Plouffe's Constant

We start with a formula which is surprising at first glance:

$$\sum_{n=0}^{\infty} \frac{\rho(a_n)}{2^{n+1}} = \frac{1}{2\cdot\pi}$$

where

$$a_n = \sin(2^n) = \begin{cases} \sin(1) & \text{if } n = 0 \\ 2\cdot a_0\cdot\sqrt{1-(a_0)^2} & \text{if } n = 1 \\ 2\cdot a_{n-1}\cdot\left[1 - 2\cdot(a_{n-2})^2\right] & \text{if } n \ge 2 \end{cases}$$

and

$$\rho(x) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x \ge 0 \end{cases}$$

In words, the binary expansion of $1/(2\cdot\pi)$ is completely determined by the sign pattern of the second-order recurrence $\{a_n\}$. The (trivial) proof uses the double angle formulas for sine and cosine:

$$\sin(4\cdot x) = 2\cdot\sin(2\cdot x)\cdot\cos(2\cdot x) = 2\cdot\sin(2\cdot x)\cdot\left(1 - 2\cdot\sin(x)^2\right)$$

and the fact that

$$\begin{aligned} \rho(a_n) = 0 \quad & \text{iff} \quad 2^n \in (2\cdot k\cdot\pi, (2\cdot k+1)\cdot\pi) \quad \text{for some integer } k \\ & \text{iff} \quad 2^{n+1}\cdot\frac{1}{2\cdot\pi} \in (2\cdot k, 2\cdot k+1) \\ & \text{iff the nth bit of } 1/(2\cdot\pi) \text{ is zero.} \end{aligned}$$

One might believe that we've uncovered here a fast way of computing the binary expansion of $1/(2\cdot\pi)$, but this would be a mistake. The reason is that we would need sin(1) to high accuracy for initialization, but computing sin(1) is no easier than computing $1/\pi$.

The double angle formula for cosine gives rise to a simpler, first-order recurrence

$$b_n = \cos\left(2^n\right) = \begin{cases} \cos(1) & \text{if } n = 0 \\ 2\cdot\left(b_{n-1}\right)^2 - 1 & \text{if } n \ge 1 \end{cases}$$

but the sum

$$\sum_{n=0}^{\infty} \frac{\rho\left(b_n\right)}{2^{n+1}} = 0.4756260767\ldots$$

doesn't appear to have a closed-form expression. The double angle formula for tangent, however, gives rise to both a first-order recursion

$$c_n = \tan\left(2^n\right) = \begin{cases} \tan(1) & \text{if } n = 0 \\ \dfrac{2\cdot c_{n-1}}{1 - \left(c_{n-1}\right)^2} & \text{if } n \ge 1 \end{cases}$$

and a closed-form expression for the sum

$$\sum_{n=0}^{\infty} \frac{\rho\left(c_n\right)}{2^{n+1}} = \frac{1}{\pi}$$

by a trivial proof like before. Again, computing tan(1) is no easier than computing $1/\pi$.

We've observed so far that, for sine and tangent, certain irrational inputs yield recognizable irrational outputs. S. Plouffe([1]) wondered if this process could be adjusted somewhat. He asked if it was possible to initialize any of these three recurrences with *rational* values, such as 1/2, and still obtain recognizable irrational binary expansions. Define

$$\alpha_n = \sin\left(2^n \cdot \arcsin\left(\frac{1}{2}\right)\right) = \begin{cases} \frac{1}{2} & \text{if } n=0 \\ \frac{\sqrt{3}}{2} & \text{if } n=1 \\ 2\cdot\alpha_{n-1}\cdot\left[1 - 2\cdot\left(\alpha_{n-2}\right)^2\right] & \text{if } n \ge 2 \end{cases}$$

$$\beta_n = \cos\left(2^n \cdot \arccos\left(\frac{1}{2}\right)\right) = \begin{cases} \frac{1}{2} & \text{if } n=0 \\ 2\cdot\left(\beta_{n-1}\right)^2 - 1 & \text{if } n \ge 1 \end{cases}$$

$$\gamma_n = \tan\left(2^n \cdot \arctan\left(\frac{1}{2}\right)\right) = \begin{cases} \frac{1}{2} & \text{if } n=0 \\ \frac{2\cdot\gamma_{n-1}}{1 - \left(\gamma_{n-1}\right)^2} & \text{if } n \ge 1 \end{cases}$$

then the first two sums turn out to be rational

$$\sum_{n=0}^{\infty} \frac{\rho\left(\alpha_n\right)}{2^{n+1}} = \frac{1}{12} \qquad \sum_{n=0}^{\infty} \frac{\rho\left(\beta_n\right)}{2^{n+1}} = \frac{1}{2}$$

but the third sum

$$C = \sum_{n=0}^{\infty} \frac{\rho\left(\gamma_n\right)}{2^{n+1}} = 0.1475836177\ldots$$

is more mysterious. Plouffe numerically determined that

$$C = \frac{1}{\pi} \cdot \arctan\left(\frac{1}{2}\right)$$

and it is reasonable to conjecture that C is irrational. A large number of decimal digits appear at the Inverse Symbolic Calculator web pages.

