-3) ...
10	9.853776194545545924780425622219189319647108247543984198(-4) ...

we obtain the generating function [18, (1.513.5)],

$$x + (1-x)\log(1-x) = \sum_{n=2}^{\infty} \frac{x^n}{n(n-1)}. \tag{35}$$

In the limit $x \to 1$ we find [18, (0.141)]

$$\sum_{n=2}^{\infty} \frac{1}{n(n-1)} = 1. \tag{36}$$

Larger powers of the first factor of the denominator display partial sums of zeta functions through partial fraction decomposition,

$$\sum_{n=2}^{\infty} \frac{1}{n^2(n-1)} = \sum_{n=2}^{\infty} \left(\frac{1}{n(n-1)} - \frac{1}{n^2} \right) = 2 - \frac{\pi^2}{6}; \tag{37}$$

$$\sum_{n=2}^{\infty} \frac{1}{n^3(n-1)} = \sum_{n=2}^{\infty} \left(\frac{1}{n(n-1)} - \frac{1}{n^2} - \frac{1}{n^3} \right) = 3 - \frac{\pi^2}{6} - \zeta(3); \tag{38}$$

$$\sum_{n=2}^{\infty} \frac{1}{n^s(n-1)} = s - \sum_{l=2}^{s} \zeta(l); \quad s \geq 1. \tag{39}$$

Examples of these numbers are collected in Table 7, A152416 and A152419. Their sum is

$$\sum_{s=1}^{\infty} \sum_{n=2}^{\infty} \frac{1}{n^s} \frac{1}{n-1} = \sum_{n=2}^{\infty} \frac{1}{(n-1)^2} = \zeta(2) = \frac{\pi^2}{6}. \tag{40}$$

Definition 4. *Restriction of the summation in (39) to k-almost primes defines a set of constants $B_{k,s}$* [19],

$$B_{k,s} \equiv \sum_{\substack{n=2 \\ \Omega(n)=k}}^{\infty} \frac{1}{n^s(n-1)} = \sum_{\substack{n=2 \\ \Omega(n)=k}}^{\infty} \sum_{l=0}^{\infty} \frac{1}{n^{s+1+l}} = \sum_{l=0}^{\infty} P_k(s+1+l). \tag{41}$$

TABLE 8. Some values of $B_{k,1}$. In accordance with (36), the series limit of the partial sums is 1 as $k \to \infty$.

k	$B_{k,1} = \sum_{n,\Omega(n)=k} 1/[n(n-1)]$
1	.773156669049795127864367459855942395618741336083186048311006 06 ...
2	.171051892979996636622202564372374213991246612035500597491079 97 ...
3	.41920339281764199227805032233471158322784525420828606710238790(-1) ...
4	.10414202346301156141109353888171559234184072973208943335673068(-1) ...
5	.25944317032356863609340108179412019910406474149863463566912649(-2) ...
6	.64712678336846601104554817217635310331423959328614964903156640(-3) ...
7	.16154547889045106884023528793253084539703632404976961733195128(-3) ...
8	.40350403394466614988860237144035458196679194641891917284153345(-4) ...
9	.10082343610557897391498490786448232831594447756388459395195738(-4) ...
10	.25198413274347214707213045392269003455525827323722877742607328(-5) ...
11	.62985737261498933173999701960384580565617680181013417876197437(-6) ...
12	.15745036385232517679881727385782020683287536986374137174089714(-6) ...
13	.39360719611599520681959076312200454332371020210585755644656168(-7) ...
14	.98399321710455906992308452734477378441132531283484928659273734(-8) ...
15	.24599505388932024146078978227171922132133360668716211893213680(-8) ...
16	.61498340075560680170866561648700452467948343944318880698907151(-9) ...
17	.15374530193448859466794967919224844842656125026606949920935630(-9) ...
18	.38436254839132442407411486355109175383764384816988280391113599(-10) ...
19	.96090546444389183823411185748926311892868554398657649851752505(-11) ...
20	.24022625018522170257126899356289067549236119426157474416944241(-11) ...
21	.60056547765687746047761256134140717295823654635402409654484759(-12) ...
22	.15014135061632779005230691663603510295794137329809014913244327(-12) ...
23	.37535335268553839790798447165432785944777132136305316859771955(-13) ...
24	.93838335149705557279342461255618055328547335958778301173823243 9(-14) ...
25	.23459583405297725332944711004418618278736884444821387508950778(-14) ...

