

Series representations for π^3 involving the golden ratio

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Abstract

Although many series exist for π and π^2 , very few are known for π^3 . In this article, we derive, using a trigonometric identity obtained by Euler, two representations of π^3 involving infinite sums and the golden ratio. The methodology can be generalized in order to obtain further series, relating by the way π^3 to other mathematical constants.

1 Introduction

Many series exist for π and π^2 [1–3]. However, there are only very few published expressions for π^3 . Besides the understanding of such a scarcity from the mathematical point of view and the derivation of summations in order to point out connections between mathematical constants, π^3 is encountered in physics for instance in the expression of the equivalent of the effective area (used for electromagnetic antennas) for gravitational-wave antennas, which is a measure of the antenna's ability to gather energy from the incident wave [4, 5]. In Ref. [6], Sun mentions that the only well-known series for π^3 is the following one

$$\pi^3 = 32 \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k+1)^3} \quad (1)$$

and in 2010 [7], the same author suggested that

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(2k+1)^3 16^k} = \frac{7\pi^3}{216}, \quad (2)$$

which was proven later [8]. In the latter paper, Pilehrood and Pilehrood obtained the following Apéry-like series:

$$\pi^3 = 32 \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{16^k (2k+1)^3} - 24 \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{16^k (2k+1)} \sum_{m=0}^{2k-1} \frac{1}{(2m+1)^2}. \quad (3)$$

In the above mentioned article [6], Sun derived the following expression

$$\sum_{k=1}^{\infty} \frac{2^k H_{k-1}^{(2)}}{k \binom{2k}{k}} = \frac{\pi^3}{48}, \quad (4)$$

where $H_n^{(2)}$ denotes, for $n \in \mathbb{N}^*$, the Harmonic number

$$H_n^{(2)} = \sum_{k=1}^n \frac{1}{k^2}. \quad (5)$$

Gupta published new series representations of π , π^3 and π^5 in terms of Euler numbers and π^2 , π^4 and π^6 in terms of Bernoulli numbers [9]. He found the following relation

$$\pi^3 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3 (2^{2k+2} - 1)} 2^{2k+4} (2k+3)! \sum_{j=0}^k \left[-\frac{1}{(2n-1)^2 \pi^2} \right]^j \frac{1}{(2k-2j+1)!} \quad (6)$$

where j, k are integers and ≥ 0 . Following his success in discovering a new formula for π , Simon Plouffe [10–12] postulated several identities which relate either π^m or $\zeta(m)$ to three infinite series. Letting

$$S_n(r) = \sum_{k=1}^{\infty} \frac{1}{k^n (e^{\pi r k} - 1)}, \quad (7)$$

the first two examples are

$$\pi = 72 S_1(1) - 96 S_1(2) + 24 S_1(4) \quad (8)$$

$$\pi^3 = 720 S_3(1) - 900 S_3(2) + 180 S_3(4). \quad (9)$$

In the present work, we show that using a trigonometric series obtained by Euler and involving the functions $x \mapsto \cot x$ as well as $x \mapsto \operatorname{cosec} x = 1/\sin x$, it is possible to derive series expansions for π^3 . We present two of them which are of particular interest since they involve the golden ratio.

2 New series for π^3 involving the golden ratio

In the book by Borwein and Borwein [1], the following formula, due to Euler

$$\pi^3 [\cot(\pi x) \operatorname{cosec}^2(\pi x)] = \sum_{n=-\infty}^{\infty} \frac{1}{(x-n)^3} \quad (10)$$

is presented (13.b, p. 382). It can be used to derive series expansions for π^3 . In particular, for $x = 1/5$ or $x = 1/10$ for instance, it is possible to obtain expressions of π^3 as series multiplied by a coefficient involving the golden ratio. Indeed, let us consider first the case $x = 1/5$. One has

$$\cos\left(\frac{\pi}{5}\right) = \frac{\phi}{2} \quad (11)$$

and

$$\sin\left(\frac{\pi}{5}\right) = \sqrt{\frac{5 - \sqrt{5}}{8}}, \quad (12)$$

where

$$\phi = \frac{1 + \sqrt{5}}{2} \quad (13)$$

is the golden ratio. One thus has

$$\cot\left(\frac{\pi}{5}\right) = \frac{\sqrt{2}\phi}{\sqrt{5 - \sqrt{5}}} = \frac{\phi}{\sqrt{3 - \phi}} \quad (14)$$

and

$$\operatorname{cosec}\left(\frac{\pi}{5}\right) = \frac{2\sqrt{2}}{\sqrt{5 - \sqrt{5}}} = \frac{2}{\sqrt{3 - \phi}}. \quad (15)$$

Therefore, using Eq. (10) for $x = 1/5$ yields

$$\boxed{\pi^3 = \frac{125}{4} \frac{(3 - \phi)^{3/2}}{\phi} \sum_{n=-\infty}^{\infty} \frac{1}{(1 - 5n)^3}} \quad (16)$$

In the same way, setting $x = 1/10$, a similar expansion can be deduced. One has indeed

$$\cos\left(\frac{\pi}{10}\right) = \frac{\sqrt{10 + 2\sqrt{5}}}{4} = \frac{\sqrt{2 + \phi}}{2} \quad (17)$$

and

$$\sin\left(\frac{\pi}{10}\right) = \frac{1}{2\phi}, \quad (18)$$

yielding

$$\cot\left(\frac{\pi}{10}\right) = \phi\sqrt{2 + \phi} \quad (19)$$

as well as

$$\operatorname{cosec}\left(\frac{\pi}{10}\right) = 2\phi \quad (20)$$

giving finally the expansion

$$\boxed{\pi^3 = \frac{250}{\phi^3 \sqrt{2 + \phi}} \sum_{n=-\infty}^{\infty} \frac{1}{(1 - 10n)^3}} \quad (21)$$

Additional expressions can of course be obtained¹ for other values of x , but only a few of them will only involve the golden ratio. As an example, using $x = \pi/15$, one has

$$\cos\left(\frac{\pi}{15}\right) = \frac{1}{8} \left(\sqrt{30 + 6\sqrt{5}} + \sqrt{5} - 1 \right) \quad (22)$$

and

$$\sin\left(\frac{\pi}{15}\right) = \frac{1}{16} \left(2\sqrt{3} - 2\sqrt{15} + \sqrt{40 + 8\sqrt{5}} \right) \quad (23)$$

which unfortunately involves $\sqrt{3}$...

¹For instance, using $x = 1/4$, since $\cot(\pi/4) = 1$ and $\operatorname{cosec}(\pi/4) = \sqrt{2}$, one gets $\pi^3 = 32 \sum_{n=-\infty}^{\infty} \frac{1}{(1-4n)^3}$.

3 Conclusion

We proposed two representations of π^3 involving infinite series and the golden ratio. Although the number of expressions of this type is probably rather limited, the technique can be easily applied to derive other series, relating π^3 to other mathematical constants. It is worth mentioning that the third power of π finds applications in several fields of physics, such as gravitational-wave antennas, in the expression of the effective area.

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