A set of formulas for primes

by
Simon Plouffe
December 31, 2018

Abstract

In 1947, W. H. Mills published a paper describing a formula that gives primes : if A = 1.30637788386308069046861449226... then $\lfloor A^{3^n} \rfloor$ is always prime, here $\lfloor x \rfloor$ is the integral part of x. Later in 1951, E.M.Wright published also a formula for primes, if $g_0 = \alpha$ and $g_{n+1} = 2^{g_n}$ then

$$\lfloor g_n \rfloor = \left\lfloor 2^{\cdots 2^{2^{\alpha}}} \right\rfloor \text{ is always prime.}$$

When $\alpha = 1.9287800$, the primes are uniquely determined, 3, 13, 16381, ...
The growth rate of these functions is very high since the fourth term of Wright formula is a 4932 digit prime and the 8'th prime of Mills formula is a 762 digit prime.

A new set of formulas are presented here, minimizing the growth rate. The first one is : if $S_0 = 43.8046877158 \ldots$ and $S_{n+1} = \{ S_n^{5/4} \}$, where $\{ x \}$ is the rounded value of x then
$S(n) = 113, 367, 102217, 1827697, 67201679, 6084503671, \ldots$ Other exponents can also give primes like $11/10$, $21/20$ or $51/50$. If S_0 is well chosen then it is conjectured that any exponent > 1 can also give an arbitrary series of primes. The method for obtaining the formulas is explained. All results are empirical.

Résumé

Une série de formules donnant un nombre arbitraire de nombres premiers est présentée. Les formules sont inspirées des résultats de W. H. Mills (1947) et E.M. Wright (1951). La première de ces formules est : si $S_0 = 43.8046877158 \ldots$ et $S_{n+1} = \{ S_n^{5/4} \}$, $\{ x \}$ étant l'arrondi de x, alors $S(n) = 113, 367, 102217, 1827697, 67201679, 6084503671, \ldots$ est une série de nombres premiers dont la croissance est nettement plus petite que les 2 formules historiques. Les premiers sont uniquement déterminés. On peut même trouver des exposants aussi petits que $11/10$, $21/20$ ou même $51/50$ en autant que S_0 soit bien choisi. Il est conjecturé que l'exposant peut être aussi près de 1 que l'on veut. Tous les résultats présentés ici sont empiriques.

Introduction

An experiment was done with a variety of formulas using a Monte-Carlo method and the Simulated Annealing method (principe du recuit simulé in french). The first one tested was $\lfloor c * n^n \rfloor$, where c is 0.26558837294314339089... and n≥ 3.

The best one found was the following series of primes for n = 3 to 22.

$\lfloor c * n^n \rfloor$ = 7, 67, 829, 12391, 218723, 4455833, 102894377, 2655883729, 75775462379, 2368012611049, 80440106764817...

But fails when n = 23.

Formulas like $\lfloor c * n^n \rfloor$, $\left\lfloor c * \frac{2^{n+2} n!}{\pi^{n+1}} \right\rfloor$ where tested but failed after less than 10 primes. The main reason is that they grow too slowly. The next step was to consider formulas like Mills or Wright but they grow too fast. The question was then : is there a way to get a useful formula that grows just enough to produce primes ?

If we consider the recurrence $a_{n+1} = a_n{}^2 - a_n + 1$ that arises in the context of Sylvester sequence. The Sylvester sequence is A000058 of the OEIS catalogue and begins like this : (2, 3, 7, 43, 1807, 3263443, 10650056950807,), that sequence has the property that

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \frac{1}{1807} + \frac{1}{3263443} + \cdots$$

The natural extension that comes next is: can we choose $a(0)$ so that $a(n)$ will always produce primes ? The answer is yes, when $a(0) = 1.6181418093242092 \ldots$ and by using the $\lfloor x \rfloor$ function we get,

$$a(n) = 2, 3, 7, 43, 1811, 3277913, 10744710357637, \ldots$$

The sequence and formula are interesting for one reason the growth rate is quite smaller than the one of Mills and Wright.

A Formula for primes

What if we choose the exponent to be as small as possible? The problem with that last one is that it is stilll growing too fast, $a(14) = 9.838\ldots \times 10^{1667}$. The size of primes doubles in length at each step.

The simplest found was $S_0 = 43.80468771580293481\ldots$ and using the $\{\,x\,\}$, rounding to the nearest integer, we get

$$S_{n+1} = \{\,S_n^{\frac{5}{4}}\,\}$$

Now, what if we carefully choose S_0 so that the exponent is smaller, would it work ? Let's try with 11/10 and start with a large number.

$$S_{n+1} = \{\,S_n^{\frac{11}{10}}\,\}$$

For example when $S_0 = 10000000000000000000000000000049.31221074776345\ldots$ and the exponent beeing 11/10 then we get the primes :

10000000000000000000000000000049
158489319246111348520210137339236753
5248074602497725973643121570227258944 01
3908408957924020300919472370957356345933709
70990461585528724931289825118059422005340095813
3438111840350699188044461057631015443312900908952333
4897246900042000942655570714250230366715503 64178496540501
…

If we want a smaller starting value then S_0 has to be bigger, I could get a series of primes when $S_0 = 10^{64} + 57 + \varepsilon$, where $0 < \varepsilon < 0.5$ chosen at random. In this case the exponent is

$$S_{n+1} = \{\,S_n^{\frac{21}{20}}\,\}$$

If we choose $S_0 = 10^{500} + 961 + \varepsilon$ then we get our formula to be.

$$S_{n+1} = \{\,S_n^{\frac{51}{50}}\,\}$$

Description of the algorithm and method

There are 3 steps

1) First we choose a starting value and exponent (preferably a rational fraction for technical reasons).
2) Use Monte-Carlo method with the Simulated Annealing, in plain english we keep only the values that show primes and ignore the rest. Once we have a series of 4-5 primes we are ready for the next step.
3) We use a formula for forward calculation and backward. The forward calculation is

Forward : Next smallest prime to $\{\ S(n)^e\ \}$.