This reduction to a geometric series and sum over the P_k has been used to calculate Table 8. This definition introduces an analog to (14),

$$\sum_{n=2}^{\infty} \frac{1}{n^s(n-1)} = \sum_{k=1}^{\infty} B_{k,s}. \tag{42}$$

Remark 9. *$B_{1,1}$ is A136141, calculated by Cohen* [13]. *$B_{1,2}$ is A152441. $B_{2,1}$ is A152447.*

Projection of (39) onto the n of a fixed $\Omega(n)$ yields

$$B_{k,s} = B_{k,1} - \sum_{l=2}^{s} P_k(l). \tag{43}$$

The benefit of this formula is that the $B_{k,s}$ can be derived from $B_{k,1}$ without accumulating the individual terms of the geometric series proposed in (41), reaching back to Tables 1 and 4 instead. Consider for example $B_{3,2} \approx 0.003404 = B_{3,1} - P_3(2) \approx 0.041920 - 0.038516$ in Table 9.

The square-free variant of (41) is

TABLE 9. Some values of $B_{k,s}$. The series limits of the partial sums $\sum_k B_{k,s}$ are in Table 7.

k	s	$B_{k,s} = \sum_{\Omega(n)=k} 1/[n^s(n-1)]$
1	2	.32090924900872962935782409502369446144550999284329362657458713 ...
2	2	.30291458630973248399451638958496960216327336030620333566930993(-1) ...
3	2	.34041462990695583149671096054107628838843579424470347730472318(-2) ...
4	2	.40476614481790532069851037008630456765933446284259484392905784(-3) ...
1	3	.14614660970928609293471078035798776047009787091714433668591512 ...
2	3	.64854251582012887207556831056607595873951575502718767213033361(-2) ...
3	3	.35478421673524536821901075833145988403566339382745677940994463(-3) ...
4	3	.20861610202178413235709527570020237570255452209283788001114446(-4) ...
5	3	.12661340580586229820433777990228912341234438356263326260613819(-5) ...
1	4	.69153469945039247992091484424829890308056811202301146420977111(-1) ...
2	4	.14907506895639490854788090560814702648930997020410089927942399(-2) ...
3	4	.40356719902303626036886094140740676728492698470022769157503418(-4) ...
4	4	.11819765739964985563009679542861280192566527050014484191210481(-5) ...
5	4	.35812310330127538835013908172912057990127682625080625278844028(-7) ...
1	5	.33398452461114990859273241885974179176359534475649814730884436(-1) ...
2	5	.35473826470759431896325286534569759919935164601490043664281503(-3) ...
3	5	.47794665316209349180059179127476174264255324471931883104976848(-5) ...
4	5	.69871075602351514235551110353718404446313416171798954710770233(-7) ...
5	5	.10577117291999709417355375910626719164172948014247187268185467(-8) ...
1	6	.16328365610478477905139568619914779966773592601105569997721067(-1) ...
2	6	.86031097146184686541514125731610045619472901302328472349204898(-4) ...
3	6	.57819099766026370361808361682441463146581149531325986267386085(-6) ...
4	6	.42224059396278682956248009245427510821206201028665096074007060(-8) ...
5	6	.31946136165040413132735812808230110887705358254036598872046674(-10)

Definition 5.

$$B_{k,s}^{(\mu)} \equiv \sum_{\substack{n=2 \\ \Omega(n)=k}}^{\infty} \frac{\mu(n)}{n^s(n-1)}. \tag{44}$$