J. M. Borwein and R. Girgensohn([2]) succeeded in proving Plouffe's formula for C and much more. They demonstrated that, given an arbitrary real value x, if

$$\xi_n = \tan\left(2^n \cdot \arctan(x)\right) = \begin{cases} x & \text{if } n = 0 \\ \text{if } n \ge 1 & \begin{cases} \dfrac{2 \cdot \xi_{n-1}}{1 - \left(\xi_{n-1}\right)^2} & \text{if } \left|\xi_{n-1}\right| \ne 1 \\ -\infty & \text{if } \left|\xi_{n-1}\right| = 1 \end{cases} \end{cases}$$

then

$$\sum_{n=0}^{\infty} \frac{\rho\left(\xi_n\right)}{2^{n+1}} = \begin{cases} \dfrac{\arctan(x)}{\pi} & \text{if } x \ge 0 \\ 1 + \dfrac{\arctan(x)}{\pi} & \text{if } x < 0 \end{cases}$$

which we call **Plouffe's recursion**.

This, however, was only one facet of their paper ([2]). It turns out to be crucial that the above sum, call it f(x), satisfies the functional equation

$$\begin{cases} 2 \cdot f(x) = f\left(\dfrac{2 \cdot x}{1 - x^2}\right) & \text{if } x \ge 0 \\ 2 \cdot f(x) - 1 = f\left(\dfrac{2 \cdot x}{1 - x^2}\right) & \text{if } x < 0 \end{cases}$$

We won't attempt to summarize [2] except to remark that Plouffe's recursion appears to be the simplest example in the theory. Here are two results, corresponding to cosine and sine, due to Borwein and Girgensohn:

- Given arbitrary $-1 \le x \le 1$, if

$$\eta_n = \begin{cases} x & \text{if } n = 0 \\ \text{if } n \ge 1 & \begin{cases} 1 - 2\cdot\left(\eta_{n-1}\right)^2 & \text{if } -1 \le \eta_{n-1} \le 0 \\ 2\cdot\left(\eta_{n-1}\right)^2 - 1 & \text{if } 0 < \eta_{n-1} \le 1 \end{cases} \end{cases}$$

  then

$$\sum_{n=0}^{\infty} \frac{\sigma\left(\eta_n\right)}{2^{n+1}} = \frac{\arccos(x)}{\pi}$$

  where

$$\sigma(x) = \begin{cases} 1 & \text{if } -1 \le x \le 0 \\ 0 & \text{if } 0 < x \le 1 \end{cases}$$

- Given arbitrary $0 \le x \le 1$, if

$$\zeta_n = \begin{cases} x & \text{if } n = 0 \\ \text{if } n \ge 1 & \begin{cases} 2\cdot\zeta_{n-1}\cdot\sqrt{1 - \left(\zeta_{n-1}\right)^2} & \text{if } 0 \le \zeta_{n-1} < \frac{1}{\sqrt{2}} \\ 2\cdot\left(\zeta_{n-1}\right)^2 - 1 & \text{if } \frac{1}{\sqrt{2}} \le \zeta_{n-1} \le 1 \end{cases} \end{cases}$$

  then

$$\sum_{n=0}^{\infty} \frac{\tau(\zeta_n)}{2^{n+1}} = \frac{2\cdot\arcsin(x)}{\pi}$$

where

$$\tau(x) = \begin{cases} 1 & \text{if } 0 \le x < \frac{1}{\sqrt{2}} \\ 0 & \text{if } \frac{1}{\sqrt{2}} \le x \le 1 \end{cases}$$

Other examples, associated with logarithmic, hyperbolic and elliptic integrals of the first kind, are presented in [2]. But suitably generalized binary expansions, given arbitrary x and extending those for the recursions $\{a_n\}$, $\{\alpha_n\}$ and $\{b_n\}$, $\{\beta_n\}$, remain undiscovered.

The Mathcad PLUS 6.0 file brwngrgs.mcd verifies the results given above. (Click here if you have 6.0 and don't know how to view web-based Mathcad files).

References

1. S. Plouffe, Home page (CECM, Simon Fraser University).
2. J. M. Borwein and R. Girgensohn, Addition theorems and binary expansions, *Canadian J. Math.* 47 (1995) 262-273.
3. N. J. A. Sloane, On-Line Encyclopedia of Integer Sequences, AT&T Research, look up sequences A004715, A004716 and A004717.

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