It is easy to find a probable prime up to thousands of digits. Maple has a limit of about 10000 digits on a Intel core i7 6700K, if I use PFGW I can get a probable prime of 1000000 digits in a matter of minutes.

Backward : (to check if the formula works)

Previous prime $=$ solve for x in $x^e - S(n+1)$. Where $S(n+1)$ is the next prime candidate. This is where e needs to be in rational form in order to solve easily in floating point to high precision using Newton-like methods.

Conclusion

There are no proofs of all this, just empirical results. In practical terms, we have now a way to generate an arbitrary series of primes with (so far) a minimal growth function. The formulas are much smaller in growth rate than of the 2 historical results of Mills and Wright. Perhaps there is even a simpler formulation, I did not find anything simpler. In the appendix, the 50'th term of the sequence beginning with $10^{500} + 961$ is given, breaking the record of known series of primes in either a polynomial (46 values) or primes in arithmetic progression (26 values).

Appendix

Value of S(0) for $S_{n+1} = \left\{ S_n^{\frac{5}{4}} \right\}$.

S(0) = (2600 digits)

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43. 80468771580293481859664562569089495081037087137495184074061328752670419506091\
366496333710781644573956794355807574665393001463642473571079065207223321969922\9
3264945523871320676837539601640703027906901627234335816037601910027530786667742\
4699462324971177019386074328699571966485110591669754207733416768610281388237076\
1457089916472910767210053536764714815683021515348888782413463002196590898912153\
7807818768276618328571083354683602327177557419282082356357338008605137724420625\
6355959550708768409691378766548276776271408551872012425050013678129199364572781\
6813484896708864364878618010738012262326340551265300995594932556249327764325520\
8194746104359540189525699560278047888104384271808675933509955483655110570975535\
0785549077202528823776864085546060978256419523714505379764311017583547117340087\
4119475081308042239895607717353189481000696610499835674474037156885696125148581\
8456576179531740469553899959017477684394053845672138911212798487820145291551046\
7629440955925010855475725385626417446232721714485692213455756668913904742940022\
8509137974545537496866153147014899471237034869279337802008797927308327868037503\
8887734734110132599896243006668693634843271039811956815726425758834972052461079\
7249503005611993311614482718587460518670384095172358532168924929832717916742877\
5928083798545071506803190331910539166466084176777828927515630775969847441245544\9
4235840260513685727576427182775664320370267582830712839903568563176995646262746\
1416682265277556094918638843788662398254929224611054007569004788919208610982286\
8199056832247214421586610021379123061307988471051671497283110156867428960680180\
5503966626145521465670913389257049550857812576171202023957487357339788493712676\
9426225000014887911004760565168355053802557466312278070529726060791106644597456\
7661803493305130350891168486525370221416972540649352435163491961811066911684870\
8294575729521396968670925609170964827383150968820007346616462008754667125912162\
9887393234505470764134731474624437303908697918037642878970154570903124414003971\
0357272367808690366664091433772132766429665349666355316481741675272445299039148\
2286893771949904585620265483705408090265640475448008548569022696419278537015273\
9052515660046613248284007441714998112109106552831729128554638499405603790836674\
4055846672832545083146225507711246665670893874389752184393456176782493932252715\1
9446656406377385343422986304586480251980132774829846894886302934727512320855666\
7311057826570702055978747565264786935758838660918835790956716726895968781893980\
6197577089835533428250070076463861672738236689824429341266143008249630480529249\
5142244721354743123104963974936146293364440867656174513881761811216195861355409\
7226590982832455160532348892270853032526217953271203753936241844699444780852653\
926712756105661
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50'th prime in the formula $S_{n+1} = \left\{ S_n^{\frac{51}{50}} \right\}$ and $S_0 = 10^{500} + 961$.

S(50)= (1320 digits).

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2546223984434301126230560405750836844181156799213460578433735627810123608333414\
6478717920178906592095759064531694091270765292863071164826802108312142369000399\
0918112200598834059063704371274865057278618525926963160906859256204291253853779\
0859263106429982680171352298624867598838188610026225438753363257569617236093303\
5597738787705978929949787991104492121070145202492140062372248031624879195606603\
9750665982863462494190199174051345073411763460889751017250124325112902369085214\
0184828179289329015864573760367060446309168404382462687293461046639169729920475\
3165541783062017852987962447647230150345731151542656737173297014806244172277491\
6879804546188079063455344339451528745750861580732838099712175302071382677624428\
4262968338987678773648741141485490879969521079400774328097643567743161861694439\
2384400589489332204990330011362168526537013264388448543980911860963264776201897\
0883314652642873229279315244591969596989887077667619942605707704348251758251233\
1172015772215487384144161658368225114390354140649309995571635019281763656568221\
8604642413102656623287079740572453580298728150605437498702948770557573781722652\
8086931990592223405655647742963234918217685490055779977425394611159718143306147\
7320512686294534342712886388641680379470072208880905199640988562813955595060316\
873702694557447026182065422852814015579611967627130232889
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