As in (41), there is a representation generated by expansion as a geometric series, and another one from the decomposition in partial fractions:

$$B_{k,s}^{(\mu)} = \sum_{l=0}^{\infty} P_k^{(\mu)}(s+1+l) = B_{k,1}^{(\mu)} - \sum_{l=2}^{s} P_k^{(\mu)}(l). \tag{45}$$

Explicit values follow in Tables 10 and 11. The special values of

$$B_{1,s}^{(\mu)} = -B_{1,s} \tag{46}$$

can be read off Table 8 and 9. Because the squared primes are those 2-almost primes which are not square-free, constants like

$$\sum_p \frac{1}{p^{2s}(p^2-1)} = \sum_{l=0}^{\infty} P(2(1+s+l)) = B_{2,s} - B_{2,s}^{(\mu)} \tag{47}$$

can be extracted subtracting values of Tables 9 and 11.

TABLE 10. Some values of (44) at $s = 1$.

k	$(-1)^k B_{k,1}^{(\mu)} = \sum_{\Omega(n)=\omega(n)=k} 1/[n(n-1)]$
1	.773156669049795127864367459855942395618741336083186048311006 06 ...
2	.716060153640629506890149052332785700329775774967647669968815 66(-1) ...
3	.376417253516777399878971446429349348845131716782847330953617 49(-2) ...
4	.105332414263703090731895610576158062026770175361747979960701 37(-4) ...
5	.166234636469136638486313598129999707580430301933255004804554 97(-5) ...
6	.161509651950074526358195106026380997615184370466023086157452 91(-7) ...
7	.103796579458318232104051271843596744184040447048937802976410 14(-9) ...
8	.466081643503390326657927250003528560595309106638172113567157 81(-12) ...
9	.152579165087340741791819169952715626128768490819828120660313 66(-14) ...
10	.376746054054628169548650369360873289300356592826866797688326 86(-17) ...
11	.721463071043588130589650676373972071425893488956132468634572 89(-20) ...
12	.109661849340687892121284402661739510919473636482173026661951 70(-22) ...
13	.134912725901803031878068618120721810499779413625692668361826 58(-25) ...
14	.136592039134269215659939917855424579971007241467426598337500 46(-28) ...
15	.115445502993058190203620748483373213473616483106202874133491 37(-31) ...
16	.824681636639460831899067124823570290723947436932810761282069 27(-35) ...
17	.503331877009651073572042716710769139323868225160307419290801 00(-38) ...
18	.264985289565306233448476680012725887780011222546187288533316 10(-41) ...
19	.121355796877961607378935913608577566023107263821400603104260 06(-44) ...
20	.487134309402437273947947602572902434006651497256039536931811 01(-48) ...
21	.172555876256607214572438936660767411999052591813094754235005 37(-51) ...
22	.542701372575587107188589098948329204845813337042087458782973 10(-55) ...
23	.152388815625808612794223561917541202032964550216720769017804 09(-58) ...
24	.383975514498094078962920974164964748946908646039053220113283 73(-62) ...
25	.872214201347970856697771646333650583589802573423791199116417 91(-66) ...

4.2. **Hurwitz Zeta Decompositions.**

Definition 6. *The Hurwitz Zeta Function is*

$$\zeta(s,a) \equiv \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} = \sum_{n=1}^{\infty} \frac{1}{(n+a-1)^s}; \quad \Re s > 1, \quad \Re a > 1. \tag{48}$$

On the trot, we project this sum onto the subspaces of k-almost primes, too,

Definition 7. *(Hurwitz Prime and Almost-Prime Zeta Functions)*

$$P_k(s,a) \equiv \sum_{\substack{n=2 \\ \Omega(n)=k}}^{\infty} \frac{1}{(a-1+n)^s}; \quad P_k(s,1) = P_k(s). \tag{49}$$

$$P_k^{(\mu)}(s,a) \equiv \sum_{\substack{n=2 \\ \Omega(n)=k}}^{\infty} \frac{\mu(n)}{(a-1+n)^s}; \quad P_k^{(\mu)}(s,1) = P_k^{(\mu)}(s). \tag{50}$$

This partitions (48) into

$$\zeta(s,a) = \frac{1}{a^s} + \sum_{k=1}^{\infty} P_k(s,a), \tag{51}$$

TABLE 11. Some values of $(-1)^k B_{k,s}^{(\mu)}$.

k	s	$(-1)^k B_{k,s}^{(\mu)}$
2	2	.78387207792864074088832757525591864200993612067276933376705990(-2) ...
3	2	.67928371714938620394179831898825372859777454798046059788912729(-4) ...
4	2	.21490577139328324666260111005953570208961711778930701363329944(-6) ...
5	2	.26416111168510261601673859617237932837795292029557704345022153(-9) ...
2	3	.11027741571509606843209931657823850007531246009234812252063101(-2) ...
3	3	.17797204685886808724500022597142572187555600706419537190829226(-5) ...
4	3	.73257817973354076886116612880679583412683957083696886124052352(-9) ...
5	3	.76323430497840111065747811948812031341440309840660902575023325(-13) ...
2	4	.16950505503145160956755825888617620122821126476435449772990259(-3) ...
3	4	.52574659216450409428743432610083961488375889775555842142889903(-7) ...
4	4	.29750108591388430703115241596312267861630644973967306755420261(-11) ...
5	4	.26684984687313228427857617894414570413989159871977420443245736(-16) ...
2	5	.27096204612077060907852768298142951836335754023449262132044308(-4) ...
3	5	.16344646176240525767383244968461834157114119071517352759281108(-8) ...
4	5	.13043647405152222773308067259925713545003355242564896189378205(-13) ...
5	5	.10128782422925704028633932747239792053264472723250762731720244(-19) ...
2	6	.44155070852131080127626208491363262363892772582004900125801062(-5) ...
3	6	.52141702169123237830700571831415136754126549873293803568336271(-10) ...
4	6	.59376215895380325925646622617419720223297866728632203157065385(-16) ...
5	6	.40105446162489087900384620760650048636932117449331297950429083(-23)

generalizing (14).

Remark 10. *By series expansion* [18, (1.112.1)]*, the sum rule associated with (41) is*

$$B_{k,s} = \sum_{\substack{n=2 \\ \Omega(n)=k}}^{\infty} \frac{1}{(n-1)^{1+s}} \frac{1}{1+\frac{1}{n-1}} = \sum_{l=0}^{\infty} (-1)^l P_k(1+s+l,0). \tag{52}$$

The reduction of (49) to the Prime Zeta Functions is obtained by the binomial expansion [18, (1.110)][7, Entry 22]

$$\begin{aligned} P_k(s,a) &= \sum_{\substack{n=2 \\ \Omega(n)=k}}^{\infty} \frac{1}{n^s} \frac{1}{\left(1-\frac{1-a}{n}\right)^s} = \sum_{\substack{n=2 \\ \Omega(n)=k}}^{\infty} \frac{1}{n^s} \sum_{l=0}^{\infty} \binom{-s}{l} (-1)^l \frac{(1-a)^l}{n^l} \\ &= \sum_{\substack{n=2 \\ \Omega(n)=k}}^{\infty} \frac{1}{n^s} \sum_{l=0}^{\infty} \frac{(s)_l}{l!} \frac{(1-a)^l}{n^l} = \sum_{l=0}^{\infty} \frac{(s)_l}{l!} (1-a)^l P_k(s+l). \end{aligned} \tag{53}$$

$$P_k^{(\mu)}(s,a) = \sum_{l=0}^{\infty} \frac{(s)_l}{l!} (1-a)^l P_k^{(\mu)}(s+l). \tag{54}$$

Pochhammer's symbol $(s)_l \equiv \Gamma(s+l)/\Gamma(s)$ is introduced to simplify the notation [1, (6.1.22)]. Brute-force accumulation of partial sums over l for the case $a = 0$

TABLE 12. Some values of $P_k(s,0)$. Cohen has reported $P_1(2,0)$ [13], which is A086242 [32].

k	s	$P_k(s,0)=\sum_{\Omega(n)=k} 1/(n-1)^s$
1	2	1.3750649947486352879172531305224396991795999601753174587091893 3 ...
2	2	.20978832394001949275536860246918923626861393292185134375281708 9 ...
3	2	.45725064947335617950946289678986796223966354508430430410489882 3(-1) ...
4	2	.10841086446746639413051146621522276428741495477518340581949743 7(-1) ...
5	2	.26450902754366179219734676915598667345768015100943909620609610 5(-2) ...
1	3	1.1475290977585800469332838062821304016447647355251122552758241 2 ...
2	3	.49761051132666598187595086637795240584134077475563781478339517 7(-1) ...
3	3	.42572897291299622510794720051946380461452546738693718720604196 5(-2) ...
4	3	.45033756160166149697050440138324577397603201882714754852315632 7(-3) ...
5	3	.51999636872926720244468211527582842969608754953180254440665364 6(-4) ...
1	4	1.0673601122715716981152740206525870389352585930455083681150824 1 ...
2	4	.14425186705012534720632177138532577465573291687174661133754006 9(-1) ...
3	4	.51159457333490112726831665038324470399751764156523619081324775 0(-3) ...
4	4	.24873230818407033306862041151012146887825764161146623665937205 2(-4) ...
5	4	.13802520386784340630985093314482234251766500445474088703351780 6(-5) ...
1	5	1.0323710059783419658517759206386829450348249693177698543469547 5 ...
2	5	.44881455331786089589045252142214145722419706205185394395231674 2(-2) ...
3	5	.67052571070664457009991790600217340512952258544340178467593752 7(-4) ...
4	5	.15094711833618483834163159714104371580744915252387501412694499 6(-5) ...
5	5	.40383271832494953449528931532282284332617790229837706544418382 8(-7) ...
1	6	1.0158920113997241100667591845732551010347357805373116863693222 6 ...
2	6	.14418186083919275820265154865452180515441808992545472607102913 8(-2) ...
3	6	.91352310781785040874898510416351034274165014034086162005145392 5(-5) ...
4	6	.95494204559184230821709623548243838952599548418059540129811585 8(-7) ...
5	6	.12332239556263507277396638246189276451856567070923595170858299 4(-8) ...

yields Table 12. Resummation in (49) imposes the sum rule

$$\sum_{k=1}^{\infty} P_k(s,0) = \zeta(s). \tag{55}$$

The derivative of (53) with respect to s is

$$\frac{\partial}{\partial s} P_k(s,a) = P_k'(s,a) = -\sum_{\substack{n=2\\ \Omega(n)=k}}^{\infty} \frac{\log(a-1+n)}{(a-1+n)^s}$$

$$= \sum_{l=0}^{\infty} \frac{(s)_l (1-a)^l}{l!} \left[(\psi(s+l)-\psi(s))P_k(s+l) + P_k'(s+l)\right] \tag{56}$$

in terms of digamma functions ψ [1, (6.3)]. Cases with $a=0$ are illustrated by Table 13.

4.3. **Logarithmic Functions.** Repeated integration of (35) with respect to x yields a ladder of functions with increasing order $l+1$ of the polynomial of n

TABLE 13. Some absolute values of (56) at $a = 0$.

k	s	$\|P'_k(s,0)\| = \sum_{\Omega(n)=k} \log(n-1)/(n-1)^s$
1	2	.41203862694845359298953672788691959310869395599327228478933425 3 . . .
2	2	.34940640284372909402185884055232841175526894585356744464594757 3 . . .
3	2	.12205387755707122925286013523447082114429866127443150239453986 0 . . .
4	2	.38132460451284098460250578361697402655973886883490127323691338 5(-1) . . .
5	2	.11345330145908190653685129380426757054609624923945631821397715 8(-1) . . .
1	3	.12249199446961189441811066412654630614897170036457460907622772 8 . . .
2	3	.64714774953307807218033332603133476909382625963039415451352120 4(-1) . . .
3	3	.93511821299604486785344482950310396068212991090002889386963953 1(-2) . . .
4	3	.13426815050192521128656175367017700548935879805444552108012769 4(-2) . . .
5	3	.19332293801196743055978270801199939526414775611810214336768401 7(-3) . . .
1	4	.50570301128204611358044936611385290799593809942637052489223910 6(-1) . . .
2	4	.17209705293432802090645408159268782334669982079390541672879346 2(-1) . . .
3	4	.10554373177063536665646770238819388838159340514225000849017322 0(-2) . . .
4	4	.70525517835888271324849807097282608249363778189007229018833430 2(-4) . . .
5	4	.49219484581585817304650538193651853442855735982793387274484292 6(-5) . . .
1	5	.23283372897235960974795474665348649346606439342430017965306654 1(-1) . . .
2	5	.51514186381684730424656578489083986565030929648997088453136699 9(-2) . . .
3	5	.13465149486296280558875245457689598957287974687823737236393017 4(-3) . . .
4	5	.41908173771400522285153354527276876819832437437581569907193790 4(-5) . . .
5	5	.14155000129099320209090371919019984428334776014905645151244061 2(-6) . . .
1	6	.11210707166945248739184883478401558535256000255235862240625892 8(-1) . . .
2	6	.16230902936673749404626391039894034790170203317277259644622063 6(-2) . . .
3	6	.18100924875792548862270211392192188522362792500530173087764897 0(-4) . . .
4	6	.26238532241109239153326304910986750913096892113717906690679485 9(-6) . . .
5	6	.42861788940397744818922016664239601121131580145601615489255183 5(-8) . . .

in the denominator [18, (1.513.7)][24],

$$x + (1-x)^2 \ln(1-x) = \frac{3x^2}{2} - 2\sum_{n=1}^{\infty} \frac{x^{n+2}}{n(n+1)(n+2)}; \tag{57}$$

$$x + (1-x)^3 \ln(1-x) = \frac{5x^2}{2} - \frac{11x^3}{6} + 6\sum_{n=1}^{\infty} \frac{x^{n+3}}{n(n+1)(n+2)(n+3)}; \tag{58}$$

$$x + (1-x)^4 \ln(1-x) = \frac{7x^2}{2} - \frac{13x^3}{3} + \frac{25x^4}{12} - 24\sum_{n=1}^{\infty} \frac{x^{n+4}}{n(n+1)(n+2)(n+3)(n+4)}; \tag{59}$$

$$x + (1-x)^l \ln(1-x) = \sum_{i=2}^{l} \tau_{i,l} x^i - (-1)^l l! \sum_{n=1}^{\infty} \frac{x^{n+l}}{n(n+1)\cdots(n+l)}. \tag{60}$$

The rational coefficients of the auxiliary polynomials obey the recurrence

$$\tau_{2,l} = l - \frac{1}{2}, \quad l \geq 2; \quad i\tau_{i,l} = (-1)^i \binom{l-1}{i-1} - l\tau_{i-1,l-1}, \quad i \geq 3. \tag{61}$$

$i!\tau_{i,l}$ are tabulated in A067176 and A093905.

Table 14. Logarithmic sums of k-almost primes defined in (62).

k	l	$L_{k,l}$
1	1	0.2643400417600216467317482459030646149252805027004715099264 4 ...
1	2	0.61210909600861648810973500903074917744536817685379132220140 ...
1	3	0.83162031622701364585608159976581294069621021299936900413624 ...
1	4	0.97814589475979949499502571065792051537852172306539544173835 ...
2	1	0.07483176753552750888163142884027057172300662738030777281173 ...
2	2	0.20274600679905285623473576314229971108548607276186973818502 ...
2	3	0.30956716856446670438488608714185233323846931207915051554876 ...
2	4	0.39967077199200952417631057083019796152634867816759135315832 ...
3	1	0.01979445800870394214182491696866714839521363828266446266352 ...
3	2	0.05673037199050316150740109374211659920215472144741775457277 ...
3	3	0.09078044492621127414186177360393579101033447804508695808347 ...
3	4	0.12224056980762694782137972349781919346596885467095976821370 ...

Substitution of inverse k-almost primes for x in (60) followed by summation defines constants $L_{k,l}$ which generalize (4),

$$L_{k,l} \equiv \sum_{\Omega(n)=k} \left[\frac{1}{n} + \left(1-\frac{1}{n}\right)^l \ln\left(1-\frac{1}{n}\right)\right]$$

$$= \sum_{s=2}^{l} \tau_{s,l} P_k(s) - (-1)^l l! \sum_{s=1}^{\infty} \frac{1}{s(s+1)\cdots(s+l)} P_k(s+l), \tag{62}$$

placed in Table 14. At $l=1$, partial fraction decomposition of $1/[s(s+1)]$ in (62) implies that $L_{1,1}$ plus the value at $u=1$ in Table 2 equals the value in (5). Series of the form $\sum_{s\geq 1} P_k(s+l)/\prod_{i=0}^{l}(s+i)$ are quickly accessible by subtracting the τ-dependent terms of (62) with the aid of Table 4. Varieties with non-contiguous factors in the denominator can be reduced to the $L_{k,l}$ basis by partial fraction synthesis: examples of this approach with one missing factor $s+2$ or two missing factors s and $s+2$ on the left hand side are

$$\frac{1}{s(s+1)(s+3)} = \frac{1}{s(s+1)(s+2)} - \frac{1}{s(s+1)(s+2)(s+3)} \tag{63}$$

and

$$\frac{1}{(s+1)(s+3)} = \frac{1}{s(s+1)} - \frac{3}{s(s+1)(s+2)} + \frac{3}{s(s+1)(s+2)(s+3)}. \tag{64}$$

Remark 11. *The case of $x=1/2$ in (34) is decomposable in k-almost primes as well* [17, 9]

$$\log 2 = \sum_{n=1}^{\infty} \frac{1}{n2^n} = \frac{1}{2} + \sum_{k=1}^{\infty} \sum_{\substack{n=1 \\ \Omega(n)=k}}^{\infty} \frac{1}{n2^n}. \tag{65}$$

The dominating components are

$$\sum_{\substack{n=1\\ \Omega(n)=k}}^{\infty} \frac{1}{n2^n} \tag{66}$$

$= 0.17408707176097936247199331662155444265874950008103306 8401\ldots (k=1)$
$= 0.18550266279949706589265485288204777430168931869275127 0328(-1)\ldots (k=2)$
$= 0.50888635346997830953864506326222795951839979087649956 7910(-3)\ldots (k=3)$
$= 0.95615827023041653295586429957085893341780180452052281 3509(-6)\ldots (k=4)$

These sums generated from $|x| < 1$ *converge rapidly on their own and do not need the acceleration strategies outlined in the first sections.*

Remark 12. *The limits* $x \to 1$ *in (57)–(60) generalize (36)* [34, (p. 42)]

$$\frac{1}{k \cdot k!} = \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)\cdots(n+k)}. \tag{67}$$

Restricted summation over k*-almost primes in the spirit of (41) would split these into another basic type of constants.*

5. Summary

Almost-prime zeta functions have been defined by restriction of the summation of the standard definition of zeta functions to k-almost primes. Their values can be bootstrapped from a multinomial overlay of the values of the ordinary prime zeta functions. Efficient schemes to compute the latter have been employed to calculate series summed over k-almost primes of some basic inverse polynomials to high accuracy.

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34. Daniel Zwillinger (ed.), *CRC standard mathematical tables and formulae*, 31 ed., Chapman & Hall/CRC, Boca Raton, FL, 2003, E: the lower limit in eqs. (11)–(14) on page 42 ought be $k = 1$, not $k = 0$.

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